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Fully Discrete A-ϕ Finite Element Method for Maxwell's Equations with a Nonlinear Boundary Condition

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Abstract. In this paper we present a fully discrete $\boldsymbol{A}$-ϕ finite element method to solve Maxwell's equations with a nonlinear degenerate boundary condition, which represents a generalization of the classical Silver-Müller condition for a non-perfect conductor. The relationship between the normal components of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ obeys a power-law nonlinearity of the type $\boldsymbol{H}\times\boldsymbol{n}=\boldsymbol{n}\times(|\boldsymbol{E}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{E}\times\boldsymbol{n})$ with $\alpha\in(0,1]$. We prove the existence and uniqueness of the solutions of the proposed $\boldsymbol{A}$-ϕ scheme and derive the error estimates. Finally, we present some numerical experiments to verify the theoretical result.

AMS subject classifications: 35Q60, 65N30, 65M60

Key words: Maxwell's equations; nonlinear Silver-Müller boundary condition; $\boldsymbol{A}$-ϕ method; full discretization; nodal elements; error estimates.

1. Introduction

We propose to study the following Maxwell's equations

$$\begin{cases}\partial_t(\varepsilon\boldsymbol{E})+\sigma\boldsymbol{E}=\nabla\times\boldsymbol{H}+\boldsymbol{J},\\ \partial_t(\mu\boldsymbol{H})+\nabla\times\boldsymbol{E}=\boldsymbol{0},\end{cases}\tag{1.1}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field and $\boldsymbol{J}$ denotes the source current term. Let $\Omega\subset\mathbb{R}^3$ be a simply-connected bounded convex polyhedron with the boundary Γ, which consists of some conducting and nonconducting domains. Assume that there exist N faces $(F_j)_{j=1,\cdots,N}$ such that $\Gamma=\cup_j\overline{F}_j$. The electric permittivity ε, the electric conductivity σ and the magnetic permeability μ are supposed to be piecewise constants, and there exist constants ε_{min}, ε_{max}, σ_{max}, μ_{min} and μ_{max} such that $0<\varepsilon_{min}\le\varepsilon\le\varepsilon_{max}$, $0\le\sigma\le\sigma_{max}$ and $0<\mu_{min}\le\mu\le\mu_{max}$.

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The problem (1.1) is accompanied with a nonlinear boundary condition between the normal components of $\boldsymbol{E}$ and $\boldsymbol{H}$, which corresponds to a non-perfect contact of different materials at the boundary. This means that the material on one side of the boundary doesn't allow the field to penetrate without losing the energy, which can be described in terms of an absorbing boundary condition. In this paper, we shall consider a power-law nonlinearity of the type

$$\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{n} \times \boldsymbol{g}(\boldsymbol{E} \times \boldsymbol{n}) = \boldsymbol{n} \times (|\boldsymbol{E} \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{n}), \quad \alpha \in (0, 1], \tag{1.2}$$

where $\boldsymbol{n}$ stands for the outward normal vector on the boundary. Then we obtain the following initial boundary value problem:

$$\begin{cases} \varepsilon\partial_{tt}\boldsymbol{E} + \sigma\partial_t\boldsymbol{E} + \nabla \times \left(\dfrac{1}{\mu}\nabla \times \boldsymbol{E}\right) = \boldsymbol{F}, & (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ \boldsymbol{n} \times \left(\dfrac{1}{\mu}\nabla \times \boldsymbol{E}\right) = \boldsymbol{n} \times \partial_t(|\boldsymbol{E} \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{n}), & (\boldsymbol{x}, t) \in \Gamma \times (0, T], \\ \boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0,\ \partial_t\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0', & \boldsymbol{x} \in \Omega. \end{cases} \tag{1.3}$$

For a more general function $\boldsymbol{g}$, we have to adopt some assumptions ensuring the monotonicity, demicontinuity and coercivity of the nonlinear operator in appropriate function spaces. In the case when $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}$, (1.2) represents the classical Silver-Müller condition, which (cf. e.g. [8, 14]) is a first order approximation of the so-called "transparent" boundary condition. Sometimes it is also called Leontovich or impedance boundary condition (cf. e.g. [13, 18]).

It is well-known that edge finite element methods and nodal finite element methods are usually used to approximate Maxwell's equations. Edge elements address the problem of discontinuity of the normal component of the field at the interface between two materials and avoid the nonphysical solutions called "spurious modes". There are a great deal of works on numerical approximation to Maxwell's equations and also on the convergence analysis and error estimates. For Maxwell's equations with perfectly conducting conditions, we refer readers to [6, 10–12, 19]. Numerically solving the full Maxwell's system with a linear Silver-Müller condition can been found in [9]. In [21, 24, 25], there have been some studies on quasi-static Maxwell's equations with a power-law nonlinear boundary. In [22], a mixed finite element scheme of the electric and magnetic field has been suggested for Maxwell's equations with a power-law nonlinear boundary condition.

To apply the nodal finite element method to solve Maxwell's equations, some potential fields are introduced and some gauges need imposing to guarantee the uniqueness of the potentials. The $\boldsymbol{A}$-ϕ finite element method (cf. e.g. [1–4, 15–17, 20]) is to decompose the electric field into summation of a vector potential and the gradient of a scalar potential, afterward to approximate the potential fields by piecewise polynomial functions. The $\boldsymbol{A}$-ϕ method has some advantages: First, although introducing the vector and scalar potentials increases the number of unknowns and equations, this seeming complication is justified by a better way of dealing with possible discontinuities

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of mediums. Second, it can be applied to the case of any simply/multiply connected domain. Third, it has good numerical accuracy. Finally, many popular nodal finite element softwares and computational techniques can be applied directly. Due to our knowledge, relevant theoretical researches on the $\boldsymbol{A}$-ϕ method for Maxwell's equations with a nonlinear boundary have not shown so far.

The main goal of this paper is to design a fully discrete $\boldsymbol{A}$-ϕ method to solve the nonlinear problem (1.3). Then we decompose the solution $\boldsymbol{E}$ of (1.3) into

$$\boldsymbol{E} = \boldsymbol{A} + \nabla\phi$$

to obtain the $\boldsymbol{A}$-ϕ equation and then take divergence on its both hand sides to obtain the following:

$$\nabla \cdot \Big(\varepsilon\partial_{tt}(\boldsymbol{A} + \nabla\phi) + \sigma\partial_t(\boldsymbol{A} + \nabla\phi)\Big) = \nabla \cdot \boldsymbol{F}.$$

By adding a penalty function term into the dominant $\boldsymbol{A}$-ϕ equation

$$-\nabla\Big(\frac{1}{\mu}\nabla \cdot \boldsymbol{A}\Big)$$

to ensure the divergence-free property of the vector $\boldsymbol{A}$ (see, e.g. [26]), thus we obtain the following $\boldsymbol{A}$-ϕ formulation:

$$\begin{cases} \varepsilon\partial_{tt}(\boldsymbol{A} + \nabla\phi) + \sigma\partial_t(\boldsymbol{A} + \nabla\phi) & \\ \quad +\nabla \times \Big(\frac{1}{\mu}\nabla \times \boldsymbol{A}\Big) - \nabla\Big(\frac{1}{\mu}\nabla \cdot \boldsymbol{A}\Big) = \boldsymbol{F}, & (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ \nabla \cdot \Big(\varepsilon\partial_{tt}(\boldsymbol{A} + \nabla\phi) + \sigma\partial_t(\boldsymbol{A} + \nabla\phi)\Big) = \nabla \cdot \boldsymbol{F}, & (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ \boldsymbol{n} \times \Big(\frac{1}{\mu}\nabla \times \boldsymbol{A}\Big) = \boldsymbol{n} \times \partial_t(|\boldsymbol{A} \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{A} \times \boldsymbol{n}), & (\boldsymbol{x}, t) \in \Gamma \times (0, T], \\ \boldsymbol{A} \cdot \boldsymbol{n} = 0,\ \phi(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \Gamma \times (0, T], \\ \boldsymbol{A}(\boldsymbol{x}, 0) = \boldsymbol{A}_0,\ \partial_t\boldsymbol{A}(\boldsymbol{x}, 0) = \boldsymbol{A}_0', & \boldsymbol{x} \in \Omega, \\ \phi(\boldsymbol{x}, 0) = \phi_0,\ \partial_t\phi(\boldsymbol{x}, 0) = \phi_0', & \boldsymbol{x} \in \Omega. \end{cases} \tag{1.4}$$

Here the functions $(\boldsymbol{A}_0, \phi_0)$ and $(\boldsymbol{A}_0', \phi_0')$ satisfy the decomposition of $\boldsymbol{E}_0 = \boldsymbol{A}_0 + \nabla\phi_0$ and $\boldsymbol{E}_0' = \boldsymbol{A}_0' + \nabla\phi_0'$ with divergence-free of $\boldsymbol{A}_0$ and $\boldsymbol{A}_0'$ respectively.

The paper is organized as follows. In Section 2, we give some notations used in this paper and discuss a time discretization variational formulation for the problem (1.4). In Section 3, we present a fully discrete $\boldsymbol{A}$-ϕ scheme based on the time discretization form and prove the existence and uniqueness of its solutions. In Section 4-5, we study some stability estimates of the approximate solutions and derive the error estimates for the fully discreted problem. In Section 6, we verify the proposed scheme by some numerical experiments. Finally, we draw some conclusions in the last section.

2. Time discretization variational formulation

For convenience of presentation, we first give some notations that will be used throughout this paper. Let $L^s(\Omega)$ $(s > 1)$ denote a s-integrable function space equipped

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with the norm $\|\cdot\|_{L^s(\Omega)}$. In the case $s = 2$, $L^2(\Omega)$ is the usual Hilbert space of square integrable functions equipped with the inner product and norm:

$$(u, v) := \int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})d\boldsymbol{x} \ \text{ and } \ \|u\| := (u, u)^{1/2}.$$

Define $H^m(\Omega) := \{v \in L^2(\Omega) : \ D^\xi v \in L^2(\Omega), \ 0 < |\xi| \le m\}$ with the following norm

$$\|u\|_{H^m(\Omega)} := \left(\sum_{|\xi|\le m} \|D^\xi u\|^2 \right)^{1/2},$$

where ξ represents a non-negative triple index. Let

$$W^{m,s}(\Omega) := \Big\{ u \in L^s(\Omega) : \ D^k u \in L^s(\Omega) \text{ for } 0 < |k| \le m \Big\}$$

stand for the Sobolev spaces with a non-negative integer m and $1 \le s < \infty$. We use boldface notations to express vector-valued quantities, such as $\boldsymbol{L}^2(\Omega) := \big(L^2(\Omega)\big)^3$. Let $\boldsymbol{L}^s(\Gamma)$ $(s > 1)$ denote a s-integrable function space defined on the boundary Γ equipped with norm $\|\cdot\|_{L^s(\Gamma)}$. The $\boldsymbol{L}^2$-inner product on Γ is written as $(\boldsymbol{u}, \boldsymbol{v})_\Gamma = \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{v}$. Let

$$\widehat{\boldsymbol{H}}_0^1(\Omega) := \Big\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \ \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{L}^{1+\alpha}(\Gamma), \ 0 < \alpha \le 1 \ \text{ and } \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \Gamma \Big\}.$$

Thus we can denote a Banach space $\boldsymbol{V} := \widehat{\boldsymbol{H}}_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the sum-norm

$$\|(\boldsymbol{Q}, \phi)\|_{\boldsymbol{V}} := \|\boldsymbol{Q}\|_{\boldsymbol{H}^1(\Omega)} + \|\nabla\phi\| + \|\boldsymbol{Q} \times \boldsymbol{n}\|_{L^{1+\alpha}(\Gamma)}.$$

The dual space of $\boldsymbol{V}$ is denoted by $\boldsymbol{V}^*$. We assume that

$$\boldsymbol{F}(x,t) \in H^1\big((0,T), \boldsymbol{L}^2(\Omega)\big), \ \ (\boldsymbol{A}_0, \phi_0) \in \boldsymbol{V}, \ \ (\boldsymbol{A}_0', \phi_0') \in \boldsymbol{L}^2(\Omega) \times H^1(\Omega). \tag{2.1}$$

In addition, we show the following lemma borrowed from [16], which will be used in the consequent sections.

Lemma 2.1. *Let Ω be a convex bounded polyhedron. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &C\Big(\|\boldsymbol{Q} + \nabla\phi\| + \|\nabla \times \boldsymbol{Q}\| + \|\nabla \cdot \boldsymbol{Q}\| \Big) \\ &\ge \|\boldsymbol{Q}\|_{\boldsymbol{H}^1(\Omega)} + \|\nabla\phi\|, \quad \forall (\boldsymbol{Q}, \phi) \in \boldsymbol{V}. \end{aligned}$$

Next, we give a time discretization variational formulation of the problem (1.4). Let n be a positive integer and $\{t_i = i\tau : \ i = 0, \cdots, n\}$ be a equidistant partition of $[0, T]$ with $\tau = T/n$. Now set

$$u_i = u(t_i), \ \ \delta u_i = \frac{u_i - u_{i-1}}{\tau}, \ \ \delta^2 u_i = \frac{\delta u_i - \delta u_{i-1}}{\tau}.$$

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We consider the following problem: Given $(\boldsymbol{A}_0, \phi_0)$ and $(\delta\boldsymbol{A}_0, \delta\phi_0) = (\boldsymbol{A}_0^{'}, \phi_0^{'})$, find $(\boldsymbol{A}_i, \phi_i) \in \boldsymbol{V}$, $1 \le i \le n$, such that

$$\begin{aligned}&\big(\varepsilon\delta^2(\boldsymbol{A}_i + \nabla\phi_i), \boldsymbol{Q} + \nabla\psi\big) + \big(\sigma\delta(\boldsymbol{A}_i + \nabla\phi_i), \boldsymbol{Q} + \nabla\psi\big) + \Big(\frac{1}{\mu}\nabla \times \boldsymbol{A}_i, \nabla \times \boldsymbol{Q}\Big)\\ &\quad + \Big(\frac{1}{\mu}\nabla \cdot \boldsymbol{A}_i, \nabla \cdot \boldsymbol{Q}\Big) + \Big(\delta(|\boldsymbol{A}_i \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_i \times \boldsymbol{n}), \boldsymbol{Q} \times \boldsymbol{n}\Big)_\Gamma\\ &= (\boldsymbol{F}_i, \boldsymbol{Q} + \nabla\psi), \quad \forall (\boldsymbol{Q}, \psi) \in \boldsymbol{V}. \end{aligned} \tag{2.2}$$

In the following theorem, we shall prove the existence and uniqueness of these approximate potential fields.

Theorem 2.1. *The variational problem* (2.2) *admits a unique solution pair* $(\boldsymbol{A}_i, \phi_i)$ *for each* $1 \le i \le n$.

Proof. We define the operator $\mathcal{L}$ from $\boldsymbol{V}$ to $\boldsymbol{V}^*$ as

$$\begin{aligned}&\big\langle \mathcal{L}(\boldsymbol{P}, \varphi), (\boldsymbol{Q}, \psi)\big\rangle\\ &= \Big(\varepsilon\frac{\boldsymbol{P} + \nabla\varphi}{\tau^2}, \boldsymbol{Q} + \nabla\psi\Big) + \Big(\sigma\frac{\boldsymbol{P} + \nabla\varphi}{\tau}, \boldsymbol{Q} + \nabla\psi\Big)\\ &\quad + \Big(\frac{1}{\mu}\nabla \times \boldsymbol{P}, \nabla \times \boldsymbol{Q}\Big) + \Big(\frac{1}{\mu}\nabla \cdot \boldsymbol{P}, \nabla \cdot \boldsymbol{Q}\Big)\\ &\quad + \frac{1}{\tau}\left(|\boldsymbol{P} \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{P} \times \boldsymbol{n}, \boldsymbol{Q} \times \boldsymbol{n}\right)_\Gamma, \quad \forall (\boldsymbol{P}, \varphi),\ (\boldsymbol{Q}, \psi) \in \boldsymbol{V}\end{aligned}$$

and the functional $\boldsymbol{f}_i \in \boldsymbol{V}^*$ as

$$\begin{aligned}&\big\langle \boldsymbol{f}_i, (\boldsymbol{Q}, \psi)\big\rangle\\ &= \Big(\boldsymbol{F}_i + \varepsilon\frac{\boldsymbol{A}_{i-1} + \nabla\phi_{i-1}}{\tau^2} + \sigma\frac{\boldsymbol{A}_{i-1} + \nabla\phi_{i-1}}{\tau} + \varepsilon\frac{\delta(\boldsymbol{A}_{i-1} + \nabla\phi_{i-1})}{\tau}, \boldsymbol{Q} + \nabla\psi\Big)\\ &\quad + \frac{1}{\tau}\left(|\boldsymbol{A}_{i-1} \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_{i-1} \times \boldsymbol{n}, \boldsymbol{Q} \times \boldsymbol{n}\right)_\Gamma, \quad \forall (\boldsymbol{Q}, \psi) \in \boldsymbol{V}.\end{aligned}$$

Clearly, the variational formulation (2.2) is equivalent to the following operator equation

$$\mathcal{L}(\boldsymbol{A}_i, \phi_i) = \boldsymbol{f}_i \qquad \text{in } \boldsymbol{V}^*.$$

Using the relation

$$(\alpha + \beta)^2(x^\alpha - y^\alpha)(x^\beta - y^\beta) \ge 4\alpha\beta\Big(x^{\frac{\alpha+\beta}{2}} - y^{\frac{\alpha+\beta}{2}}\Big)^2 \tag{2.3}$$

for all positive α, β and all non-negative real numbers x and y leads to

$$\begin{aligned}&(|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u} - |\boldsymbol{v}|^{\alpha-1}\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_\Gamma\\ &\ge C\int_\Gamma (|\boldsymbol{u}|^\alpha - |\boldsymbol{v}|^\alpha)(|\boldsymbol{u}| - |\boldsymbol{v}|)\, d\boldsymbol{x}\\ &\ge C\int_\Gamma \Big(|\boldsymbol{u}|^{\frac{\alpha+1}{2}} - |\boldsymbol{v}|^{\frac{\alpha+1}{2}}\Big)^2 d\boldsymbol{x} \ge 0, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2(\Gamma).\end{aligned} \tag{2.4}$$

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By (2.4) and Lemma 2.1, we obtain that for any $(\boldsymbol{P},\varphi),(\boldsymbol{Q},\psi)\in\boldsymbol{V}$,

$$\begin{aligned}
&\big\langle\mathcal{L}(\boldsymbol{P},\varphi)-\mathcal{L}(\boldsymbol{Q},\psi),(\boldsymbol{P},\varphi)-(\boldsymbol{Q},\psi)\big\rangle\\
\geq&\frac{\varepsilon_{min}}{\tau^2}\left\|(\boldsymbol{P}+\nabla\varphi)-(\boldsymbol{Q}+\nabla\psi)\right\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{1}{\mu_{max}}\left\|\nabla\times(\boldsymbol{P}-\boldsymbol{Q})\right\|^2+\frac{1}{\mu_{max}}\left\|\nabla\cdot(\boldsymbol{P}-\boldsymbol{Q})\right\|^2\\
&+\frac{1}{\tau}\Big(\left|\boldsymbol{P}\times\boldsymbol{n}\right|^{\alpha-1}\boldsymbol{P}\times\boldsymbol{n}-\left|\boldsymbol{Q}\times\boldsymbol{n}\right|^{\alpha-1}\boldsymbol{Q}\times\boldsymbol{n},\boldsymbol{P}\times\boldsymbol{n}-\boldsymbol{Q}\times\boldsymbol{n}\Big)_{\Gamma}\\
\geq&\frac{\varepsilon_{min}}{\tau^2}\left\|(\boldsymbol{P}+\nabla\varphi)-(\boldsymbol{Q}+\nabla\psi)\right\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{1}{\mu_{max}}\left\|\nabla\times(\boldsymbol{P}-\boldsymbol{Q})\right\|^2\\
&+\frac{1}{\mu_{max}}\left\|\nabla\cdot(\boldsymbol{P}-\boldsymbol{Q})\right\|^2+\frac{C}{\tau}\int_{\Gamma}\Big(\left|\boldsymbol{P}\times\boldsymbol{n}\right|^{\frac{\alpha+1}{2}}-\left|\boldsymbol{Q}\times\boldsymbol{n}\right|^{\frac{\alpha+1}{2}}\Big)^2\\
\geq&C(\tau)\Big(\left\|\boldsymbol{P}-\boldsymbol{Q}\right\|^2_{\boldsymbol{H}^1(\Omega)}+\left\|\nabla\varphi-\nabla\psi\right\|^2\Big).
\end{aligned}$$

Further, we get from this that

$$\big\langle\mathcal{L}(\boldsymbol{P},\varphi)-\mathcal{L}(\boldsymbol{Q},\psi),(\boldsymbol{P},\varphi)-(\boldsymbol{Q},\psi)\big\rangle=0\quad\Rightarrow\quad(\boldsymbol{P},\varphi)=(\boldsymbol{Q},\psi)\quad\text{a.e. in }\ \Omega.$$

Thus the operator $\mathcal{L}$ is strictly monotone. Moreover,

$$\begin{aligned}
&\frac{\big\langle\mathcal{L}(\boldsymbol{P},\varphi),(\boldsymbol{P},\varphi)\big\rangle}{\left\|(\boldsymbol{P},\varphi)\right\|_{\boldsymbol{V}}}\\
\geq&\frac{\frac{\varepsilon_{min}}{\tau^2}\left\|\boldsymbol{P}+\nabla\varphi\right\|^2+\frac{1}{\mu_{max}}\left\|\nabla\times\boldsymbol{P}\right\|^2+\frac{1}{\mu_{max}}\left\|\nabla\cdot\boldsymbol{P}\right\|^2+\frac{1}{\tau}\|\boldsymbol{P}\times\boldsymbol{n}\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}}{\left\|(\boldsymbol{P},\varphi)\right\|_{\boldsymbol{V}}}\\
\geq&C(\tau)\frac{\left\|\boldsymbol{P}\right\|^2_{\boldsymbol{H}^1(\Omega)}+\left\|\nabla\varphi\right\|^2+\|\boldsymbol{P}\times\boldsymbol{n}\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)}}{\left\|\boldsymbol{P}\right\|_{\boldsymbol{H}^1(\Omega)}+\left\|\nabla\varphi\right\|+\|\boldsymbol{P}\times\boldsymbol{n}\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)}}\to\infty\qquad\text{as }\ \left\|(\boldsymbol{P},\varphi)\right\|_{\boldsymbol{V}}\to\infty,
\end{aligned}$$

from which we conclude that the operator $\mathcal{L}$ is coercive. Due to the demi-continuity of the operator $\mathcal{L}$, we obtain by Theorem 18.2 in [23] that the equation $\mathcal{L}(\boldsymbol{A}_i,\phi_i)=\boldsymbol{f}_i$ has a unique solution $(\boldsymbol{A}_i,\phi_i)\in\boldsymbol{V}$ since $\boldsymbol{f}_i\in\boldsymbol{V}^*$.

3. Full discretization

Now we analyze a fully discrete approximation scheme based on nodal finite elements. Let $\mathcal{T}^h$ be a standard tetrahedral triangulation of Ω with a mesh size h. We define

$$\begin{aligned}
\boldsymbol{Y}_h^0&:=\left\{\boldsymbol{Q}^h\in\widehat{\boldsymbol{H}}_0^1(\Omega):\boldsymbol{Q}^h|_{\mathcal{K}}\in(\mathcal{P}_1)^3,\ \forall\mathcal{K}\in\mathcal{T}^h\right\},\\
W_h^0&:=\left\{\phi^h\in H_0^1(\Omega):\phi^h|_{\mathcal{K}}\in\mathcal{P}_1,\ \forall\mathcal{K}\in\mathcal{T}^h\right\},
\end{aligned}$$

where $\mathcal{P}_1$ is the space of linear polynomials. Let $\boldsymbol{V}_h:=\boldsymbol{Y}_h^0\times W_h^0$. We denote Π^h as the standard interpolation operator for $\mathcal{T}^h$. Thus the interpolation error estimates (see [7])

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are given by

$$\begin{cases} \|u-\Pi^h u\|_{W^{0,s}(\Omega)} \le Ch\|u\|_{W^{1,s}(\Omega)}, \\ \|u-\Pi^h u\|_{W^{0,s}(\Omega)} + h\|u-\Pi^h u\|_{W^{1,s}(\Omega)} \le Ch^2\|u\|_{W^{2,s}(\Omega)} \end{cases} \tag{3.1}$$

for any $u \in W^{2,s}(\Omega)$ with $1 \le s < \infty$. With the finite element setting, we can formulate the fully discretized problem to read: Find $(\boldsymbol{A}_i^h, \phi_i^h) \in \boldsymbol{V}_h$, $1 \le i \le n$, such that

$$\begin{cases} \big(\varepsilon\delta^2(\boldsymbol{A}_i^h + \nabla\phi_i^h), \boldsymbol{Q}^h + \nabla\psi^h\big) + \big(\sigma\delta(\boldsymbol{A}_i^h + \nabla\phi_i^h), \boldsymbol{Q}^h + \nabla\psi^h\big) \\ \quad + \Big(\frac{1}{\mu}\nabla\times\boldsymbol{A}_i^h, \nabla\times\boldsymbol{Q}^h\Big) + \Big(\frac{1}{\mu}\nabla\cdot\boldsymbol{A}_i^h, \nabla\cdot\boldsymbol{Q}^h\Big) \\ \quad + \big(\delta(|\boldsymbol{A}_i^h\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_i^h\times\boldsymbol{n}), \boldsymbol{Q}^h\times\boldsymbol{n}\big)_\Gamma \\ = \Big(\boldsymbol{F}_i, \boldsymbol{Q}^h + \nabla\psi^h\Big), \quad \forall(\boldsymbol{Q}^h, \psi^h) \in \boldsymbol{V}_h, \\ \boldsymbol{A}_0^h = \Pi^h\boldsymbol{A}_0,\ \phi_0^h = \Pi^h\phi_0, \quad \boldsymbol{x}\in\Omega, \\ \delta\boldsymbol{A}_0^h = \Pi^h\boldsymbol{A}_0',\ \delta\phi_0^h = \Pi^h\phi_0', \quad \boldsymbol{x}\in\Omega. \end{cases} \tag{3.2}$$

In order to prove the solvability of the discretized problem (3.2) for each time step, we need the following lemma ([23], Lemma 18.2).

Lemma 3.1. *Let D be an open bounded set in Euclidean space E, containing zero, and G a continuous mapping of the closure $\bar{D}$ into E. If*

$$\big(G(x), x\big) > 0,$$

on the boundary of D, then the equation $G(x) = 0$ has at least one solution $x_0 \in D$.

Theorem 3.1. *The fully discretized system* (3.2) *has a solution $(\boldsymbol{A}_i^h, \phi_i^h) \in \boldsymbol{V}_h$ on each time step.*

Proof. On each time step we need to solve the following stationary problem: Find $(\boldsymbol{A}^h, \phi^h) \in \boldsymbol{V}_h$:

$$\begin{aligned} &\frac{1}{\tau^2}\big(\varepsilon(\boldsymbol{A}^h + \nabla\phi^h), \boldsymbol{Q}^h + \nabla\psi^h\big) + \frac{1}{\tau}\big(\sigma(\boldsymbol{A}^h + \nabla\phi^h), \boldsymbol{Q}^h + \nabla\psi^h\big) + \Big(\frac{1}{\mu}\nabla\times\boldsymbol{A}^h, \nabla\times\boldsymbol{Q}^h\Big) \\ &\quad + \Big(\frac{1}{\mu}\nabla\cdot\boldsymbol{A}^h, \nabla\cdot\boldsymbol{Q}^h\Big) + \frac{1}{\tau}\Big(|\boldsymbol{A}^h\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}^h\times\boldsymbol{n}, \boldsymbol{Q}^h\times\boldsymbol{n}\Big)_\Gamma \\ &= \Big(\boldsymbol{f}, \boldsymbol{Q}^h + \nabla\psi^h\Big) + \frac{1}{\tau}\big(|\boldsymbol{h}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{Q}^h\times\boldsymbol{n}\big)_\Gamma, \quad \forall(\boldsymbol{Q}^h, \psi^h) \in \boldsymbol{V}_h, \end{aligned}$$

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which is equivalent to the following two equations:

$$\begin{aligned}
&\frac{1}{\tau^2}\big(\varepsilon(\boldsymbol{A}^h+\nabla\phi^h),\boldsymbol{Q}^h\big)+\frac{1}{\tau}\big(\sigma(\boldsymbol{A}^h+\nabla\phi^h),\boldsymbol{Q}^h\big)+\Big(\frac{1}{\mu}\nabla\times\boldsymbol{A}^h,\nabla\times\boldsymbol{Q}^h\Big)\\
&\quad+\Big(\frac{1}{\mu}\nabla\cdot\boldsymbol{A}^h,\nabla\cdot\boldsymbol{Q}^h\Big)+\frac{1}{\tau}\Big(|\boldsymbol{A}^h\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}^h\times\boldsymbol{n},\boldsymbol{Q}^h\times\boldsymbol{n}\Big)_\Gamma\\
&=\Big(\boldsymbol{f},\boldsymbol{Q}^h\Big)+\frac{1}{\tau}\big(|\boldsymbol{h}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{Q}^h\times\boldsymbol{n}\big)_\Gamma,\\
&\frac{1}{\tau^2}\big(\varepsilon(\boldsymbol{A}^h+\nabla\phi^h),\nabla\psi^h\big)+\frac{1}{\tau}\big(\sigma(\boldsymbol{A}^h+\nabla\phi^h),\nabla\psi^h\big)\\
&=\Big(\boldsymbol{f},\nabla\psi^h\Big),\quad\forall(\boldsymbol{Q}^h,\psi^h)\in\boldsymbol{V}_h,
\end{aligned}$$

where $\boldsymbol{h}\in\widehat{\boldsymbol{H}}_0^1(\Omega)$. Let $\varphi_1,\dots,\varphi_k$ be the nodal basis functions of $\mathcal{T}^h(\Omega)$. If we write $\boldsymbol{A}^h=\sum_{i=1}^k\boldsymbol{\alpha}_i\varphi_i$ with $\boldsymbol{\alpha_i}=(\alpha_{ix},\alpha_{iy},\alpha_{iz})$ and $\phi^h=\sum_{i=1}^k\beta_i\varphi_i$, then we need to solve this problem to find $\boldsymbol{\gamma}=\big(\boldsymbol{\alpha}_1,\beta_1,\dots,\boldsymbol{\alpha}_k,\beta_k\big)$. We define the operator $\mathcal{L}$ from $\mathbb{R}^{4k}$ to $\mathbb{R}^{3k}$: $\mathcal{L}(\boldsymbol{\gamma})=\big(\mathcal{L}_1(\boldsymbol{\gamma}),\cdots,\mathcal{L}_k(\boldsymbol{\gamma})\big)$ with

$$\begin{aligned}
\mathcal{L}_j(\boldsymbol{\gamma})=&\frac{1}{\tau^2}\Big(\varepsilon\sum_{i=1}^k(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i),\varphi_j\Big)+\frac{1}{\tau}\Big(\sigma\sum_{i=1}^k(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i),\varphi_j\Big)\\
&+\Big(\frac{1}{\mu}\nabla\times\sum_{i=1}^k\boldsymbol{\alpha}_i\varphi_i,\mathbf{M}_j\Big)+\Big(\frac{1}{\mu}\nabla\cdot\sum_{i=1}^k\boldsymbol{\alpha}_i\varphi_i,\nabla\varphi_j\Big)+\frac{1}{\tau}\int_\Gamma\boldsymbol{n}\\
&\times\Big[\Big|\big(\sum_{i=1}^k\boldsymbol{\alpha}_i\varphi_i\big)\times\boldsymbol{n}\Big|^{\alpha-1}\big(\sum_{i=1}^k\boldsymbol{\alpha}_i\varphi_i\big)\times\boldsymbol{n}\Big]\varphi_j\\
&-(\boldsymbol{f},\varphi_j)-\frac{1}{\tau}\int_\Gamma\boldsymbol{n}\times\big(|\boldsymbol{h}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{h}\times\boldsymbol{n}\big)\,\varphi_j,\quad j=1,\dots,k,
\end{aligned}$$

where

$$\mathbf{M}_j=\begin{pmatrix}0&\frac{\partial\varphi_j}{\partial z}&-\frac{\partial\varphi_j}{\partial y}\\-\frac{\partial\varphi_j}{\partial z}&0&\frac{\partial\varphi_j}{\partial x}\\\frac{\partial\varphi_j}{\partial y}&-\frac{\partial\varphi_j}{\partial x}&0\end{pmatrix}$$

is a 3×3 coefficient matrix such that $\boldsymbol{\alpha}_j\mathbf{M}_j=\nabla\times(\boldsymbol{\alpha}_j\varphi_j)$, and the operator ℓ from $\mathbb{R}^{4k}$ to $\mathbb{R}^k$: $\ell(\gamma)=\big(\ell_1(\boldsymbol{\gamma}),\cdots,\ell_k(\boldsymbol{\gamma})\big)$ with

$$\begin{aligned}
\ell_j(\boldsymbol{\gamma})=&\frac{1}{\tau^2}\Big(\varepsilon\sum_{i=1}^k(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i),\nabla\varphi_j\Big)+\frac{1}{\tau}\Big(\sigma\sum_{i=1}^k(\boldsymbol{\alpha}_i\varphi_i\\
&+\beta_i\nabla\varphi_i),\nabla\varphi_j\Big)-(\boldsymbol{f},\nabla\varphi_j),\quad j=1,\dots,k.
\end{aligned}$$

Denote the operator $\mathbf{L}=(\mathcal{L}_1,\ell_1,\cdots,\mathcal{L}_k,\ell_k)$. The problem is now reduced to solving the nonlinear algebraic equation $\mathbf{L}(\boldsymbol{\gamma})=\mathbf{0}$. Then, using the continuous imbedding of

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$\boldsymbol{L}^2(\Gamma)$ in $\boldsymbol{L}^{\alpha+1}(\Gamma)$ for $0<\alpha\le 1$, Hölder inequality and Lemma 2.1, we have

$$
\begin{aligned}
&\mathbf{L}(\boldsymbol{\gamma})\cdot\boldsymbol{\gamma}\\
&\ge \frac{\varepsilon_{min}}{\tau^2}\Big\|\sum_{i=1}^{k}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i)\Big\|^2+\frac{1}{\mu_{max}}\Big\|\sum_{i=1}^{k}\nabla\times(\boldsymbol{\alpha}_i\varphi_i)\Big\|^2\\
&\quad+\frac{1}{\mu_{max}}\Big\|\sum_{i=1}^{k}\nabla\cdot(\boldsymbol{\alpha}_i\varphi_i)\Big\|^2+\frac{1}{\tau}\int_\Gamma\Big|(\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i)\times\boldsymbol{n}\Big|^{1+\alpha}\\
&\quad-\Big(\boldsymbol{f},\sum_{i=1}^{k}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i)\Big)-\frac{1}{\tau}\int_\Gamma|\boldsymbol{h}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{h}\times\boldsymbol{n}\,(\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i)\times\boldsymbol{n}\\
&\ge C(\tau)\Big(\Big\|\sum_{i=1}^{k}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i)\Big\|^2+\Big\|\sum_{i=1}^{k}\nabla\times(\boldsymbol{\alpha}_i\varphi_i)\Big\|^2+\Big\|\sum_{i=1}^{k}\nabla\cdot(\boldsymbol{\alpha}_i\varphi_i)\Big\|^2\Big)\\
&\quad-C\,\|\boldsymbol{f}\|^2-\frac{1}{\tau}\big\||\boldsymbol{h}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{h}\times\boldsymbol{n}\big\|_{\boldsymbol{L}^{\frac{\alpha+1}{\alpha}}(\Gamma)}\cdot\Big\|(\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i)\times\boldsymbol{n}\Big\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)}\\
&\ge C(\tau)\Big(\Big\|\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i\Big\|^2_{\boldsymbol{H}^1(\Omega)}+\Big\|\sum_{i=1}^{k}\beta_i\varphi_i\Big\|^2_{H^1(\Omega)}-\|\boldsymbol{f}\|^2\Big)\\
&\quad-\frac{1}{\tau}\big\|\boldsymbol{h}\times\boldsymbol{n}\big\|^{\alpha}_{\boldsymbol{L}^{\alpha+1}(\Gamma)}\cdot\Big\|(\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i)\times\boldsymbol{n}\Big\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)}\\
&\ge C(\tau)\Big(\Big\|\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i\Big\|^2_{\boldsymbol{H}^1(\Omega)}+\Big\|\sum_{i=1}^{k}\beta_i\varphi_i\Big\|^2_{H^1(\Omega)}-\|\boldsymbol{f}\|^2\\
&\quad-\Big(\sum_{k=1}^{N}\big\|\boldsymbol{h}\times\boldsymbol{n}\big\|^2_{\boldsymbol{L}^2(F_k)}\Big)^{\frac{\alpha}{2}}\cdot\Big\|(\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i)\times\boldsymbol{n}\Big\|_{\boldsymbol{L}^2(\Gamma)}\Big)\\
&\ge C(\tau)\Big(\Big\|\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i\Big\|^2_{\boldsymbol{H}^1(\Omega)}+\Big\|\sum_{i=1}^{k}\beta_i\varphi_i\Big\|^2_{H^1(\Omega)}-\|\boldsymbol{f}\|^2-\big\|\boldsymbol{h}\big\|^{\alpha}_{\boldsymbol{H}^1(\Omega)}\cdot\Big\|\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i\Big\|_{\boldsymbol{H}^1(\Omega)}\Big)\\
&\ge C(\tau)\Big(\Big\|\sum_{i=1}^{k}\boldsymbol{\alpha}_i\varphi_i\Big\|^2+\Big\|\sum_{i=1}^{k}\beta_i\varphi_i\Big\|^2-\|\boldsymbol{f}\|^2-\big\|\boldsymbol{h}\big\|^{2\alpha}_{\boldsymbol{H}^1(\Omega)}\Big)\\
&\ge C(\tau)\Big(\,|\boldsymbol{\gamma}|^2-\|\boldsymbol{f}\|^2-\big\|\boldsymbol{h}\big\|^{2\alpha}_{\boldsymbol{H}^1(\Omega)}\Big).
\end{aligned}
$$

From Lemma 3.1, we conclude that the equation $\mathbf{L}(\boldsymbol{\gamma})=\mathbf{0}$ has at least one solution in the set $\{\boldsymbol{\gamma}\in\mathbb{R}^{4k}:\ |\boldsymbol{\gamma}|\le r\}$ if $r>(\|\boldsymbol{f}\|^2+\|\boldsymbol{h}\|^{2\alpha}_{\boldsymbol{H}^1(\Omega)})^{1/2}$. □

It is clear that the equation system (3.2) has a unique solution $(\boldsymbol{A}_i^h,\phi_i^h)\in\boldsymbol{V}_h$ from Theorem 2.1 and Theorem 3.1.

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4. Stability

In the following lemmas we study some stability estimates for the approximate potential fields. These *a priori* estimates will serve as uniform bounds in order to prove the error estimates.

Lemma 4.1. *There exist positive constants τ_0 and C such that for $1 \le j \le n$,*

$$\begin{aligned}
&\left\|\sqrt{\varepsilon}(\boldsymbol{A}_j^h + \nabla\phi_j^h)\right\|^2 + \sum_{i=1}^{j}\left\|\sqrt{\varepsilon}(\boldsymbol{A}_i^h + \nabla\phi_i^h) - \sqrt{\varepsilon}(\boldsymbol{A}_{i-1}^h + \nabla\phi_{i-1}^h)\right\|^2 \\
&\quad + \sum_{i=1}^{j}\tau\left\|\sqrt{\sigma}(\boldsymbol{A}_i^h + \nabla\phi_i^h)\right\|^2 + \left\|\sum_{i=1}^{j}\frac{\tau}{\sqrt{\mu}}\nabla\times\boldsymbol{A}_i^h\right\|^2 + \sum_{i=1}^{j}\left\|\frac{\tau}{\sqrt{\mu}}\nabla\times\boldsymbol{A}_i^h\right\|^2 \\
&\quad + \left\|\sum_{i=1}^{j}\frac{\tau}{\sqrt{\mu}}\nabla\cdot\boldsymbol{A}_i^h\right\|^2 + \sum_{i=1}^{j}\left\|\frac{\tau}{\sqrt{\mu}}\nabla\cdot\boldsymbol{A}_i^h\right\|^2 + \sum_{i=1}^{j}\tau\left\|\boldsymbol{A}_i^h\times\boldsymbol{n}\right\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)}^{\alpha+1} \\
&\le C, \qquad \forall\, 0 < \tau < \tau_0.
\end{aligned} \tag{4.1}$$

Proof. The first step of the proof consists of multiplying the equation of (3.2) by τ and summing for $i = 1, \dots, k$. If we define the sequence $\boldsymbol{b}_k = \sum_{i=1}^{k}\tau\nabla\times\boldsymbol{A}_i^h$ with $\boldsymbol{b}_0 = \boldsymbol{0}$ and $c_k = \sum_{i=1}^{k}\tau\nabla\cdot\boldsymbol{A}_i^h$ with $c_0 = 0$, this results in

$$\begin{aligned}
&\left(\varepsilon\delta(\boldsymbol{A}_k^h + \nabla\phi_k^h), \boldsymbol{Q}^h + \nabla\psi^h\right) + \left(\sigma(\boldsymbol{A}_k^h + \nabla\phi_k^h), \boldsymbol{Q}^h + \nabla\psi^h\right) + \left(\frac{1}{\mu}\boldsymbol{b}_k, \nabla\times\boldsymbol{Q}^h\right) \\
&\quad + \left(\frac{1}{\mu}c_k, \nabla\cdot\boldsymbol{Q}\right) + \left(|\boldsymbol{A}_k^h\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_k^h\times\boldsymbol{n}, \boldsymbol{Q}^h\times\boldsymbol{n}\right)_\Gamma \\
&= \left(\sum_{i=1}^{k}\tau\boldsymbol{F}_i, \boldsymbol{Q}^h + \nabla\psi^h\right) + \left(\varepsilon\delta(\boldsymbol{A}_0^h + \nabla\phi_0^h), \boldsymbol{Q}^h + \nabla\psi^h\right) \\
&\quad + \left(\sigma(\boldsymbol{A}_0^h + \nabla\phi_0^h), \boldsymbol{Q}^h + \nabla\psi^h\right) + \left(|\boldsymbol{A}_0^h\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_0^h\times\boldsymbol{n}, \boldsymbol{Q}^h\times\boldsymbol{n}\right)_\Gamma
\end{aligned} \tag{4.2}$$

for any $(\boldsymbol{Q}^h, \psi^h) \in \boldsymbol{V}_h$. Then we put $(\boldsymbol{Q}^h, \psi^h) = (\boldsymbol{A}_k^h, \phi_k^h)$ and obtain

$$\begin{aligned}
&\left(\varepsilon\delta(\boldsymbol{A}_k^h + \nabla\phi_k^h), \boldsymbol{A}_k^h + \nabla\phi_k^h\right) + \left\|\sqrt{\sigma}(\boldsymbol{A}_k^h + \nabla\phi_k^h)\right\|^2 + \left(\frac{1}{\mu}\boldsymbol{b}_k, \delta\boldsymbol{b}_k\right) \\
&\quad + \left(\frac{1}{\mu}c_k, \delta c_k\right) + \left\|\boldsymbol{A}_k^h\times\boldsymbol{n}\right\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)}^{\alpha+1} \\
&= \left(\sum_{i=1}^{k}\tau\boldsymbol{F}_i, \boldsymbol{A}_k^h + \nabla\phi_k^h\right) + \left(\varepsilon\delta(\boldsymbol{A}_0^h + \nabla\phi_0^h), \boldsymbol{A}_k^h + \nabla\phi_k^h\right) \\
&\quad + \left(\sigma(\boldsymbol{A}_0^h + \nabla\phi_0^h), \boldsymbol{A}_k^h + \nabla\phi_k^h\right) + \left(|\boldsymbol{A}_0^h\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_0^h\times\boldsymbol{n}, \boldsymbol{A}_k^h\times\boldsymbol{n}\right)_\Gamma
\end{aligned} \tag{4.3}$$

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We multiply this by τ and sum for $k = 1, \ldots, j$. The first, third and fourth term can be rewritten using Abel's summation rule and estimated as follows:

$$\begin{aligned}
&\sum_{k=1}^{j} \tau \big(\varepsilon\delta(\boldsymbol{A}_k^h + \nabla\phi_k^h), \boldsymbol{A}_k^h + \nabla\phi_k^h\big) \\
&= \frac{1}{2}\sum_{k=1}^{j} \left\| \sqrt{\varepsilon}(\boldsymbol{A}_k^h + \nabla\phi_k^h) - \sqrt{\varepsilon}(\boldsymbol{A}_{k-1}^h + \nabla\phi_{k-1}^h) \right\|^2 \\
&\quad + \frac{1}{2}\left\| \sqrt{\varepsilon}(\boldsymbol{A}_j^h + \nabla\phi_j^h) \right\|^2 - \frac{1}{2}\left\| \sqrt{\varepsilon}(\boldsymbol{A}_0^h + \nabla\phi_0^h) \right\|^2, \\
&\sum_{k=1}^{j} \tau \left(\frac{1}{\mu}\boldsymbol{b}_k, \delta\boldsymbol{b}_k \right) = \frac{1}{2}\sum_{k=1}^{j} \left\| \frac{1}{\sqrt{\mu}}(\boldsymbol{b}_k - \boldsymbol{b}_{k-1}) \right\|^2 + \frac{1}{2}\left\| \frac{1}{\sqrt{\mu}}\boldsymbol{b}_j \right\|^2 - \frac{1}{2}\left\| \frac{1}{\sqrt{\mu}}\boldsymbol{b}_0 \right\|^2, \\
&\sum_{k=1}^{j} \tau \left(\frac{1}{\mu}c_k, \delta c_k \right) = \frac{1}{2}\sum_{k=1}^{j} \left\| \frac{1}{\sqrt{\mu}}(c_k - c_{k-1}) \right\|^2 + \frac{1}{2}\left\| \frac{1}{\sqrt{\mu}}c_j \right\|^2 - \frac{1}{2}\left\| \frac{1}{\sqrt{\mu}}c_0 \right\|^2.
\end{aligned}$$

For the first term on the right-hand side, we apply Cauchy's and Young's inequalities together with the assumptions (2.1) on the source function

$$\begin{aligned}
&\sum_{k=1}^{j} \tau \left(\sum_{i=1}^{k} \tau \boldsymbol{F}_i, \boldsymbol{A}_k^h + \nabla\phi_k^h \right) \\
&\leq \sum_{k=1}^{j} \tau^2 \sum_{i=1}^{k} \left(\frac{1}{2}\left\|\boldsymbol{F}_i\right\|^2 + \frac{1}{2\,\varepsilon_{min}} \left\| \sqrt{\varepsilon}(\boldsymbol{A}_k^h + \nabla\phi_k^h) \right\|^2 \right) \\
&\leq C + C\sum_{k=1}^{j} \tau \left\| \sqrt{\varepsilon}(\boldsymbol{A}_k^h + \nabla\phi_k^h) \right\|^2.
\end{aligned}$$

The second and third term on the right-hand side is easily bounded using Cauchy's and Young's inequalities as

$$\begin{aligned}
\big(\varepsilon\delta(\boldsymbol{A}_0^h + \nabla\phi_0^h), \boldsymbol{A}_k^h + \nabla\phi_k^h\big) &\leq \left\| \sqrt{\varepsilon}\delta(\boldsymbol{A}_0^h + \nabla\phi_0^h) \right\|^2 + \frac{1}{4}\left\| \sqrt{\varepsilon}(\boldsymbol{A}_k^h + \nabla\phi_k^h) \right\|^2, \\
\big(\sigma(\boldsymbol{A}_0^h + \nabla\phi_0^h), \boldsymbol{A}_k^h + \nabla\phi_k^h\big) &\leq \left\| \sqrt{\sigma}(\boldsymbol{A}_0^h + \nabla\phi_0^h) \right\|^2 + \frac{1}{4}\left\| \sqrt{\sigma}(\boldsymbol{A}_k^h + \nabla\phi_k^h) \right\|^2.
\end{aligned}$$

For the last term, we will need both Hölder's and Young's inequalities to obtain

$$\begin{aligned}
&\big(|\boldsymbol{A}_0^h \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_0^h \times \boldsymbol{n}, \boldsymbol{A}_k^h \times \boldsymbol{n}\big)_\Gamma \\
&\leq \left\| |\boldsymbol{A}_0^h \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_0^h \times \boldsymbol{n} \right\|_{\boldsymbol{L}^{\frac{\alpha+1}{\alpha}}(\Gamma)} \left\| \boldsymbol{A}_k^h \times \boldsymbol{n} \right\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)} \\
&= \left\| \boldsymbol{A}_0^h \times \boldsymbol{n} \right\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)}^{\alpha} \left\| \boldsymbol{A}_k^h \times \boldsymbol{n} \right\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)} \\
&\leq \frac{\alpha}{\alpha+1}\left\| \boldsymbol{A}_0^h \times \boldsymbol{n} \right\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)}^{\alpha+1} + \frac{1}{\alpha+1}\left\| \boldsymbol{A}_k^h \times \boldsymbol{n} \right\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)}^{\alpha+1}.
\end{aligned}$$

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Finally, we arrive at the following inequality:

$$\begin{aligned}
&\left\|\sqrt{\varepsilon}(\boldsymbol{A}_j^h+\nabla\phi_j^h)\right\|^2+\sum_{i=1}^{j}\left\|\sqrt{\varepsilon}(\boldsymbol{A}_i^h+\nabla\phi_i^h)-\sqrt{\varepsilon}(\boldsymbol{A}_{i-1}^h+\nabla\phi_{i-1}^h)\right\|^2\\
&\quad+\sum_{i=1}^{j}\tau\|\sqrt{\sigma}(\boldsymbol{A}_i^h+\nabla\phi_i^h)\|^2+\left\|\frac{1}{\sqrt{\mu}}\boldsymbol{b}_j\right\|^2+\sum_{i=1}^{j}\left\|\frac{1}{\sqrt{\mu}}(\boldsymbol{b}_i-\boldsymbol{b}_{i-1})\right\|^2\\
&\quad+\left\|\frac{1}{\sqrt{\mu}}c_j\right\|^2+\sum_{i=1}^{j}\left\|\frac{1}{\sqrt{\mu}}(c_i-c_{i-1})\right\|^2+\sum_{i=1}^{j}\tau\left\|\boldsymbol{A}_i^h\times\boldsymbol{n}\right\|^{\alpha+1}_{\boldsymbol{L}^{\alpha+1}(\Gamma)}\\
&\leq C_1+C_2\sum_{i=1}^{j}\tau\left\|\sqrt{\varepsilon}(\boldsymbol{A}_i^h+\nabla\phi_i^h)\right\|^2,
\end{aligned}$$

where the constant C_1 depends on norms $\left\|\sqrt{\varepsilon}(\Pi^h\boldsymbol{A}_0+\nabla\Pi^h\phi_0)\right\|$, $\left\|\sqrt{\varepsilon}(\Pi^h\boldsymbol{A}_0'+\nabla\Pi^h\phi_0')\right\|$, $\left\|\Pi^h\boldsymbol{A}_0+\nabla\Pi^h\phi_0\right\|_{\boldsymbol{L}^{\alpha+1}(\Omega)}$, $\left\|\Pi^h\boldsymbol{A}_0\times\boldsymbol{n}\right\|_{\boldsymbol{L}^{\alpha+1}(\Gamma)}$ and $\|\boldsymbol{F}\|_{L^2((0,T),\boldsymbol{L}^2(\Omega))}$. Application of Gronwall's inequality concludes the proof. □

Lemma 4.2. *There exist positive constants τ_0 and C such that for $1\leq j\leq n$,*

$$\begin{aligned}
&\left\|\sqrt{\varepsilon}\delta(\boldsymbol{A}_j^h+\nabla\phi_j^h)\right\|^2+\sum_{i=1}^{j}\left\|\sqrt{\varepsilon}\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)-\sqrt{\varepsilon}\delta(\boldsymbol{A}_{i-1}^h+\nabla\phi_{i-1}^h)\right\|^2\\
&\quad+\sum_{i=1}^{j}\tau\left\|\sqrt{\sigma}\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)\right\|^2+\left\|\frac{1}{\sqrt{\mu}}\nabla\times\boldsymbol{A}_j^h\right\|^2+\sum_{i=1}^{j}\left\|\frac{1}{\sqrt{\mu}}\nabla\times(\boldsymbol{A}_i^h-\boldsymbol{A}_{i-1}^h)\right\|^2\\
&\quad+\left\|\frac{1}{\sqrt{\mu}}\nabla\cdot\boldsymbol{A}_j^h\right\|^2+\sum_{i=1}^{j}\left\|\frac{1}{\sqrt{\mu}}\nabla\cdot(\boldsymbol{A}_i^h-\boldsymbol{A}_{i-1}^h)\right\|^2\\
&\leq\ C,\qquad\forall\,0<\tau<\tau_0.
\end{aligned}\tag{4.4}$$

Proof. Setting $(\boldsymbol{Q}^h,\psi^h)=\tau(\delta\boldsymbol{A}_i^h,\delta\phi_i^h)$ in the equation of (3.2) and summing for $i=1,\ldots,j$ yield

$$\begin{aligned}
&\sum_{i=1}^{j}\tau\big(\varepsilon\delta^2(\boldsymbol{A}_i^h+\nabla\phi_i^h),\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)\big)+\sum_{i=1}^{j}\tau\big(\sigma\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h),\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)\big)\\
&\quad+\sum_{i=1}^{j}\tau\left(\frac{1}{\mu}\nabla\times\boldsymbol{A}_i^h,\nabla\times\delta\boldsymbol{A}_i^h\right)+\sum_{i=1}^{j}\tau\left(\frac{1}{\mu}\nabla\cdot\boldsymbol{A}_i^h,\nabla\cdot\delta\boldsymbol{A}_i^h\right)\\
&\quad+\sum_{i=1}^{j}\tau\big(\delta(|\boldsymbol{A}_i^h\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_i^h\times\boldsymbol{n}),\delta\boldsymbol{A}_i^h\times\boldsymbol{n}\big)_\Gamma=\sum_{i=1}^{j}\tau\big(\boldsymbol{F}_i,\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)\big).
\end{aligned}$$

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We also apply Abel's summation rule to the first, third and fourth term to have

$$\begin{aligned}
&\sum_{i=1}^{j}\tau\big(\varepsilon\delta^2(\boldsymbol{A}_i^h+\nabla\phi_i^h),\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)\big)\\
&=\frac{1}{2}\sum_{i=1}^{j}\left\|\sqrt{\varepsilon}\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)-\sqrt{\varepsilon}\delta(\boldsymbol{A}_{i-1}^h+\nabla\phi_{i-1}^h)\right\|^2\\
&\quad+\frac{1}{2}\left\|\sqrt{\varepsilon}\delta(\boldsymbol{A}_j^h+\nabla\phi_j^h)\right\|^2-\frac{1}{2}\left\|\sqrt{\varepsilon}\delta(\boldsymbol{A}_0^h+\nabla\phi_0^h)\right\|^2,\\
&\sum_{i=1}^{j}\tau\left(\frac{1}{\mu}\nabla\times\boldsymbol{A}_i^h,\nabla\times\delta\boldsymbol{A}_i^h\right)\\
&=\frac{1}{2}\sum_{i=1}^{j}\left\|\frac{1}{\sqrt{\mu}}\nabla\times(\boldsymbol{A}_i^h-\boldsymbol{A}_{i-1}^h)\right\|^2+\frac{1}{2}\left\|\frac{1}{\sqrt{\mu}}\nabla\times\boldsymbol{A}_j^h\right\|^2-\frac{1}{2}\left\|\frac{1}{\sqrt{\mu}}\nabla\times\boldsymbol{A}_0^h\right\|^2,\\
&\sum_{i=1}^{j}\tau\left(\frac{1}{\mu}\nabla\cdot\boldsymbol{A}_i^h,\nabla\cdot\delta\boldsymbol{A}_i^h\right)\\
&=\frac{1}{2}\sum_{i=1}^{j}\left\|\frac{1}{\sqrt{\mu}}\nabla\cdot(\boldsymbol{A}_i^h-\boldsymbol{A}_{i-1}^h)\right\|^2+\frac{1}{2}\left\|\frac{1}{\sqrt{\mu}}\nabla\cdot\boldsymbol{A}_j^h\right\|^2-\frac{1}{2}\left\|\frac{1}{\sqrt{\mu}}\nabla\cdot\boldsymbol{A}_0^h\right\|^2.
\end{aligned}$$

For the fifth term, we apply the inequality (2.4) to get

$$\begin{aligned}
&\sum_{i=1}^{j}\tau\big(\delta(|\boldsymbol{A}_i^h\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_i^h\times\boldsymbol{n}),\delta\boldsymbol{A}_i^h\times\boldsymbol{n}\big)_\Gamma\\
&\geq C\sum_{i=1}^{j}\frac{1}{\tau}\int_\Gamma\big(|\boldsymbol{A}_i^h\times\boldsymbol{n}|^\alpha-|\boldsymbol{A}_{i-1}^h\times\boldsymbol{n}|^\alpha\big)\big(|\boldsymbol{A}_i^h\times\boldsymbol{n}|-|\boldsymbol{A}_{i-1}^h\times\boldsymbol{n}|\big)\\
&\geq C\sum_{i=1}^{j}\frac{1}{\tau}\int_\Gamma\Big(|\boldsymbol{A}_i^h\times\boldsymbol{n}|^{\frac{\alpha+1}{2}}-|\boldsymbol{A}_{i-1}^h\times\boldsymbol{n}|^{\frac{\alpha+1}{2}}\Big)^2\geq 0.
\end{aligned}$$

For the right hand side, we use Cauchy's and Young's inequalities together with the assumption (2.1) to obtain

$$\sum_{i=1}^{j}\tau\big(\boldsymbol{F}_i,\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)\big)\leq\ C_1+C_2\sum_{i=1}^{j}\tau\left\|\sqrt{\varepsilon}\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)\right\|^2.$$

Eventually, we arrive at the following inequality:

$$\begin{aligned}
&\left\|\sqrt{\varepsilon}\delta(\boldsymbol{A}_j^h+\nabla\phi_j^h)\right\|^2+\sum_{i=1}^{j}\left\|\sqrt{\varepsilon}\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)-\sqrt{\varepsilon}\delta(\boldsymbol{A}_{i-1}^h+\nabla\phi_{i-1}^h)\right\|^2\\
&+\sum_{i=1}^{j}\tau\left\|\sqrt{\sigma}\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)\right\|^2+\left\|\frac{1}{\sqrt{\mu}}\nabla\times\boldsymbol{A}_j^h\right\|^2+\sum_{i=1}^{j}\left\|\frac{1}{\sqrt{\mu}}\nabla\times(\boldsymbol{A}_i^h-\boldsymbol{A}_{i-1}^h)\right\|^2
\end{aligned}$$

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$$+\left\|\frac{1}{\sqrt{\mu}}\nabla\cdot\boldsymbol{A}_j^h\right\|^2+\sum_{i=1}^{j}\left\|\frac{1}{\sqrt{\mu}}\nabla\cdot(\boldsymbol{A}_i^h-\boldsymbol{A}_{i-1}^h)\right\|^2$$
$$\leq\ C_1+C_2\sum_{i=1}^{j}\tau\left\|\sqrt{\varepsilon}\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h)\right\|^2.$$

Application of Gronwall's inequality yields the inequality (4.4), where the constant C depends on $\left\|\sqrt{\varepsilon}(\Pi^h\boldsymbol{A}_0^{'}+\nabla\Pi^h\phi_0^{'})\right\|$, $\left\|\frac{1}{\sqrt{\mu}}\nabla\times\Pi^h\boldsymbol{A}_0\right\|$, $\left\|\frac{1}{\sqrt{\mu}}\nabla\cdot\Pi^h\boldsymbol{A}_0\right\|$, $\|\Pi^h\boldsymbol{A}_0^{'}\times\boldsymbol{n}\|_{L^{1+\alpha}(\Gamma)}$ and $\|\boldsymbol{F}\|_{L^2((0,T),\boldsymbol{L}^2(\Omega))}$. □

Next, in order to obtain the bound of the term of the second-order time derivative, we introduce the operator $\boldsymbol{S}$ from $\boldsymbol{V}$ to $\boldsymbol{V}^*$ as

$$\begin{aligned}&\big\langle\boldsymbol{S}(\boldsymbol{P}_i,\varphi_i),(\boldsymbol{Q},\psi)\big\rangle\\&=\Big(\varepsilon\delta(\boldsymbol{P}_i+\nabla\varphi_i)+\sigma(\boldsymbol{P}_i+\nabla\varphi_i),\boldsymbol{Q}+\nabla\psi\Big)\\&\quad+\big(|\boldsymbol{P}_i\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{P}_i\times\boldsymbol{n},\boldsymbol{Q}\times\boldsymbol{n}\big)_\Gamma,\qquad\forall(\boldsymbol{P}_i,\varphi_i),\ (\boldsymbol{Q},\psi)\in\boldsymbol{V}.\end{aligned}$$

Lemma 4.3. *There exists a positive constant C such that for every $i=1,\ldots,n$,*

$$\left|\big\langle\delta\boldsymbol{S}(\boldsymbol{A}_i^h,\phi_i^h),(\boldsymbol{Q}^h,\psi^h)\big\rangle\right|\leq C\left\|(\boldsymbol{Q}^h,\psi^h)\right\|_{\boldsymbol{V}},\qquad\forall(\boldsymbol{Q}^h,\psi^h)\in\boldsymbol{V}_h.$$

Proof. From the weak formulation for $(\boldsymbol{A}_i^h,\phi_i^h)$, Lemma 2.1 and Lemma 4.2, we know that

$$\begin{aligned}&\left|\big\langle\delta\boldsymbol{S}(\boldsymbol{A}_i^h,\phi_i^h),(\boldsymbol{Q}^h,\psi^h)\big\rangle\right|\\&=\left|\Big(\varepsilon\delta^2(\boldsymbol{A}_i^h+\nabla\phi_i^h)+\sigma\delta(\boldsymbol{A}_i^h+\nabla\phi_i^h),\boldsymbol{Q}^h+\nabla\psi^h\Big)\right.\\&\quad\left.+\Big(\delta(|\boldsymbol{A}_i^h\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_i^h\times\boldsymbol{n}),\boldsymbol{Q}^h\times\boldsymbol{n}\Big)_\Gamma\right|\\&=\left|\Big(\boldsymbol{F}_i,\boldsymbol{Q}^h+\nabla\psi^h\Big)-\Big(\frac{1}{\mu}\nabla\times\boldsymbol{A}_i^h,\nabla\times\boldsymbol{Q}^h\Big)-\Big(\frac{1}{\mu}\nabla\cdot\boldsymbol{A}_i^h,\nabla\cdot\boldsymbol{Q}^h\Big)\right|\\&\leq C\Big(\|\boldsymbol{F}_i\|+\left\|\nabla\times\boldsymbol{A}_i^h\right\|+\left\|\nabla\cdot\boldsymbol{A}_i^h\right\|\Big)\\&\quad\cdot\Big(\left\|\boldsymbol{Q}^h+\nabla\psi^h\right\|+\left\|\nabla\times\boldsymbol{Q}^h\right\|+\left\|\nabla\cdot\boldsymbol{Q}^h\right\|+\left\|\boldsymbol{Q}^h\times\boldsymbol{n}\right\|_{L^{1+\alpha}(\Gamma)}\Big)\\&\leq C\left\|(\boldsymbol{Q}^h,\psi^h)\right\|_{\boldsymbol{V}}.\end{aligned}$$

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5. Error estimates

First, we use the fields $(\boldsymbol{A}_i^h, \phi_i^h)$ to define the piecewise linear interpolations and the piecewise constants in time by

$$\begin{cases} \boldsymbol{A}_n^h(t) = \boldsymbol{A}_{i-1}^h + (t-t_{i-1})\delta\boldsymbol{A}_i^h, & t \in (t_{i-1}, t_i], \\ \boldsymbol{A}_n^h(0) = \boldsymbol{A}_0^h, \end{cases} \qquad \begin{cases} \bar{\boldsymbol{A}}_n^h(t) = \boldsymbol{A}_i^h, & t \in (t_{i-1}, t_i], \\ \bar{\boldsymbol{A}}_n^h(0) = \boldsymbol{A}_0^h, \end{cases}$$

$$\begin{cases} \phi_n^h(t) = \phi_{i-1}^h + (t-t_{i-1})\delta\phi_i^h, & t \in (t_{i-1}, t_i], \\ \phi_n^h(0) = \phi_0^h, \end{cases} \qquad \begin{cases} \bar{\phi}_n^h(t) = \phi_i^h, & t \in (t_{i-1}, t_i], \\ \bar{\phi}_n^h(0) = \phi_0^h. \end{cases}$$

Furthermore, we define

$$\begin{cases} \quad \big\langle \bar{\boldsymbol{S}}_n^h(t), (\boldsymbol{Q}^h, \psi^h) \big\rangle \\ := \big\langle \boldsymbol{S}(\bar{\boldsymbol{A}}_n^h(t), \bar{\phi}_n^h(t)), (\boldsymbol{Q}^h, \psi^h) \big\rangle = \Big(\varepsilon\delta(\boldsymbol{A}_i^h + \nabla\phi_i^h) + \sigma(\boldsymbol{A}_i^h + \nabla\phi_i^h), \boldsymbol{Q}^h + \nabla\psi^h \Big) \\ \qquad + \Big(|\boldsymbol{A}_i^h \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_i^h \times \boldsymbol{n}, \boldsymbol{Q}^h \times \boldsymbol{n} \Big)_\Gamma, \quad t \in (t_{i-1}, t_i], \\ \quad \big\langle \bar{\boldsymbol{S}}_n^h(0), (\boldsymbol{Q}^h, \psi^h) \big\rangle \\ := \big\langle \boldsymbol{S}(\bar{\boldsymbol{A}}_n^h(0), \bar{\phi}_n^h(0)), (\boldsymbol{Q}^h, \psi^h) \big\rangle = \Big(\varepsilon(\Pi^h \boldsymbol{A}_0' + \nabla\Pi^h\phi_0') \\ \qquad + \sigma(\Pi^h\boldsymbol{A}_0 + \nabla\Pi^h\phi_0), \boldsymbol{Q}^h + \nabla\psi^h \Big) + \Big(|\Pi^h\boldsymbol{A}_0 \times \boldsymbol{n}|^{\alpha-1}\Pi^h\boldsymbol{A}_0 \times \boldsymbol{n}, \boldsymbol{Q}^h \times \boldsymbol{n} \Big)_\Gamma, \end{cases}$$

$$\begin{cases} \quad \big\langle \boldsymbol{S}_n^h(t), (\boldsymbol{Q}^h, \psi^h) \big\rangle \\ := \big\langle \boldsymbol{S}(\boldsymbol{A}_n^h(t), \phi_n^h(t)), (\boldsymbol{Q}^h, \psi^h) \big\rangle \\ = \Big(\varepsilon\delta(\boldsymbol{A}_{i-1}^h + \nabla\phi_{i-1}^h) + \sigma(\boldsymbol{A}_{i-1}^h + \nabla\phi_{i-1}^h) + (t-t_{i-1})\delta\big(\varepsilon\delta(\boldsymbol{A}_i^h + \nabla\phi_i^h) \\ \qquad + \sigma(\boldsymbol{A}_i^h + \nabla\phi_i^h)\big), \boldsymbol{Q}^h + \nabla\psi^h \Big) + \Big((|\boldsymbol{A}_{i-1}^h \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_{i-1}^h \times \boldsymbol{n}) \\ \qquad + (t-t_{i-1})\delta(|\boldsymbol{A}_i^h \times \boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_i^h \times \boldsymbol{n}), \boldsymbol{Q}^h \times \boldsymbol{n} \Big)_\Gamma, \quad t \in (t_{i-1}, t_i], \\ \quad \big\langle \boldsymbol{S}_n^h(0), (\boldsymbol{Q}^h, \psi^h) \big\rangle \\ := \big\langle \boldsymbol{S}(\boldsymbol{A}_n^h(0), \phi_n^h(0)), (\boldsymbol{Q}^h, \psi^h) \big\rangle = \Big(\varepsilon(\Pi^h \boldsymbol{A}_0' + \nabla\Pi^h\phi_0') \\ \qquad + \sigma(\Pi^h\boldsymbol{A}_0 + \nabla\Pi^h\phi_0), \boldsymbol{Q}^h + \nabla\psi^h \Big) + \Big(|\Pi^h\boldsymbol{A}_0 \times \boldsymbol{n}|^{\alpha-1}\Pi^h\boldsymbol{A}_0 \times \boldsymbol{n}, \boldsymbol{Q}^h \times \boldsymbol{n} \Big)_\Gamma, \end{cases}$$

for all $(\boldsymbol{Q}^h, \psi^h) \in \boldsymbol{V}_h$. With these fields, we can reformulate the fully discretized problem (3.2) as follows:

$$\begin{aligned} &\big\langle \partial_t \boldsymbol{S}_n^h, (\boldsymbol{Q}^h, \psi^h) \big\rangle + \Big(\frac{1}{\mu}\nabla \times \bar{\boldsymbol{A}}_n^h, \nabla \times \boldsymbol{Q}^h \Big) + \Big(\frac{1}{\mu}\nabla \cdot \bar{\boldsymbol{A}}_n^h, \nabla \cdot \boldsymbol{Q}^h \Big) \\ &= \Big(\bar{\boldsymbol{F}}_n, \boldsymbol{Q}^h + \nabla\psi^h \Big), \quad \forall (\boldsymbol{Q}^h, \psi^h) \in \boldsymbol{V}_h. \end{aligned} \tag{5.1}$$

Now, we have the following main result on the error estimate.

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Theorem 5.1. *Let* $(\boldsymbol{A},\phi)$ *be the solution of the problem* (1.4). *Suppose that the assumption* (2.1) *holds and*

$$(\boldsymbol{A},\phi)\in L^2\big((0,T),\boldsymbol{H}^2(\Omega)\times H^2(\Omega)\big)\cap H^2\big((0,T),\boldsymbol{L}^2(\Omega)\times H^1(\Omega)\big),$$
$$(\boldsymbol{A}_0,\phi_0)\in \boldsymbol{W}^{2,\frac{\alpha+1}{\alpha}}(\Omega)\times H^2(\Omega),\ (\boldsymbol{A}_0',\phi_0')\in \boldsymbol{H}^1(\Omega)\times H^2(\Omega).$$

Then the fully discretized problem (3.2) *satisfies the following error estimate:*

$$\begin{aligned}&\max_{t\in[0,T]}\Big\|\sqrt{\varepsilon}\big(\boldsymbol{A}_n^h+\nabla\phi_n^h\big)-\sqrt{\varepsilon}\big(\boldsymbol{A}+\nabla\phi\big)\Big\|^2+\int_0^\xi\Big\|\sqrt{\sigma}(\bar{\boldsymbol{A}}_n^h+\nabla\bar{\phi}_n^h)-\sqrt{\sigma}(\boldsymbol{A}+\nabla\phi)\Big\|^2\\&+\Big\|\int_0^\xi\frac{1}{\sqrt{\mu}}\nabla\times(\bar{\boldsymbol{A}}_n^h-\boldsymbol{A})\Big\|^2+\Big\|\int_0^\xi\frac{1}{\sqrt{\mu}}\nabla\cdot(\bar{\boldsymbol{A}}_n^h-\boldsymbol{A})\Big\|^2\\&+\int_0^\xi\Big\||\bar{\boldsymbol{A}}_n^h\times\boldsymbol{n}|^{\frac{\alpha+1}{2}}-|\boldsymbol{A}\times\boldsymbol{n}|^{\frac{\alpha+1}{2}}\Big\|^2_{\boldsymbol{L}^2(\Gamma)}\le C(\tau+h),\qquad\forall\xi\in(0,T].\end{aligned}\tag{5.2}$$

Proof. We first multiply the first and second equation of (1.4), respectively, by any $\boldsymbol{Q}^h\in\boldsymbol{Y}_h^0$ and $\phi^h\in W_h^0$ and then integrate both results over Ω. Applying Green's formulation and the boundary conditions to them and summing together, we obtain

$$\begin{aligned}&\big(\varepsilon\partial_{tt}(\boldsymbol{A}+\nabla\phi)+\sigma\partial_t(\boldsymbol{A}+\nabla\phi),\boldsymbol{Q}^h+\nabla\psi^h\big)+\Big(\frac{1}{\mu}\nabla\times\boldsymbol{A},\nabla\times\boldsymbol{Q}^h\Big)\\&\quad+\Big(\frac{1}{\mu}\nabla\cdot\boldsymbol{A},\nabla\cdot\boldsymbol{Q}^h\Big)+\big(\partial_t(|\boldsymbol{A}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}\times\boldsymbol{n}),\boldsymbol{Q}^h\times\boldsymbol{n}\big)_\Gamma\\&=\Big(\boldsymbol{F},\boldsymbol{Q}^h+\nabla\psi^h\Big),\quad\forall(\boldsymbol{Q}^h,\psi^h)\in\boldsymbol{V}_h.\end{aligned}\tag{5.3}$$

After time integration for any $t\in(0,T]$, we have

$$\begin{aligned}&\big(\varepsilon\partial_t(\boldsymbol{A}+\nabla\phi)+\sigma(\boldsymbol{A}+\nabla\phi)-\varepsilon(\boldsymbol{A}_0'+\nabla\phi_0')-\sigma(\boldsymbol{A}_0+\nabla\phi_0),\boldsymbol{Q}^h+\nabla\psi^h\big)\\&\quad+\big(|\boldsymbol{A}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}\times\boldsymbol{n}-|\boldsymbol{A}_0\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_0\times\boldsymbol{n},\boldsymbol{Q}^h\times\boldsymbol{n}\big)_\Gamma\\&\quad+\Big(\int_0^t\frac{1}{\mu}\nabla\times\boldsymbol{A},\nabla\times\boldsymbol{Q}^h\Big)+\Big(\int_0^t\frac{1}{\mu}\nabla\cdot\boldsymbol{A},\nabla\cdot\boldsymbol{Q}^h\Big)\\&=\Big(\int_0^t\boldsymbol{F},\boldsymbol{Q}^h+\nabla\psi^h\Big),\ \ \forall(\boldsymbol{Q}^h,\psi^h)\in\boldsymbol{V}_h.\end{aligned}\tag{5.4}$$

Next, integrating (5.1) in time yields

$$\begin{aligned}&\big\langle\boldsymbol{S}_n^h-\boldsymbol{S}_n^h(0),(\boldsymbol{Q}^h,\psi^h)\big\rangle+\Big(\int_0^t\frac{1}{\mu}\nabla\times\bar{\boldsymbol{A}}_n^h,\nabla\times\boldsymbol{Q}^h\Big)+\Big(\int_0^t\frac{1}{\mu}\nabla\cdot\bar{\boldsymbol{A}}_n^h,\nabla\cdot\boldsymbol{Q}^h\Big)\\&=\Big(\int_0^t\bar{\boldsymbol{F}}_n,\boldsymbol{Q}^h+\nabla\psi^h\Big),\ \ \forall(\boldsymbol{Q}^h,\psi^h)\in\boldsymbol{V}_h.\end{aligned}\tag{5.5}$$

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Then, subtracting (5.4) from (5.5) leads to

$$\begin{aligned}
&\langle \boldsymbol{S}_n^h, (\boldsymbol{Q}^h, \psi^h)\rangle - \big(\varepsilon\partial_t(\boldsymbol{A}+\nabla\phi) + \sigma(\boldsymbol{A}+\nabla\phi), \boldsymbol{Q}^h + \nabla\psi^h\big)\\
&\quad - \big(|\boldsymbol{A}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}\times\boldsymbol{n}, \boldsymbol{Q}^h\times\boldsymbol{n}\big)_\Gamma + \Big(\int_0^t \frac{1}{\mu}\nabla\times(\bar{\boldsymbol{A}}_n^h - \boldsymbol{A}), \nabla\times\boldsymbol{Q}^h\Big)\\
&\quad + \Big(\int_0^t \frac{1}{\mu}\nabla\cdot(\bar{\boldsymbol{A}}_n^h - \boldsymbol{A}), \nabla\cdot\boldsymbol{Q}^h\Big)\\
&= \Big(\int_0^t \bar{\boldsymbol{F}}_n - \boldsymbol{F}, \boldsymbol{Q}^h + \nabla\psi^h\Big) + \Big(\varepsilon(\Pi^h\boldsymbol{A}_0' + \nabla\Pi^h\phi_0') - \varepsilon(\boldsymbol{A}_0' + \nabla\phi_0'), \boldsymbol{Q}^h + \nabla\psi^h\Big)\\
&\quad + \Big(\sigma(\Pi^h\boldsymbol{A}_0 + \nabla\Pi^h\phi_0) - \sigma(\boldsymbol{A}_0+\nabla\phi_0), \boldsymbol{Q}^h + \nabla\psi^h\Big)\\
&\quad + \Big(|\Pi^h\boldsymbol{A}_0\times\boldsymbol{n}|^{\alpha-1}\Pi^h\boldsymbol{A}_0\times\boldsymbol{n} - |\boldsymbol{A}_0\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_0\\
&\quad \times\boldsymbol{n}, \boldsymbol{Q}^h\times\boldsymbol{n}\Big)_\Gamma, \quad \forall(\boldsymbol{Q}^h,\psi^h)\in\boldsymbol{V}_h. \qquad (5.6)
\end{aligned}$$

Putting $(\boldsymbol{Q}^h,\psi^h) = (\bar{\boldsymbol{A}}_n^h - \Pi^h\boldsymbol{A}, \bar{\phi}_n^h - \Pi^h\phi)$ and integrating in time over $(0,\xi)$ again, we get

$$\begin{aligned}
&\int_0^\xi \big(\varepsilon\partial_t(\boldsymbol{A}_n^h + \nabla\phi_n^h) - \varepsilon\partial_t(\boldsymbol{A}+\nabla\phi), \boldsymbol{A}_n^h + \nabla\phi_n^h - (\boldsymbol{A}+\nabla\phi)\big)\\
&\quad + \int_0^\xi \big(\sigma(\bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h) - \sigma(\boldsymbol{A}+\nabla\phi), \bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h - (\boldsymbol{A}+\nabla\phi)\big)\\
&\quad + \int_0^\xi \Big(|\bar{\boldsymbol{A}}_n^h\times\boldsymbol{n}|^{\alpha-1}\bar{\boldsymbol{A}}_n^h\times\boldsymbol{n} - |\boldsymbol{A}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}\times\boldsymbol{n}, (\bar{\boldsymbol{A}}_n^h - \boldsymbol{A})\times\boldsymbol{n}\Big)_\Gamma\\
&\quad + \int_0^\xi \Big(\int_0^t \frac{1}{\mu}\nabla\times(\bar{\boldsymbol{A}}_n^h - \boldsymbol{A}), \nabla\times(\bar{\boldsymbol{A}}_n^h - \boldsymbol{A})\Big)\\
&\quad + \int_0^\xi \Big(\int_0^t \frac{1}{\mu}\nabla\cdot(\bar{\boldsymbol{A}}_n^h - \boldsymbol{A}), \nabla\cdot(\bar{\boldsymbol{A}}_n^h - \boldsymbol{A})\Big)\\
&= \int_0^\xi \Big(\int_0^t \bar{\boldsymbol{F}}_n - \boldsymbol{F}, \bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h - (\Pi^h\boldsymbol{A} + \nabla\Pi^h\phi)\Big)\\
&\quad + \int_0^\xi \langle \bar{\boldsymbol{S}}_n^h - \boldsymbol{S}_n^h, (\bar{\boldsymbol{A}}_n^h - \Pi^h\boldsymbol{A}, \bar{\phi}_n^h - \Pi^h\phi)\rangle\\
&\quad + \int_0^\xi \big(\varepsilon\partial_t(\boldsymbol{A}_n^h + \nabla\phi_n^h) - \varepsilon\partial_t(\boldsymbol{A}+\nabla\phi), \boldsymbol{A}_n^h + \nabla\phi_n^h - (\bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h)\big)\\
&\quad + \int_0^\xi \big(\varepsilon\partial_t(\boldsymbol{A}_n^h + \nabla\phi_n^h) - \varepsilon\partial_t(\boldsymbol{A}+\nabla\phi), (\Pi^h\boldsymbol{A} + \nabla\Pi^h\phi) - (\boldsymbol{A}+\nabla\phi)\big)\\
&\quad + \int_0^\xi \big(\sigma(\bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h) - \sigma(\boldsymbol{A}+\nabla\phi), \Pi^h\boldsymbol{A} + \nabla\Pi^h\phi - (\boldsymbol{A}+\nabla\phi)\big)
\end{aligned}$$

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$$
\begin{aligned}
&+\int_0^\xi \big(\varepsilon(\Pi^h \boldsymbol{A}_0' + \nabla \Pi^h \phi_0') - \varepsilon(\boldsymbol{A}_0' + \nabla \phi_0'), \bar{\boldsymbol{A}}_n^h + \nabla \bar{\phi}_n^h - (\Pi^h \boldsymbol{A} + \nabla \Pi^h \phi)\big) \\
&+\int_0^\xi \big(\sigma(\Pi^h \boldsymbol{A}_0 + \nabla \Pi^h \phi_0) - \sigma(\boldsymbol{A}_0 + \nabla \phi_0), \bar{\boldsymbol{A}}_n^h + \nabla \bar{\phi}_n^h - (\Pi^h \boldsymbol{A} + \nabla \Pi^h \phi)\big) \\
&+\int_0^\xi \Big(|\bar{\boldsymbol{A}}_n^h \times \boldsymbol{n}|^{\alpha-1} \bar{\boldsymbol{A}}_n^h \times \boldsymbol{n} - |\boldsymbol{A} \times \boldsymbol{n}|^{\alpha-1} \boldsymbol{A} \times \boldsymbol{n}, (\Pi^h \boldsymbol{A} - \boldsymbol{A}) \times \boldsymbol{n}\Big)_\Gamma \\
&+\int_0^\xi \Big(|\Pi^h \boldsymbol{A}_0 \times \boldsymbol{n}|^{\alpha-1} \Pi^h \boldsymbol{A}_0 \times \boldsymbol{n} - |\boldsymbol{A}_0 \times \boldsymbol{n}|^{\alpha-1} \boldsymbol{A}_0 \times \boldsymbol{n}, (\bar{\boldsymbol{A}}_n^h - \Pi^h \boldsymbol{A}) \times \boldsymbol{n}\Big)_\Gamma \\
&+\int_0^\xi \Big(\int_0^t \frac{1}{\mu} \nabla \times (\bar{\boldsymbol{A}}_n^h - \boldsymbol{A}), \nabla \times (\Pi^h \boldsymbol{A} - \boldsymbol{A})\Big) \\
&+\int_0^\xi \Big(\int_0^t \frac{1}{\mu} \nabla \cdot (\bar{\boldsymbol{A}}_n^h - \boldsymbol{A}), \nabla \cdot (\Pi^h \boldsymbol{A} - \boldsymbol{A})\Big) \\
=:& \sum_{i=1}^{11} I_i. \qquad (5.7)
\end{aligned}
$$

The left hand side of the equality (5.7) is

$$
\begin{aligned}
&\frac{1}{2}\Big\|\sqrt{\varepsilon}\big(\boldsymbol{A}_n^h(\xi) + \nabla \phi_n^h(\xi)\big) - \sqrt{\varepsilon}\big(\boldsymbol{A}(\xi) + \nabla \phi(\xi)\big)\Big\|^2 \\
&\quad - \frac{1}{2}\Big\|\sqrt{\varepsilon}(\Pi^h \boldsymbol{A}_0 + \nabla \Pi^h \phi_0) - \sqrt{\varepsilon}(\boldsymbol{A}_0 + \nabla \phi_0)\Big\|^2 \\
&\quad + \int_0^\xi \Big\|\sqrt{\sigma}(\bar{\boldsymbol{A}}_n^h + \nabla \bar{\phi}_n^h) - \sqrt{\sigma}(\boldsymbol{A} + \nabla \phi)\Big\|^2 + \frac{1}{2}\Big\|\int_0^\xi \frac{1}{\sqrt{\mu}} \nabla \times (\bar{\boldsymbol{A}}_n^h - \boldsymbol{A})\Big\|^2 \\
&\quad + \frac{1}{2}\Big\|\int_0^\xi \frac{1}{\sqrt{\mu}} \nabla \cdot (\bar{\boldsymbol{A}}_n^h - \boldsymbol{A})\Big\|^2 + \int_0^\xi \int_\Gamma \big[|\bar{\boldsymbol{A}}_n^h \times \boldsymbol{n}|^{\frac{\alpha+1}{2}} - |\boldsymbol{A} \times \boldsymbol{n}|^{\frac{\alpha+1}{2}}\big]^2.
\end{aligned}
$$

Next, of (5.7) as follows. For the estimation of I_1, we need the assumption (2.1) on $\boldsymbol{F} \in H^1\big((0,T), \boldsymbol{L}^2(\Omega)\big)$. From Cauchy's and Young's inequalities, the triangle inequality and Lemma 4.2, we obtain (here $a \lesssim b$ denotes $a \le Cb$)

$$
\begin{aligned}
I_1 &\le \int_0^\xi \Big\|\int_0^t \bar{\boldsymbol{F}}_n - \boldsymbol{F}\Big\| \cdot \Big\|\bar{\boldsymbol{A}}_n^h + \nabla \bar{\phi}_n^h - (\Pi^h \boldsymbol{A} + \nabla \Pi^h \phi)\Big\| \\
&\lesssim \tau^2 + \int_0^\xi \Big\|\sqrt{\varepsilon}(\bar{\boldsymbol{A}}_n^h + \nabla \bar{\phi}_n^h) - \sqrt{\varepsilon}(\boldsymbol{A}_n^h + \nabla \phi_n^h)\Big\|^2 + \int_0^\xi \Big\|\sqrt{\varepsilon}(\boldsymbol{A}_n^h + \nabla \phi_n^h) \\
&\quad - \sqrt{\varepsilon}(\boldsymbol{A} + \nabla \phi)\Big\|^2 + \int_0^\xi \Big\|\sqrt{\varepsilon}(\boldsymbol{A} + \nabla \phi) - \sqrt{\varepsilon}(\Pi^h \boldsymbol{A} + \nabla \Pi^h \phi)\Big\|^2 \\
&\lesssim \tau^2 + \tau^2 \int_0^\xi \Big\|\sqrt{\varepsilon}\partial_t(\boldsymbol{A}_n^h + \nabla \phi_n^h)\Big\|^2 + \int_0^\xi \Big\|\sqrt{\varepsilon}(\boldsymbol{A}_n^h + \nabla \phi_n^h) - \sqrt{\varepsilon}(\boldsymbol{A} + \nabla \phi)\Big\|^2
\end{aligned}
$$

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$$
\begin{aligned}
&+ h^2 \int_0^T \Big(\|\boldsymbol{A}\|^2_{i\boldsymbol{H}^1(\Omega)} + \|\phi\|^2_{H^2(\Omega)} \Big) \\
&\lesssim \tau^2 + \int_0^\xi \Big\| \sqrt{\varepsilon}(\boldsymbol{A}_n^h + \nabla \phi_n^h) - \sqrt{\varepsilon}(\boldsymbol{A} + \nabla \phi) \Big\|^2 + h^2 \int_0^T \Big(\|\boldsymbol{A}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\phi\|^2_{H^2(\Omega)} \Big).
\end{aligned}
$$

For I_2, we need Lemma 4.3, yielding

$$
\begin{aligned}
I_2 &= \int_0^\xi \langle \bar{\boldsymbol{S}}_n^h - \boldsymbol{S}_n^h, (\bar{\boldsymbol{A}}_n^h - \Pi^h \boldsymbol{A}, \bar{\phi}_n^h - \Pi^h \phi) \rangle \\
&\le \tau \int_0^\xi \Big| \big\langle \partial_t \boldsymbol{S}_n^h, (\bar{\boldsymbol{A}}_n^h - \Pi^h \boldsymbol{A}, \bar{\phi}_n^h - \Pi^h \phi) \big\rangle \Big| \\
&\lesssim \tau \int_0^\xi \Big\| (\bar{\boldsymbol{A}}_n^h - \Pi^h \boldsymbol{A}, \bar{\phi}_n^h - \Pi^h \phi) \Big\|_{\boldsymbol{V}} \\
&\lesssim \tau \int_0^\xi \Big(\Big\| (\bar{\boldsymbol{A}}_n^h, \bar{\phi}_n^h) \Big\|_{\boldsymbol{V}} + \|(\boldsymbol{A}, \phi)\|_{\boldsymbol{V}} \Big).
\end{aligned}
$$

Applying Lemmas 2.1, 4.1 and 4.2 lead to

$$
\begin{aligned}
&\tau \int_0^\xi \Big\| (\bar{\boldsymbol{A}}_n^h, \bar{\phi}_n^h) \Big\|_{\boldsymbol{V}} \\
&\lesssim \tau \sum_{i=1}^j \tau \Big(\Big\| \boldsymbol{A}_i^h \Big\|_{\boldsymbol{H}^1(\Omega)} + \Big\| \nabla \phi_i^h \Big\| + \Big\| \boldsymbol{A}_i^h \times \boldsymbol{n} \Big\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)} \Big) \\
&\lesssim \tau \sum_{i=1}^j \tau \Big(\Big\| \boldsymbol{A}_i^h + \nabla \phi_i^h \Big\| + \Big\| \nabla \times \boldsymbol{A}_i^h \Big\| + \Big\| \nabla \cdot \boldsymbol{A}_i^h \Big\| + \Big\| \boldsymbol{A}_i^h \times \boldsymbol{n} \Big\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)} \Big) \\
&\lesssim \tau + \tau \sum_{i=1}^j \tau \Big\| \boldsymbol{A}_i^h \times \boldsymbol{n} \Big\|_{\boldsymbol{L}^{1+\alpha}(\Gamma)} \\
&\lesssim \tau + \tau^{1+\frac{\alpha}{1+\alpha}} \Big(\sum_{i=1}^j \tau \Big\| \boldsymbol{A}_i^h \times \boldsymbol{n} \Big\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} \Big)^{\frac{1}{1+\alpha}} \\
&\lesssim \tau + \tau^{1+\frac{\alpha}{1+\alpha}} \lesssim \tau.
\end{aligned}
$$

Thus we obtain

$$
I_2 \lesssim \tau \int_0^\xi \Big(\Big\| (\bar{\boldsymbol{A}}_n^h, \bar{\phi}_n^h) \Big\|_{\boldsymbol{V}} + \|(\boldsymbol{A}, \phi)\|_{\boldsymbol{V}} \Big) \lesssim \tau.
$$

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For I_3 - I_5, we use Cauchy's inequality and Lemma 4.2 to obtain

$$
\begin{aligned}
I_3 &\le \left(\int_0^\xi \left\| \sqrt{\varepsilon}\partial_t(\boldsymbol{A}_n^h + \nabla\phi_n^h) - \sqrt{\varepsilon}\partial_t(\boldsymbol{A} + \nabla\phi) \right\|^2 \right)^{\frac{1}{2}} \\
&\qquad \left(\tau^2 \int_0^\xi \left\| \sqrt{\varepsilon}\partial_t(\boldsymbol{A}_n^h + \nabla\phi_n^h) \right\|^2 \right)^{\frac{1}{2}} \\
&\le \left(\sum_i \tau \left\| \sqrt{\varepsilon}\delta(\boldsymbol{A}_i^h + \nabla\phi_i^h \right\|^2 + \int_0^\xi \left\| \sqrt{\varepsilon}\partial_t(\boldsymbol{A} + \nabla\phi) \right\|^2 \right)^{\frac{1}{2}} \\
&\qquad \left(\tau^2 \sum_i \tau \left\| \sqrt{\varepsilon}\delta(\boldsymbol{A}_i^h + \nabla\phi_i^h) \right\|^2 \right)^{\frac{1}{2}} \lesssim \tau, \\
I_4 &\lesssim \left(\int_0^\xi \left\| \sqrt{\varepsilon}\partial_t(\boldsymbol{A}_n^h + \nabla\phi_n^h) - \sqrt{\varepsilon}\partial_t(\boldsymbol{A} + \nabla\phi) \right\|^2 \right)^{\frac{1}{2}} \\
&\qquad \left(\int_0^\xi \left\| \sqrt{\varepsilon}(\Pi^h\boldsymbol{A} + \nabla\Pi^h\phi) - \sqrt{\varepsilon}(\boldsymbol{A} + \nabla\phi) \right\|^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^\xi \left\| \sqrt{\varepsilon}(\Pi^h\boldsymbol{A} + \nabla\Pi^h\phi) - \sqrt{\varepsilon}(\boldsymbol{A} + \nabla\phi) \right\|^2 \right)^{\frac{1}{2}} \\
&\lesssim h\left(\int_0^T \left(\|\boldsymbol{A}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\phi\|^2_{H^2(\Omega)} \right) \right)^{\frac{1}{2}}, \\
I_5 &\lesssim \int_0^\xi \left\| \sqrt{\sigma}(\bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h) - \sqrt{\sigma}(\boldsymbol{A} + \nabla\phi) \right\|^2 \\
&\qquad + \int_0^\xi \left\| \sqrt{\sigma}(\boldsymbol{A} + \nabla\phi) - \sqrt{\sigma}(\Pi^h\boldsymbol{A} + \nabla\Pi^h\phi) \right\|^2 \\
&\lesssim \int_0^\xi \left\| \sqrt{\sigma}(\bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h) - \sqrt{\sigma}(\boldsymbol{A} + \nabla\phi) \right\|^2 + h^2 \int_0^T \left(\|\boldsymbol{A}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\phi\|^2_{H^2(\Omega)} \right).
\end{aligned}
$$

For I_6 and I_7, we apply Cauchy's, Young's and the triangle inequalities and Lemma 4.2 again to obtain

$$
\begin{aligned}
I_6 \lesssim & \int_0^\xi \left\| \sqrt{\varepsilon}(\Pi^h\boldsymbol{A}_0' + \nabla\Pi^h\phi_0') - \sqrt{\varepsilon}(\boldsymbol{A}_0' + \nabla\phi_0') \right\|^2 \\
& + \int_0^\xi \left\| \sqrt{\varepsilon}(\bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h) - \sqrt{\varepsilon}(\boldsymbol{A}_n^h + \nabla\phi_n^h) \right\|^2 \\
& + \int_0^\xi \left\| \sqrt{\varepsilon}(\boldsymbol{A}_n^h + \nabla\phi_n^h) - \sqrt{\varepsilon}(\boldsymbol{A} + \nabla\phi) \right\|^2 \\
& + \int_0^\xi \left\| \sqrt{\varepsilon}(\boldsymbol{A} + \nabla\phi) - \sqrt{\varepsilon}(\Pi^h\boldsymbol{A} + \nabla\Pi^h\phi) \right\|^2
\end{aligned}
$$

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$$\lesssim h^2\int_0^T \Big(\|\boldsymbol{A}_0'\|^2_{\boldsymbol{H}^1(\Omega)} + \|\phi_0'\|^2_{H^2(\Omega)} + \|\boldsymbol{A}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\phi\|^2_{H^2(\Omega)} \Big)$$
$$+ \tau^2 + \int_0^\xi \Big\| \sqrt{\varepsilon}(\boldsymbol{A}_n^h + \nabla\phi_n^h) - \sqrt{\varepsilon}(\boldsymbol{A} + \nabla\phi) \Big\|^2,$$
$$I_7 \lesssim \int_0^\xi \Big\| \sqrt{\sigma}(\Pi^h\boldsymbol{A}_0 + \nabla\Pi^h\phi_0) - \sqrt{\sigma}(\boldsymbol{A}_0 + \nabla\phi_0) \Big\|^2 + \int_0^\xi \Big\| \sqrt{\sigma}(\bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h)$$
$$- \sqrt{\sigma}(\boldsymbol{A} + \nabla\phi) \Big\|^2 + \int_0^\xi \Big\| \sqrt{\sigma}(\boldsymbol{A} + \nabla\phi) - \sqrt{\sigma}(\Pi^h\boldsymbol{A} + \nabla\Pi^h\phi) \Big\|^2$$
$$\lesssim h^2\Big(\|\boldsymbol{A}_0\|^2_{\boldsymbol{H}^1(\Omega)} + \|\phi_0\|^2_{H^2(\Omega)} \Big) + \int_0^\xi \Big\| \sqrt{\sigma}(\bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h) - \sqrt{\sigma}(\boldsymbol{A} + \nabla\phi) \Big\|^2$$
$$+ h^2\int_0^T \Big(\|\boldsymbol{A}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\phi\|^2_{H^2(\Omega)} \Big).$$

For I_8 and I_9, we use Hölder's and Young's inequalities and Lemma 4.1 to get

$$I_8 \lesssim \sum_{k=1}^N \int_0^\xi \big\| |\bar{\boldsymbol{A}}_n^h \times \boldsymbol{n}|^{\alpha-1}\bar{\boldsymbol{A}}_n^h \times \boldsymbol{n} - |\boldsymbol{A}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}\times\boldsymbol{n} \big\|_{\boldsymbol{L}^{\frac{\alpha+1}{\alpha}}(F_k)} \big\|$$
$$\Pi^h\boldsymbol{A}\times\boldsymbol{n} - \boldsymbol{A}\times\boldsymbol{n} \big\|_{\boldsymbol{L}^{\alpha+1}(F_k)}$$
$$\lesssim \sum_{k=1}^N \left(\int_0^\xi \Big\| |\bar{\boldsymbol{A}}_n^h \times \boldsymbol{n}|^{\alpha-1}\bar{\boldsymbol{A}}_n^h \times \boldsymbol{n} - |\boldsymbol{A}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}\times\boldsymbol{n} \Big\|^{\frac{\alpha+1}{\alpha}}_{\boldsymbol{L}^{\frac{\alpha+1}{\alpha}}(F_k)} \right)^{\frac{\alpha}{\alpha+1}}$$
$$\cdot \left(\int_0^\xi \Big\| \Pi^h\boldsymbol{A}\times\boldsymbol{n} - \boldsymbol{A}\times\boldsymbol{n} \Big\|^{\alpha+1}_{\boldsymbol{L}^{\alpha+1}(F_k)} \right)^{\frac{1}{\alpha+1}}$$
$$\lesssim \sum_{k=1}^N \left(\int_0^\xi \big\| \bar{\boldsymbol{A}}_n^h \times \boldsymbol{n} \big\|^{\alpha+1}_{\boldsymbol{L}^{\alpha+1}(F_k)} + \int_0^\xi \big\| \boldsymbol{A}\times\boldsymbol{n} \big\|^{\alpha+1}_{\boldsymbol{L}^{\alpha+1}(F_k)} \right)^{\frac{\alpha}{\alpha+1}}$$
$$\left(\int_0^\xi \Big\| \Pi^h\boldsymbol{A}\times\boldsymbol{n} - \boldsymbol{A}\times\boldsymbol{n} \Big\|^{\alpha+1}_{\boldsymbol{L}^{\alpha+1}(F_k)} \right)^{\frac{1}{\alpha+1}}$$
$$\lesssim \left(\sum_{i=1}^j \tau\|\boldsymbol{A}_i^h \times \boldsymbol{n}\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Gamma)} + \int_0^\xi \big\| \boldsymbol{A}\times\boldsymbol{n} \big\|^{\alpha+1}_{\boldsymbol{L}^2(\Gamma)} \right)^{\frac{\alpha}{\alpha+1}}$$
$$\sum_{k=1}^N \left(\int_0^\xi \Big\| \Pi^h\boldsymbol{A}\times\boldsymbol{n} - \boldsymbol{A}\times\boldsymbol{n} \Big\|^{\alpha+1}_{\boldsymbol{L}^2(F_k)} \right)^{\frac{1}{\alpha+1}}$$
$$\lesssim \sum_{k=1}^N \left(\int_0^T \Big\| \Pi^h\boldsymbol{A} - \boldsymbol{A} \Big\|^{\alpha+1}_{\boldsymbol{H}^1(\Omega)} \right)^{\frac{1}{\alpha+1}} \lesssim h\left(\int_0^T \Big\| \boldsymbol{A} \Big\|^{\alpha+1}_{\boldsymbol{H}^2(\Omega)} \right)^{\frac{1}{\alpha+1}},$$

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$$
\begin{aligned}
I_9 &\lesssim \sum_{k=1}^{N}\int_0^{\xi}\left|\left(|\Pi^h\boldsymbol{A}_0\times\boldsymbol{n}|^{\alpha-1}\Pi^h\boldsymbol{A}_0\times\boldsymbol{n}-|\boldsymbol{A}_0\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{A}_0\times\boldsymbol{n},(\bar{\boldsymbol{A}}_n^h-\Pi^h\boldsymbol{A})\times\boldsymbol{n}\right)_{F_k}\right| \\
&\lesssim \sum_{k=1}^{N}\int_0^{\xi}\left|\left(\Pi^h\boldsymbol{A}_0\times\boldsymbol{n}-\boldsymbol{A}_0\times\boldsymbol{n},(\bar{\boldsymbol{A}}_n^h-\Pi^h\boldsymbol{A})\times\boldsymbol{n}\right)_{F_k}\right| \\
&\qquad \text{(the demicontinuity of the function } |\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}) \\
&\lesssim \sum_{k=1}^{N}\left(\int_0^{\xi}\left\|\Pi^h\boldsymbol{A}_0\times\boldsymbol{n}-\boldsymbol{A}_0\times\boldsymbol{n}\right\|^{\frac{\alpha+1}{\alpha}}_{\boldsymbol{L}^{\frac{\alpha+1}{\alpha}}(F_k)}\right)^{\frac{\alpha}{\alpha+1}} \\
&\qquad \left(\int_0^{\xi}\left\|\bar{\boldsymbol{A}}_n^h\times\boldsymbol{n}-\Pi^h\boldsymbol{A}\times\boldsymbol{n}\right\|^{\alpha+1}_{\boldsymbol{L}^{\alpha+1}(F_k)}\right)^{\frac{1}{\alpha+1}} \\
&\lesssim \sum_{k=1}^{N}\left(\int_0^{\xi}\left\|\Pi^h\boldsymbol{A}_0\times\boldsymbol{n}-\boldsymbol{A}_0\times\boldsymbol{n}\right\|^{\frac{\alpha+1}{\alpha}}_{\boldsymbol{L}^{\frac{\alpha+1}{\alpha}}(F_k)}\right)^{\frac{\alpha}{\alpha+1}} \\
&\qquad \left(\int_0^{\xi}\left\|\bar{\boldsymbol{A}}_n^h\times\boldsymbol{n}\right\|^{\alpha+1}_{\boldsymbol{L}^{\alpha+1}(F_k)}+\int_0^{\xi}\left\|\Pi^h\boldsymbol{A}\times\boldsymbol{n}\right\|^{\alpha+1}_{\boldsymbol{L}^2(F_k)}\right)^{\frac{1}{\alpha+1}} \\
&\lesssim \sum_{k=1}^{N}\left(\int_0^{T}\left\|\Pi^h\boldsymbol{A}_0\times\boldsymbol{n}-\boldsymbol{A}_0\times\boldsymbol{n}\right\|^{\frac{\alpha+1}{\alpha}}_{\boldsymbol{L}^{\frac{\alpha+1}{\alpha}}(F_k)}\right)^{\frac{\alpha}{\alpha+1}} \\
&\qquad \left(\sum_{i=1}^{j}\tau\|\boldsymbol{A}_i^h\times\boldsymbol{n}\|^{\alpha+1}_{\boldsymbol{L}^{\alpha+1}(\Gamma)}+\int_0^{T}\|\boldsymbol{A}\|^{\alpha+1}_{\boldsymbol{H}^1(\Omega)}\right)^{\frac{1}{\alpha+1}} \\
&\lesssim \left\|\Pi^h\boldsymbol{A}_0-\boldsymbol{A}_0\right\|_{\boldsymbol{W}^{1,\frac{\alpha+1}{\alpha}}(\Omega)}\lesssim h\left\|\boldsymbol{A}_0\right\|_{\boldsymbol{W}^{2,\frac{\alpha+1}{\alpha}}(\Omega)}.
\end{aligned}
$$

Finally, I_{10} and I_{11} can be estimated as follows:

$$
\begin{aligned}
I_{10} &\lesssim \int_0^{\xi}\left\|\int_0^{t}\frac{1}{\sqrt{\mu}}\nabla\times(\bar{\boldsymbol{A}}_n^h-\boldsymbol{A})\right\|^2_{\boldsymbol{L}^2(\Omega)}+\int_0^{\xi}\left\|\frac{1}{\sqrt{\mu}}\nabla\times(\Pi^h\boldsymbol{A}-\boldsymbol{A})\right\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\lesssim \int_0^{\xi}\left\|\int_0^{t}\frac{1}{\sqrt{\mu}}\nabla\times(\bar{\boldsymbol{A}}_n^h-\boldsymbol{A})\right\|^2_{\boldsymbol{L}^2(\Omega)}+h^2\int_0^{T}\|\boldsymbol{A}\|^2_{\boldsymbol{H}^2(\Omega)}\,, \\
I_{11} &\lesssim \int_0^{\xi}\left\|\int_0^{t}\frac{1}{\sqrt{\mu}}\nabla\cdot(\bar{\boldsymbol{A}}_n^h-\boldsymbol{A})\right\|^2_{L^2(\Omega)}+\int_0^{\xi}\left\|\frac{1}{\sqrt{\mu}}\nabla\cdot(\Pi^h\boldsymbol{A}-\boldsymbol{A})\right\|^2_{L^2(\Omega)} \\
&\lesssim \int_0^{\xi}\left\|\int_0^{t}\frac{1}{\sqrt{\mu}}\nabla\cdot(\bar{\boldsymbol{A}}_n^h-\boldsymbol{A})\right\|^2_{L^2(\Omega)}+h^2\int_0^{T}\|\boldsymbol{A}\|^2_{\boldsymbol{H}^2(\Omega)}\,.
\end{aligned}
$$

Thus, we use Gronwall's inequality to arrive at the error estimate for the fully discretized problem, which completes the proof of this theorem. □

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Remark 5.1. *Using* Theorem 5.1 *and denoting* $\boldsymbol{E}_n^h = \boldsymbol{A}_n^h + \nabla\phi_n^h$ *and* $\bar{\boldsymbol{E}}_n^h = \bar{\boldsymbol{A}}_n^h + \nabla\bar{\phi}_n^h$, *we conclude that*

$$\begin{aligned}
&\max_{t\in[0,T]} \left\|\sqrt{\varepsilon}\big(\boldsymbol{E}_n^h - \boldsymbol{E}\big)\right\|^2 + \int_0^\xi \left\|\sqrt{\sigma}(\bar{\boldsymbol{E}}_n^h - \boldsymbol{E})\right\|^2 \\
&\quad + \left\|\int_0^\xi \frac{1}{\sqrt{\mu}}\nabla\times(\bar{\boldsymbol{E}}_n^h - \boldsymbol{E})\right\|^2 + \int_0^\xi \left\||\bar{\boldsymbol{E}}_n^h\times\boldsymbol{n}|^{\frac{\alpha+1}{2}} - |\boldsymbol{E}\times\boldsymbol{n}|^{\frac{\alpha+1}{2}}\right\|^2_{\boldsymbol{L}^2(\Gamma)} \\
&\le C(\tau+h), \qquad \forall\xi\in(0,T].
\end{aligned} \tag{5.8}$$

6. Numerical experiments

The purpose of this section is to validate the result of Theorem 5.1 with several numerical tests. For simplicity, we consider the nonlinear system (1.4) on the entire conducting domain $[0,1]^3\times[0,T]$ and solve it numerically using the proposed approximation scheme (3.2) with $\varepsilon = 10^{-4}$, $\sigma = 10^{-2}$ and $\mu = 10^{-4}$. The nonlinearity is solved iteratively using Newton's method. In order to compute the approximation error, we introduce a field $\boldsymbol{E} = \boldsymbol{E}_{ex}$ and calculate the function $\boldsymbol{f}$ by the following equation

$$\varepsilon\partial_{tt}\boldsymbol{E}_{ex} + \sigma\partial_t\boldsymbol{E}_{ex} + \nabla\times\Big(\frac{1}{\mu}\nabla\times\boldsymbol{E}_{ex}\Big) = \boldsymbol{f}. \tag{6.1}$$

Then we decompose $\boldsymbol{E}_{ex} = \boldsymbol{A}_{ex} + \nabla\phi_{ex}$ satisfying $\nabla\cdot\boldsymbol{A}_{ex} = 0$ and denote the function

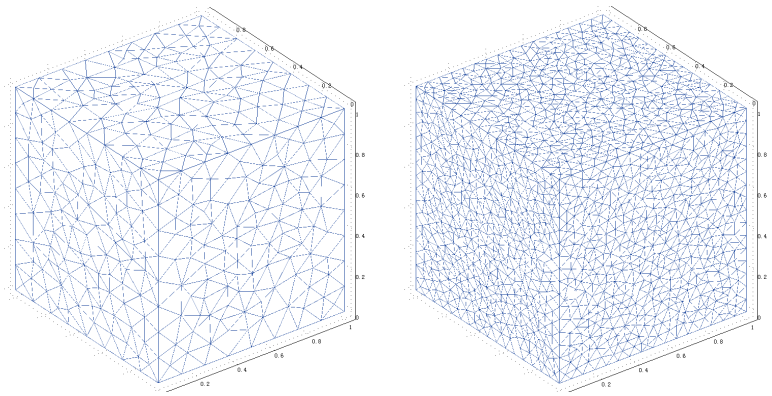
$$\boldsymbol{h} := \frac{1}{\mu}\nabla\times\boldsymbol{E}_{ex} - \partial_t\big(|\boldsymbol{E}_{ex}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{E}_{ex}\times\boldsymbol{n}\big).$$

We need to solve the following problem to get $\boldsymbol{A}$ and ϕ:

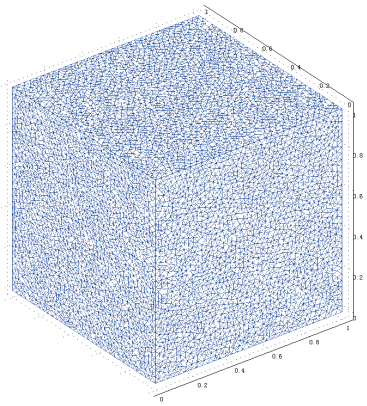
$$\begin{cases}
\varepsilon\partial_{tt}(\boldsymbol{A}+\nabla\phi) + \sigma\partial_t(\boldsymbol{A}+\nabla\phi) & \\
\quad +\nabla\times\Big(\frac{1}{\mu}\nabla\times\boldsymbol{A}\Big) - \nabla\Big(\frac{1}{\mu}\nabla\cdot\boldsymbol{A}\Big) = \boldsymbol{f}, & (\boldsymbol{x},t)\in\Omega\times(0,T], \\
\nabla\cdot\Big(\varepsilon\partial_{tt}(\boldsymbol{A}+\nabla\phi) + \sigma\partial_t(\boldsymbol{A}+\nabla\phi)\Big) = \nabla\cdot\boldsymbol{f}, & (\boldsymbol{x},t)\in\Omega\times(0,T], \\
\boldsymbol{n}\times\Big(\frac{1}{\mu}\nabla\times\boldsymbol{A}\Big) = \boldsymbol{n}\times\partial_t\big(|(\boldsymbol{A}+\nabla\phi) & \\
\qquad\qquad \times\boldsymbol{n}|^{\alpha-1}(\boldsymbol{A}+\nabla\phi)\times\boldsymbol{n}\big) + \boldsymbol{n}\times\boldsymbol{h}, & (\boldsymbol{x},t)\in\Gamma\times(0,T], \\
\boldsymbol{A}\cdot\boldsymbol{n} = \boldsymbol{A}_{ex}\cdot\boldsymbol{n},\ \phi(\boldsymbol{x},t) = \phi_{ex}, & (\boldsymbol{x},t)\in\Gamma\times(0,T], \\
\boldsymbol{A}(\boldsymbol{x},0) = \boldsymbol{A}_{ex}(0),\ \phi(\boldsymbol{x},0) = \phi_{ex}(0), & \boldsymbol{x}\in\Omega.
\end{cases} \tag{6.2}$$

by the fully discrete scheme (3.2) and compute the error between the numerical result and the exact solution in the appropriate function space. The theoretical results remain valid for the nonhomogeneous boundary condition. Space discretization is based on the first-order nodal elements as implemented in the COMSOL software in accordance with

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(a) $h = 1/8$ (b) $h = 1/16$



(c) $h = 1/32$

Figure 1: Tetrahedral meshes on which the problem is solved, constructed in COMSOL. (a) 8434 tetrahedra; (b) 70418 tetrahedra; (c) 573076 tetrahedra.

the theory from Section 5. For every experiment the time step τ is varied as 2^{-n} with $n = 2, \ldots, n_{\max}$ and the error

$$err = \max_{t\in[0,T]} \left\| \sqrt{\varepsilon}\big(\boldsymbol{E}_n^h - \boldsymbol{E}\big) \right\|^2 \tag{6.3}$$

is computed. This procedure is repeated for the three meshes in Figure 1 in order to evaluate the dependence of the error on the mesh parameter h. The mesh in Figure 1 (b and c) are obtained by refinement (by splitting) of the first mesh.

Experiment 6.1. We choose the following exact solution:

$$\boldsymbol{E}_{ex}(\boldsymbol{x}, t) = \big(\sin(\pi t) + 1\big) \begin{pmatrix} z - y \\ x - z \\ y - x \end{pmatrix}$$

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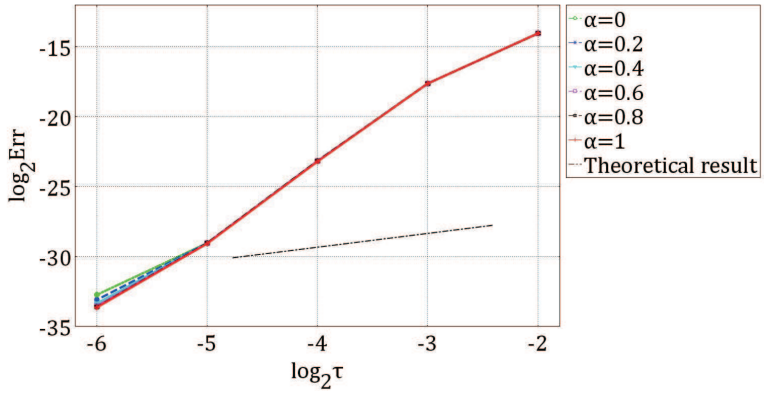


Figure 2: Convergence rate for Experiment 1 with α between 0 and 1. The slope of the black dashed (theoretical result) equals to 1.

with $t \in [0,1]$. Decompose the field $\boldsymbol{E}_{ex}$ to obtain

$$\boldsymbol{A}_{ex}(\boldsymbol{x},t) = \big(\sin(\pi t)+1\big)\begin{pmatrix} z-y-\dfrac{1}{8} \\ x-z-\dfrac{1}{8} \\ y-x-\dfrac{1}{8} \end{pmatrix}$$

and $\phi_{ex}(\boldsymbol{x},t) = \frac{1}{8}(\sin(\pi t)+1)(x+y+z+1)$ with the form $\mathcal{P}_1$ in the spatial part, which is exactly represented by the finite element basis functions. Thus the error will only depend on the time step τ (the calculations can be done on the coarse mesh). We calculate the error err in (6.3) as a function of the time step τ for several values of the exponent α of the power law. The results are presented in Figure 2, where $\log_2(err)$ is plotted as a function of $\log_2 \tau$. Figure 2 gives the convergence rate for $0 \le \alpha \le 1$. We obtain the almost same convergence rate for all values of α at the preceding iteration step, faster than the estimated $\mathcal{O}(\tau)$ rate.

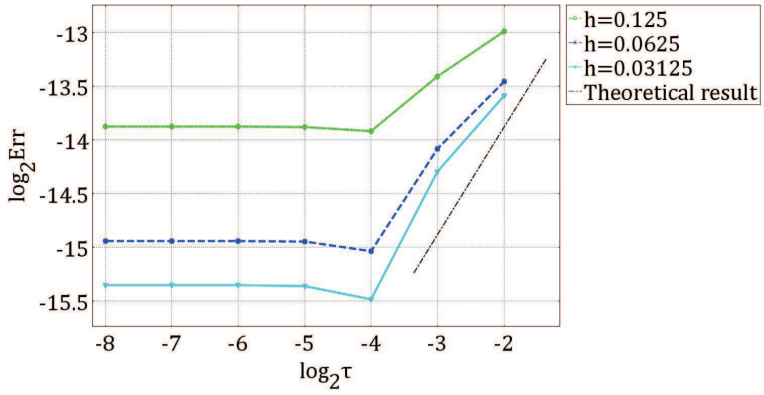
Experiment 6.2. As a second experiment we take

$$\boldsymbol{E}_{ex}(\boldsymbol{x},t) = (t^4+1)\begin{pmatrix} xy-xz+x \\ yz-xy+y \\ xz-yz+z \end{pmatrix},$$

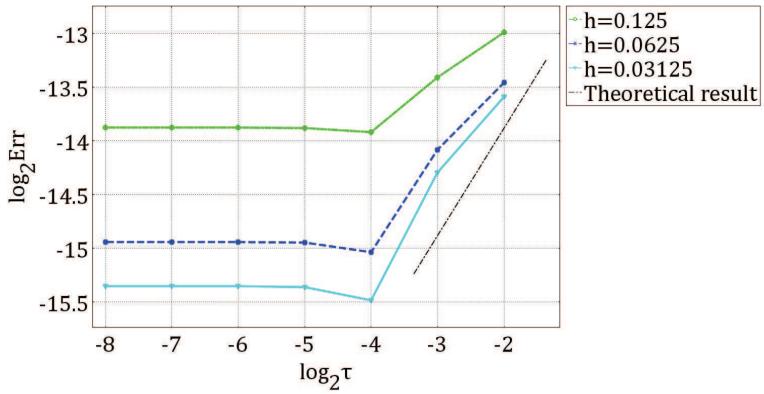
as an exact solution. Decompose the field $\boldsymbol{E}_{ex}$ to obtain

$$\boldsymbol{A}_{ex}(\boldsymbol{x},t) = \big(t^4+1\big)\begin{pmatrix} xy-xz \\ yz-xy \\ xz-yz \end{pmatrix}$$

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(a) $\alpha = 0.5$

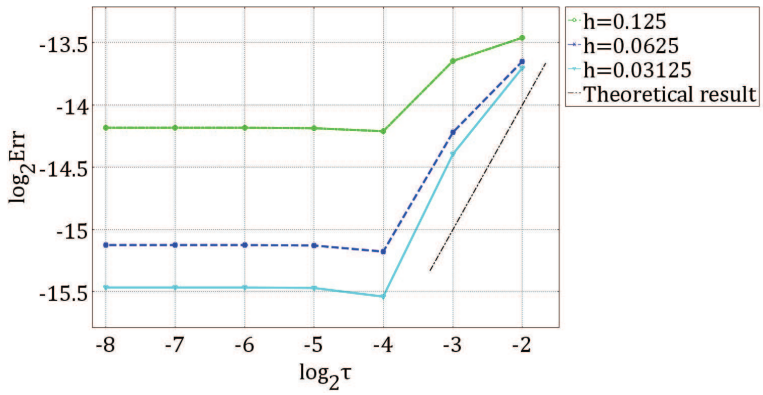


(b) $\alpha = 0.9$

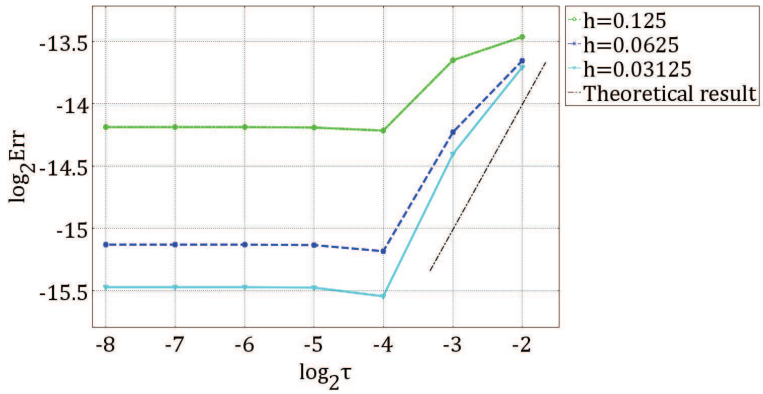
Figure 3: Convergence rate for Experiment 2. The slope of the black dashed (theoretical result) equals to 1.

and $\phi_{ex}(\boldsymbol{x}, t) = \frac{1}{2}(t^4+1)(x^2+y^2+z^2)$. Now the mesh parameter will play an important role in the error and the calculations are repeated for the three meshes in Figure 1. The results for $\alpha = 0.5$ and $\alpha = 0.9$ are shown in Figure 3 respectively. The error decreases with the time step, but eventually attains a constant value only dependent on the mesh parameter. This is in accordance with the theoretical result of Theorem 5.1, where the error is bounded by two terms. For large time steps, the error due to space discretization is much smaller than the error of time discretization and decreasing the time step will decrease the total error. However, for small time steps, the error of space discretization will dominate. Refining the mesh twice decreases the total error as can be clearly seen in the results. We also observe that the first refinement leads to a better convergence rate and the second refinement a less convergence rate than the predicted $\mathcal{O}(\tau)$ rate.

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(a) $\alpha = 0.5$



(b) $\alpha = 0.9$

Figure 4: Convergence rate for Experiment 3. The slope of the black dashed (theoretical result) equals to 1.

Experiment 6.3. Finally, we take the following exact solution

$$\boldsymbol{E}_{ex}(\boldsymbol{x},t) = (t^4+1)\begin{pmatrix} \sin(x) - x\cos(y) + \dfrac{3}{2}\cos(x) \\ \sin(y) - y\cos(z) + \dfrac{3}{2}\cos(y) \\ \sin(z) - z\cos(x) + \dfrac{3}{2}\cos(z) \end{pmatrix}.$$

Decompose the field $\boldsymbol{E}_{ex}$ to obtain

$$\boldsymbol{A}_{ex}(\boldsymbol{x},t) = (t^4+1)\begin{pmatrix} \sin(x) - x\cos(y) \\ \sin(y) - y\cos(z) \\ \sin(z) - z\cos(x) \end{pmatrix}$$

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and $\phi_{ex}(\boldsymbol{x},t)=\frac{3}{2}(t^4+1)(\sin(x)+\sin(y)+\sin(z))$.The results are shown in Figure 4 for $\alpha=0.5$ and $\alpha=0.9$ respectively and confirm the conclusions of Experiment 6.2.

7. Conclusions

We have studied Maxwell's equations with a nonlinear degenerate boundary condition by means of the $\boldsymbol{A}$-ϕ formulation. We first design a nonlinear time-discrete scheme for approximation in suitable function spaces and prove the existence and uniqueness of the approximate solutions. Then we present a fully discrete $\boldsymbol{A}$-ϕ scheme to solve this type of Maxwell's equations based on backward Euler discretization in time and nodal elements in space. We discuss its error estimate and support the theoretical result by some numerical experiments.

The obtained error estimate is suboptimal for both time and space discretization. As we know, the optimality of the backward Euler method for the presented problem cannot be expected a priori due to the nonlinearity of the problem. The suboptimality of space discretization is due to the estimation of I_8. If the nonlinear function is replaced by a Lipschitz continuous approximation, then we could use Young's inequality to estimate I_8 and obtain optimality with respect to h.

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