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On Mill's Ratio

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Abstract. The inequalities related to Mill's Ratio were conjectured true by Birnbaum (2) and were proved by Murty (4) for sufficiently large $x > 0$. In the present paper, these are proved for all finite x and have been used to calculate an upper limit for Mill's Ratio over the range $x > -1$.

1. *Introduction.* Gordon (3) and Birnbaum (1) have given, respectively, upper and lower limits for Mill's Ratio R_x as

$$\tfrac{1}{2}(\sqrt{(4+x^2)}-x) < R_x < 1/x \quad (x > 0), \tag{1·1}$$

where

$$R_x = e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}u^2}du. \tag{1·2}$$

Using successive convergents of Laplace's expression for the normal integral as a continued fraction, Murty (4) has derived limits to $R_x (x > 0)$ of any required degree of accuracy.

If the functions $f(x)$ and $\alpha(x)$ are defined by

$$f(x) = 1/R_x \quad \text{and} \quad \alpha(x) = f'(x) = f(x)\{f(x)-x\},$$

the inequalities

$$0 < \alpha(x) < 1, \tag{1·3}$$

$$\alpha'(x) = f(x)[\{f(x)-x\}\{2f(x)-x\}-1] > 0, \tag{1·4}$$

are of importance in the theory of truncated normal data. The result (1·4) was conjectured as true for positive x by Birnbaum (2) while obtaining the expression for the third moment of X (k-dimensional random variable), truncating the distribution in Y (1-dimensional random variable) to the set $Y \geqslant x > 0$, and was proved by Murty (4) for sufficiently large positive x. In this paper it was proved for all finite x and is used to calculate an upper limit for R_x over the range $x > -1$. The result (1·3) is also proved for all finite x. The upper limit is equivalent to the lower limit in (1·1), which is thus valid for $x < 0$ as well as for $x > 0$.

2. *Inequality on* $\alpha(x)$. *Proof.* The function

$$e^{-\frac{1}{2}u^2}\Big/\int_x^\infty e^{-\frac{1}{2}u^2}du$$

is a probability density function over the range $x \leqslant u < \infty$, and its variance is $1-f(x)\{f(x)-x\}$. The upper limit in (1·3) follows trivially as variance must be positive for finite x. Again, since $f(x) > 0$ by definition, $\{f(x)-x\} > 0$ for all $x \leqslant 0$, and by (1·1) for $x > 0$, and hence the lower limit follows.

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3. *Inequality on* $\alpha'(x)$. *Proof.* The result (1·4) is equivalent to

$$F = \{f(x)-x\}\{2f(x)-x\} > 1. \tag{3·1}$$

An expansion by parts for positive x gives

$$\int_x^\infty e^{-\frac{1}{2}u^2}du = (1-1/x^2+R)e^{-\frac{1}{2}x^2}/x,$$

where $R = O(1/x^4)$.

Therefore, F reduces to

$$F = \{1+1/x^2-O(1/x^2)\}/(1-1/x^2+R)^2 \to 1 \quad \text{as} \quad x\to\infty. \tag{3·2}$$

Also as $x\to-\infty, f(x)\to 0$, and $\qquad F \approx x^2\to\infty.$ (3·3)

Now suppose there exists a finite point ξ_1 such that

$$F(\xi_1) = 1. \tag{3·4}$$

Since F is a continuous and differentiable function, equations (3·2), (3·3) and (3·4) imply the existence of a finite point ξ_2 for which

$$\left.\begin{aligned} F(\xi_2) &\leqslant 1,\\ F'(\xi_2) &= 0.\end{aligned}\right\} \tag{3·5}$$

But
$$\begin{aligned} F' &= \{\alpha(x)-1\}\{2f(x)-x\}+\{f(x)-x\}\{2\alpha(x)-1\}\\ &= f(x)(F-1)+2\{f(x)-x\}\{\alpha(x)-1\},\end{aligned}$$

implies $F' < f(x)(F-1)$, for finite x, and hence the result (1·4) follows.

The quadratic $F = 1$ has solution

$$f(x) = \tfrac{1}{4}\{3x \pm \sqrt{(x^2+8)}\}, \tag{3·6}$$

as $f(x)$ is known to be greater than x, only the positive sign in (3·6) is considered. The result so obtained is everywhere greater than x and is positive for all $x > -1$. Then, the upper limit of R_x is $\quad R_x < 4/\{3x+\sqrt{(x^2+8)}\} \quad (x > -1).$

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