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A note on the queueing system $GI/G/\infty$

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1. *Introduction.* Consider a queueing system $GI/G/\infty$ in which (i) the inter-arrival times are distributed with distribution function $A(t)$ $(A(0+)=0)$ (ii) the service times have distribution function $B(t)$ such that the expected value of the service time is β $(<\infty)$.

It follows from Kiefer and Wolfowitz (1) that the stationary distribution of the rth largest unexpired service time just before an arrival always exists. We may, however, note that the analysis given by them is complicated.

In this note we shall give a simple approach to obtain this result. Furthermore we establish here that the stationary distribution of the queue length just before an arrival always exists.

2. *The main results.* Let $W_r(n)$ $(r \geqslant 1,\ n \geqslant 0)$ be the rth largest unexpired service time (we assume the unexpired service time to be $= 0$ if it is not > 0) just before the $(n+1)$th arrival. Furthermore, let t_n and z_n represent respectively the time separating the nth arrival and the $(n+1)$th arrival, and the service time of the nth customer. We have the following:

THEOREM 1. $\lim_{n\to\infty} \Pr(W_r(n) > x)$ $(r \geqslant 1)$ *exists independently of the distribution of* $W_k(0)$ $(k \geqslant 1)$.

Proof. Let $x_r(n)$ be the unexpired service time of the rth $(1 \leqslant r \leqslant n)$ customer just before the $(n+1)$th $(n \geqslant 1)$ arrival. Then defining by $x(k,r,n)$ $(k \geqslant 1, r \geqslant 0, n \geqslant 1)$ the kth largest value amongst $x_{r+1}(n+r), \ldots, x_{n+r}(n+r)$ if $k \leqslant n$ and 0 if $n < k$, we see that for $n \geqslant 1$

$$x(k,0,n+1) \geqslant x(k,1,n),$$

and hence

$$\Pr(x(k,0,n+1) \leqslant y) \leqslant \Pr(x(k,1,n) \leqslant y)$$
$$= \Pr(x(k,0,n) \leqslant y).$$

This shows that $\lim_{n\to\infty} \Pr(x(k,0,n) \leqslant y)$ $(k=1,2,\ldots)$ exists. Moreover, as there exists an integer m such that

$$E(t_1+\ldots+t_m) = mE(t_1) > C \quad (C<\infty)$$

and as

$$\Pr(t_1\ldots+t+_{mr+s} \leqslant C)$$
$$\leqslant \Pr(t_1+\ldots+t_{mr} \leqslant C)$$
$$\leqslant (\Pr(t_1+\ldots+t_m \leqslant C))^r \quad (r \geqslant 1, 0 \leqslant s < m),$$

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we have that $\lim_{n\to\infty} \Pr(t_1+\ldots+t_n \leqslant C) = 0$, and hence that

$$\lim_{n\to\infty} \Pr(t_1+\ldots+t_n \leqslant W_1(0)) = 0.$$

Therefore the fact that $W_k(n) = x(k,0,n)$ when $t_1+\ldots+t_n > W_1(0)$ shows that

$$\lim_{n\to\infty} \Pr(W_k(n) \leqslant y) = \lim_{n\to\infty} \Pr(x(k,0,n) \leqslant y).$$

This proves the theorem.

THEOREM 2. *Defining* $\lim_{n\to\infty} \Pr(W_r(n) > x)$ *by* $F_r(x)$, *we have*

$$\lim_{r\to\infty} F_r(0) = 0 \tag{1}$$

and

$$\lim_{x\to\infty} F_r(x) = 0. \tag{2}$$

Proof. We have readily that

$W_1(n) = \max(0, z_n - t_n, z_{n-1} - (t_n + t_{n-1}), \ldots, z_1 - (t_1+\ldots+t_n), W_1(0) - (t_1+\ldots+t_n))$.

Hence

$$\begin{aligned}\lim_{x\to\infty}\lim_{n\to\infty} \Pr(W_1(n) \leqslant x) &= \lim_{x\to\infty} \Pr(z_1 - t_1 \leqslant x, z_2 - (t_1+t_2) \leqslant x, \ldots)\\ &= \lim_{x\to\infty} \Pr(z_m - (t_1+\ldots+t_m) \leqslant x, z_{m+1} - (t_1+\ldots+t_{m+1}) \leqslant x, \ldots)\\ &= \lim_{x\to\infty} \Pr(\xi_r \leqslant x \quad (r=1,2,\ldots,m)),\end{aligned}$$

where m is such that $mE(t_i) > E(z_i)$ and $\xi_r = \sup_{s\geqslant 1}\left(z_{ms+r-1} - \sum_{i=1}^{ms+r-1} t_i\right)$. It follows readily that

$$\begin{aligned}\Pr(\xi_m \leqslant x) &\geqslant \Pr(\xi_{m-1} \leqslant x)\\ &\geqslant \Pr(\xi_1 \leqslant x)\\ &\geqslant \Pr\left(\sum_{i=1}^{s} z_i - \sum_{i=1}^{ms} t_i \leqslant x \quad (s=1,2,\ldots)\right)\end{aligned}$$

From Lindley (1952)

$$\lim_{x\to\infty} \Pr\left(\sum_{i=1}^{s} z_i - \sum_{i=1}^{ms} t_i \leqslant x \quad (s=1,2,\ldots)\right) = 1,$$

and hence

$$\lim_{x\to\infty} \Pr(\xi_i \leqslant x) = 1 \quad (i=1,2,\ldots,m).$$

Thus we see that

$$\lim_{x\to\infty}\lim_{n\to\infty} \Pr(W_1(n) \leqslant x) = \lim_{x\to\infty} \Pr(\xi_r \leqslant x (r=1,2,\ldots,m)) = 1.$$

This establishes (2) at once. Furthermore, as

$$\Pr(W_{r+1}(n+r) \leqslant x) \geqslant \Pr(W_1(n) \leqslant x + t_{n+1} + \ldots + t_{n+r}),$$

we have

$$\lim_{r\to\infty}\lim_{n\to\infty} \Pr(W_r(n) = 0) = \lim_{x\to\infty}\lim_{n\to\infty} \Pr(W_1(n) \leqslant x) = 1,$$

which completes the proof of the theorem.

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COROLLARY. *The stationary distribution of the queue length just before an arrival always exists.*

The corollary follows because the limiting probability that the queue length is greater than i is $F_{i+1}(0)$.

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