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Regular and non-regular solutions of the Riemann problem in ideal magnetohydrodynamics

K. TAKAHASHI[1] and S. YAMADA[2,3]

[1]Department of Physics, Waseda University, 3-4-1 Okubo, Shinjuku, 169-8555, Japan
(ktakahashi@heap.phys.waseda.ac.jp)
[2]Department of Science & Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, 169-8555, Japan
[3]Advanced Research Institute for Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, 169-8555, Japan

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Abstract. We have built a code to numerically solve the Riemann problem in ideal magnetohydrodynamics (MHD) for an arbitrary initial condition to investigate a variety of solutions more thoroughly. The code can handle not only regular solutions, in which no intermediate shocks are involved, but also all types of non-regular solutions if any. As a first application, we explored the neighborhood of the initial condition that was first picked up by Brio and Wu (1988) (Brio, M. and Wu, C. C. 1988 An upwind differencing scheme for the equation of ideal magnetohydrodynamics. *J. Comput. Phys.* **75**, 400–422) and has been frequently employed in the literature as a standard problem to validate numerical codes. Contrary to the conventional wisdom that there will always be a regular solution, we found an initial condition for which there is no regular solution but a non-regular one. The latter solution has only regular solutions in its neighborhood and actually sits on the boundary of regular solutions. This implies that the regular solutions are not sufficient to solve the ideal MHD Riemann problem and suggests that at least some types of non-regular solutions are physical. We also demonstrate that the non-regular solutions are not unique. In fact, we found for the Brio and Wu initial condition that there are uncountably many non-regular solutions. This poses an intriguing question: Why a particular non-regular solution is always obtained in numerical simulations? This has important ramifications to the discussion of which intermediate shocks are really admissible.

1. Introduction

The Riemann problem for a system of 1st order, hyperbolic partial differential equations is an initial value problem, in which two distinct constant states are initially separated by a discontinuity. By hyperbolic we mean that all eigenvalues of the Jacobian matrix that characterizes small perturbations are real. It is well known that the Riemann problem in ideal magnetohydrodynamics (MHD) has in general non-unique solutions for a given initial condition even if one discards manifestly unphysical ones such as those including rarefaction shock waves in which entropy is decreased. This is in sharp contrast to the hydrodynamical counterparts and is due to the facts that the system of equations that describes ideal MHD is not strictly hyperbolic, i.e., some of the eigenvalues of the Jacobian matrix are coincident with each other, and that the characteristic fields are neither linear nor genuinely nonlinear (see Sec. 2 and for example Lax 1957; Jeffrey and Taniuti 1964; Polovin and Demutskii 1990). In fact, the MHD Riemann problem has a far greater variety of solutions, including the so-called intermediate shock waves. Although these intermediate shocks are deemed to be unphysical in some text books, some interplanetary experiments have reported the detection of intermediate shocks (Chao 1995; Feng and Wang 2008; Feng et al. 2009) and they are actually realized in almost all numerical solutions. Their reality is hence still an issue of controversy, as will be described below in more detail.

Solutions of the Riemann problem consist of discontinuities and centered self-similar simple waves in general (see Sec. 2 for more details). Various shock waves appearing in the solutions of the Riemann problem are usually characterized by the flow speeds upstream and downstream in the shock-rest frame: In ideal MHD, for example, the fast shock has a flow velocity that exceeds the fast-wave velocity (termed as super-fast hereafter) upstream whereas the downstream flow velocity is sub-fast, i.e., lower than the fast-wave velocity, but super-Alfvénic, i.e., higher than the Alfvén-wave velocity (see Sec. 3 for the definitions of these wave velocities); assigning the numbers 1, 2, 3, and 4 to the super-fast, sub-fast and super-Alfvénic, sub-Alfvénic and super-slow, and sub-slow velocities, respectively, we designate the fast shock as $1 \rightarrow 2$ shock; other shocks are referred to just in the same way.

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The intermediate shock waves in ideal MHD have coplanar configurations, i.e., the upstream transverse magnetic field is anti-parallel to the downstream one and are characterized by the flow velocities that are super-Alfvénic upstream and sub-Alfvénic downstream. These are normally assigned to one of the following four types: $1 \rightarrow 3$, $1 \rightarrow 4$, $2 \rightarrow 3$, or $2 \rightarrow 4$ shock. There are some boundary types as well: for example, the $2 \rightarrow 3,4$ shock is a boundary type that has a downstream flow speed equal to the slow speed as indicated by its name. As mentioned above, these intermediate shock waves are considered to be unphysical in some textbooks (see, e.g., Lax 1957; Jeffrey and Taniuti 1964; Polovin and Demutskii 1990) because they do not satisfy the so-called evolutionary conditions, which require the existence of perturbed states of the same type; for ordinary shock waves (such as the fast shocks in ideal MHD), one of the characteristics runs into the discontinuity from both sides; this is not the case for some of the intermediate shocks; for instance, the $1 \rightarrow 3$ intermediate shock is a shock wave whose flow speeds in the shock-rest frame are super-fast upstream and sub-Alfvénic and super-slow downstream; hence the fast and Alfvén characteristics go into the shock both upstream and downstream; in the case of the $2 \rightarrow 3$ shock, only the Alfvén characteristic converges to the shock wave; however, another evolutionary condition on the linear independence of eigenfunctions is violated and it is classified as an unphysical shock.

In spite of the evolutionary conditions, the intermediate shocks are commonly observed as a stable constituent wave in numerical solutions of the MHD Riemann problems (Wu 1987, 1988a,b, 1990; Brio and Wu 1988; Wu and Kennel 1992). One of the best known cases is the one discussed by Brio and Wu (1988), in which the numerical solution contains the so-called compound wave that consists of a $2 \rightarrow 3,4$ shock followed by a slow rarefaction wave. Wu (1987) also showed numerically that the $2 \rightarrow 4$ shock-like structure could be obtained by steepening of a continuous wave, an indication that the resultant structure is physical. In addition, he demonstrated that a perturbed state obtained for a slightly different initial condition retains a similar structure. Moreover, Wu (1988b) computed interactions between the $2 \rightarrow 4$ shock-like structure and an Alfvén wave and found that the structure remains for a while before it breaks to other waves. These results indicate that the intermediate shock is stable to the perturbation in a sense. Wu (1988b) found that the shock-like structures observed in the computations with perturbations are time-dependent and in fact do not satisfy the Rankine–Hugoniot condition. He claimed that these new types of shocks are the neighboring states for intermediate shocks. In yet another simulation, Wu (1988a) showed that rotational discontinuities, which are not intermediate, break into some other waves that include an intermediate shock. This is a suggestion that it is not intermediate shocks but particular types of evolutionary discontinuities that are unstable and unphysical. On the other hand, Wu (1990) studied the structures inside shock waves by introducing dissipations and found that they are not uniquely determined by the asymptotic states and there remain extra degrees of freedom. He concluded that the stability of the intermediate shocks to the interactions with Alfvén waves is originated from these degrees of freedom. Incidentally, it was also demonstrated that all types of intermediate shocks can be produced by steepening processes in dissipative MHD.

Completely opposite and equally convincing arguments were made by Falle and Komissarov (1997, 2001). They pointed out that if perfect planarity or coplanarity is imposed initially, the solution retains the symmetry and magnetic fields cannot rotate in evolutions; then the problem is equivalent to solving the reduced system of MHD equations, in which it is assumed that magnetic fields are confined in a plane and, as a consequence, the Alfvén waves do not exist; this changes the number of characteristics of the system and modifies the evolutionary conditions; in fact, the 180° rotational discontinuity becomes non-evolutionary whereas $1 \rightarrow 3$ and $2 \rightarrow 4$ shocks become evolutionary for this reduced system. They also demonstrated numerically that the intermediate shock does not emerge if coplanarity of the solution is broken by inserting a thin layer within which the magnetic field rotates continuously. Barmin et al. (1996), on the other hand, found numerically that the compound wave breaks into a rotational discontinuity and a slow shock if the exact coplanarity is perturbed. Although Falle and Komissarov (2001) agreed with Wu (1990) that the temporary survival of $1 \rightarrow 3$, $2 \rightarrow 4$, and $1 \rightarrow 4$ shocks in their interactions with Alfvén waves is due to non-unique shock structures, it was claimed that they should be regarded as transients.

The evolutionary conditions themselves have been reconsidered in the mean time (Hada 1994; Markovskii 1998; Inoue and Inutsuka 2007). These authors took dissipations into account, considering that the waves that emerge in dissipative MHD should be physical. With dissiptations shocks are no longer discontinuities and the analysis is facilitated. They found new modes that do not exist in ideal MHD and argued that on account of these modes the intermediate shocks are indeed evolutionary.

Putting aside the reality of the intermediate shocks, Torrilhon (2002, 2003) studied when the intermediate shocks appear in solutions of the MHD Riemann problem more in detail. In his paper, Torrilhon (2003) defined at first the regular and non-regular shocks: the former has $N + 1$ incoming characteristics, where N is the number of equations whereas the latter includes all the rest. The definition is a bit vague for those shocks that have a characteristic speed equal to the shock velocity, since it is not obvious in this case that the characteristic is incoming or outgoing and, in fact, there has been a controversy in the literature (Jeffrey and Taniuti 1964).

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Although there is no explicit mention on this issue, Torrilhon (2002, 2003) regarded them as non-regular. We will follow this practice hereafter in this paper.[1] Torrilhon (2003) referred to the solutions without non-regular shocks as regular solutions and all the others are called non-regular solutions. Believing that there is always a regular solution, he concluded that if a certain condition is satisfied, there is a unique regular solution to the ideal MHD Riemann problem. Note, however, his analysis was not complete, since he assumed that only a left-going or right-going $1 \to 3$ or $2 \to 4$ intermediate shock exists and he ignored the cases in which the transverse magnetic field vanishes. Furthermore, he did not prove that there is always a unique regular solution.

We address these issues in this paper, extending the approach in Torrilhon (2002, 2003). We believe that the detailed study of particular solutions of the MHD Riemann problem together with their neighboring ones are also helpful to get some more insights into the reality of intermediate shocks. For this purpose we coded a program to produce both regular and non-regular solutions for arbitrary initial conditions. We have extended the algorithm proposed by Torrilhon (2002, 2003) so that we could handle all combinations of constituent waves, including all types of intermediate shocks. Note that Andreev et al. (2008) are providing a code to solve the MHD Riemann problem that is available online. However, it can handle only regular solutions. It should be emphasized that numerical codes to compute temporal evolutions are useless for the current purpose, since they produce a single solution for a given initial condition and we do not know how numerical dissipations that are inevitably inherent to numerical simulations affect the outcomes. Hence, we need to construct solutions in all possible ways to satisfy the given jump condition.

The parameter space of the MHD Riemann problem is vast (15 dimensions indeed) and exploring the entire space is almost impossible. Hence, we need to pay attention to those initial conditions and their neighborhoods that appear to be of particular interest in the above-explained context. In this paper we investigated two cases: The first one is the initial condition first picked up by Brio and Wu (1988) that includes the compound wave in one of the solutions as mentioned earlier; we found that there is a regular solution in this case and that there are in fact uncountably many non-regular solutions as well; this poses an interesting question: Why the non-regular solution with the compound wave is always realized in numerical simulations? In the second case, we found that there is no regular solution; this provides a counter-example to the conventional wisdom that there is always a regular solution. Note that the regular solutions normally consist of fast waves, slow waves, rotational discontinuities, and a contact surface with an exception that there exit the so-called switch-off rarefaction waves that nullify transverse magnetic fields. Then rotational discontinuity does not appear in the solutions.

The paper is organized as follows. In Sec. 2, we briefly review the general theory of the Riemann problem. In Sec. 3, the shock waves and simple waves in ideal MHD, which are constituent waves in the solutions of the MHD Riemann problem, are presented. In Sec. 4, we describe how to construct various solutions that satisfy jump conditions. In Sec. 5, we apply the code to two particular initial conditions mentioned above and show both regular and non-regular solutions. We discuss possible implications of the results in Sec. 6.

2. Theory of Riemann problem

2.1. Hyperbolic equations and weak solutions

In this section we consider general conservation equations in one spatial dimension. Since the Riemann problem assumes plane-symmetry, this is sufficient. Then the coupled equations are expressed as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}}{\partial x} = \boldsymbol{0}, \tag{2.1}$$

where $\boldsymbol{u}$ is a vector of conserved variables and $\boldsymbol{f}(\boldsymbol{u})$ is a flux vector. The system (2.1) is called hyperbolic if the Jacobian matrix, which is defined as

$$A_{ij} := \frac{\partial f_i}{\partial u_j}, \tag{2.2}$$

has N real eigenvalues, $\lambda_k(\boldsymbol{u})$ $(k = 1, 2, \ldots, N)$, and corresponding N linearly independent right eigenvectors, $\boldsymbol{r}_k(\boldsymbol{u})$ (e.g. Falle and Komissarov 2001). In particular, if all eigenvalues are different from each other for any $\boldsymbol{u}$, the system is called strictly hyperbolic.

The kth characteristic field is called genuinely nonlinear if the following condition is satisfied for any $\boldsymbol{u}$:

$$\boldsymbol{r}_k \cdot \nabla_{\boldsymbol{u}} \lambda_k \neq 0. \tag{2.3}$$

Here $\nabla_{\boldsymbol{u}}$ denotes the operator, ${}^t(\partial/\partial u_1, \ldots, \partial/\partial u_N)$, where ${}^t(\ldots)$ stands for transposition. On the other hand, The kth characteristic field is called linear if

$$\boldsymbol{r}_k \cdot \nabla_{\boldsymbol{u}} \lambda_k = 0 \tag{2.4}$$

for all $\boldsymbol{u}$. Note that the characteristic field may be neither linear nor genuinely nonlinear.

In the system described by (2.1), discontinuities are ubiquitous. Indeed, they may be spontaneously produced from continuous initial conditions by wave-steepening. Since (2.1) is a differential equation, these discontinuities cannot be treated as they are and some extensions are needed. The weak solutions of (2.1) are defined as the functions that satisfy the following equations instead of (2.1) in a finite domain $D(x, t)$ (e.g., Jeffrey and Taniuti

[1] More precisely, we define regular shocks as the shocks that satisfy the evolutionary condition (see Sec. 2.3). We treat those characteristics that have the same velocity as the shock velocity as incoming. All other shocks are defined to be non-regular. The solutions are referred to as regular if they include no non-regular shocks, otherwise these are called non-regular.

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1964):

$$\int_{-\infty}^{\infty} \boldsymbol{w}(x,0)\cdot\boldsymbol{\phi}(x)\mathrm{d}x + \int_{0}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x\left(\frac{\partial \boldsymbol{w}}{\partial t}\cdot\boldsymbol{u}+\frac{\partial \boldsymbol{w}}{\partial x}\cdot\boldsymbol{f}\right) = 0, \tag{2.5}$$

where $\boldsymbol{\phi}(x)$ is an initial condition and $\boldsymbol{w}(x,t)$ is a vector of test functions, which are differentiable with respect to x and t as many times as is required and are identically zero outside $D(x,t)$. This equation is derived by multiplying the original equations by $\boldsymbol{w}$ and integrating the product by parts. Note that differentiability of weak solutions is not required. If the solution is differentiable and satisfies (2.1), then (2.5) is also satisfied. Hence, ordinary solutions are indeed weak solutions. It should be remarked that uniqueness of weak solutions for a given initial condition is no longer guaranteed.

2.2. *Solutions of Riemann problem: simple waves and discontinuities*

The Riemann problem is an initial value problem for a system of conservation equations in which two constant states are separated initially by a discontinuity, which is located, for example, at $x = 0$:

$$\boldsymbol{u}(x,0) = \begin{cases} \boldsymbol{u}_{\mathrm{R}} & (x>0) \\ \boldsymbol{u}_{\mathrm{L}} & (x<0) \end{cases}. \tag{2.6}$$

We seek weak solutions to this problem. As remarked at the end of the previous section, the weak solutions are not unique in general and it is one of the important issues in the Riemann problem to determine which solution is physically meaningful. Since there is no typical time and length scales in the problem, the solution should be self-similar. It is well known that solutions of the Riemann problem consist of a couple of centered simple waves and discontinuities. The latter satisfies the Rankine–Hugoniot relations (e.g., Jeffrey and Taniuti 1964).

The simple waves are defined as waves, in which there is only one independent component in $\boldsymbol{u}$ and all the other components are its functions. They are related with the eigenvalues of the Jacobian matrix (2.2) in such a way that all quantities in the simple wave are constant along each characteristic given by

$$x = \lambda_k t + \text{constant}. \tag{2.7}$$

The simple wave which corresponds to eigenvalue λ_k is referred to as the kth simple wave.

Associated with the kth simple wave are the so-called the kth Riemann invariants denoted by $J_l^k (l = 1,2,\ldots,N-1)$, which are constant across the kth simple wave. They are defined to be the quantities that satisfy the following relation with the kth right eigenvector, $\boldsymbol{r}_k$:

$$\boldsymbol{r}_k \cdot \nabla_{\boldsymbol{u}} J_l^k = 0. \tag{2.8}$$

The kth Riemann invariants define $N-1$ hypersurfaces in phase space and their intersection is a line, to which the kth right eigenvector, $\boldsymbol{r}_k$, is tangential, and is called the locus of the kth simple wave. See a schematic picture drawn in Fig. 1.

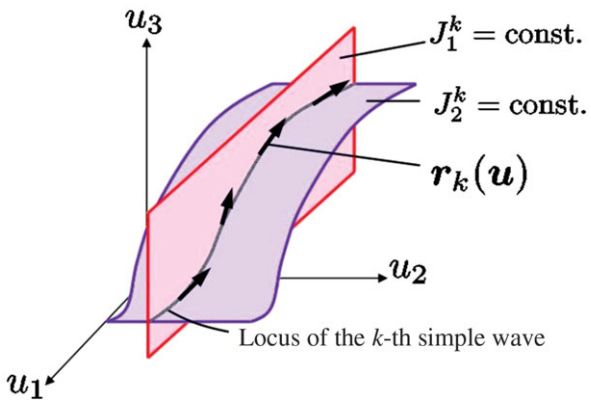


Figure 1. (Colour online) A schematic picture of the kth Riemann invariants and corresponding locus of the kth simple wave in the case $N = 3$. There are two kth Riemann invariants: J_1^k and J_2^k. The red surface is defined by $J_1^k = \text{const.}$ and the blue one is given by $J_2^k = \text{const.}$ The intersection is a line called the locus of the kth simple wave, to which the kth right eigenvector, $\boldsymbol{r}_k(\boldsymbol{u})$, is tangential.

The Rankine–Hugoniot relations, which are satisfied by the quantities on both sides of a discontinuity, are expressed as

$$s[\![\boldsymbol{u}]\!] - [\![\boldsymbol{f}]\!] = \boldsymbol{0}, \tag{2.9}$$

where s is the velocity of the discontinuity and $[\![X]\!] := X_1 - X_0$ denotes a jump in quantity X across the discontinuity and X_0 and X_1 are the values ahead of and behind the discontinuity respectively. Since this is a system of N equations, we obtain a one-parameter family of solutions for a state given on one side of the discontinuity. Not all solutions are physical as mentioned repeatedly. Rarefaction shock waves for hydrodynamical equations are well-known examples. Since entropy decreases across such shock waves, they are certainly unphysical and rejected. This entropy condition is not sufficient to eliminate all unphysical solutions in general and the so-called evolutionary conditions are introduced.

2.3. *Evolutionary conditions*

Intuitively put, the evolutionary conditions require that physical solutions should be stable against splitting into other waves (see, e.g., Lax 1957; Jeffrey and Taniuti 1964; Landau et al. 1984; Polovin and Demutskii 1990; Falle and Komissarov 2001). To be more precise, when a small-amplitude wave is injected to the discontinuity, the resultant state, which is determined uniquely, is required to still consist of the same type of discontinuity and outgoing small-amplitude waves. This is satisfied if the number of characteristics emanating from the discontinuity is $N+1$ and the initial jump $[\![\boldsymbol{u}]\!]$ is linearly independent of the eigenvectors that correspond to the outgoing characteristics.

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3. Ideal MHD

In plane symmetry, the ideal MHD equations are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_n) = 0, \quad (3.1)$$

$$\frac{\partial}{\partial t}(\rho v_n) + \frac{\partial}{\partial x}\left(\rho v_n^2 + p + \frac{\boldsymbol{B}_t^2}{2}\right) = 0, \quad (3.2)$$

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}_t) + \frac{\partial}{\partial x}(\rho v_n \boldsymbol{v}_t - B_n \boldsymbol{B}_t) = \boldsymbol{0}, \quad (3.3)$$

$$\frac{\partial \boldsymbol{B}_t}{\partial t} + \frac{\partial}{\partial x}(v_n \boldsymbol{B}_t - B_n \boldsymbol{v}_t) = \boldsymbol{0}, \quad (3.4)$$

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x}\left[\left(e + p + \frac{\boldsymbol{B}^2}{2}\right) v_n - B_n \boldsymbol{B}\cdot\boldsymbol{v}\right] = 0, \quad (3.5)$$

where ρ, p, $\boldsymbol{v}$, and $\boldsymbol{B}$ are density, pressure, flow velocity, and magnetic field respectively (Landau et al. 1984). The subscripts n and t indicate the normal component, i.e., x-component, and transverse component, i.e., y- or z-component respectively. The total energy density is denoted by $e = p/(\gamma-1) + \rho\boldsymbol{v}^2/2 + \boldsymbol{B}^2/2$, where the equation of state for ideal gas is assumed and γ is the ratio of specific heats. The normal component of magnetic field, B_n, is constant owing to the divergence-free condition.

3.1. Simple waves

The eigenvalues of the Jacobian matrix for the system (3.1)–(3.5) are

$$v_n \mp c_f, \quad v_n \mp c_A, \quad v_n \mp c_s, \quad v_n, \quad (3.6)$$

where c_f, c_A, and c_s are called the fast, Alfvén, and slow speeds respectively, and are expressed as

$$c_{f,s} := \left[\frac{1}{2}\left(\frac{\boldsymbol{B}^2}{\rho} + a^2\right) \pm \sqrt{\frac{1}{4}\left(\frac{\boldsymbol{B}^2}{\rho} + a^2\right)^2 - a^2\frac{B_n^2}{\rho}}\right]^{1/2}, \quad (3.7)$$

$$c_A := \sqrt{\frac{B_n^2}{\rho}}. \quad (3.8)$$

In the above expressions, $a = \sqrt{\gamma p/\rho}$ is the acoustic speed. In (3.6), the minus (plus) sign is applied to the left-going (right-going) waves. The simple waves corresponding to these eigenvalues are referred to as the fast, Alfvén, slow, and entropy waves respectively. The right eigenvectors for fast and slow waves are given as

$$\boldsymbol{r}_{f,s}^{\mp} = \xi_{f,s}\begin{bmatrix} -\rho \\ -\gamma p \\ \pm c_{f,s} \\ \pm\dfrac{c_{f,s}}{1-(c_{f,s}/c_A)^2}\dfrac{\boldsymbol{B}_t}{B_n} \\ \dfrac{\boldsymbol{B}_t}{(c_A/c_{f,s})^2 - 1} \end{bmatrix}, \quad (3.9)$$

in which the new variables, ξ_f and ξ_s, are introduced as follows:

$$\xi_f := \sqrt{\frac{a^2 - c_s^2}{c_f^2 - c_s^2}}, \quad \xi_s := \sqrt{\frac{c_f^2 - a^2}{c_f^2 - c_s^2}}. \quad (3.10)$$

These factors are necessary to ensure that the eigenvectors do not vanish for any $\boldsymbol{u}$. In deriving the above expressions of right eigenvectors, we assume that $\boldsymbol{u} = (\rho, p, v_n, \boldsymbol{v}_t, \boldsymbol{B}_t)$. Since we do not use the right eigenvectors for the Alfvén and entropy waves in solving the MHD Riemann problem, we do not give their explicit forms here. Note that the Alfvén and entropy waves are linear whereas the fast and slow waves are neither linear nor genuinely nonlinear.

The eigenvalues are degenerate in the following two cases:

$$B_n = 0: \quad c_f = \sqrt{a^2 + \frac{\boldsymbol{B}_t^2}{\rho}}, \quad c_s = c_A = 0, \quad (3.11)$$

$$B_n \neq 0, \ \boldsymbol{B}_t = \boldsymbol{0}: \quad c_f = \max(a, c_A), \quad c_s = \min(a, c_A). \quad (3.12)$$

In the latter case, the limits of the right eigenvectors for fast and slow waves as $\boldsymbol{B}_t \to \boldsymbol{0}$ depend on the magnitudes of the acoustic and Alfvén speeds. For $a > c_A$ we obtain

$$\boldsymbol{r}_f^{\mp} \to \begin{bmatrix} -\rho \\ -\gamma p \\ \pm a \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{r}_s^{\mp} \to a\begin{bmatrix} 0 \\ 0 \\ 0 \\ \pm\mathrm{sgn}(B_n)\,\boldsymbol{e}_t \\ \sqrt{\rho}\,\boldsymbol{e}_t \end{bmatrix}, \quad (3.13)$$

where $\boldsymbol{e}_t$ is a unit vector that has the same direction as the transverse magnetic field. Note that $\boldsymbol{r}_f^{\mp}$ is reduced to the eigenvectors for the rarefaction waves in ordinary hydrodynamics. In the opposite case, i.e., $a < c_A$, we get

$$\boldsymbol{r}_f^{\mp} \to a\begin{bmatrix} 0 \\ 0 \\ 0 \\ \mp\mathrm{sgn}(B_n)\boldsymbol{e}_t \\ -\sqrt{\rho}\boldsymbol{e}_t \end{bmatrix}, \quad \boldsymbol{r}_s^{\mp} \to \begin{bmatrix} -\rho \\ -\gamma p \\ \pm a \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \quad (3.14)$$

in which $\boldsymbol{r}_s^{\mp}$ is reduced to the eigenvectors for the ordinary rarefaction waves in hydrodynamics. Finally, in the case of $a = c_A$, we find

$$\boldsymbol{r}_f^{\mp} \to \frac{1}{\sqrt{2}}\begin{bmatrix} -\rho \\ -\gamma p \\ \pm a \\ \mp a\boldsymbol{e}_t \\ -a\sqrt{\rho}\boldsymbol{e}_t \end{bmatrix}, \quad \boldsymbol{r}_s^{\mp} \to \frac{1}{\sqrt{2}}\begin{bmatrix} -\rho \\ -\gamma p \\ \pm a \\ \pm a\boldsymbol{e}_t \\ a\sqrt{\rho}\boldsymbol{e}_t \end{bmatrix}. \quad (3.15)$$

As mentioned earlier, the right eigenvectors are chosen in our code so that these degenerate cases could be properly handled as the limits of non-degenerate cases.

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In the fast rarefaction wave the magnitude of transverse magnetic field is decreased and, as a limiting case, it vanishes behind the so-called switch-off rarefaction wave. Since the fast rarefaction wave cannot reverse the direction of the transverse magnetic field, the switch-off rarefaction is the end point of the fast rarefaction locus.

3.2. Discontinuities

As mentioned earlier, discontinuities are another important element in the solutions of Riemann problem. The quantities on both sides of a discontinuity satisfy the Rankine–Hugoniot relations, which in ideal MHD are expressed as

$$m = \text{const.}, \quad (3.16)$$

$$m^2[\![v]\!] + \left[\!\!\left[p + \frac{\boldsymbol{B}_t^2}{2}\right]\!\!\right] = 0, \quad (3.17)$$

$$m[\![\boldsymbol{v}_t]\!] - B_n[\![\boldsymbol{B}_t]\!] = \boldsymbol{0}, \quad (3.18)$$

$$m[\![v\boldsymbol{B}_t]\!] - B_n[\![\boldsymbol{v}_t]\!] = \boldsymbol{0}, \quad (3.19)$$

$$m\left(\left[\!\!\left[\frac{pv}{\gamma-1}\right]\!\!\right] + \langle p\rangle[\![v]\!] + \frac{1}{4}[\![v]\!][\![\boldsymbol{B}_t]\!]^2\right) = 0, \quad (3.20)$$

in the rest frame of the discontinuity. In the above expressions, $m := \rho_0 v_{n0} = \rho_1 v_{n1}$ is the mass flux, $v := 1/\rho$ is the specific volume, and $\langle X\rangle := (X_0 + X_1)/2$ stands for the arithmetic mean of upstream and downstream quantities. The solutions can be classified into the fast, slow, or intermediate shock or into the rotational, contact, or tangential discontinuity. Details of the classification are found in the literature (e.g., Jeffrey and Taniuti 1964; Polovin and Demutskii 1990; Torrilhon 2002, 2003). In what follows, we summarize only those features that are needed for later discussions.

Following Torrilhon (2002, 2003), we normalize all quantities with those upstream as

$$\hat{v} := \frac{v_1}{v_0}, \quad \hat{p} := \frac{p_1}{p_0}, \quad \hat{\boldsymbol{B}}_t := \frac{\boldsymbol{B}_{t1}}{\sqrt{p_0}}, \quad (3.21)$$

$$\boldsymbol{A} := \frac{\boldsymbol{B}_{t0}}{\sqrt{p_0}}, \quad B := \frac{B_n}{\sqrt{p_0}}, \quad M_0 := \frac{v_{n0}}{a_0}, \quad (3.22)$$

and employ in the following the dimensionless MHD Rankine–Hugoniot relations, which are obtained by substituting (3.21) and (3.22) and eliminating $[\![\boldsymbol{v}_t]\!]$ in (3.16)–(3.20):

$$\hat{p} - 1 + \gamma M_0^2(\hat{v} - 1) + \frac{1}{2}\left(\hat{\boldsymbol{B}}_t^{\,2} - \boldsymbol{A}^2\right) = 0, \quad (3.23)$$

$$\gamma M_0^2(\hat{v}\hat{\boldsymbol{B}}_t - \boldsymbol{A}) - B^2(\hat{\boldsymbol{B}}_t - \boldsymbol{A}) = 0, \quad (3.24)$$

$$M_0\left[\frac{1}{\gamma-1}(\hat{p}\hat{v} - 1) + \frac{1}{2}(\hat{v} - 1)(\hat{p} + 1) + \frac{1}{4}(\hat{v} - 1)(\hat{\boldsymbol{B}}_t - \boldsymbol{A})^2\right] = 0. \quad (3.25)$$

Fixing the upstream quantities, A, B, and M_0, we solve (3.23)–(3.25) and use (3.21) and (3.22) to obtain $\hat{v}$, $\hat{p}$, and $\hat{\boldsymbol{B}}_t$. The other downstream quantities can be calculated as

$$v_{n1} = \hat{v}v_{n0}, \quad (3.26)$$

$$\boldsymbol{v}_{t1} = \boldsymbol{v}_{t0} \pm \frac{a_0 B}{\gamma M_0}[\![\hat{\boldsymbol{B}}_t]\!]. \quad (3.27)$$

In (3.27), the plus and minus signs correspond to the left- and right-going discontinuities respectively.

3.2.1. Contact, tangential, and rotational discontinuities. The solutions of (3.23)–(3.25) that have a vanishing mass flux, i.e., $M_0 = 0$ but a non-vanishing normal component of magnetic field, i.e., $B \neq 0$, are called the contact discontinuity and satisfy the following relations:

$$\hat{v} = \text{arbitrary}, \quad \hat{p} = 1, \quad \hat{\boldsymbol{B}}_t = \boldsymbol{A}, \quad (3.28)$$

$$[\![\boldsymbol{v}]\!] = \boldsymbol{0}. \quad (3.29)$$

The solutions with $M_0 = 0$ and $B = 0$, on the other hand, are named the tangential discontinuity, for which the following relations hold:

$$\hat{v} = \text{arbitrary}, \quad \hat{p} - 1 + \frac{1}{2}(\hat{\boldsymbol{B}}_t^{\,2} - \boldsymbol{A}^2) = 0, \quad (3.30)$$

$$[\![v_n]\!] = 0, \quad [\![\boldsymbol{v}_t]\!] = \text{arbitrary}. \quad (3.31)$$

The solutions with $M_0 \neq 0$ and $B \neq 0$ are either a linear wave ($\hat{v} = 1$) or a shock wave ($\hat{v} > 1$). The former is referred to as the rotational discontinuity, since the transverse component of magnetic field rotates, not varying its magnitude during its passage. The rotational discontinuities meet the following conditions:

$$\hat{v} = 1, \quad \hat{p} = 1, \quad \hat{\boldsymbol{B}}_t^{\,2} = \boldsymbol{A}^2, \quad M_0^2 = \frac{B^2}{\gamma}, \quad (3.32)$$

$$[\![v_n]\!] = 0, \quad [\![\boldsymbol{v}_t]\!] = \pm\frac{1}{\sqrt{\rho}}[\![\boldsymbol{B}_t]\!], \quad (3.33)$$

where the plus and minus signs correspond to the left- and right-going waves respectively. The last relation in Eq. (3.32) implies the upstream Mach number is equal to the ratio of the Alfvén velocity to the acoustic speed there.

All the above discontinuities satisfy the evolutionary conditions except for the rotational discontinuity in which the transverse magnetic field rotates by 180°. The latter is sometimes called weakly evolutionary in the literature (Jeffrey and Taniuti 1964), since the neighboring rotational discontinuities are all evolutionary. Following this point of view, we treat solutions with weakly evolutionary discontinuities as regular solutions in this paper.[2]

3.2.2. Shock waves. The solutions of (3.23)–(3.25), for which $M_0 \neq 0$ and $\hat{v} > 1$, i.e., matter is compressed as it passes through the discontinuities, are called shock

[2] This may need reconsideration, however, and will be addressed elsewhere.

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waves. Their notable feature is that magnetic fields are either planar or coplanar. This is apparent from (3.24). Indeed, recalling $\hat{v} > 1$, we obtain

$$\hat{\boldsymbol{B}}_t = \frac{\gamma M_0^2 - B^2}{\gamma M_0^2 \hat{v} - B^2} \boldsymbol{A}. \tag{3.34}$$

Substituting $v_{n1} = \hat{v} v_{n0}$ to eliminate $\hat{v}$, we also obtain the following relation:

$$\hat{\boldsymbol{B}}_t = \frac{v_{n0}^2 - c_{A0}^2}{v_{n1}^2 - c_{A1}^2} \boldsymbol{A}, \tag{3.35}$$

which shows immediately that transverse magnetic fields are coplanar if and only if the upstream flow velocity is super-Alfvénic whereas the downstream speed is sub-Alfvénic.

The shocks with planar transverse magnetic fields are either fast or slow shocks, the former amplifies the magnitude of transverse magnetic fields whereas the latter reduces it. The shocks that change the direction of transverse magnetic fields are referred to as intermediate shocks. The terminology employed in the literature is rather confusing. It is remarked that the intermediate shock in this paper is different from the intermediate discontinuity defined in Jeffrey and Taniuti (1964). In fact, the latter is used for the discontinuities that satisfy the evolutionary conditions whereas the intermediate shocks in this paper are non-evolutionary as will be evident shortly.

Recalling $c_f \geqslant c_A \geqslant c_s$, we assign 1 to the states with super-fast velocities, 2 to those with sub-fast and super-Alfvénic velocities, 3 to those with sub-Alfvénic and super-slow velocities, and 4 to the states with sub-slow velocities in the shock-rest frame. With this allocation, the fast shock is denoted by $1 \to 2$ shock, since the upstream velocity is super-fast (state 1) whereas the downstream speed is sub-fast and super-Alfvénic (state 2). Similarly the slow shock is designated as $3 \to 4$ shocks. The intermediate shocks normally belong to one of the following four types: $1 \to 3$, $1 \to 4$, $2 \to 3$, and $2 \to 4$ shocks. The $1 \to 3$ and $2 \to 4$ intermediate shocks are called over-compressive shocks and 9 out of 14 ($= 7 \times 2$) characteristics run into these shock waves. To the $1 \to 3$ and $2 \to 4$ shocks converge the fast and Alfvén characteristics and the Alfvén and slow characteristics respectively. The $1 \to 4$ intermediate shock is doubly over-compressive and 10 characteristics of all types go into the shock. In the case of $2 \to 3$ shock, only the Alfvénic characteristic converges to the shock wave and the number of in- and out-going waves are right. However, the other evolutionary condition on the linear independence of eigenfunctions is violated and hence the shock is classified as non-evolutionary. In some cases, the flow velocity coincides with one of the characteristic velocities. We employ pairs of numbers to specify those states: '1, 2', '2, 3', and '3, 4' represent those states whose flow speed is equal to the fast, Alfvén, and slow speeds respectively. The shock wave with the upstream velocity being super-Alfvénic and the downstream speed being equal to the slow velocity, for example, is designated as the $2 \to 3, 4$ shock. Of our special concern among these intermediate shocks are the so-called switch-on ($1 \to 2, 3$) and switch-off ($2, 3 \to 4$) shocks, the details of which will be given shortly.

3.2.3. Fast and slow loci. We regard the shock solutions of (3.23)–(3.25) as functions of the upstream Mach number (M_0), normal (B_n or B), and transverse ($\hat{\boldsymbol{B}}_t$ or $\boldsymbol{A}$) components of magnetic field, divide them into two families, and look into their loci in some detail. Since magnetic fields in shock waves are either planar or coplanar as pointed out earlier, in the following we assume without loss of generality that magnetic fields are confined in the (x, y)-plane and treat $\hat{B}_t$ and A ($\geqslant 0$) as scalar variables.

Eliminating $\hat{p}$ and $\hat{B}_t$ from (3.23)–(3.25), we obtain the following cubic equation for the specific volume, $\hat{v}$:

$$\begin{aligned}(\gamma M_0^2 \hat{v})^3 - \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\gamma M_0^2 + 2B^2 + \frac{\gamma A^2}{\gamma+1}\right](\gamma M_0^2 \hat{v})^2 \\ + \left[2B^2\left(\frac{2\gamma}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\gamma M_0^2\right) + B^4 - \frac{2-\gamma}{\gamma+1}\gamma M_0^2 A^2 \right. \\ \left. + A^2 B^2\right]\gamma M_0^2 \hat{v} - \left[B^4\left(\frac{2\gamma}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\gamma M_0^2\right) \right. \\ \left. + \frac{\gamma-1}{\gamma+1}\gamma M_0^2 A^2 B^2\right] = 0.\end{aligned} \tag{3.36}$$

In deriving this equation, we used the assumption of $\hat{v} \neq 1$. The family of the fast shock is the solution characterized by the feature that matter is compressed and the transverse magnetic field is amplified by the passage of shock wave. It is then found that this branch of solutions satisfies the inequality $\hat{v}_{\min} < \hat{v} < 1$, where the minimum is given by

$$\hat{v}_{\min} = \max\left(\frac{B^2}{\gamma M_0^2}, \frac{\gamma-1}{\gamma+1}\right). \tag{3.37}$$

The loci of the solutions are shown in Fig. 2 as a function of the upstream Mach number (M_0) for some combinations of the upstream normal (B) and transverse (A) components of magnetic field.

As seen in the right panel of Fig. 2, the fast locus is divided into two branches for the special case of $A = 0$, i.e., the vanishing upstream transverse magnetic field, with Mach numbers satisfying the following inequalities:

$$\hat{c}_{f0} < M_0 < \sqrt{\frac{\gamma+1}{\gamma-1}\frac{B^2}{\gamma} - \frac{2}{\gamma-1}}, \tag{3.38}$$

where $\hat{c}_{f0} := c_{f0}/a_0$ is the normalized fast velocity. One of the branches that generates non-vanishing transverse magnetic fields by the shock passage is called the switch-on shock branch and the other is referred to as the Euler shock branch, in which the transverse component of magnetic field remains zero. The post-shock specific

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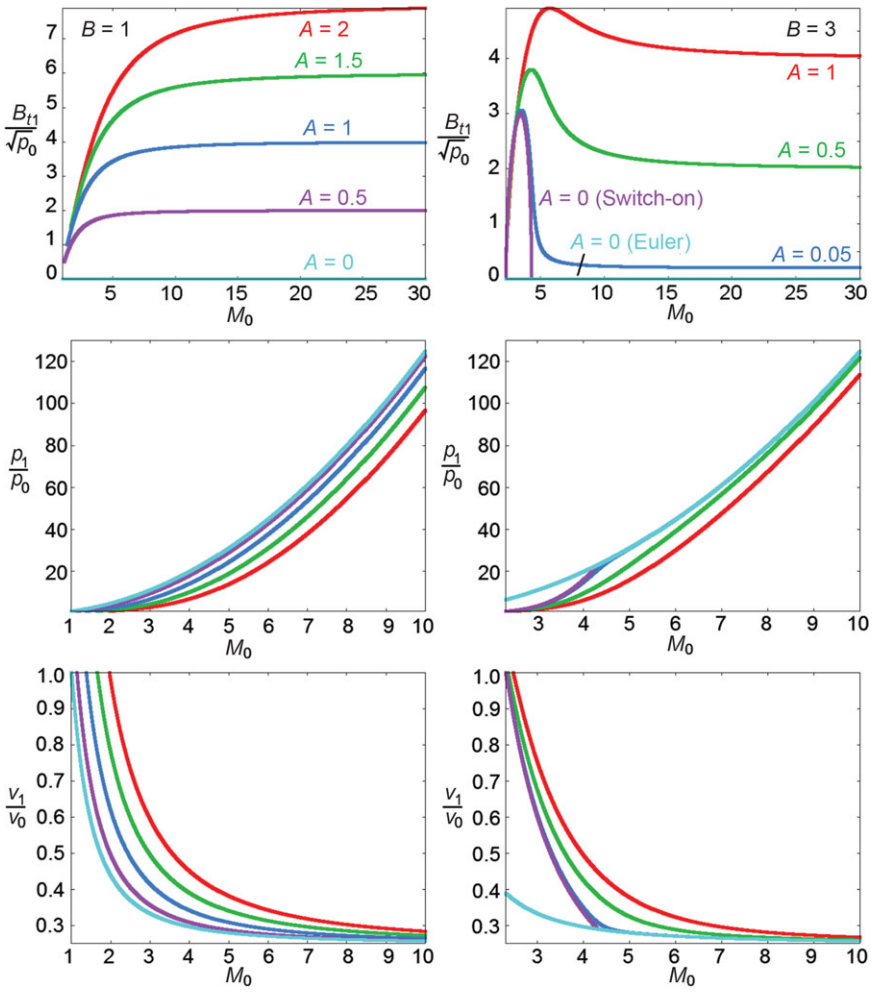


Figure 2. (Colour online) The fast loci for different combinations of normal (B) and transverse (A) components of magnetic field. The left panels: $B = 1$ and $A = 2$ (red), 1.5 (green), 1 (blue), 0.5 (purple), and 0 (light blue). The right panels: $B = 3$ and $A = 1$ (red), 0.5 (green), 0.05 (blue), 0 (purple, switch-on shock), and 0 (light blue, Euler shock). The switch-on shock does not exist for $B = 1$. See text for details.

volume and transverse magnetic field are given by

$$\hat{v} = \frac{2 + (\gamma - 1)M_0^2}{(\gamma + 1)M_0^2}, \quad \frac{B^2}{\gamma M_0^2}, \tag{3.39}$$

$$\hat{B}_t = 0, \quad \sqrt{\frac{\gamma M_0^2 - B^2}{B^2}\left[(\gamma - 1)\left(\frac{\gamma + 1}{\gamma - 1}B^2 - \gamma M_0^2\right) - 2\gamma\right]}, \tag{3.40}$$

respectively. In the above expressions, the first options correspond to the Euler shock and the second one to the switch-on shock. The requirement that the quantity in the square root be non-negative gives inequality (3.38). It is noted that the flow speed behind the switch-on shock is equal to the Alfvén speed. Hence, the switch-on shock is designated as the $1 \rightarrow 2,3$ shock and is non-regular. It is also noteworthy that the Euler shock is essentially a hydrodynamical shock wave and its locus is extended to $M_0 < \hat{c}_{f0}$, where it is smoothly connected to the slow-shock counterpart. The Euler shock is evolutionary except the range within which the switch-on shock branch appears, i.e., the range satisfying inequality (3.38).

The slow family is characterized by the feature that matter is compressed but the transverse magnetic field is reduced and in some cases reversed by the shock passage. The slow loci are shown in Fig. 3 as a function of the upstream Mach number (M_0) for a number of combinations of the upstream normal (B) and transverse (A) components of magnetic field. It is evident that some loci are two-valued as a function of M_0. There is a maximum Mach number in each locus, at which the downstream flow speed becomes equal to the the slow speed. In some cases the minimum values appear in the transverse magnetic fields for given A and B.

The intermediate shocks are those that give negative downstream transverse magnetic fields. The shocks that nullify the transverse magnetic field are called the switch-off shock ($2,3 \rightarrow 4$ shock). It is apparent that the switch-off shocks are located at the boundary between the regular slow shocks and the intermediate shocks, the fact which will be important later. Each locus is terminated at the point that corresponds to a rotational discontinuity,

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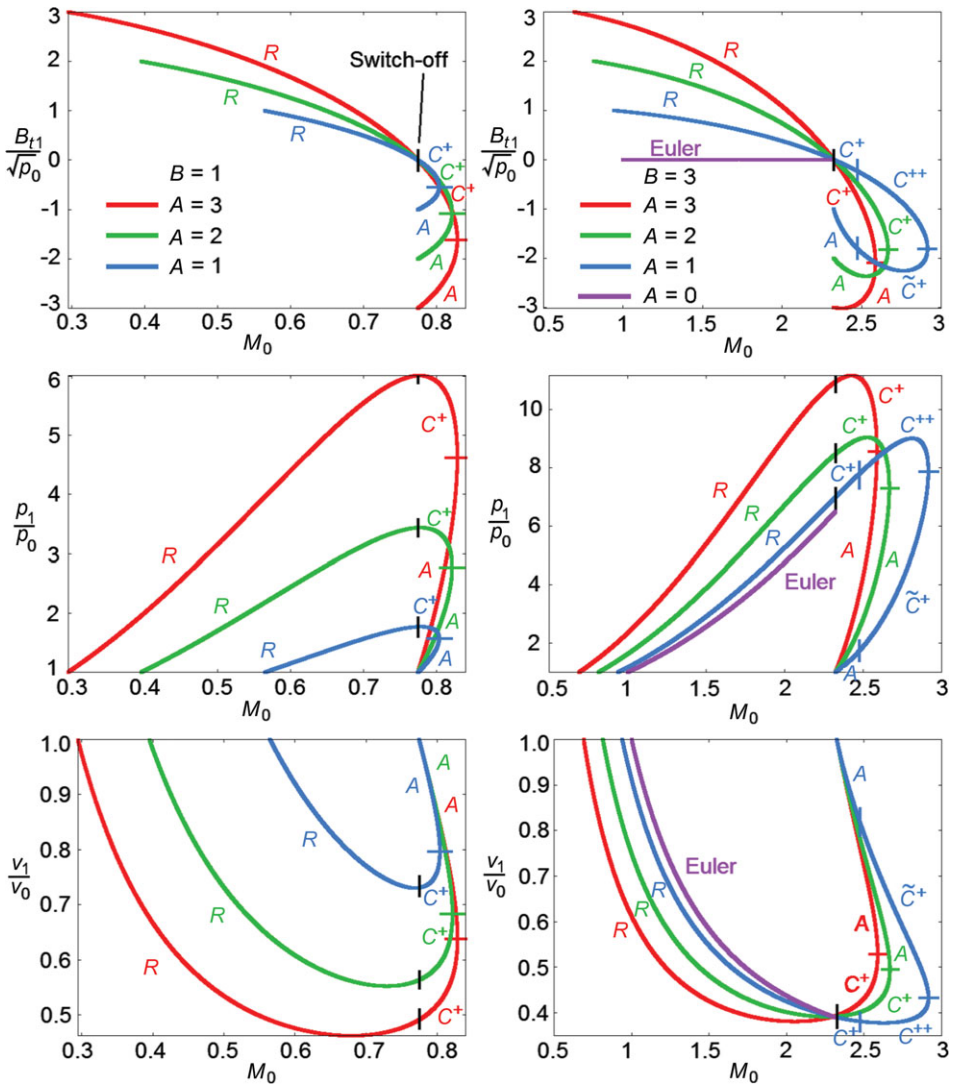


Figure 3. (Colour online) The slow loci for different combinations of normal (B) and transverse (A) components of magnetic field. The left panels: $B = 1$ and $A = 3$ (red), 2 (green), and 1 (blue). The vertical black dashes indicate the points at which $M_0 = \hat{c}_{A0}$ and switch-off shocks ($2, 3 \to 4$ shocks) occur. Since $\hat{c}_{A0}$ is independent of A, the Mach numbers, M_0s, at the points for all loci coincide with one another. These points mark the boundary between the regular slow shocks and the non-regular intermediate shocks. The characters, R, C^+, and A, attached to each locus, stand for the regular slow ($3 \to 4$), $2 \to 4$ intermediate, and $2 \to 3$ intermediate shocks respectively. The horizontal dash on each locus shows the point at which the Mach number reaches its maximum on the locus and a $2 \to 3, 4$ shock occurs. This is the boundary between the $2 \to 4$ (C^+) and $2 \to 3$ (A) intermediate shocks. The right panels: $B = 3$ and $A = 3$ (red), 2 (green), 1 (blue), and 0 (purple, Euler shocks). The vertical black dashes again give the boundary between the regular and non-regular shocks at which switch-off shocks ($2, 3 \to 4$ shocks) occur. The characters R, C^+, and A have the same meaning as in the left panels whereas C^{++} and $\tilde{C}^+$, which emerge only for small As, represent the $1 \to 4$ and $1 \to 3$ intermediate shocks respectively. The horizontal dash on each locus marks again the point at which the maximum Mach number is reached. On the other hand, the two vertical blue dashes on each blue locus indicate the points at which $M_0 = \hat{c}_{f0}$. A $1, 2 \to 4$ shock occurs at the point closer to the vertical black dashes whereas a $1, 2 \to 3$ shock emerges at the other point. A $1 \to 3, 4$ shock occurs at the point indicated by the horizontal blue dash. Note in passing that the locus vanishes at $B = 0$.

which rotates magnetic field by π radians, i.e., $\hat{B}_t = -A$. In the limit of $B \to 0$ or $A \to 0$, the slow branch vanishes, i.e., there is no solution that satisfies (3.36) and the inequalities $\hat{c}_{s0} < M_0 < \hat{c}_{f0}$ and $0 < \hat{v} < 1$ except when $A = 0$ and the upstream Alfvén speed is larger than the acoustic speed, i.e., $\hat{c}_A (:= c_{A0}/a_0) > 1$, in which case the Euler shock branch takes its place. This branch is extended to the regime of $M_0 > \hat{c}_{f0}$ and is connected smoothly to the fast-shock counterpart as mentioned earlier.

4. Solving the MHD Riemann problem

The basics for finding solutions for the MHD Riemann problem are essentially the same as that of Torrilhon (2002). However, we have extended these substantially so that we could obtain all types of solutions for arbitrary initial conditions, including non-regular ones. In the following we briefly describe the basic ideas, and the technical details will be published elsewhere (Takahashi and Yamada 2012).

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First we consider the regular solutions that were treated by Torrilhon (2002). It is assumed that both normal and transverse components of magnetic field are non-vanishing. This implies that the degenerate case mentioned in Sec. 3.1 is ignored. As a definition of the regular solution, no intermediate shock is involved. It is further assumed that there is no switch-off rarefaction wave either. Since the switch-off rarefaction wave is regular, this is another genuine restriction on the solutions that can be handled. Thanks to these assumptions, the types of waves generated and their order are known *a priori*: the fast, Alfvén, and slow waves are fanning out in this order on both sides of the contact discontinuity. Since each regular wave forms a one-parameter family, the regular solutions considered now are specified by the seven parameters that are associated with the waves involved in the solution. At the contact discontinuity, the six quantities other than density are continuous. Hence, the six parameters out of seven that correspond to regular waves other than contact discontinuity are determined by the requirement that the six quantities other than density be continuous at the contact discontinuity. Then the last parameter that specifies the contact discontinuity is automatically obtained. As pointed out by Torrilhon (2002), further simplification arises from the observation that only the rotational discontinuities rotate transverse magnetic fields. In fact, the sum of rotation angles at the left- and right-going rotational discontinuities is equal to the angle that are made by the transverse components of magnetic fields on both sides of initial discontinuity. Hence, we have only to solve one of the rotation angles.

We thus arrive at the reduced problem, in which the remaining five parameters should be determined. The system of equations is expressed symbolically as

$$\boldsymbol{F}^s_L(\psi^s_L; \boldsymbol{u}^r_L(\psi^r; \boldsymbol{u}^f_L(\psi^f_L; \boldsymbol{u}_L))) - \boldsymbol{F}^s_R(\psi^s_R; \boldsymbol{u}^r_R(\psi^r; \boldsymbol{u}^f_R(\psi^f_R; \boldsymbol{u}_R))) = \boldsymbol{0}, \tag{4.1}$$

where ψ^f_L, ψ^s_L, ψ^f_R, ψ^s_R, and ψ^r are the five parameters that specify jumps at the left-going fast and slow waves, right-going fast and slow waves, and rotational discontinuities respectively. The initial left and right states are given by $\boldsymbol{u}_{L,R} = {}^t(\rho_{L,R}, p_{L,R}, \boldsymbol{v}_{L,R}, \boldsymbol{B}_{tL,R})$. The downstream states of the left- and right-going fast waves are expressed as $\boldsymbol{u}^f_{L,R}$, which are the functions of upstream states $\boldsymbol{u}_{L,R}$, and parameters $\psi^f_{L,R}$. Similarly, $\boldsymbol{u}^r_{L,R}$ represents the downstream states of left- and right-going rotational discontinuities. The five downstream quantities, p, $\boldsymbol{v}$, and $|\boldsymbol{B}_t|$, of the left- and right-going slow waves, which should be continuous at the contact discontinuity, are denoted by $\boldsymbol{F}^s_{L,R}$. These equations are numerically solved by the Newton–Raphson method in our code.

As mentioned already, if the switch-off rarefaction is involved in the solution, the above method cannot be applied as it is, since we do not know *a priori* which waves emerge in the solution. Torrilhon (2002, 2003) neglected such possibilities by assuming that both normal and transverse components of magnetic fields are non-vanishing. This is also the case for the code developed by Andreev et al. (2008) and available online, which appears to be based on Torrilhon (2002, 2003). Hence, their codes cannot handle all regular solutions in addition to non-regular ones, which they did not take into account from the beginning.

On the contrary, our code can treat all these cases: vanishing normal or transverse component of magnetic field with a possible existence of the switch-off rarefaction waves as well as the emergence of intermediate shocks, including the switch-on and switch-off shocks. All intermediate shocks other than switch-on shocks are parametrized as slow shocks because they are parts of slow loci. The switch-on shocks and the switch-off rarefactions, on the other hand, are treated as a part of fast shock and rarefaction respectively. As already mentioned, since we do not know *a priori* which waves are generated, we need to try all possible combinations. For example, if the transverse magnetic fields are non-vanishing initially, we first attempt to find solutions that do not include a fast wave on one side of the contact discontinuity, i.e., solution with either a '$1 \to 3$', '$1 \to 4$', '$1 \to 3,4$', '$1,2 \to 3$', '$1,2 \to 4$', or '$1,2 \to 3,4$' intermediate shock being included. We then try to obtain a solution that does include a fast wave but not a rotational discontinuity, i.e., the solution with either a '$2 \to 3$', '$2 \to 4$', or '$2 \to 3,4$' intermediate shock or a switch-off shock. Finally, regular solutions are sought. The technical details of these procedures will be described in a separate paper (Takahashi and Yamada 2012).

5. Results

In this section we apply our new code to two specific initial conditions and their neighbors. One of them is first picked up by Brio and Wu (1988) and is known to have a non-regular solution with a compound wave. This is one of the initial conditions in the MHD Riemann problem that are best known to developers of numerical MHD codes. As demonstrated by Andreev et al. (2008), there is also a regular solution to this problem. The controversy in the literature is why the former is almost always obtained in numerical solutions. After confirming the above regular and non-regular solutions by our code, we show that there are actually uncountably many non-regular solutions to this problem. By studying the properties of these solutions as well as exploring the neighboring solutions, we obtain new insights into the controversy.

In the second application, we present the initial condition, for which there seems to be no regular solution. In fact, the non-regular solution we found is a limit of a sequence of regular solutions and include a switch-off shock wave. We investigate neighboring solutions in detail and reveal how the solution is approached by

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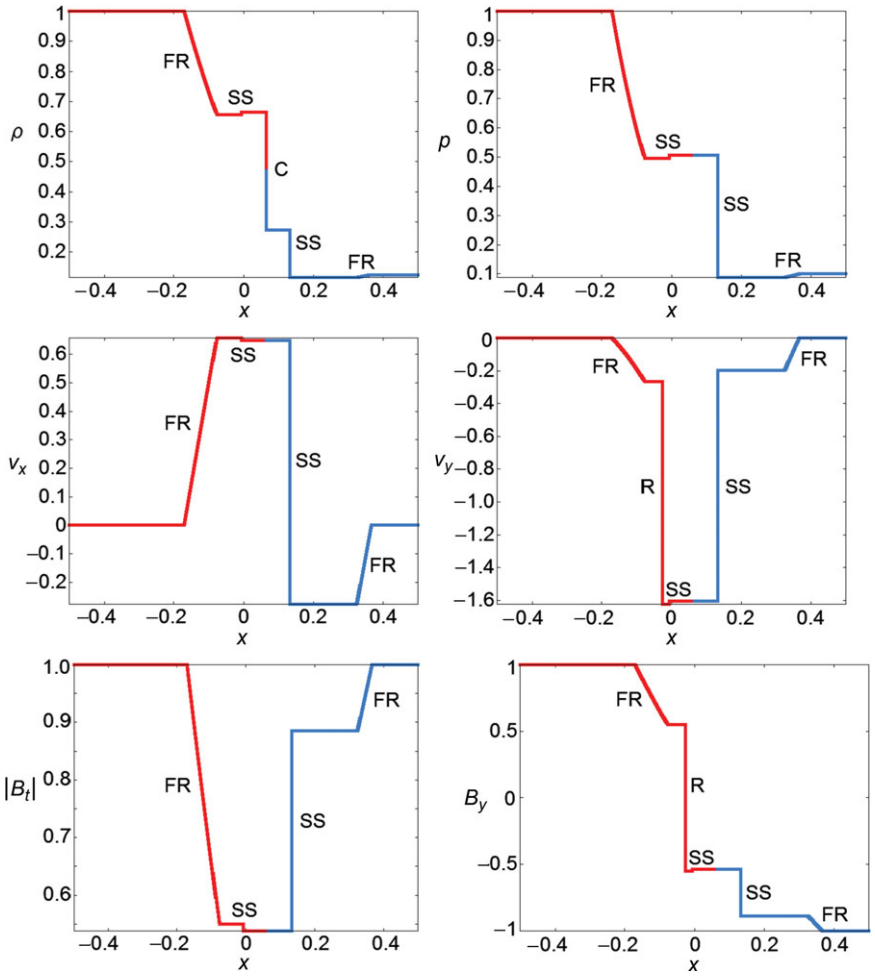


Figure 4. (Colour online) The regular solution for the Brio and Wu (1988) problem. Waves in red and blue portions that are separated by contact discontinuity are left- and right-going respectively. The designations FR, R, SS, and C represent fast rarefaction wave, rotational discontinuity, slow shock, and contact discontinuity respectively.

various sequences of solutions. The result forces us to rethink interpretations of regular waves and switch-off shocks.

5.1. *Brio and Wu problem: uncountably many non-regular solutions*

Here we present solutions for the initial condition in the MHD Riemann problem solved numerically by Brio and Wu (1988). The left and right states in the initial condition are given by

$$(\rho_L,\ p_L,\ v_{xL},\ v_{yL},\ v_{zL},\ B_{yL},\ B_{zL}) = (1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0), \tag{5.1}$$

$$(\rho_R,\ p_R,\ v_{xR},\ v_{yR},\ v_{zR},\ B_{yR},\ B_{zR}) = (0.125,\ 0.1,\ 0,\ 0,\ 0,-1,\ 0). \tag{5.2}$$

The normal component of magnetic field is $B_n = 0.75$ and we take for the adiabatic index $\gamma = 5/3$ instead of $\gamma = 2$, the value adopted by Brio and Wu (1988). This is because the Riemann invariants are singular at $\gamma = 2$ (e.g., Takahashi and Yamada 2012) and cannot be used for check as done usually. Note, however, even in this case there is no problem to find solutions by our code. In fact, we confirmed that the solutions are not different qualitatively between $\gamma = 5/3$ and $\gamma = 2$. This is a coplanar problem, for which both regular and non-regular solutions are expected. It is also noted that this can be regarded as a Riemann problem of the reduced MHD system.

We begin with the regular solution. In Fig. 4 we show various quantities in the solution at $t = 0.1$. v_z and B_z are identically zero and omitted in the figure. The solution consists of a fast rarefaction wave (denoted by FR in the figure), rotational discontinuity (R), and slow shock (SS) going leftwards and a fast rarefaction wave, slow shock, and contact discontinuity running in the right direction in addition to a contact discontinuity (C). No right-going rotational discontinuity exists because the left-going rotational discontinuity has already rotated the magnetic field by π radians. This is the same solution as the one obtained by Andreev et al. (2008).

Next we present a non-regular solution, various quantities in which are displayed in Fig. 5 at $t = 0.1$. As is evident, this solution contains a compound wave (denoted by $2 \rightarrow 3,4$ IS and SR in the figure) and is

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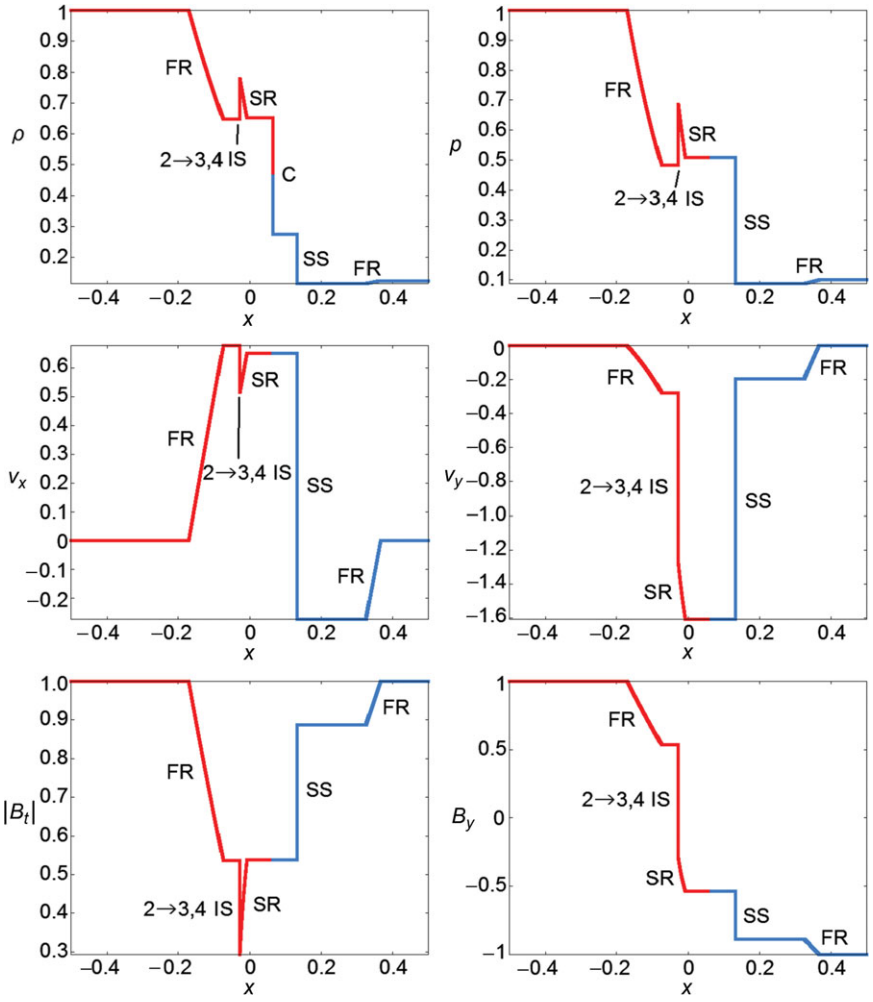


Figure 5. (Colour online) The non-regular solution for the Brio and Wu (1988) problem that contains a compound wave. The notation is the same as in Fig. 4 except for the designations SR and IS, which stand for slow rarefaction wave and intermediate shock respectively.

exactly the solution that Brio and Wu (1988) obtained in their numerical simulations. The compound wave is a $2 \rightarrow 3,4$ intermediate shock, to which a slow rarefaction wave is attached. Other waves that constitute the solution are a left-going fast rarefaction wave and a contact discontinuity, slow shock, and fast rarefaction wave going rightward. Note that the compound wave is responsible for changing the direction of magnetic field. It should be also pointed out that the strengths of the waves other than the compound wave are different between regular and non-regular solutions. It is the merit of our code that these non-regular solutions can be handled unlike the one developed and made available for public use by Andreev et al. (2008), which can treat only regular solutions.

Another advantage of our code is that it attempts to find all solutions, trying all combinations of constituent waves, including intermediate shocks. In its application to the current problem, we found that other non-regular solutions exist, which involve various $2 \rightarrow 3$ intermediate shocks. Some of such solutions are shown in Fig. 6 at $t = 0.1$. The solution given in the top panel resembles the non-regular solution with the compound wave presented above. A closer look (see the inset), however, reveals a difference: a $2 \rightarrow 3$ intermediate shock is closely followed by a slow rarefaction wave but they are detached. The separation of the two waves is more apparent in the middle panel, in which the strengths of the intermediate shock and the rarefaction wave are also weaker than in the top panel. In the bottom panel, the solution does not include a slow rarefaction wave but a slow shock instead. The $2 \rightarrow 3$ intermediate shock is even weaker in this case. It is interesting that the strengths of other waves other than the intermediate shock do not change very much. It should be emphasized here that these are the solutions to the same initial condition.

As a matter of fact, these solutions form a one-parameter family, with the Brio and Wu (1988) solution with the compound wave at one end and the regular solution with a rotational discontinuity at the other end point of the sequence. The parameter that characterizes the solution family is a magnitude of jump in the transverse magnetic field across the $2 \rightarrow 3$ intermediate shock. Since the sequence of solutions is obtained by gradually changing this parameter, we can deduce that in fact there are uncountably many solutions. As already

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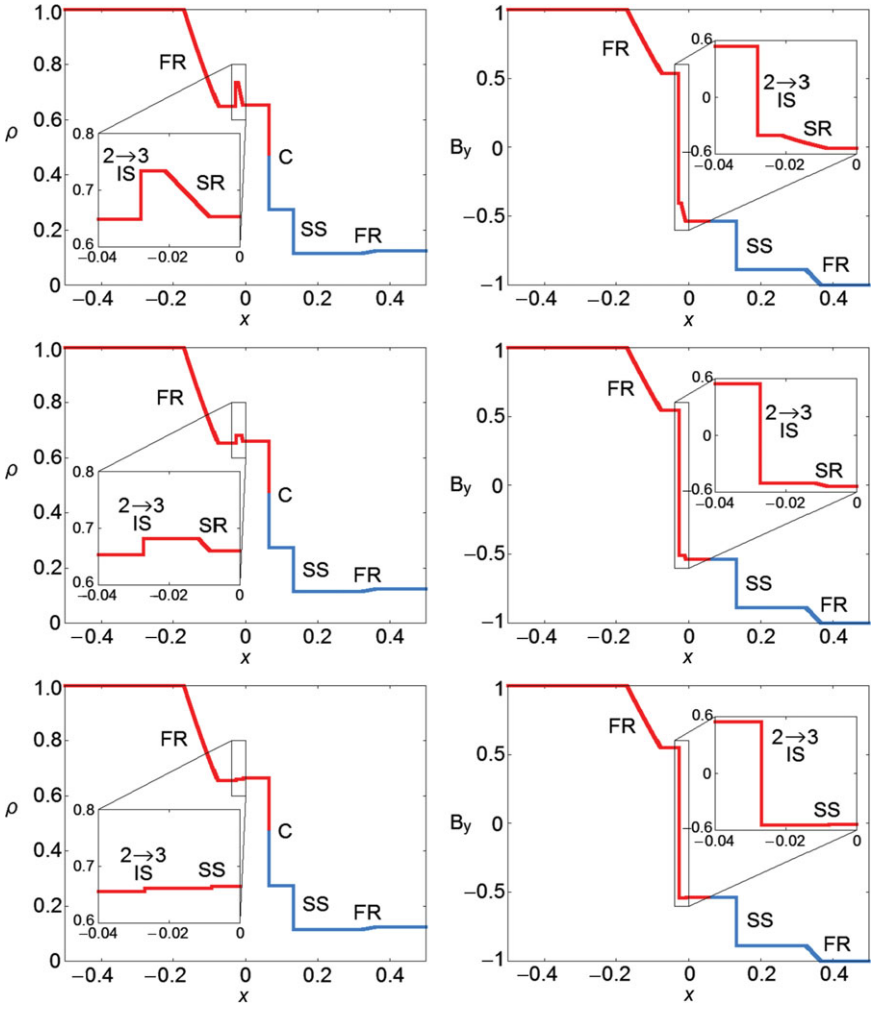


Figure 6. (Colour online) Some of the uncountably many non-regular solutions for the Brio and Wu (1988) problem that contain a $2 \rightarrow 3$ intermediate shock. The notation is the same as in Fig. 5. The insets are the close-ups of indicated regions.

mentioned, of these non-regular solutions and the unique regular solution, it is the Brio and Wu (1988) non-regular solution that numerical simulations almost always obtain. From the point of view of the evolutionary conditions, the regular solution is the only solution that contains evolutionary shocks alone. Both $2 \rightarrow 3,4$ and $2 \rightarrow 3$ shocks are non-evolutionary. Hence, numerical simulations seem to prefer the non-evolutionary solution to the evolutionary solution. This is the original controversy. Now we have added another question: Why is the Brio and Wu (1988) solution is singled out from the uncountably many non-regular solutions with non-evolutionary shocks?

Since the evolutionary conditions are related with the neighboring solutions, we consider the initial conditions that are close to the original one. Both non-coplanar and coplanar perturbations are discussed in turn separately. As a non-coplanar initial configuration we take the following:

$$(\rho_L,\ p_L,\ v_{xL},\ v_{yL},\ v_{zL},\ B_{yL},\ B_{zL}) = (1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0), \tag{5.3}$$

$$(\rho_R,\ p_R,\ v_{xR},\ v_{yR},\ v_{zR},\ B_{yR},\ B_{zR}) = (0.125,\ 0.1,\ 0,\ 0,\ 0, \cos 3.0,\ \sin 3.0). \tag{5.4}$$

The transverse magnetic field in the right state is rotated slightly with other quantities being unchanged. In accord to the evolutionary condition, we found only a regular solution, which is shown in Fig. 7. The solution includes left- and right-going rotational discontinuities, which are responsible for the rotation of magnetic fields. As evident from the comparison (see Fig. 4), this solution is indeed close to the regular solution of the original problem and can be regarded as the perturbed solution. On the other hand, no neighboring non-regular solution exists in this case. Wu (1988a,b), however, observed in his dissipative MHD simulations a new type of non-coplanar shock-like structures that do not satisfy the Rankine–Hugoniot relations. They concluded that this time-dependent 'intermediate shocks' are the neighboring states to the ordinary intermediate shocks. It is however noted that the time-dependent non-coplanar structures of these new 'intermediate shocks' depend upon dissipation coefficients as shown in Wu (1988a,b)

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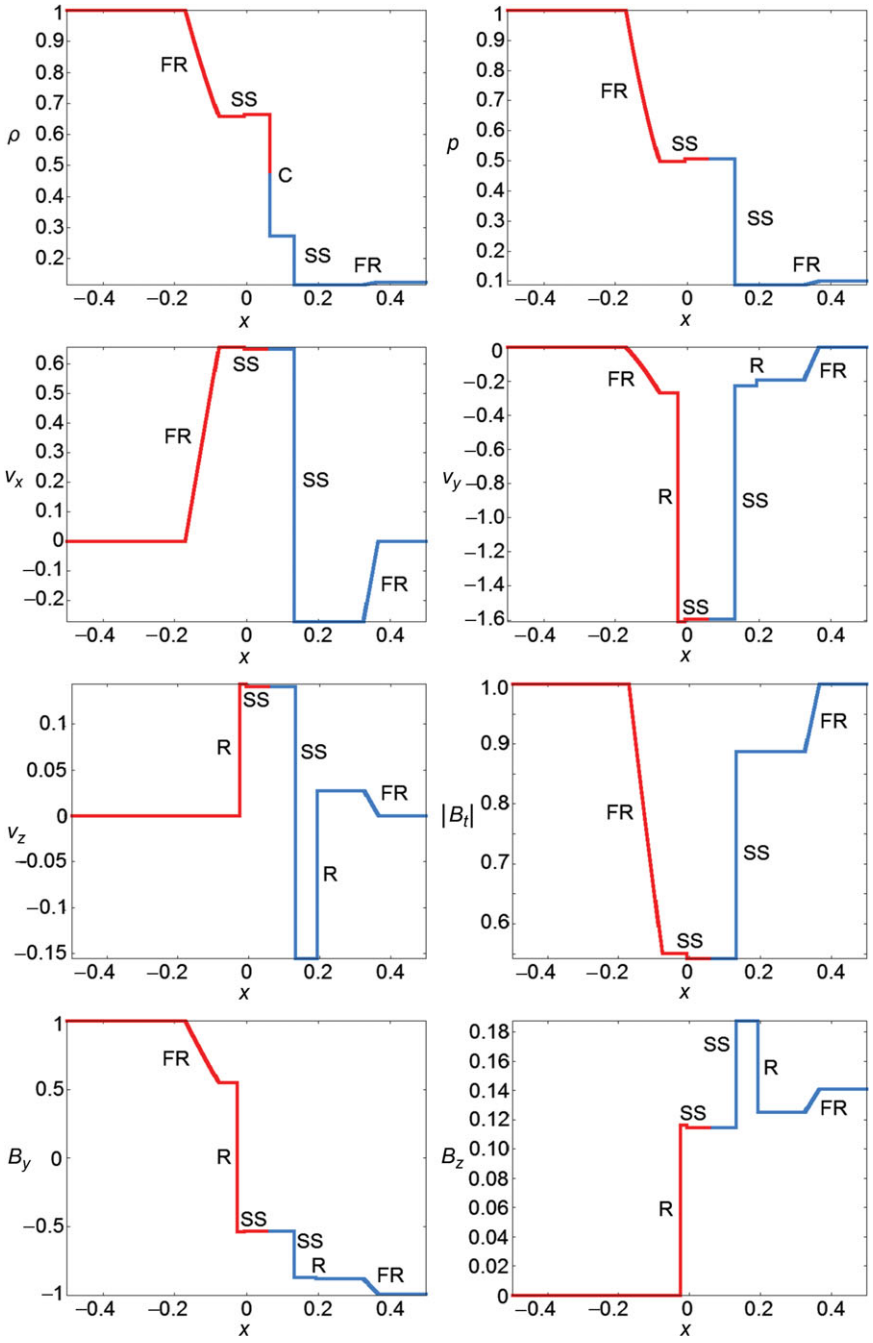


Figure 7. (Colour online) A non-coplanar neighboring solutions to the Brio and Wu (1988) problem. The initial magnitude of the transverse magnetic field of the right state is the same as the original value. The notation is the same as in Fig. 4.

and his succeeding one (Wu 1990) and may not be realized in ideal MHD.

It is of great interest to investigate coplanar perturbations, since coplanarity is maintained even in numerical simulations unless it is broken explicitly. If the symmetry is retained indeed, we are essentially dealing with the reduced MHD system, in which the Alfvén characteristics do not exist. Then it should be noted that the evolutionary conditions will be modified as pointed out by Falle and Komissarov (2001). For example, the $1 \rightarrow 3$ and $2 \rightarrow 4$ shocks become evolutionary and the rotational discontinuities become non-evolutionary. On the other hand, the $2 \rightarrow 3$ shock remains non-evolutionary with no converging characteristic. It is noteworthy that the $2 \rightarrow 3,4$ shock is evolutionary, since the slow characteristics are marginally converging and the linear independence of eigenfunctions of outgoing waves and the initial jumps is recovered. As a consequence of these changes, the Brio and Wu solution (1988) should be regarded as a regular solution in the reduced system whereas other solutions with $2 \rightarrow 3$ shocks are still non-regular even in this system. Falle and Komissarov (1997, 2001) claimed that this is the reason why numerical simulations almost always obtain the Brio and Wu (1988) solution. The problem may be a bit more complicated, however, as will be demonstrated shortly.

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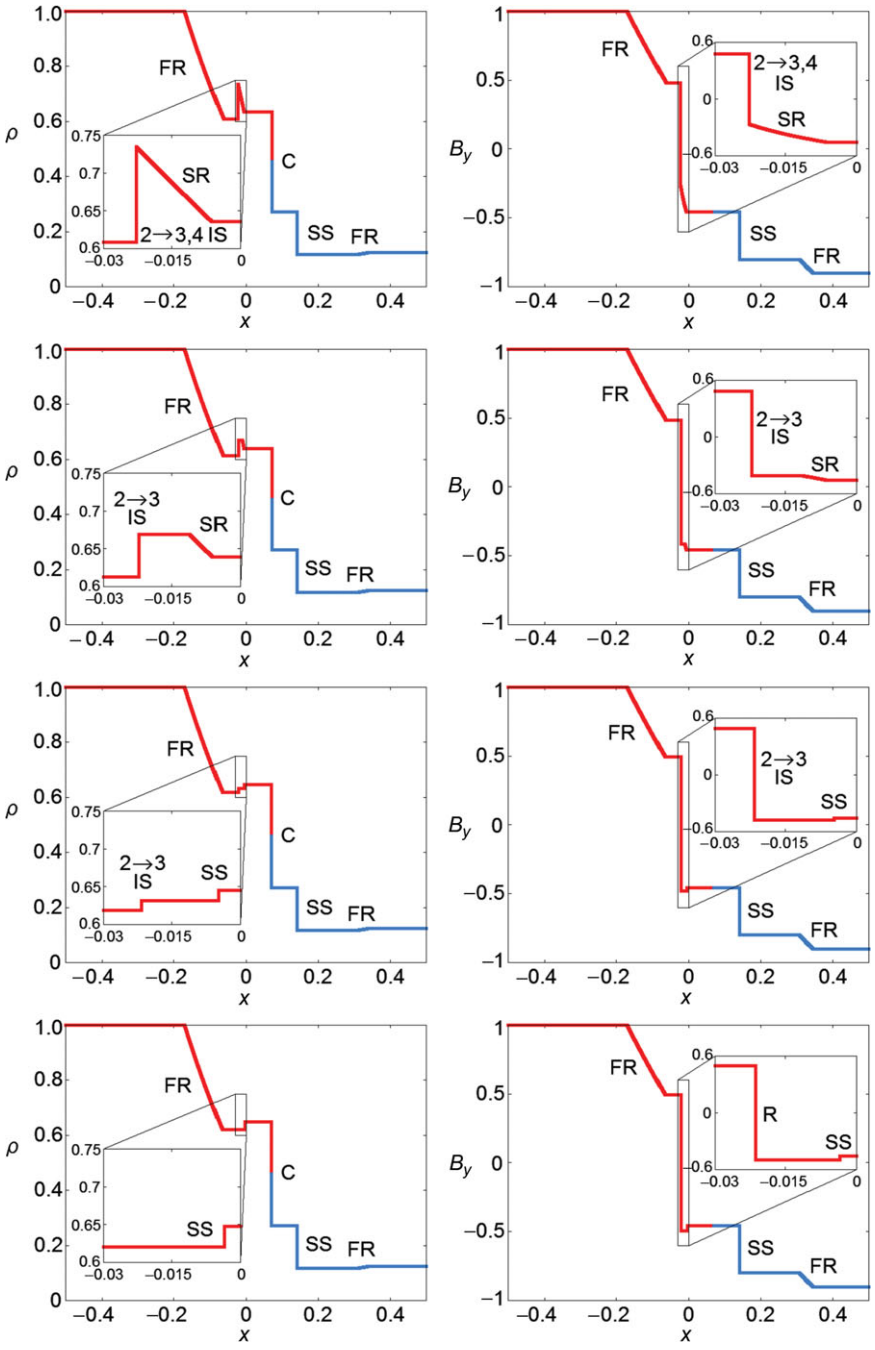


Figure 8. (Colour online) Some coplanar neighboring solutions to the Brio and Wu (1988) problem. The initial magnitude of the transverse magnetic field of the right state is reduced from the original value. The notation is the same as in Fig. 5. The insets are the close-ups of indicated regions.

In Fig. 8 we present the solutions that we found with our code for the following initial condition:

$$(\rho_L,\ p_L,\ v_{xL},\ v_{yL},\ v_{zL},\ B_{yL},\ B_{zL}) = (1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0), \tag{5.5}$$

$$(\rho_R,\ p_R,\ v_{xR},\ v_{yR},\ v_{zR},\ B_{yR},\ B_{zR}) = (0.125,\ 0.1,\ 0,\ 0,\ 0,-0.95,\ 0). \tag{5.6}$$

This time we reduced slightly the transverse magnetic field in the right state, retaining the coplanarity. We then found essentially the same types of solutions as in the original problem: (1) solution with a rotational discontinuity,[3] (2) solution with the compound wave consisting of a $2 \to 3,4$ shock and slow rarefaction wave, and (3) solution with a $2 \to 3$ shock. As mentioned, the second solution is regular and the first and the third are now non-regular in the reduced MHD system. The

[3] The rotational discontinuity may be an inappropriate term in the reduced system, in which the Alfvén characteristic ceases to exist. However, we use it to make clear the correspondence to the full system.

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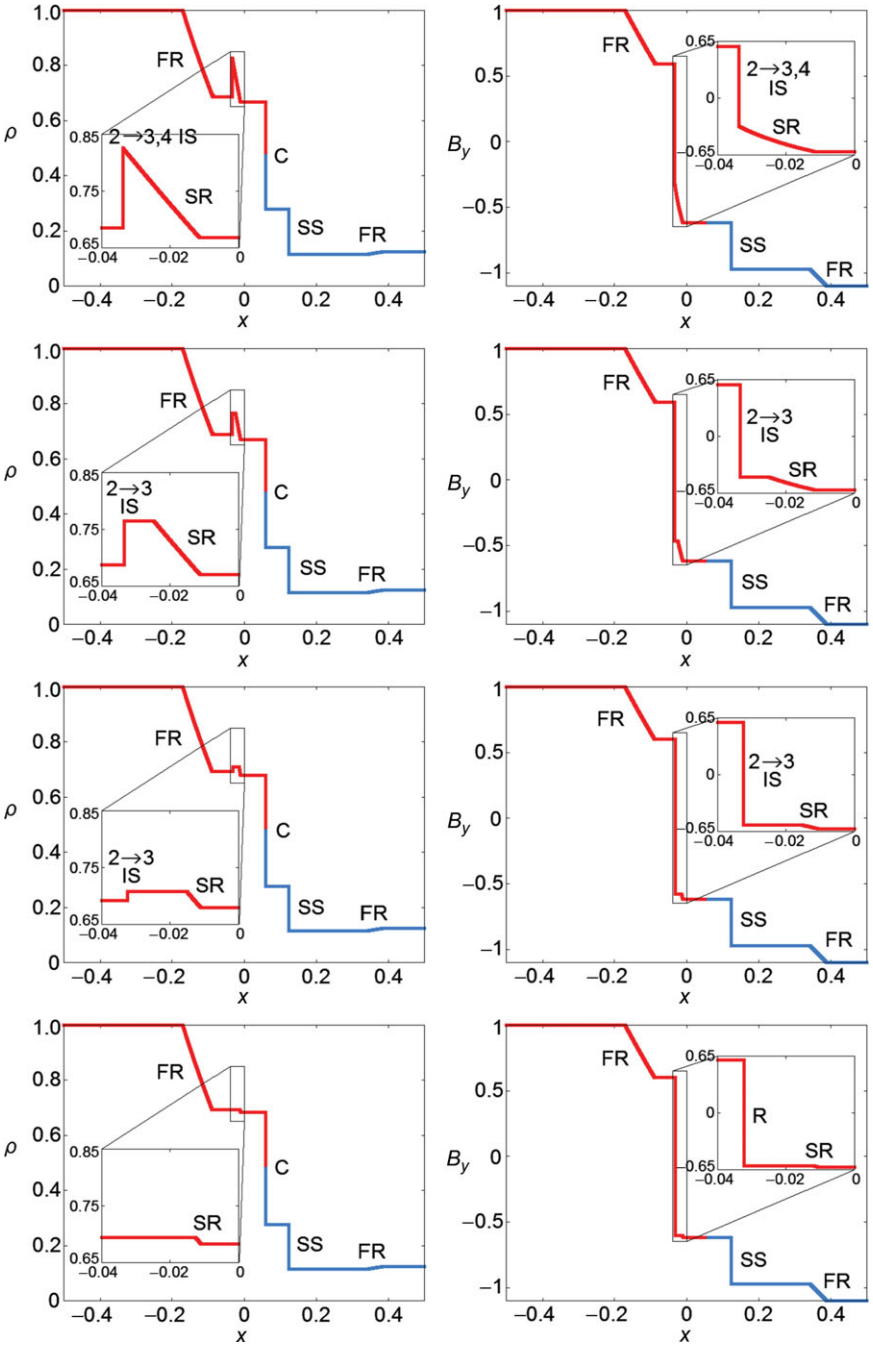


Figure 9. (Colour online) Some coplanar neighboring solutions to the Brio and Wu (1988) problem. The initial magnitude of the transverse magnetic field of the right state is increased from the original value. The notation is the same as in Fig. 5. The insets are the close-ups of indicated regions.

three types of solutions actually form a one-parameter solution family with the first and the second solutions being the end points. As shown in Fig. 9 this is also the case for the following initial condition:

$$(\rho_L,\ p_L,\ v_{xL},\ v_{yL},\ v_{zL},\ B_{yL},\ B_{zL}) = (1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0), \tag{5.7}$$

$$(\rho_R,\ p_R,\ v_{xR},\ v_{yR},\ v_{zR},\ B_{yR},\ B_{zR}) = (0.125,\ 0.1,\ 0,\ 0,\ 0,-1.05,\ 0), \tag{5.8}$$

in which the transverse magnetic field in the right state is slightly amplified.

These results show clearly that the Brio and Wu (1988) solution, or the regular solution with a compound wave, has a unique neighboring solution of the same type as expected from the evolutionary conditions in the reduced system. In the case of non-regular solutions either with the $2 \rightarrow 3$ shock or with the rotational discontinuity, however, the perturbed solutions are not uniquely determined, since there are too many outgoing characteristics. And the existence of uncountably

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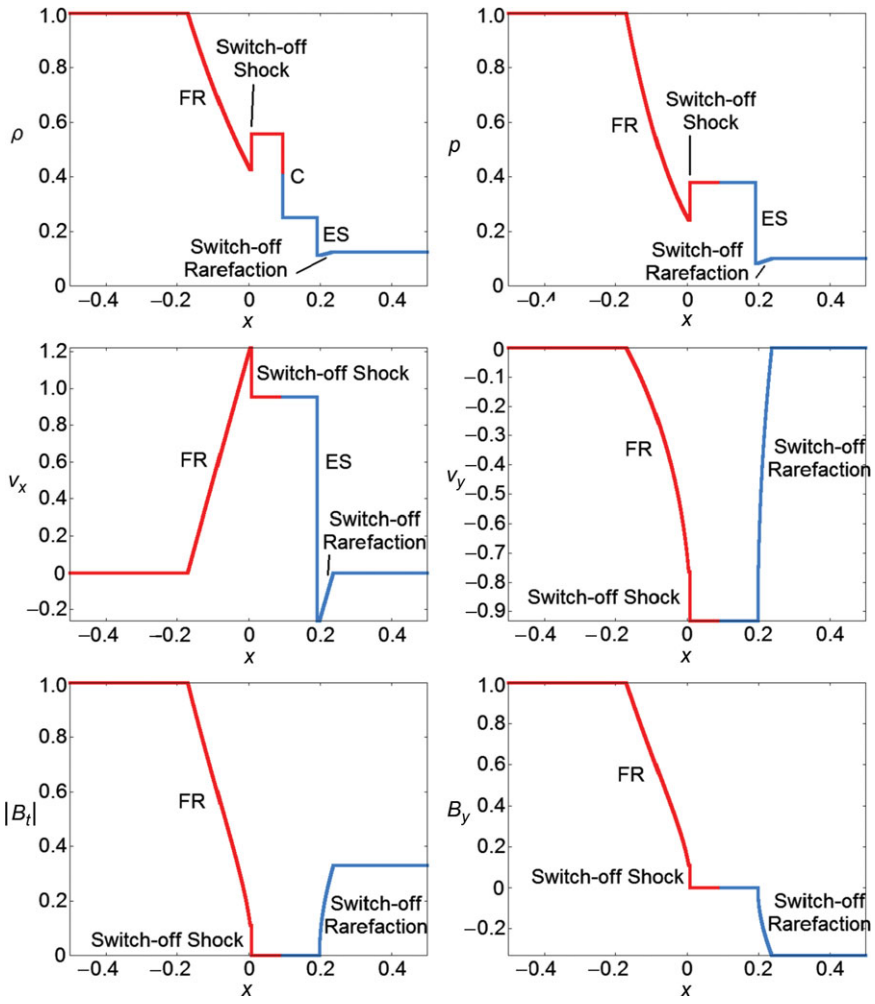


Figure 10. (Colour online) The non-regular solution with both switch-off shock and switch-off rarefaction wave. The notation is the same as in Fig. 5 except for the designation ES, which stands for Euler shock. The initial condition is the same as for the Brio and Wu (1988) problem except for the strength of the transverse magnetic field of the right state. See Sec. 5.2 of text for precise value.

many non-regular solutions is a manifestation of this under-determination. Unlike the over-determination and as a consequence the non-existence of neighboring solution, the meaning of the non-uniqueness of perturbed solutions is rather obscure. The results of numerical simulations however indicate that the evolutionary conditions in the reduced system appear to work in selecting the solution.

5.2. *An initial condition with no regular solution*

In the continuous modifications of the Brio and Wu (1988) problem, we ran into an interesting initial condition, for which there seems to be no regular solution. If true, this is a counter-example to conventional expectation that a unique regular solution always exists. This initial condition is truly a special point unlike the Brio and Wu problem. In fact, the regular and non-regular solutions of the Brio and Wu problem are shared by neighboring coplanar initial conditions as demonstrated above. This is not the case for the problem considered here.

The initial condition is given by the following:

$$(\rho_L,\ p_L,\ v_{xL},\ v_{yL},\ v_{zL},\ B_{yL},\ B_{zL}) = (1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0), \tag{5.9}$$

$$(\rho_R,\ p_R,\ v_{xR},\ v_{yR},\ v_{zR},\ B_{yR},\ B_{zR}) = (0.125,\ 0.1,\ 0,\ 0,\ 0,-0.329875,\ 0), \tag{5.10}$$

in addition to $B_n = 0.75$ and $\gamma = 5/3$. This differs from the Brio and Wu problem only on the magnitude of transverse magnetic field ($-B_{yR} = B_{\rm crit} = 0.329875$) of the right state.[4] This is again a coplanar configuration. The only solution that our code obtained is a non-regular solution, which is presented in Fig. 10 at $t = 0.1$. Note that v_z and B_z are identically zero in this solution. As can be seen, the solution consists of a fast rarefaction wave and a switch-off shock, both of which go leftward, and a switch-off rarefaction wave and Euler

[4] The ratio of specific heats is not set to 2, the value adopted by Brio and Wu (1988), but to 5/3 for the same reason as given for the first case.

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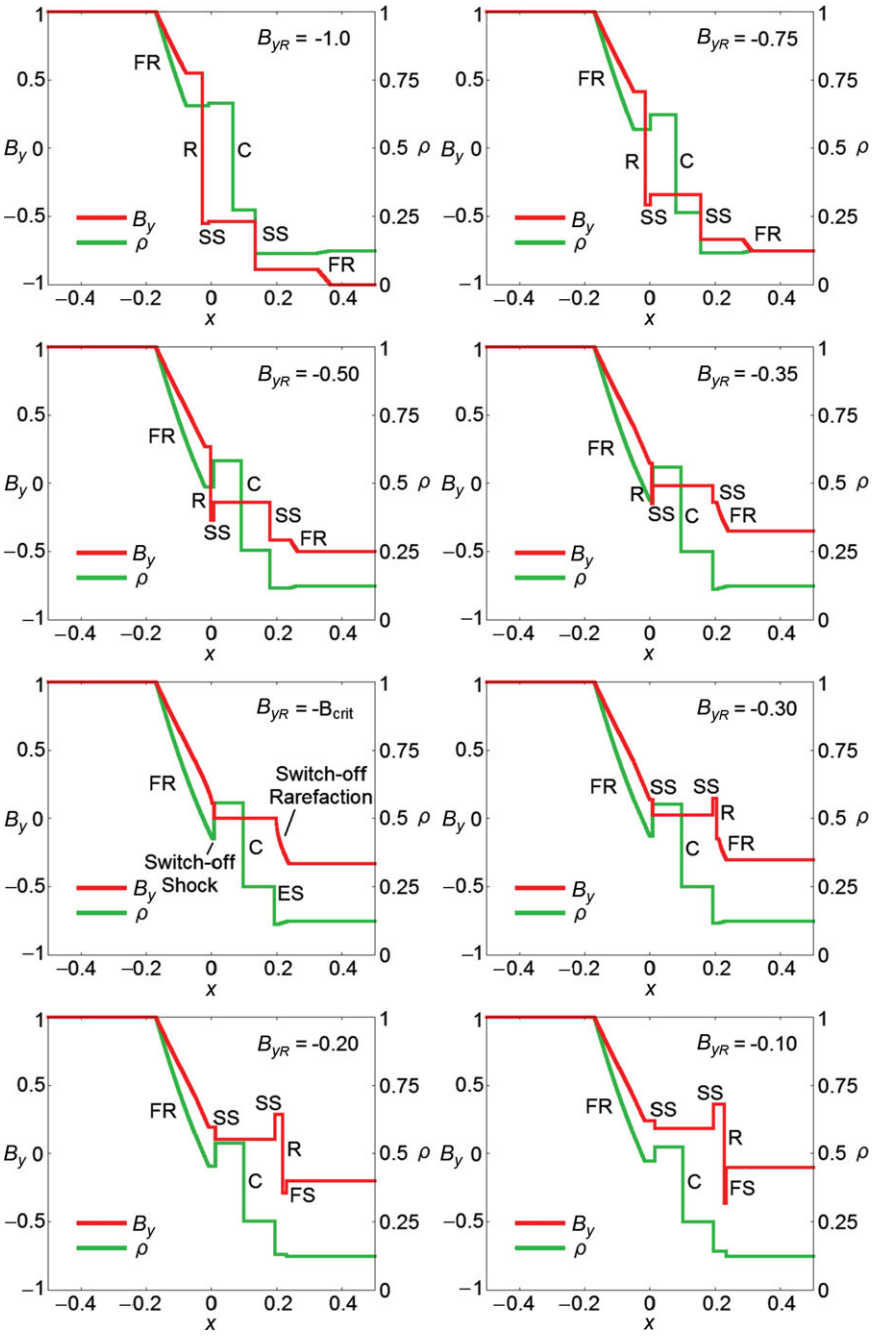


Figure 11. (Colour online) Coplanar neighboring solutions to the solution presented in Fig. 10. The red line represents the transverse magnetic field whereas the green line displays the density. Only the strength of the transverse magnetic field of the right state is varied. See Sec. 5.2 of the text for details.

shock, which proceed rightward, as well as a contact discontinuity. Although it may seem that shocks and rarefaction waves form compound waves on both sides of the contact discontinuity, they are actually detached. As a matter of fact, it is theoretically shown that the switch-off waves, irrespective of shock or rarefaction, never form compound waves. As mentioned earlier, the switch-off shock is a $2,3 \rightarrow 4$ shock, with the Alfvén speed being equal to the upstream flow speed. Preceded by the switch-off rarefaction, which is actually one of the fast rarefaction waves, the fast speed and Alfvén speeds are degenerate ahead of the Euler shock, which is itself an evolutionary shock not only in the full system but also in the reduced MHD system in this case although it is possible to be non-evolutionary in principle. It should be stressed that it is the advantage of our code that can treat these degenerate cases. In fact, other codes, for example developed by Andreev et al. (2008), ignore the switch-off waves entirely. It is normally impossible to prove the non-existence of particular solutions by numerical computations. We think, however, that no regular solution indeed exists to the current problem. In order to demonstrate why we think so and elucidate the special place that the current initial condition

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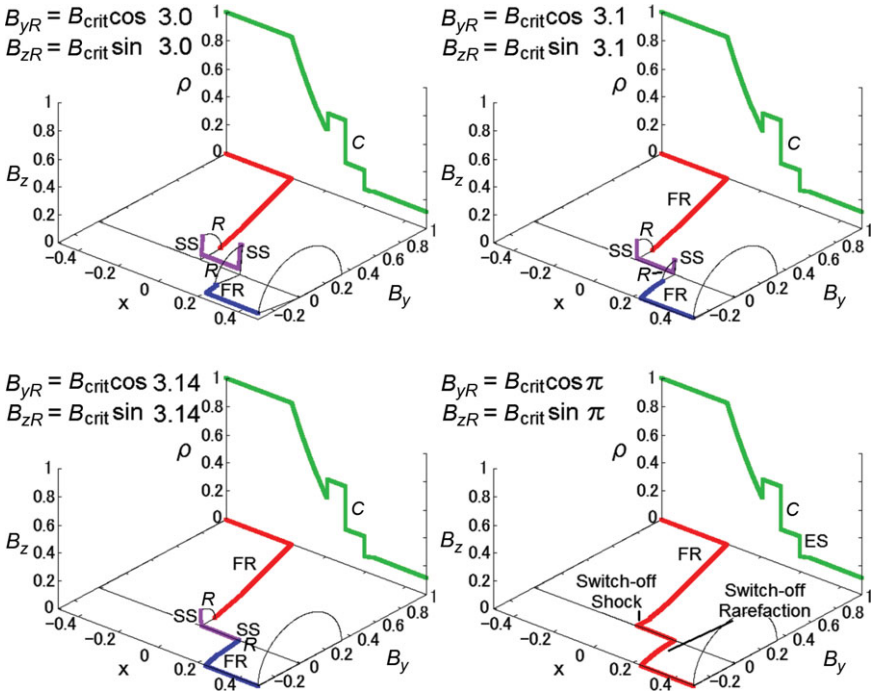


Figure 12. (Colour online) Non-coplanar neighboring solutions to the solution presented in Fig. 10 as well as in the bottom right panel of this figure. The green line displays the density whereas the red, purple, and blue lines represent transverse magnetic fields, each on a different plane, the angle of which is shown by an arc. Only the direction of the initial transverse magnetic field of the right state is varied in these computations. See Sec. 5.2 of the text for details.

occupies, we study the neighborhood of this problem in detail.

Presented in Fig. 11 are the solutions to coplanar neighboring problems. More specifically, we vary the strength of the transverse magnetic field of the right state, B_{yR}. The critical value $B_{\rm crit}$ is approached from both above and below. Here we are concerned only with regular solutions. As shown in figures, in the case of $|B_{yR}| > B_{\rm crit}$, the solutions consist of a fast rarefaction wave, rotational discontinuity, and slow shock going leftward and a fast rarefaction wave and slow shock going to the right, in addition to a contact discontinuity. As the initial strength of the transverse magnetic field in the right state becomes smaller, the left-going rotational discontinuity and slow shock come closer to each other. Among the right-going waves, the fast rarefaction wave approaches the slow shock. At the critical point the left-going rotational discontinuity and slow shock merge into a switch-off shock, and the right-going fast rarefaction wave and slow shock become the switch-off rarefaction wave and the Euler shock respectively.

As the initial strength of the transverse magnetic field becomes even smaller and $|B_{yR}| < B_{\rm crit}$ is satisfied, the right-going rotational discontinuity emerges and is responsible for the inversion of the transverse magnetic field. The Euler shock is replaced by the rotational discontinuity and a slow shock whereas the switch-off rarefaction is modified to an ordinary fast rarefaction wave again at first as long as $|B_{yR}|$ is not very different from $B_{\rm crit}$. A fast shock appears instead of the fast rarefaction wave when $|B_{yR}|$ becomes smaller than a certain value. On the left side of the contact discontinuity, on the other hand, the switch-off shock is changed to an ordinary slow shock and no rotational discontinuity exists. It is stressed that the unique non-regular solution at the critical point is connected continuously to the sequences of regular solutions obtained for $|B_{yR}| > B_{\rm crit}$ and $|B_{yR}| < B_{\rm crit}$ respectively.

Next we turn to non-coplanar neighboring problems. This time we change the direction of the initial transverse magnetic field in the right state, keeping its strength at the critical value, $B_{tR} = B_{\rm crit}$. In this case we always found regular solutions alone, which are displayed in Fig. 12. As is evident, there appear left- and right-going rotational discontinuities when transverse magnetic fields are misaligned initially. They are preceded by a fast rarefaction wave and followed by a slow shock running in each direction. As the misalignment gets smaller, or the critical point is approached, the amplitudes of transverse magnetic fields rotated by these rotational discontinuities become smaller. And they disappear at the critical point and we recover the solution, including the switch-off shock and rarefaction wave presented above. Again this clearly demonstrates that the non-regular solution at the critical point is a limit of the sequence of regular solutions. Put another way, there are only regular solutions in the sufficiently close vicinity of a non-regular solution at the critical point. This is the main reason why we think that there is no regular solution at the critical point. Note, however, that there might be another regular solution that is not included in the neighborhood and has eluded our search.

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It is interesting to see how a coplanar neighboring solution is approached by non-coplanar solutions. Note that they are both regular solutions. As observed above, the left- and right-going rotational discontinuities cooperate to rotate the transverse magnetic field by the angle imposed by the initial condition. It is found that it is always the right-going rotational discontinuity that rotates the transverse magnetic field by a larger angle if $0 < B_{tR} < B_{\rm crit}$ and vice versa. As the misalignment becomes smaller and the coplanar configuration is approached, one of the rotational discontinuities that gives a smaller rotation gets weaker and disappears for the coplanar configuration. For a non-vanishing but very small misalignment, the relative amplitude of two rotational discontinuities as a function of B_{tR} changes very rapidly near $B_{tR} = B_{\rm crit}$. In this sense the non-coplanar solutions shown in Fig. 12 for $B_{tR} = B_{\rm crit}$ are rather special themselves.

Finally, it is legitimate to ask: What has become of the coplanar non-regular solutions including $2 \rightarrow 3$ intermediate shocks that are observed near the Brio and Wu (1988) problem in the previous section? This is an interesting issue in its own right indeed. As long as $|B_{yR}| > B_{\rm crit}$ is satisfied, we find uncountably many non-regular solutions that include a $2 \rightarrow 3$ intermediate shock. The sequence of solutions is terminated at one end with a regular solution, including a rotational discontinuity. The other end of the sequence, on the other hand, is initially a $2 \rightarrow 3,4$ intermediate shock. As $|B_{yR}|$ becomes smaller, however, the $2 \rightarrow 3,4$ intermediate shock cannot be reached at some point and the sequence is terminated with a $2 \rightarrow 3$ intermediate shock. At the same time, another non-regular solution with a $2 \rightarrow 4$ intermediate shock appears. This solution moves up the slow locus as $|B_{yR}|$ gets smaller, becomes a $1,2 \rightarrow 4$ intermediate shock eventually at $B_{yR} \approx -B_S = -0.50303$, and disappears. In the other case of $|B_{yR}| < B_{\rm crit}$, there are again uncountably many non-regular solutions, which consist of a right-going $2 \rightarrow 3$ intermediate shock and a slow shock, in addition to regular solutions that have right-going rotational discontinuities. The latter is an end point of the sequence of the former. No solution with a compound wave is found in this case and the other end point of the sequence is the $2 \rightarrow 3$ intermediate shock, which approaches a regular solution as $|B_{yR}|$ increases and seems to merge at $B_{yR} \approx -B_Q = -0.25$. Thereafter only regular solutions exist. Note finally that all these solutions disappear at $B_{yR} = -B_{\rm crit}$ and only the non-regular solution with the switch-off shock waves exists.

6. Summary and discussions

In order to explore non-unique solutions to the MHD Riemann problem, we have developed the numerical code to obtain weak solutions that satisfy the entropy condition, the minimum requirement that the physically relevant solutions have to meet. Unlike the preceding ones (e.g., Andreev et al. 2008), our code can treat both regular and non-regular solutions and can also handle the initial condition, in which no transverse magnetic field exists. We have confirmed that the shock waves included in solution satisfy the Rankine–Hugoniot jump conditions and the rarefaction waves indeed keep the Riemann invariants constant. Details of the code will be presented elsewhere (Takahashi and Yamada 2012). The code will be useful not only for the study of various solutions as demonstrated in this paper but also for numerical simulations, in which the code will provide exact solutions to be compared with and may be implemented to evaluate numerical fluxes. In this paper we have applied the new code to two initial conditions, which we believe are useful for the consideration of which weak solutions are really physical, the problem with a long history of controversy.

The first problem is one of the best known problems for developers of MHD codes: the Brio and Wu (1988) problem. This particular problem has been frequently utilized for the validation of newly developed numerical codes partly because it has a coplanar configuration and two-dimensional codes can be applied. It has been also known for a long time that numerical solutions always involve a compound wave, which is a $2 \rightarrow 3,4$ intermediate shock, to which a slow rarefaction wave is attached. This is a non-regular solutions, in which the compound wave does not satisfy the evolutionary condition and its reality has been a subject of debate. There are two competing arguments so far. Wu and his collaborators (Wu 1987; Brio and Wu 1988; Wu 1988a,b, 1990; Wu and Kennel 1992) insist that intermediate shocks are stable and it is the evolutionary conditions to blame. On the other hand, Falle and Komissarov (1997, 2001) claim that the existence of intermediate shocks in numerical solutions are due to coplanarity, which is inherent in intermediate shocks. Unless the symmetry is broken somehow, the solution will retain the symmetry (coplanarity or planarity) and, as a consequence, we are dealing with the reduced MHD system. Then the point is that the $2 \rightarrow 3,4$ intermediate shock that comprises the compound wave in the numerical solution of the Brio and Wu (1988) problem becomes regular whereas the regular solution that contains rotational discontinuities, which was shown to exist by Andreev et al. (2008) and has been also confirmed in this paper, is now non-regular in the reduced MHD system.

It seems that the existence of uncountably many other solutions that are non-regular both in full and reduced systems support the claim of Falle and Komissarov (1997, 2001), since they have never been realized in numerical simulations. It is intriguing to point out, however, that these solutions do not satisfy the evolutionary conditions not because the number of outgoing characteristics is wrong but because the requirement on the linear independence is not met. This implies that there exit too many neighboring solutions to these non-regular solutions and we cannot single

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out a physically relevant one. This under-determinacy is certainly different from over-determinacy that we commonly find for non-evolutionary shocks and needs further investigations. It should be also mentioned that some authors (Hada 1994; Markovskii 1998; Inoue and Inutsuka 2007) insist that eventually all intermediate shocks satisfy evolutionary conditions if dissipations are taken into account. Although this may explain the realization of a compound wave in the solution of the Brio and Wu (1988) problem, it remains to be addressed why other non-regular solutions that are also evolutionary in their analyses but are not produced numerically. They may be linearly unstable. Further investigations are certainly needed.

In the second application we have picked up initial condition, which appears to possess no regular solution. This initial condition was found by continuously modifying the original Brio and Wu (1988) problem in various ways. In fact, the initial magnetic field has a coplanar configuration and is obtained from that in the Brio and Wu problem by reducing the strength of the transverse magnetic field in the right state. We have obtained two sequences of regular solutions as we approach this initial condition from both directions, keeping the coplanarity. They are continuously connected at this point by the non-regular solution, which includes both switch-off shock and rarefaction waves. The switch-off shock is non-regular and is located at the boundary between regular slow shocks and intermediate shocks on slow locus. It is true that we cannot prove the non-existence of regular solutions by these numerical calculations but the fact that all the neighboring regular solutions, coplanar or non-coplanar, are terminated at this non-regular solution suggests strongly that regular solution is unlikely to exist. If true, the conventional expectation that there is always a unique regular solution is wrong. We are certainly interested in whether this non-regular solution is indeed realized for this initial condition. This will be a difficult task for numerical simulations, however, since the initial condition has to be prepared precisely.

This initial condition is singular also in the following sense: As the coplanar configurations with $|B_{yR}| > B_{\rm crit}$ approach this critical initial condition, the rotational discontinuity is always left-going whereas the right-going one emerges in the opposite case, $|B_{yR}| < B_{\rm crit}$, and at the critical condition these rotational discontinuities disappear and the switch-off shock and rarefaction wave occur; if the initial transverse magnetic fields are misaligned slightly, the sequence of regular solutions exhibits a rapid and drastic change of configuration in the vicinity of the critical initial condition: the left- and right-going rotational discontinuities exchange their roles. Such sudden changes of configuration may have a ramification for the stability of regular solution for non-coplanar initial configurations with $B_{tR} \approx B_{\rm crit}$.

We have also found that the sequence of non-regular solutions behaves in an intriguing manner. When we approach the critical initial condition from the regime with $|B_{yR}| > B_{\rm crit}$, we always find uncountably many non-regular solutions, including a $2 \rightarrow 3$ intermediate shock, which is terminated at one end with the regular solution that consists of a rotational discontinuity. The other end point is initially the $2 \rightarrow 3,4$ intermediate shock that forms a compound wave. For smaller $|B_{yR}|$, however, the end point becomes a $2 \rightarrow 3$ intermediate shock. At the same time another non-regular solution, including a $2 \rightarrow 4$ intermediate shock, emerges, moving along the slow Hugoniot loci toward a $1,2 \rightarrow 4$ intermediate shock, at which this type of solution ceases to exist. When the critical initial condition is approached from the opposite direction, $|B_{yR}| < B_{\rm crit}$, we again find uncountably many non-regular solutions, including a $2 \rightarrow 3$ intermediate shock, unless $|B_{yR}|$ is very close to $B_{\rm crit}$. In contrast to the opposite case, no $2 \rightarrow 3,4$ intermediate shock is realized. The sequence of non-regular solutions shrinks as $|B_{yR}|$ gets larger and at some point it reduces to a regular solution with a rotational discontinuity and no non-regular solution exists any longer. For the critical initial condition even this regular solution disappears and only the non-regular solution with the switch-off shock exists. These interesting behaviors certainly warrant further investigations.

After all, it seems that the stability of various intermediate shocks needs major reanalysis. The conventional evolutionary conditions are certainly unsatisfactory. Maybe the introduction of dissipations and its zero limits should be scrutinized. In so doing our new code will be useful to generate all possible solutions. We believe that the survey and investigation of other initial conditions and the corresponding solutions, which manifest some singularities, will provide us with new insights and eventually a clue to the understanding as to which solutions are physically relevant.

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