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Random Sampling of Plane Partitions

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This article presents uniform random generators of plane partitions according to size (the number of cubes in the 3D interpretation). Combining a bijection of Pak with the method of Boltzmann sampling, we obtain random samplers that are slightly superlinear: the complexity is $O(n(\ln n)^3)$ in approximate-size sampling and $O(n^{4/3})$ in exact-size sampling (under a real-arithmetic computation model). To our knowledge, these are the first polynomial-time samplers for plane partitions according to size (there exist polynomial-time samplers of another type, which draw plane partitions that fit inside a fixed bounding box). The same principles yield efficient samplers for $(a \times b)$-boxed plane partitions (plane partitions with two dimensions bounded), and for skew plane partitions. The random samplers allow us to perform simulations and observe limit shapes and frozen boundaries, which have been analysed recently by Cerf and Kenyon for plane partitions, and by Okounkov and Reshetikhin for skew plane partitions.

1. Introduction

Plane partitions, originally introduced by A. Young [30], constitute a natural generalization of integer partitions in the plane, as they consist of a matrix of integers that are non-increasing in both dimensions (whereas an integer partition is an array of non-increasing integers). In addition, they also have a nice interpretation in 3D space as a heap of cubes (see Figure 1). Plane partitions have motivated a huge literature in numerous fields of mathematics [1, 11, 12, 19, 29] and statistical physics [16, 27], and have provided crucial insight for solving challenging problems in combinatorics [31]; see [2] for a detailed historical account. The problem of enumerating plane partitions was solved by MacMahon [15], who proved the beautiful formula

$$P(x) = \prod_{r \geqslant 1} \frac{1}{(1-x^r)^r} \tag{1.1}$$

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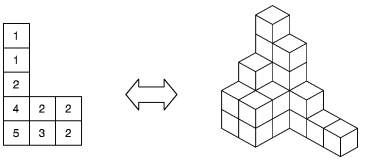


Figure 1. Plane partition of size 22 and its 3D representation.

for the generating function. The simplicity of the formula asks for a combinatorial interpretation. A first direct bijective proof has been given by Krattenthaler [13]. The principle is inspired by the seminal bijection of Novelli, Pak and Stoyanovskii [21], giving an interpretation of the hook length formula. In [13], Krattenthaler also discusses as an application of his bijection a polynomial-time algorithm for the random generation of plane partitions in a given $a \times b \times c$ box. Upon looking at the heap of cubes in the $(1, 1, 1)$ direction, this task is equivalent to sampling tilings of a hexagon of side lengths (a, b, c, a, b, c) by rhombi; there also exist random samplers for such tilings, which rely either on 'coupling from the past principles' [26] or on 'determinant algorithms' [28]. In contrast, we are interested here in sampling plane partitions uniformly at random with respect to *size*, defined as the sum of the matrix entries. For this purpose, we use another bijective interpretation of MacMahon's formula recently given by Pak [24].

Let us briefly mention the motivations for having a random sampler of plane partitions according to size. Size is a natural parameter, as it corresponds to the volume of the plane partition (number of cubes) in the 3D interpretation. Recently, several authors have studied the statistical properties of plane partitions with respect to size. In particular, under fixed-size distribution, Mutafchiev [18] has shown a limit law for the maximal entry, and Cerf and Kenyon [3] have determined the asymptotic shape (the asymptotic shape in the boxed framework – hexagon tilings – is due to Cohn, Larsen and Propp [4]). Even more recently, Okounkov and Reshetikhin, using a method based on Schur processes, have rediscovered the limit shape of Cerf and Kenyon [22]. They have studied in a subsequent article [23] the local correlations and limit shapes for plane partitions under a mixed model: the plane partition is constrained to a 2-coordinate $a \times b$ box and is drawn under the Boltzmann model with respect to size. (We will also describe random samplers for this mixed model.) In addition, physicists have developed new models relying on plane partitions, giving rise to a simplified version of the 3-dimensional models of lattice vesicles [14]. Plane partitions are also related to the 3-dimensional Ising model in the cubic lattice [3]. In general, physicists are interested in checking experimentally or conjecturing some limit properties of these models, by generating very large random objects.

For this purpose, this paper introduces efficient samplers for plane partitions. Our approach combines methods from bijective combinatorics and symbolic combinatorics. More precisely, a minor reformulation of Pak's bijection maps a multiset of integer pairs (the class is denoted by $\mathcal{M}$) to a plane partition with the same size. (Since the class $\mathcal{M}$ has generating function $\prod_{r \geqslant 1}(1 - x^r)^{-r}$, this gives a direct proof of (1.1).) Our aim here is to take advantage of this bijection for random sampling. Indeed, Pak's bijection reduces the task of finding a sampler for plane partitions to the task of finding a

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sampler for $\mathcal{M}$. As the class $\mathcal{M}$ has an explicit simple combinatorial decomposition, it is amenable to random sampling methods from symbolic combinatorics. At present there is the *recursive method* [20, 9], based on counting sequences, and *Boltzmann samplers*, based on generating functions, as introduced in [5] and further developed in [6]. We adopt here the framework of Boltzmann samplers, which tend to be more efficient as they avoid the costly precomputations of coefficients required by the recursive method.

As opposed to the recursive method – which produces exact-size samplers – the probability distribution in Boltzmann sampling is spread over the whole class; more precisely, an object of size n has probability proportional to x^n, where x is a fixed real parameter. In particular, as two objects having the same size have equal probability, the probability distribution restricted to a given size n is uniform. As we are interested in generating very large plane partitions, the Boltzmann framework is suitable, due to the gain obtained by relaxing the exact-size constraint. The articles [5] and [6] provide a collection of rules for building a sampler for a class admitting a decomposition involving classical constructions. Using these rules, the decomposition of $\mathcal{M}$ is readily translated into a Boltzmann sampler. This yields, via Pak's bijection, a Boltzmann sampler for plane partitions. In addition, as the size distribution of plane partitions – under the Boltzmann model – has good concentration properties, it is possible to 'tune' the parameter x so as to draw objects of size around (or exactly at) a given target value n. With the parameter x suitably tuned and a rejection loop targeted at the size, we obtain a *quasi-linear-time* approximate-size sampler for plane partitions: for any tolerance ratio $\varepsilon \in (0, 1)$, our sampler draws a plane partition of size in $[n(1-\varepsilon), n(1+\varepsilon)]$ with an expected running time $O(n(\ln n)^3)$. The same principles, with the rejection loop running until a given size n is attained, yields an exact-size sampler for plane partitions, with expected running time $O(n^{4/3})$. To our knowledge, our algorithm is the first exact-size sampler for plane partitions with expected polynomial running time. This allows us to generate objects of size up to 10^7 in a few minutes on a PC. The same principles (*i.e.*, Pak's bijection + Boltzmann samplers) yield efficient Boltzmann samplers for $(a \times b)$-boxed plane partitions (plane partitions whose non-zero entries lie in an $(a \times b)$ rectangle), which are those considered by Okounkov and Reshetikhin. We obtain for boxed plane partitions an approximate-size sampler with expected running time $O_{a,b,\varepsilon}(1)$ and an exact-size sampler of expected running time $O_{a,b}(n)$, where ε is a tolerance ratio on the size (for approximate-size sampling) and where n is the target size (the dependency in a, b, ε of the asymptotic constants in the big Os are stated precisely in Theorem 5.6).

Proving the correct complexity orders of the samplers is the major technical difficulty we have to deal with. At first we have to analyse the expected running time of the Boltzmann sampler for the multiset class $\mathcal{M}$, as well as the size distribution on $\mathcal{M}$ under the Boltzmann model. All this is done using the Mellin transform. Second, we study the complexity of Pak's bijection, which depends on a natural *length parameter* of a plane partition, which is the maximum hook length (abscissa + ordinate + 1) over all non-zero entries of the matrix. Let us finally mention that, for the sake of simplicity, all complexity results are stated and proved with the O notation (upper bounds). With a little more care one could prove that all the stated complexity results hold in fact with a Θ notation, *i.e.*, an upper and a lower bound.

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Table 1. Some constructions on combinatorial classes.

Class	Generating function	Definition
$\mathcal{C} = \mathcal{E}$	$C(z) = 1$	neutral object of size 0
$\mathcal{C} = \mathcal{Z}$	$C(z) = z$	atom of size 1
$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$C(z) = A(z) + B(z)$	disjoint union
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(z) = A(z) \times B(z)$	Cartesian product
$\mathcal{C} = \textsc{Seq}(\mathcal{A})$	$C(z) = (1 - A(z))^{-1}$	$\mathcal{E} + \mathcal{A} + \mathcal{A} \times \mathcal{A} + \mathcal{A} \times \mathcal{A} \times \mathcal{A} + \cdots$
$\mathcal{C} = \textsc{MSet}(\mathcal{A})$	$C(z) = \exp(\sum A(z^k)/k)$	a multiset of elements of $\mathcal{A}$

Outline of the paper. After some definitions in Section 2 about combinatorial classes and plane partitions, we present in Section 3 a slight reformulation (more algorithmic) of Pak's bijection. The bijection induces a combinatorial isomorphism between the set of plane partitions and the class $\mathcal{M} := \textsc{MSet}(\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2)$. Section 4 recalls basic principles of Boltzmann sampling, in particular the sampling rules associated to the constructions appearing in the specification of $\mathcal{M}$. The Boltzmann sampler for $\mathcal{M}$, as well as $\mathcal{M}_{a,b} := \textsc{MSet}(\mathcal{Z} \times \textsc{Seq}_{<a}(\mathcal{Z}) \times \textsc{Seq}_{<b}(\mathcal{Z}))$, is derived in Section 5, giving rise to Boltzmann samplers for plane partitions and $(a \times b)$-boxed plane partitions. We then explain briefly how the principles extend to obtain Boltzmann samplers for so-called skew plane partitions. In Section 5.4, using suitable choices of the parameter x in the Boltzmann samplers, we obtain efficient samplers for plane partitions targeted exactly or approximately at a given size n (precise statements are given in Theorems 5.5 and 5.6). The expected running times of the targeted samplers are then analysed in Section 6.

2. Definitions

A *combinatorial class* is a pair $(\mathcal{A}, |\cdot|)$ where $\mathcal{A}$ is a set and $|\cdot|$ is a function from $\mathcal{A}$ to $\mathbb{N}$, called the *size function*, such that the number of elements of any given size is finite. Using the size function, we can graduate $\mathcal{A}$ as $\mathcal{A} = \bigcup_n \mathcal{A}_n$, where $\mathcal{A}_n$ is the set of objects of $\mathcal{A}$ that have size n. Hereafter, we let A_n denote the cardinality of $\mathcal{A}_n$. To each combinatorial class $\mathcal{A}$, we associate the *generating function* $A(z) = \sum A_n z^n$.

Two combinatorial classes $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ and $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ are said to be *combinatorially isomorphic*, $\mathcal{A} \simeq \mathcal{B}$, if and only if there exists a one-to-one mapping from $\mathcal{A}$ to $\mathcal{B}$ that preserves the size. Let us notice that two classes $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if their generating functions are equal.

Here are some classical constructions on combinatorial classes that will be used in this paper. Notation and rules are summarized in Table 1 (a more general presentation can be found in [8]).

- $\mathcal{E}$ and $\mathcal{Z}$ are *atoms* of size 0 and 1.
- *Disjoint union* $\mathcal{A} + \mathcal{B}$: the union of two copies of $\mathcal{A}$ and $\mathcal{B}$ made disjoint.
- *Cartesian product* $\mathcal{A} \times \mathcal{B}$: the set of pairs (α, β) where $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$.

Given a class $\mathcal{A}$ not containing empty atoms:

- *Sequence*: $\textsc{Seq}(\mathcal{A})$ is the class of finite sequences of objects of $\mathcal{A}$.
- *Multiset*: $\textsc{MSet}(\mathcal{A})$ is the class of finite sets of objects of $\mathcal{A}$, with repetitions allowed.

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In all these constructions, the size of an object in the composed class is naturally defined as the sum of the sizes of the components (*e.g.*, the size of a sequence $\gamma_1, \ldots, \gamma_k$ is $|\gamma_1| + \cdots + |\gamma_k|$). Observe that, in a multiset $\mu \in \textsc{MSet}(\mathcal{A})$, each element $\alpha \in \mathcal{A}$ has a *multiplicity* $c_\alpha \geqslant 0$. Hence, if $\mathcal{A}$ is a finite set,

$$\textsc{MSet}(\mathcal{A}) \simeq \prod_{\alpha \in \mathcal{A}} \textsc{Seq}(\{\alpha\}). \tag{2.1}$$

A *plane partition* (Figure 1) of n is a two-dimensional array of non-negative integers $(a_{i,j})_{\mathbb{N}^2}$ that are non-increasing both from left to right and bottom to top and that add up to n. In other words,

$$\forall (i,j) \in \mathbb{N}^2 \quad a_{i,j} \geqslant a_{i,j+1}, \qquad a_{i,j} \geqslant a_{i+1,j} \quad \text{and} \quad \sum_{i,j} a_{i,j} = n. \tag{2.2}$$

We let $\mathcal{P}$ denote the combinatorial class of plane partitions, endowed with the size function $|(a_{i,j})_{\mathbb{N}^2}| = \sum_{i,j} a_{i,j}$. Plane partitions have a natural representation in 3D space as a heap of cubes with non-increasing height in the direction of the x-axis and y-axis: see Figure 1. Observe that the size of the plane partition exactly corresponds to the number of cubes in the 3D representation.

The *bounding rectangle* of a plane partition $(a_{i,j})_{\mathbb{N}^2}$ is the smallest double range $R = [0..\ell - 1] \times [0..w - 1]$ such that $a_{i,j} = 0$ for all index pairs (i,j) outside R. An $(a \times b)$-*boxed plane partition* is a plane partition whose bounding rectangle is at most $a \times b$. Equivalently, $a_{i,j}$ is null for any (i,j) such that $i \geqslant a$ or $j \geqslant b$. We let $\mathcal{P}_{a,b}$ denote the class of $(a \times b)$-boxed plane partitions. Define the two combinatorial classes $\mathcal{M}$ and $\mathcal{M}_{a,b}$ as follows, where $\textsc{Seq}_{<d}(\mathcal{A})$ denotes the class of sequences of at most $d-1$ elements of $\mathcal{A}$:

$$\mathcal{M} := \textsc{MSet}(\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2), \tag{2.3}$$

$$\mathcal{M}_{a,b} := \textsc{MSet}(\mathcal{Z} \times \textsc{Seq}_{<a}(\mathcal{Z}) \times \textsc{Seq}_{<b}(\mathcal{Z})). \tag{2.4}$$

Classically, $\textsc{Seq}(\mathcal{Z})$ is identified with the class of non-negative integers, so that we can specify $\mathcal{M}$ with the following simplified notation:

$$\mathcal{M} \simeq \textsc{MSet}(\mathcal{Z} \times \mathbb{N}^2). \tag{2.5}$$

In the next section we explain how Pak's bijection yields an explicit combinatorial isomorphism between $\mathcal{P}$ and $\mathcal{M}$. For this purpose, we introduce some more terminology. The *diagram* D of an element $M \in \mathcal{M}$ is a two-dimensional array $(m_{i,j})_{\mathbb{N}^2}$ (with $(0,0)$ at the bottom left) where $m_{i,j}$ is the multiplicity of $(\mathcal{Z}, i, j)$ in M (see the first two pictures of Figure 2). The *size* of D is defined as $|D| = \sum_{i,j} m_{i,j}(i + j + 1)$, so that it corresponds to the size of the multiset in $\mathcal{M}$. The *bounding rectangle* of M is defined similarly as for plane partitions: it is the smallest double range $R = [0..\ell - 1] \times [0..w - 1]$ such that all entries of D outside of R are zero. The integers ℓ and w are respectively called the *length* and the *width* of D. Note that, for fixed integers a and b, the diagrams of elements in

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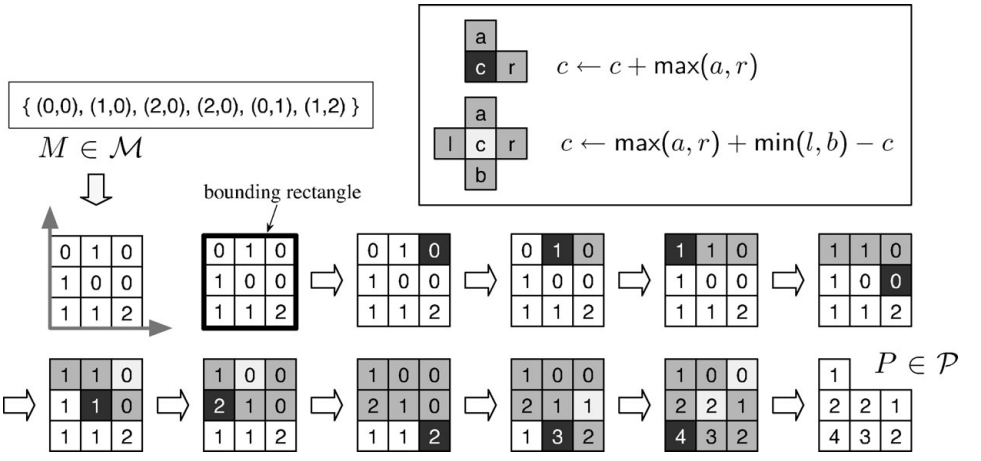


Figure 2. An example of Pak's bijection.

$\mathcal{M}_{a,b}$ are constrained to have their bounding rectangle $\subseteq [0..a-1] \times [0..b-1]$. Therefore $\mathcal{M}_{a,b}$ is called the class of $(a \times b)$-*boxed multisets.*

3. Pak's bijection

In [24], Pak presents a bijection between plane partitions bounded in a shape μ (μ being a Ferrers diagram) and fillings of the entries of μ with non-negative integers. We reformulate this bijection as an algorithm, Algorithm 1 below, which realizes explicitly the combinatorial isomorphism $\mathcal{M} \simeq \mathcal{P}$: see Figure 2 for an example.

Algorithm 1: From the diagram of a multiset to a plane partition

Input : The diagram D of a multiset in $\mathcal{M}$.
Output: a plane partition.
Let ℓ be the length and w be the width of D.
for $i \leftarrow \ell - 1$ **downto** 0 **do**
  for $j \leftarrow w - 1$ **downto** 0 **do**
    $D[i, j] \leftarrow D[i, j] + \max(D[i+1, j]), D[i, j+1])$;
    for $c \leftarrow 1$ **to** $\min(w - 1 - i, \ell - 1 - j)$ **do**
      $x \leftarrow i + c$; $y \leftarrow j + c$;
      $D[x, y] \leftarrow \max(D[x+1, y], D[x, y+1]) + \min(D[x-1, y], D[x, y-1]) - D[x, y]$;
return D;

Proposition 3.1 (Pak [24]). *Algorithm 1 yields an explicit size-preserving bijection between the class $\mathcal{M}_{a,b}$ and the class of $(a \times b)$-boxed plane partitions. In other words, the algorithm realizes the combinatorial isomorphism*

$$\mathcal{P}_{a,b} \simeq \textsc{MSet}(\mathcal{Z} \times \textsc{Seq}_{<a}(\mathcal{Z}) \times \textsc{Seq}_{<b}(\mathcal{Z})). \tag{3.1}$$

Proof. See [24]. □

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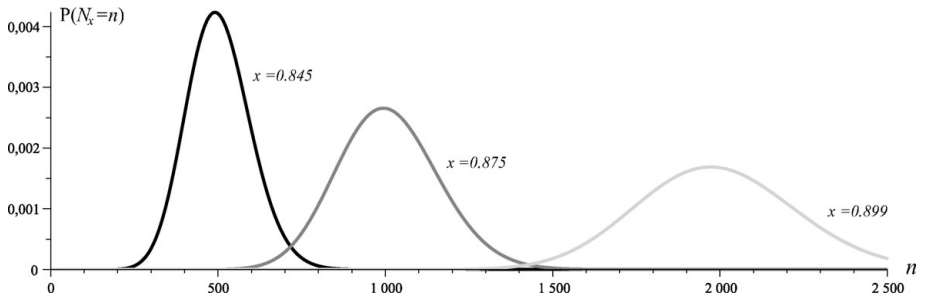


Figure 3. Probability distribution of sizes for plane partitions under the Boltzmann model, with different values of the parameter x.

Proposition 3.2. *Algorithm 1 realizes the combinatorial isomorphism*

$$\mathcal{P} \simeq \textsc{MSet}(\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2). \tag{3.2}$$

Proof. Take the limit $a \to \infty$ and $b \to \infty$ in Proposition 3.1. □

4. Boltzmann sampling

This section recalls basic principles of approximate-size sampling under the Boltzmann model [5, 6].

Definition (Boltzmann model). Let $\mathcal{C}$ be a combinatorial class and let $C(x) := \sum_{\gamma \in \mathcal{C}} x^{|\gamma|}$ be its generating function. Given a coherent positive real value of x, *i.e.*, chosen within the disk of convergence of $C(x)$), the *Boltzmann model* of parameter x assigns to any element $\gamma \in \mathcal{C}$ the following probability:

$$\mathbb{P}_x(\gamma) = \frac{x^{|\gamma|}}{C(x)}.$$

A *Boltzmann sampler* $\Gamma C(x)$ for $\mathcal{C}$ is an algorithm that produces objects of $\mathcal{C}$ at random under the Boltzmann model. As elements of the same size have the same weight, the probability induced by a Boltzmann sampler on any given size n is uniform. The size of the output is a random variable N_x satisfying

$$\mathbb{P}(N_x = n) = \frac{C_n x^n}{C(x)}.$$

Figure 3 shows this probability distribution for plane partitions. When a target size n has to be achieved, the idea is to tune the parameter x so that $\mathbb{E}(N_x) = n$ (see Section 6).

Table 2 briefly summarizes how to obtain samplers for the constructions used in the specification of $\mathcal{M}$ (see details in [6]); the rules can be combined to build a generator for any class specified with these constructions, in particular the class $\mathcal{M}$. The sampling rules make use of simple auxiliary generators: Geom(p) generates integers under the geometric law $\mathbb{P}(k) = p^k(1-p)$, Pois(λ) generates integers under the Poisson law $\mathbb{P}(k) = e^{-\lambda}\frac{\lambda^k}{k!}$, and $\text{Pois}_{\geqslant 1}(\lambda)$ generates integers under the positive Poisson law $\mathbb{P}(k) = (e^{\lambda}-1)^{-1}\frac{\lambda^k}{k!}$ for $k > 0$. Such generators are classically realized by simple iterative loops, the complexity of generating an integer k being $O(k)$; see [5] for a discussion.

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Table 2. Sampling rules associated to Boltzmann samplers for some combinatorial constructions.

$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$\Gamma C(x) := \langle \Gamma A(x), \Gamma B(x) \rangle$
$\mathcal{C} = \textsc{Seq}(\mathcal{A})$	$\Gamma C(x) := [\text{Geom}(A(x)) \Longrightarrow \Gamma A(x)]$ where $G \Longrightarrow \Gamma Y$ means 'return G independent calls to ΓY'.

$\mathcal{C} = \textsc{MSet}(\mathcal{A})$	Define the probability distribution relative to $\mathcal{A}$ and x:
	$$\Pr(K \leqslant k) = \prod_{j>k} \exp\Big(-\frac{1}{j}A(x^j)\Big).$$
	Let $\textsc{Max_Index}(A; x)$ be a generator according to this distribution.
	$\Gamma C(x)$: $\gamma \leftarrow \varnothing$; $k_0 \leftarrow \textsc{Max_Index}(A; x)$;
	if $k_0 \neq 0$ **then**
	for j **from** 1 **to** $k_0 - 1$ **do**
	$p \leftarrow \text{Pois}\big(\frac{A(x^j)}{j}\big)$;
	for i **from** 1 **to** p **do**
	$\gamma \leftarrow \gamma, \text{copy}(\Gamma A(x^j)\ j \text{ times})$
	$p \leftarrow \text{Pois}_{\geqslant 1}\big(\frac{A(x^{k_0})}{k_0}\big)$;
	for i **from** 1 **to** p **do**
	$\gamma \leftarrow \gamma, \text{copy}(\Gamma A(x^{k_0})\ k_0 \text{ times})$
	return γ.

Proposition 4.1 (Flajolet, Fusy and Pivoteau [6]). *Given two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ endowed with Boltzmann samplers $\Gamma A(x)$ and $\Gamma B(x)$, the sampler $\Gamma C(x)$, as defined in the first entry of Table 2, is a Boltzmann sampler for $\mathcal{A} \times \mathcal{B}$. Given a class $\mathcal{A}$ not containing the empty atom and endowed with a Boltzmann sampler $\Gamma A(x)$, the samplers $\Gamma C(x)$, as defined in the second and third entries*[1] *of Table 2, are, respectively, Boltzmann samplers for* $\textsc{Seq}(\mathcal{A})$ *and for* $\textsc{MSet}(\mathcal{A})$.

From these sampling rules, a class $\mathcal{C}$ recursively specified from atomic sets in terms of these constructions can be endowed with a Boltzmann sampler $\Gamma C(x)$. The complexity of generating an object $\gamma \in \mathcal{C}$ is $O(|\gamma|)$.

Complexity model. Let us say a few words on the specific real-arithmetic complexity model used for Boltzmann samplers. First, notice that the samplers given in Table 2 draw integers according to distributions (Geom, Pois, $\textsc{Max_Index}$) that require the exact values of the generating functions of the classes involved. Hence, such generating functions should be evaluated. The complexity model we adopt, as already defined in [5], relies on the *oracle* assumption. This assumption allows us to separate the combinatorial complexity of the sampler from the complexity of evaluating the generating functions (there are already some results [25] and work in progress dedicated to the latter issue). Given any combinatorial class $\mathcal{C}$ specified recursively using the constructions of Table 2, and given a value $x > 0$ within the disk of convergence of $C(x)$, we assume that an *oracle* provides, at unit cost, the exact values at x of the generating functions for all classes intervening in the decomposition of $\mathcal{C}$. In practice, we work with a fixed precision (*e.g.*, 20 digits) and precompute the values

[1] We hereby correct an omission in the definition of the sampler for $\textsc{MSet}(\mathcal{A})$ given in [6], namely the test $k_0 \neq 0$.

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of generating functions used by the Boltzmann sampler. Finally, let us come back to the complexity of drawing an integer K under a certain discrete distribution

$$\mathbb{P}(K = k) = p_k,$$

where the constant p_k are known (if p_k involves exact values of generating functions, the oracle provides the values of p_k). As discussed in [5], drawing an integer K under this distribution is done by a simple loop.

Algorithm 2: Drawing an integer K under an arbitrary distribution

$U :=$ uniform(0,1); $S := 0$; $k := 0$;
while $U < S$ **do**
  $S := S + p_k$; $k := k + 1$;
return k;

Thus, the cost of drawing K is of the order of the value k that is finally assigned to K. An exception is the case of the geometric law, which is simpler. Indeed, to draw K under Geom(x) (with $x \in [0, 1]$), it is enough to set $K = \lfloor \ln(U)/\ln(x) \rfloor$, where U is uniform in $(0, 1)$.

Hence, the cost of drawing a geometric law is $O(1)$.

5. Samplers for plane partitions

5.1. Boltzmann sampler for plane partitions

The explicit bijection between $\mathcal{M}$ and $\mathcal{P}$ allows us to design a simple Boltzmann sampler for plane partitions, consisting of two steps: (i) generate a multiset in $\mathcal{M}$ under the Boltzmann model, (ii) apply Algorithm 1 (Pak's bijection) to the diagram of the multiset generated.

Algorithm 3: $\Gamma M(x)$ [Boltzmann sampler for $\mathcal{M}$]

M is the diagram of the multiset to be generated
$\forall (x, y)\ M[x, y] \leftarrow 0$;
$k_0 \leftarrow$ MAX_INDEX($A; x$), where $A(x) = x/(1-x)^2$;
if $k_0 \neq 0$ **then**
  for $k \leftarrow 1$ **to** $k_0 - 1$ **do**
    $p \leftarrow \text{Pois}\left(\frac{x^k}{k(1-x^k)^2}\right)$;
    for $i \leftarrow 1$ **to** p **do**
      $x \leftarrow \text{Geom}(x^k)$; $y \leftarrow \text{Geom}(x^k)$;
      $M[x, y] \leftarrow M[x, y] + k$;
  $p \leftarrow \text{Pois}_{\geqslant 1}\left(\frac{x^{k_0}}{k_0(1-x^{k_0})^2}\right)$;
  for $i \leftarrow 1$ **to** p **do**
    $x \leftarrow \text{Geom}(x^{k_0})$; $y \leftarrow \text{Geom}(x^{k_0})$;
    $M[x, y] \leftarrow M[x, y] + k_0$;
return M;

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Table 3. Time per generation for different sizes of plane partitions.

Approx. size	10^3	10^4	10^5	10^6	10^7
Generation	$\sim$ 0.1 sec	$\sim$ 0.5 sec	$\sim$ 2–3 sec	$\sim$ 10 sec	$\sim$ 60 sec
Pak's transform	$\sim$ 0.1 sec	$\sim$ 0.3 sec	$\sim$ 2 sec	$\sim$ 20–30 sec	$\sim$ 8–9 min

Lemma 5.1. *Given $0 < x < 1$, the generator $\Gamma M(x)$, as defined in Algorithm 3, is a Boltzmann sampler for $\mathcal{M}$.*

Proof. The specification of $\mathcal{M}$, given in equation (2.5), is translated to a Boltzmann sampler using the rules of Table 2. The translation is carried out directly on the diagram of the multiset (recall that the entry (i, j) of the diagram corresponds to the multiplicity of $(\mathcal{Z}, i, j)$ in the multiset). □

Since Pak's bijection preserves the size, Algorithm 4 is a Boltzmann sampler for plane partitions.

Algorithm 4: $\Gamma P(x)$, with $0 < x < 1$ [Boltzmann sampler for plane partitions]

Compute $\mu \leftarrow \Gamma M(x)$;
Apply Algorithm 1 (Pak's bijection) to μ;
return μ;

Table 3 shows computation times[2] of $\Gamma P(x)$ for sizes up to 10^7: the first line gives the time of generation of the multiset ($\Gamma M(x)$) and the second line gives the computation time of Pak's bijection. The sampler was implemented in Maple and the bijection in OCaml. As we can see, the complexity is dominated by Pak's bijection for objects of large size. This is confirmed by the analysis to be given in Section 6: the complexity of drawing a multiset of size around n is of order $n^{2/3}$, while the expected running time of Pak's bijection applied to a random multiset of size n is of order $n(\ln n)^3$. Figure 4 shows two large plane partitions generated by $\Gamma P(x)$ for x close to 1, $x = 0.947$ and $x = 0.9866$.

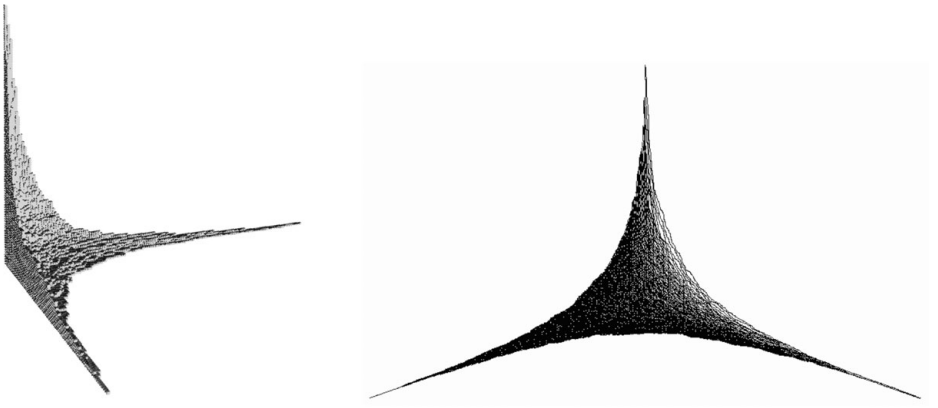
5.2. Boltzmann sampler for $(a \times b)$-boxed plane partitions

According to the equivalence with the definition in terms of diagrams, an element of $\mathcal{M}_{a,b}$ is a multiset of pairs (i, j), with $0 \leqslant i < a$ and $0 \leqslant j < b$, each element (i, j) having size $(i + j + 1)$. The set of such pairs being finite, equation (2.1) yields

$$\mathcal{M}_{a,b} = \prod_{\substack{0 \leqslant i < a \\ 0 \leqslant j < b}} \textsc{Seq}(\mathcal{Z}^{i+j+1}). \tag{5.1}$$

[2] Computations were performed on a Mac OS X Power PC G4 1.42 GHz, with 1 GB of RAM and 512 kB of cache.

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(a) A random plane partition of size 15,256, generated by $\Gamma P(0.947)$.

(b) A random plane partition of size $1,005,749$ generated by $\Gamma P(0.9866)$, seen from the direction $(1,1,1)$.

Figure 4. (a) A random plane partition of size 15,256, generated by $\Gamma P(0.947)$. (b) A random plane partition of size 1 005 749 generated by $\Gamma P(0.9866)$, seen from the direction $(1,1,1)$.

Lemma 5.2. *Given $0 < x < 1$, the generator $\Gamma M_{a,b}(x)$, as defined in Algorithm 5, is a Boltzmann sampler for $\mathcal{M}_{a,b}$.*

Proof. Translate the specification (5.1) to a Boltzmann sampler for $\mathcal{M}_{a,b}$ using the rules of Table 2. □

Algorithm 5: $\Gamma M_{a,b}(x)$ [Boltzmann sampler for $\mathcal{M}_{a,b}$]

```
M is the diagram of the multiset to be generated
for i ← 0 to a − 1 do
    for j ← 0 to b − 1 do
        M[i, j] ← Geom(x^{i+j+1});
return M;
```

Again, since Pak's bijection preserves the size, the following generator is a Boltzmann sampler for $(a \times b)$-boxed plane partitions.

Algorithm 6: $\Gamma P_{a,b}(x)$, with $0 < x < 1$ [Boltzmann sampler for boxed plane partitions]

```
Compute μ ← ΓM_{a,b}(x);
Apply Algorithm 1 (Pak's bijection) to μ;
return μ;
```

5.3. Extension to skew plane partitions

We consider here a natural generalization of $(a \times b)$-boxed plane partitions, called $(a \times b)$-boxed skew plane partitions. An $(a \times b)$-boxed *skew plane partition* is given by an *index*

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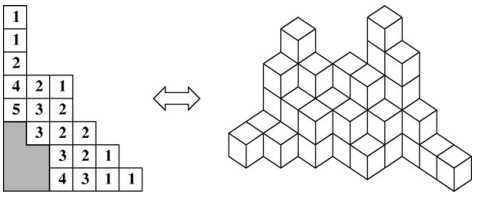


Figure 5. A skew plane partition of size 43.

domain $D \subset [0..a-1] \times [0..b-1]$, such that D is obtained from $[0..a-1] \times [0..b-1]$ by removing rectangles of the form $[0..a'-1] \times [0..b'-1]$, with $a' \leqslant a$ and $b' \leqslant b$. Each truncation by a smaller rectangle makes an *outer corner* appear in the index domain (*e.g.*, the partition of Figure 5 has 2 outer corners). Let us denote by $\mathcal{P}_D$ the class of all such partitions for a given domain D. For this new class of partitions, we need to define the hook length of a pair (i, j) in the domain D. Let $\ell(i)$ be the minimum abscissa such that $(\ell(i), j) \in D$ and let $d(j)$ be the minimum ordinate such that $(i, d(j)) \in D$. The hook length of (i, j) in D is then $h(i, j) = (i - \ell(i)) + (j - d(j)) + 1$, which is exactly $i + j + 1$ when D is $[0..a-1] \times [0..b-1]$. In [24], Pak's bijection is most generally described for skew plane partitions, which leads to the following combinatorial isomorphism:

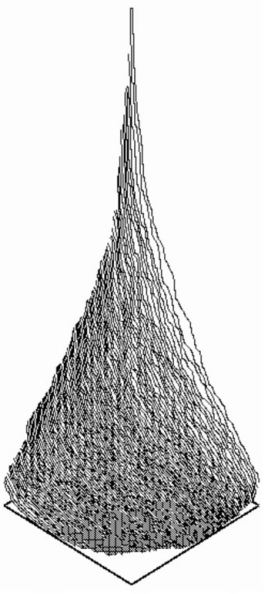
$$\mathcal{P}_D \simeq \prod_{(i,j)\in D} \textsc{Seq}(\mathcal{Z}^{h(i,j)}).$$

The Boltzmann sampler for $(a \times b)$-boxed plane partitions extends directly to a Boltzmann sampler for skew plane partitions as follows: to sample a diagram, draw the value at each point (i, j) in D according to a geometric law of parameter $x^{h(i,j)}$; then apply Algorithm 1 (Pak's bijection) to the multiset generated, with the difference that the domain scanned by (i, j) is D.

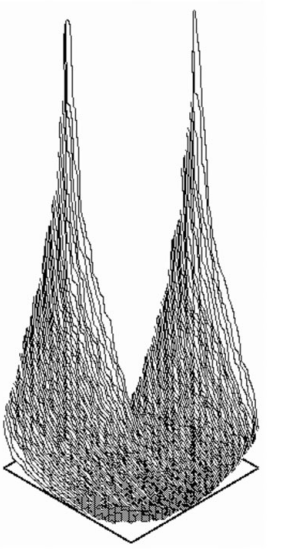
Okounkov and Reshetikhin [23] have studied the limit shape of a skew plane partition under the Boltzmann distribution, with the Boltzmann parameter x tending to 1. If the lengths of the rectangles are of order $(1-x)^{-1}$, some interesting phenomena are to be observed regarding the typical shape of a random skew plane partition. Using a technique based on Schur processes, the authors of [23] provide a precise analysis of these phenomena. They prove that the (rescaled) limit shape of a skew plane partition has a frozen boundary that satisfies explicit equations, and they classify the non-smooth points of the boundary as *turning points* and *cusps*. Turning points always appear, even for a boxed domain $(a \times b)$; they correspond to points of tangency of the frozen boundary with the delimiting 3D box $\{(x, y, z) \in \mathbb{R}_+^3, \text{ s.t. } (x, y) \in D\}$. If the index domain has outer corners, some cusp points possibly appear at each of the outer corners.

Our random sampler for skew plane partitions makes it possible to perform simulations and observe these asymptotic phenomena. Figure 6 shows a (100×100)-boxed plane partition of size 999 400. A frozen boundary appears that meets the delimiting 3D box in a tangential way (these points of tangency are precisely the turning points in the terminology of Okounkov and Reshetikhin). And Figure 6 shows a skew plane partition of size 1 005 532 on the index domain $(100 \times 100) \backslash (50 \times 50)$, which has an outer corner

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(a) A (100×100)-boxed plane partition of size 999,400 drawn under Boltzmann distribution at $x = 0.9931$.



(b) A $[\![0..99]\!] \times [\![0..99]\!] \backslash [\![0..49]\!] \times [\![0..49]\!]$ skew plane partition of size $1,005,532$ drawn under Boltzmann distribution at $x = 0.9942$.

Figure 6. (a) A (100×100)-boxed plane partition of size 999 400 drawn under the Boltzmann distribution at $x = 0.9931$. (b) A $[\![0..99]\!] \times [\![0..99]\!] \backslash [\![0..49]\!] \times [\![0..49]\!]$ skew plane partition of size 1 005 532 drawn under the Boltzmann distribution at $x = 0.9942$.

at $(50, 50)$; accordingly a cusp point appears on the boundary of the limit shape at the point above $(50, 50)$.

Note that the typical shape of a large unconstrained random plane partition, as shown in Figure 4, has different features: there are 3 'legs' – one in each axis direction – whose lengths tend to infinity even when the plane partition is rescaled to have unit volume (which essentially corresponds to rescaling by a factor $(1 - x)^{-1}$ in each dimension).

5.4. Samplers targeted around a given size

Given a class $\mathcal{C} = \cup_n \mathcal{C}_n$, an *exact-size sampler* is a procedure that, for any given $n \geqslant 1$ (called the *target size*), outputs an object of $\mathcal{C}_n$ uniformly at random. An *approximate-size sampler* is a procedure that, for any given $n \geqslant 1$ and $\varepsilon \in (0, 1)$ (called the *tolerance ratio*), outputs an object of $\mathcal{C}$ of size in $[n(1 - \varepsilon), n(1 + \varepsilon)]$ and such that two objects of the same size have the same chance of being drawn (hence the distribution induced on each size $k \in [n(1 - \varepsilon), n(1 + \varepsilon)]$ is uniform).

Such procedures are easily obtained if $\mathcal{C}$ is endowed with a Boltzmann sampler $\Gamma C(x)$. Fix a suitable value of x and repeat calling $\Gamma C(x)$ until the size is in the desired size-range Ω; $\Omega = \{n\}$ for exact-size sampling and $\Omega = [n(1 - \varepsilon), n(1 + \varepsilon)]$ for approximate-size

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sampling. The exact-size sampler and approximate-size sampler defined in this way are denoted $\textsc{Sample}\mathcal{C}(x;n)$ and $\textsc{Sample}\mathcal{C}(x;n,\epsilon)$, respectively.

In general one chooses x so that the expected size $\Lambda C(x)$ of the output of $\Gamma C(x)$ – which satisfies $\Lambda C(x) = xC'(x)/C(x)$ as proved in [5] – is equal to n, or at least asymptotically equal to n, that is, one looks for an exact or an approximate solution of the so-called *target size equation*:

$$x\frac{C'(x)}{C(x)} = n. \tag{5.2}$$

As we show in Section 6, the expected size $\Lambda M(x)$ of the output of $\Gamma M(x)$ satisfies

$$\Lambda M(x) \sim \frac{2\zeta(3)}{(1-x)^3},$$

where $\zeta(\cdot)$ is the Riemann zeta function, so a suitable value of the parameter x to reach a target size n is $\xi_n = 1-(2\zeta(3)/n)^{1/3}$, since $2\zeta(3)/(1-\xi_n)^3 = n$; and we show in Section 6 that the expected size $\Lambda M_{a,b}(x)$ of the output of $\Gamma M_{a,b}(x)$ satisfies

$$\Lambda M_{a,b}(x) \sim \frac{ab}{1-x},$$

so a suitable value of the parameter x to reach target size n is $\xi_n^{a,b} = 1 - ab/n$.

Lemma 5.3 (Targeted samplers for the multiset class $\mathcal{M}$). *Define*

$$\xi_n := 1-(2\zeta(3)/n)^{1/3}.$$

Then, under the oracle assumption, the expected running time of $\textsc{Sample}\mathcal{M}(\xi_n;n)$ *is* $O(n^{4/3})$*; and, for fixed* $\varepsilon \in (0,1)$*, the expected running time of* $\textsc{Sample}\mathcal{M}(\xi_n;n,\varepsilon)$ *is* $O(n^{2/3})$ *as* $n \to \infty$*, the constant in the big O being independent of* ε.[3]

In order to state the expected running times of the targeted samplers for $(a \times b)$-boxed multisets, we define the following functions:

$$\phi(\alpha) = \frac{(\alpha/e)^\alpha}{\Gamma(\alpha)}, \quad \Phi(\alpha,\varepsilon) := \frac{(\alpha/e)^\alpha}{\Gamma(\alpha)}\int_{-\varepsilon}^{\varepsilon}(1+s)^{\alpha-1}e^{-\alpha s}ds.$$

Lemma 5.4 (Targeted samplers for $(a \times b)$-boxed multisets). *Define*

$$\xi_n^{a,b} := 1 - ab/n.$$

Then, for $n \geqslant 1$*, the expected running time of* $\textsc{Sample}\mathcal{M}_{a,b}(\xi_n^{a,b};n)$ *is equivalent to* $\phi(ab)/n$ *as* $n \to \infty$*. For fixed* (a,b,ε)*, the expected running time of* $\textsc{Sample}\mathcal{M}_{a,b}(\xi_n^{a,b};n,\varepsilon)$ *converges to the constant* $ab/\Phi(ab,\varepsilon)$ *as* $n \to \infty$.

[3] More precisely, for each fixed ϵ there is an $n_0(\epsilon)$ such that the running time is at most $c\,n^{2/3}$ for $n \geqslant n_0(\epsilon)$, where the constant c is independent of ϵ.

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Theorem 5.5 (Targeted samplers for plane partitions). *For $n \geqslant 1$, define the algorithm* SamplePartitions$[n]$ *as the procedure that calls* Sample$\mathcal{M}(\xi_n; n)$ *and applies Algorithm 1 (Pak's bijection) to the generated diagram.*

Then SamplePartitions$[n]$ *is an exact-size sampler for plane partitions, of expected running time $O(n^{4/3})$.*

For $n \geqslant 1$ and $\varepsilon \in (0,1)$, define SamplePartitions$[n,\varepsilon]$ *as the algorithm that calls* Sample$\mathcal{M}(\xi_n; n, \varepsilon)$ *and applies Algorithm 1 to the generated diagram. Then algorithm* SamplePartitions$[n,\varepsilon]$ *is an approximate-size sampler for plane partitions, of expected running time $O(n(\ln n)^3)$ as $n \to \infty$ (with ε fixed), the asymptotic constant in the big O not depending on ε.*

The proofs of the expected running times announced in Theorem 5.5 and Theorem 5.6 (given next) are delayed to Section 6.

To state the expected running times of the targeted samplers for $(a \times b)$-boxed plane partitions, we define the following function:

$$\psi(a,b) := \tfrac{1}{2}L\ell(\ell+1) - \tfrac{1}{6}(\ell^3 - 1), \quad \text{where } L = \max(a,b),\ \ell = \min(a,b).$$

Theorem 5.6 (Targeted samplers for $(a \times b)$-boxed plane partitions). *For $n \geqslant 1$, define* SamplePartitions$_{a,b}[n]$ *as the algorithm that calls* Sample$\mathcal{M}_{a,b}(\xi_n^{a,b}; n)$ *and applies Algorithm 1 (Pak's bijection) to the generated diagram.*

Then SamplePartitions$_{a,b}[n]$ *is an exact-size sampler for $(a \times b)$-boxed plane partitions, of expected running time equivalent to $\phi(ab)/n$ as $n \to \infty$.*

For $n \geqslant 1$ and $\varepsilon \in (0,1)$, define SamplePartitions$_{a,b}[n,\varepsilon]$ *as the algorithm that calls* Sample$\mathcal{M}_{a,b}(\xi_n^{a,b}; n, \varepsilon)$ *and applies Algorithm 1 to the generated diagram.*

Then SamplePartitions$_{a,b}[n,\varepsilon]$ *is an approximate-size sampler for $(a \times b)$-boxed plane partitions, of expected running time equivalent to the constant $\psi(a,b) + ab/\Phi(ab,\varepsilon)$ as $n \to \infty$ (with a, b, ε fixed).*

Let us mention that the targeted samplers for $(a \times b)$-boxed plane partitions are easily extended to the framework of $(a \times b)$-boxed *skew* plane partitions. For a fixed admissible index domain $D \subset [0..a-1] \times [0..b-1]$, the appropriate value to reach a target size n exactly (or approximately) is $\xi_n^{(D)} := 1 - |D|/n$, where $|D|$ is the cardinality of D.

Another important remark is that the value $\xi_n^{(D)}$ works well in the asymptotic regime, that is, when $n \gg |D|$. If not in the asymptotic regime (say one generates plane partitions of size 10 000 constrained to a rectangular box 100×100) one has to consider the target size equation more closely. The generating function for multisets with support in D is

$$M_D(x) = \prod_{(i,j)\in D} \frac{1}{1 - x^{i+j+1}}.$$

Hence the target size equation, $xM_D'(x)/M_D(x) = n$, is

$$\sum_{(i,j)\in D} \frac{(i+j+1)x^{i+j+1}}{1 - x^{i+j+1}} = n,$$

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which is to be solved exactly if n is not in the asymptotic regime for the domain D. (The solution is asymptotically $1 - |D|/n$, but the rate of convergence is slow when $|D|$ is large.)

6. Analysis of the complexity

This section is dedicated to proving the expected running times of the random samplers, as stated in Theorem 5.5 and Theorem 5.6. Since most of the difficulty is in proving Theorem 5.5 (unconstrained plane partitions), the proof of Theorem 5.6 is only given in the very last subsection (Section 6.6) and is kept short.

Recall that the random samplers consist of two steps: generate a diagram under the Boltzmann model until the size is in the desired target domain, and then apply Algorithm 1 (Pak's bijection) to the diagram so as to output a random plane partition. Accordingly, the complexity of generation is obtained by adding up the cost of generating a diagram and the cost of Pak's bijection.

The expected costs of generating diagrams under the Boltzmann model are naturally expressed as certain infinite sums, which are best handled by the Mellin transform, recalled next. On the other hand, Pak's bijection has complexity cubic in a certain parameter called the maximum hook length, which is the maximal value of the hook length (abscissa + ordinate + 1) over all non-zero entries. Therefore, we need to find the asymptotic order of the maximum hook length under the uniform distribution at size n.

6.1. The Mellin transform

The Mellin transform is a powerful technique to derive asymptotic estimates of expressions involving specific infinite sums (see [7] for a detailed survey), which occur recurrently in the analysis of our samplers. Given a continuous function $f(t)$ defined on $\mathbb{R}^+$, the *Mellin transform* of $f(t)$ is the function

$$f^*(s) := \int_0^\infty f(t)t^{s-1}dt. \tag{6.1}$$

For instance, the Euler Gamma function $\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt$ is the Mellin transform of e^{-t}. If $f(t) = O(t^{-a})$ as $t \to 0^+$ and $f(t) = O(t^{-b})$ as $t \to +\infty$, then $f^*(s)$ is an analytic function defined on the *fundamental domain* $a < \mathrm{Re}(s) < b$. In addition, $f^*(s)$ is in most cases continuable to a meromorphic function in the whole complex plane (for instance, $\Gamma(s)$ is continuable to a meromorphic function having its poles at negative integers). In a similar way to the Fourier transform, the Mellin transform is almost involutive, the function $f(t)$ being recovered from $f^*(s)$ using the *inversion formula*

$$f(t) = \int_{c-i\infty}^{c+i\infty} f^*(s)t^{-s}ds, \quad \text{for any } c \in (a, b). \tag{6.2}$$

From the inversion formula and the residue theorem, the asymptotic expansion of $f(t)$ as $t \to 0^-$ can be derived from the poles of $f^*(s)$ on the left of the fundamental domain, the rightmost such pole giving the dominant term of the asymptotic expansion. If $f^*(s)$ is decreasing very fast as $\mathrm{Im}(s) \to \infty$ (which occurs in all the series to be analysed next, based on the fact that $\Gamma(s)$ is decaying fast and $\zeta(s)$ is of moderate growth as $\mathrm{Im}(s) \to \infty$),

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then we have the following *transfer rule* [7]: a pole of $f^*(s)$ of order $k+1$ ($k \geqslant 0$),

$$f^*(s) \underset{s\to\alpha}{\sim} \lambda_\alpha \frac{(-1)^k k!}{(s-\alpha)^{k+1}},$$

yields a term

$$\lambda_\alpha t^{-\alpha} (\ln t)^k$$

in the singular expansion of $f(t)$ around 0. In particular, a simple pole $\lambda_\alpha/(s-\alpha)$ yields a term λ_α/t^α.

Another fundamental property of the Mellin transform is to *factorize* sums of a certain form:

$$g(t) = \sum_{k\geqslant 1} a_k f(\mu_k t) \;\Rightarrow\; g^*(s) = \left(\sum_{k\geqslant 1} a_k \mu_k^{-s}\right) f^*(s). \tag{6.3}$$

6.2. Complexity of the Boltzmann samplers for multisets

In this section, we analyse the complexity of the free Boltzmann sampler $\Gamma M(x)$ for the multiset class $\mathcal{M}$, not yet studying the rejection cost when targeting at a certain size domain. In general, given a combinatorial class $\mathcal{C}$ for which an explicit Boltzmann sampler $\Gamma C(x)$ is designed, we write $\Lambda C(x)$ for the expected running time of a call to $\Gamma C(x)$. More generally, we use thereafter the letter Λ as a prefix to denote the expected running time of a random generator. As we are interested in drawing large plane partitions, which requires us to let x tend to 1, we analyse the asymptotic order of $\Lambda M(x)$ as $x \to 1^-$. Recall that $\mathcal{A} := \mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2$, with generating function $A(x) = x/(1-x)^2$. By definition, the first step of the Boltzmann sampler $\Gamma M(x)$ is to draw an integer K under the probability distribution

$$\mathbb{P}(K \leqslant k) = \prod_{j>k} \exp\left(-\tfrac{1}{j} A(x^j)\right). \tag{6.4}$$

Under the oracle assumption discussed in Section 4 and described in detail in [5], the complexity of drawing K is thus of the order of the value k that is finally assigned to K. Hence the expected running time of drawing K is of the same order as the expected value of K under the above given distribution.

Lemma 6.1. *The expectation $\mathbb{E}_x(K)$ of K under the distribution* (6.4) *satisfies*

$$\mathbb{E}_x(K) = O((1-x)^{-1}\ln(1-x)) \quad \textit{as } x \to 1^-.$$

Proof. Fix $x \in (0,1)$. Let $r = r(x)$ be the smallest integer such that $x^r < (1-x)/2$, *i.e.*, $r = \lfloor \ln((1-x)/2)/\ln(x) \rfloor + 1$. Note that $x^i \leqslant 1/2$ for $i \geqslant r$. Hence, for $i \geqslant r$, $A(x^i) \leqslant 4x^i$. Further, for $k \geqslant r$,

$$\sum_{i>k} \frac{1}{i} A(x^i) \leqslant 4 \sum_{i>k} \frac{x^i}{i} \leqslant \frac{4x^k}{1-x} \leqslant 2x^{k-r}.$$

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Hence, for $k \geqslant r$,

$$\mathbb{P}(K > k) = 1 - \mathbb{P}(K \leqslant k) \leqslant 1 - \exp(-2x^{k-r}) \leqslant 2x^{k-r}.$$

We thus obtain

$$\mathbb{E}_x(K) = \sum_{k \geqslant 0} \mathbb{P}(K > k) \leqslant r + 2\sum_{k \geqslant r} x^{k-r} \leqslant r + \frac{2}{1-x},$$

which concludes the proof since $r = r(x)$ is $O((1-x)^{-1}\ln(1-x))$ as $x \to 1^-$. □

Once the integer K is drawn, the Boltzmann sampler $\Gamma M(x)$ draws Poisson laws and geometric laws (a call to $\Gamma A(x)$ consists of two calls to geometric laws). Precisely, for each $i \geqslant 1$, the number of calls to $\Gamma A(x^i)$ follows a Poisson law $\mathrm{Pois}(A(x^i)/i)$. Since $\mathbb{E}(\mathrm{Pois}(\lambda)) = \lambda$, the expected number of calls to $\Gamma A(x^i)$ is $A(x^i)/i$. In addition, each call to ΓA takes constant time, since it consists of two calls to geometric laws. Hence

$$\Lambda M(x) = O\big(\mathbb{E}_x(K)\big) + O\left(\sum_{i \geqslant 1} A(x^i)/i\right) = O\left(\frac{\ln(1-x)}{1-x}\right) + O\left(\sum_{i \geqslant 1} A(x^i)/i\right). \tag{6.5}$$

Lemma 6.2. *The expected running time of the Boltzmann sampler $\Gamma M(x)$ satisfies*

$$\Lambda M(x) = \underset{x \to 1^-}{O}\big((1-x)^{-2}\big).$$

Proof. By equation (6.5), it is enough to show that $F(x) := \sum_{i \geqslant 1} A(x^i)/i$ is $O((1-x)^{-2})$ as $x \to 1^-$. This is the first instance where the Mellin transform can be successfully applied (more elementary approaches would work in this simple case). Define $L(t) := F(e^{-t})$. Then

$$L(t) = \sum_{r \geqslant 1} \frac{e^{-rt}}{r(1-e^{-rt})^2} = \sum_{r \geqslant 1} \frac{1}{r} f(rt), \quad \text{where } f(t) := \frac{e^{-t}}{(1-e^{-t})^2}.$$

The factorization property of the Mellin transform, equation (6.3), yields

$$L^*(s) = \left(\sum_{r \geqslant 1} \frac{1}{r} r^{-s}\right) f^*(s) = \zeta(s+1) f^*(s),$$

where $\zeta(s) := \sum_{r \geqslant 1} r^{-s}$ is the Riemann zeta function. Since $f(t) = \sum_{n \geqslant 1} n e^{-nt}$, the factorization property yields

$$f^*(s) = \left(\sum_{n \geqslant 1} n n^{-s}\right) \Gamma(s) = \zeta(s-1)\Gamma(s).$$

Thus, $L^*(s) = \zeta(s+1)\zeta(s-1)\Gamma(s)$. It is easily checked that $L(t) = O(1/t^2)$ as $t \to 0^+$ and $L(t) = O(e^{-t})$ as $t \to \infty$, so that the fundamental domain of $L^*(s)$ is $\mathrm{Re}(s) > 2$. Hence, to determine the asymptotic behaviour of $L(t)$ as $t \to 0^+$, we have to find the rightmost poles of $L^*(s)$ such that $\mathrm{Re}(s) \leqslant 2$. The function $\zeta(s)$ has a unique pole at $s = 1$ with coefficient 1, and the function $\Gamma(s)$ has its poles at non-positive integers. Hence $L^*(s)$ has a simple pole at $s = 2$, with coefficient $\zeta(3)$, and no other pole for $\mathrm{Re}(s) \geqslant 1$, so that the transfer

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rule of the Mellin transform yields

$$L(t) = \frac{\zeta(3)}{t^2} + O\left(\frac{1}{t}\right) \quad \text{as } t \to 0^+.$$

The change of variable $t = -\ln(x)$ yields

$$F(x) = \frac{\zeta(3)}{(1-x)^2} + O\left(\frac{1}{1-x}\right) \quad \text{as } x \to 1^-.$$

As a consequence, $F(x) = O((1-x)^{-2})$ as $x \to 1^-$. □

6.3. Analysis of the size of a multiset in $\mathcal{M}$ under the Boltzmann model

Given $0 < x < 1$, let N_x denote the random variable giving the size of the output of $\Gamma M(x)$ (which is also the size of a plane partition under the Boltzmann model at x); Figure 3 shows plots of N_x for several values of x. As the Boltzmann probability of an object of size n is $x^n/M(x)$, the expectation and variance of N_x satisfy (see [5] for details):

$$\mathbb{E}(N_x) = \sum_{n\geqslant 1} nM_n \frac{x^n}{M(x)} = x\frac{M'(x)}{M(x)}, \quad \mathbb{V}(N_x) = \sum_{n\geqslant 1} n^2 M_n \frac{x^n}{M(x)} - \mathbb{E}(N_x)^2 = x\frac{d\mathbb{E}(N_x)}{dx}.$$

Lemma 6.3. *The expectation and variance of the size of a multiset $\mu \in \mathcal{M}$ drawn under the Boltzmann model satisfy*

$$\mathbb{E}(N_x) = \frac{2\zeta(3)}{(1-x)^3} + \underset{x\to 1^-}{O}\left(\frac{1}{(1-x)^2}\right), \quad \mathbb{V}(N_x) = \frac{6\zeta(3)}{(1-x)^4} + \underset{x\to 1^-}{O}\left(\frac{1}{(1-x)^3}\right).$$

Proof. We once again use the Mellin transform to derive the asymptotic estimates. Observe that $M'(x)/M(x)$ is the logarithmic derivative of $M(x)$, hence the expression (1.1) of $M(x) = P(x)$ yields

$$\mathbb{E}(N_x) = x\sum_{r\geqslant 1} r\frac{rx^{r-1}}{1-x^r} = \sum_{r\geqslant 1} r^2 \frac{x^r}{1-x^r}.$$

Define $L(t) := \mathbb{E}(N_{e^{-t}})$. Then

$$L(t) = \sum_{r\geqslant 1} r^2 \frac{e^{-rt}}{1-e^{-rt}} = \sum_{r\geqslant 1} r^2 f(rt),$$

where $f(t) := e^{-t}/(1-e^{-t}) = \sum_{n\geqslant 1} e^{-nt}$. Hence

$$L^*(s) = \sum_{r\geqslant 1} (r^2 r^{-s}) f^*(s) = \zeta(s-2) f^*(s) = \zeta(s-2)\sum_{n\geqslant 1} n^{-s}\Gamma(s) = \zeta(s-2)\zeta(s)\Gamma(s).$$

The function $L^*(s)$ is defined on the fundamental domain $\mathrm{Re}(s) > 3$. The rightmost pole such that $\mathrm{Re}(s) \leqslant 3$ is at $s = 3$, where $L^*(s) \sim 2\zeta(3)/(s-3)$. As there are no other poles for $\mathrm{Re}(s) \geqslant 2$, the transfer rule yields

$$L(t) = \frac{2\zeta(3)}{t^3} + O(t^{-2}) \quad \text{as } t \to 0^+.$$

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Hence the change of variable $x = -\ln(t)$ gives

$$\mathbb{E}(N_x) = \frac{2\zeta(3)}{(1-x)^3} + O\left(\frac{1}{(1-x)^2}\right) \quad \text{as } x \to 1^-.$$

The variance is treated similarly:

$$\mathbb{V}(N_x) = x\frac{d\mathbb{E}(N_x)}{dx} = \sum_{r\geqslant 1} r^3 \frac{x^r}{(1-x^r)^2}.$$

Hence the function $L(t) := \mathbb{V}(N_{e^{-t}})$ satisfies $L(t) = \sum_{r\geqslant 1} r^3 g(rt)$, where $g(t) = e^{-t}/(1-e^{-t})^2 = \sum_{n\geqslant 1} n e^{-nt}$. Thus, $L^*(s) = \zeta(s-3)\zeta(s-1)\Gamma(s)$. The location of the poles of $L^*(s)$ and the transfer rule yields

$$L(t) = \frac{6\zeta(3)}{t^4} + O(t^{-2}) \quad \text{as } t \to 0^+,$$

giving

$$\mathbb{V}(N_x) = \frac{6\zeta(3)}{(1-x)^4} + O\left(\frac{1}{(1-x)^3}\right) \quad \text{as } x \to 1^-. \qquad \square$$

6.4. Complexity of the targeted samplers for multisets

Recall that the targeted samplers for the multiset class $\mathcal{M}$ repeatedly call the Boltzmann sampler $\Gamma M(x)$ with a suitable value of x until the size is in the target domain Ω; $\Omega = \{n\}$ for exact-size sampling and $\Omega = [n(1-\varepsilon), n(1+\varepsilon)]$ for approximate-size sampling.

Lemma 6.4. *For $n \geqslant 1$, let ξ_n be the solution of $2\zeta(3)/(1-x)^3 = n$, i.e.,*

$$\xi_n = 1 - (2\zeta(3)/n)^{1/3}.$$

Define π_n to be the probability that the output of $\Gamma M(\xi_n)$ has size n. For any $\varepsilon \in (0,1)$, define $\pi_{n,\varepsilon}$ to be the probability that the size of the output of $\Gamma M(\xi_n)$ is in the range $[n(1-\varepsilon), n(1+\varepsilon)]$. Then, $\pi_n \sim Cn^{-2/3}$ as $n \to \infty$, with $C \approx 0.1082$; and, for fixed $\varepsilon \in (0,1)$, $\pi_{n,\varepsilon} \to 1$ as $n \to \infty$.

Proof. As ξ_n is the solution of $2\zeta(3)/(1-x)^3 = n$, Lemma 6.3 ensures that $\mathbb{E}(N_{\xi_n}) = n + O(n^{2/3})$ as $n \to \infty$, *i.e.*, there exists $C > 0$ such that $|\mathbb{E}(N_{\xi_n}) - n| \leqslant Cn^{2/3}$. Hence Chebyshev's inequality gives, for any $\varepsilon \in (0,1)$,

$$\begin{aligned} 1 - \pi_{n,\varepsilon} &= \mathbb{P}(|N_{\xi_n} - n| > \varepsilon n) \\ &\leqslant \mathbb{P}\big(|N_{\xi_n} - \mathbb{E}(N_{\xi_n})| > (\varepsilon n - Cn^{2/3})\big) \leqslant \frac{\mathbb{V}(N_{\xi_n})}{(\varepsilon n - Cn^{2/3})^2}. \end{aligned}$$

Given the fact that $\mathbb{V}(N_{\xi_n}) = O((1-\xi_n)^{-4}) = O(n^{4/3})$, we have $1 - \pi_{n,\varepsilon} \to 0$ as $n \to \infty$.

Next we prove the estimate of π_n. Note that

$$\pi_n = M_n \cdot (\xi_n)^n / M(\xi_n) = P_n \cdot (\xi_n)^n / P(\xi_n).$$

Hence it is enough to obtain the asymptotics of P_n and $(\xi_n)^n$ as $n \to \infty$, and of $P(x)$ as $x \to 1^-$. These were first found by Wright [29] and later by Meinardus in a more general

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framework [17] relying on the saddle point method (a detailed and accessible presentation of the saddle point method is given in [8, Chapter VIII]; partitions are studied in the 6th section of the chapter). We briefly review the main ingredients. To find the asymptotics of $P(x)$ as $x \to 1^-$, we apply the Mellin transform techniques to the series $L(t) = \ln(P(e^{-t}))$, and find

$$P(x) \sim C_1(1-x)^{1/12} \exp\left(\zeta(3)\frac{x}{(1-x)^2}\right) \quad \text{as } x \to 1^-, \tag{6.6}$$

with C_1 an explicit constant, $C_1 \approx 0.9368$. Define $c := (2\zeta(3))^{1/3}$, so $\zeta(3) = c^3/2$ and $\xi_n = 1 - cn^{-1/3}$. From (6.6) we obtain

$$P(\xi_n) \sim C_1' n^{-1/36} \exp\left(\tfrac{1}{2}cn^{2/3} - \tfrac{1}{2}c^2 n^{1/3}\right),$$

with $C_1' = C_1 c^{1/12} \approx 0.9599$. The asymptotics of $(\xi_n)^n$ is easy to obtain. Since $\log(1 - cn^{-1/3}) = -cn^{-1/3} - \frac{1}{2}c^2 n^{-2/3} - \frac{c^3}{3}n^{-1} + o(n^{-1})$, we obtain

$$(\xi_n)^n = \exp(n\log(1 - cn^{-1/3})) \sim C_2 \exp\left(-cn^{2/3} - \tfrac{1}{2}c^2 n^{1/3}\right),$$

with $C_2 = \exp(-c^3/3) \approx 0.4487$. Finally, the asymptotics of P_n was obtained by Wright [29] using the saddle point method and the estimate (6.6). The idea is to use Cauchy's formula,

$$P_n = \frac{1}{2i\pi}\int_{C(0,\xi_n)} P(z) z^{-n-1} dz,$$

with the circle of radius ξ_n centred at 0 as the integration contour. Using the estimate (6.6) (more precisely, we need the fact that this estimate holds in an open cone centred at 1 and containing the line $z < 1$), we show that the main contribution of the integral is on a small arc of $C(0, \xi_n)$ around the origin, and obtain

$$P_n \underset{n\to\infty}{\sim} C_3 n^{-25/36} \exp\left(\tfrac{3}{2}cn^{2/3}\right),$$

with $C_3 \approx 0.2315$. Thus, from the estimates of $P(\xi_n)$, $(\xi_n)^n$, and P_n, we find

$$\pi_n \sim Cn^{-2/3},$$

with $C \approx 0.1082$. □

It is easily checked that the expected running time of a rejection sampler is the expected running time of the sampler times the expected number of calls (which is the inverse of the probability of success), therefore

$$\Lambda\big(\textsc{Sample}\mathcal{M}[\xi_n; n]\big) = \frac{\Lambda M(\xi_n)}{\pi_n}, \quad \Lambda\big(\textsc{Sample}\mathcal{M}[\xi_n; n, \varepsilon]\big) = \frac{\Lambda M(\xi_n)}{\pi_{n,\varepsilon}}.$$

Since $\Lambda M(x) = O((1-x)^{-2})$ and $1 - \xi_n = O(n^{-1/3})$, we have $\Lambda M(\xi_n) = O(n^{2/3})$. Moreover, $1/\pi_n = O(n^{2/3})$ and $1/\pi_{n,\varepsilon} \to 1$ (for fixed $\varepsilon \in (0,1)$) by Lemma 6.4. Hence $\Lambda(\textsc{Sample}\mathcal{M}(\xi_n; n)) = O(n^{4/3})$ and $\Lambda(\textsc{Sample}\mathcal{M}(\xi_n; n, \epsilon)) = O(n^{2/3})$, which concludes the proof of Lemma 5.3. □

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6.5. Complexity of Pak's bijection

The aim of this section is to provide an O bound on the expected running time of Algorithm 1 for a multiset $\mu \in \mathcal{M}$ of size n taken uniformly at random. As we will see, the complexity of Algorithm 1 applied to a multiset is expressed in terms of the width and length of the bounding rectangle of μ. These parameters have been recently studied by Mutafchiev in [18]: using the saddle point method he shows that the width (similarly the length), after suitable normalization, converges weakly to an explicit distribution. Since we are only interested in a big O, we only need upper bounds, which are much simpler to show. Therefore we prefer to provide our own simple self-contained analysis.

For a multiset $\mu \in \mathcal{M} := \textsc{MSet}(\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2)$ represented by its diagram, let w and h be the width and height of the bounding rectangle of μ. Pak's bijection scans the double range $[0 \leqslant i \leqslant w-1, 0 \leqslant j \leqslant h-1]$; when a square (i,j) is treated, the squares that are updated are those on the up-right diagonal $\{(i+c, j+c) \text{ s.t. } i+c \leqslant w,\ j+c \leqslant h\}$; each update of an entry consists of a fixed number of operations involving $\{+, -, \max, \min\}$. The sum of the lengths of the up-right diagonals over the squares of the bounding rectangle is $\sum_{i=1}^{\min(w,h)} i(w-i+h-i+1)$, which is equal to

$$\psi(w,h) := \tfrac{1}{2}L\ell(\ell+1) - \tfrac{1}{6}(\ell^3-1), \quad \text{where } L = \max(w,h),\ \ell = \min(w,h). \tag{6.7}$$

This quantity is clearly $O(L^3)$, so that the complexity of Algorithm 1 is cubic in L.

We introduce a parameter that will crucially simplify the analysis. Given $\mu \in \mathcal{M}$ represented as a diagram, the *hook length* of an entry (i,j) of the diagram is defined as $h(i,j) := i + j + 1$, *i.e.*, $h(i,j)$ is the size of (i,j) seen as an element of $\mathcal{A} = \mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2$. The *maximum hook length* of μ, denoted by $k(\mu)$, is the maximum value of the hook length over all non-zero entries of the diagram of μ.

Lemma 6.5. *Given an element μ in $\mathcal{M} := \textsc{MSet}(\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2)$, the complexity of Pak's bijection applied to μ is $O([k(\mu)]^3)$, where $k(\mu)$ is the maximal hook length of μ.*

Proof. The maximal hook length $k(\mu)$ is at least equal to the width w and to the height h of the bounding rectangle of μ. Hence the complexity of Pak's bijection, which is $O([\max(w,h)]^3)$, is also $O([k(\mu)]^3)$. □

Hence, to have a big O bound on the expected running time of Pak's bijection, we need to bound the expected value of $k(\mu)^3$ for a multiset $\mu \in \mathcal{M}$ of size n taken uniformly at random. First, for a series $C(x) = \sum_n c_n x^n$ with non-negative coefficients and with radius of convergence $\rho > 0$, we recall the trivial bound

$$c_n \leqslant C(x)x^{-n}, \quad \text{for any } x \in (0,\rho). \tag{6.8}$$

Lemma 6.6 (Expected running time of Pak's bijection at a fixed size). *For $n \geqslant 1$, let $\mu \in \mathcal{M}$ be a multiset of size n taken uniformly at random. Then the expected running time of Algorithm 1 applied to μ is $O(n(\ln n)^3)$.*

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Proof. Let H_n denote the expectation of $k(\mu)^3$ for a multiset $\mu \in \mathcal{M}$ of size n taken uniformly at random. By Lemma 6.5, the expected running time of Algorithm 1 under the uniform distribution (on $\mathcal{M}$) at size n is $O(H_n)$. Hence, to prove the lemma we just have to show that $H_n = O(n(\ln n)^3)$. For $k \geqslant 1$, let $\mathcal{M}^{(k)}$ denote the family of multisets in $\mathcal{M}$ with maximal hook length equal to k, and let $M^{(k)}(x)$ denote the series of $\mathcal{M}^{(k)}$. Define

$$K(x) := \sum_{k \geqslant 1} k^3 M^{(k)}(x).$$

Note that $H_n = [x^n]K(x)/[x^n]M(x) = [x^n]K(x)/P_n$, with P_n the number of plane partitions of size n. For $k \geqslant 1$, define a *k-pointed multiset* as a multiset $\mu \in \mathcal{M}$, where a non-zero entry at hook length k is marked in the diagram of μ. Note that the series of k-pointed multisets is $kx^k M(x)$, where the factor k counts the possible places to mark an entry and where the factor x^k takes account of the fact that the marked entry is non-zero. Since k-pointed multisets form a superfamily of $\mathcal{M}^{(k)}$, we have

$$M^{(k)}(x) \leqslant kx^k M(x) = kx^k P(x), \quad \text{for any } 0 < x < 1.$$

Let B be a constant whose value is to be fixed later, and define $u_n := \lfloor Bn^{1/3} \log(n) \rfloor$. We have

$$H_n = \frac{1}{P_n} \sum_{k=1}^{n} k^3 [x^n] M^{(k)}(x) \leqslant (u_n)^3 + \frac{n^3}{P_n} \sum_{k=u_n}^{n} [x^n] M^{(k)}(x).$$

As $M^{(k)}(x) \leqslant kx^k P(x)$, the bound (6.8) ensures that, for $u_n \leqslant k \leqslant n$,

$$[x^n] M^{(k)}(x) \leqslant M^{(k)}(\xi_n) \cdot (\xi_n)^{-n} \leqslant k(\xi_n)^k \cdot P(\xi_n) \cdot (\xi_n)^{-n} \leqslant n(\xi_n)^{u_n} \cdot P(\xi_n) \cdot (\xi_n)^{-n}.$$

Hence

$$H_n \leqslant (u_n)^3 + n^5 (\xi_n)^{u_n} \frac{P(\xi_n)}{P_n \cdot (\xi_n)^n}.$$

Let $c := (2\zeta(3))^{1/3}$. We have

$$(\xi_n)^{u_n} = \exp\bigl(u_n \log(1 - cn^{-1/3})\bigr) \sim \exp(-cu_n n^{-1/3}) \sim n^{-cB}.$$

Moreover, according to Lemma 6.4,

$$\frac{P(\xi_n)}{P_n \cdot (\xi_n)^n} = \frac{1}{\pi_n} = O(n^{2/3}).$$

Hence

$$H_n = O(n(\ln n)^3) + O(n^5 n^{-cB} n^{2/3}).$$

Taking the constant B sufficiently large so that $cB > 4 + 2/3$ (*e.g.*, $B = 5$), we obtain

$$H_n = O(n(\ln n)^3). \qquad \square$$

Proposition 6.7. *For any $\varepsilon \in (0, 1)$, the expected running time of* SamplePartitions$[n, \varepsilon]$ *satisfies*

$$\Lambda\bigl(\textsc{SamplePartitions}[n, \varepsilon]\bigr) = O(n(\ln n)^3) \quad \text{as } n \to \infty,$$

the asymptotic constant in the big O being independent of ε.

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The expected running time of SamplePartitions$[n]$ *satisfies*

$$\Lambda\big(\textsc{SamplePartitions}[n]\big) = O(n^{4/3}) \quad \text{as } n \to \infty.$$

Proof. Start with the proof of the exact-size sampler. By definition, we have

$$\Lambda\big(\textsc{SamplePartitions}[n]\big) = \Lambda\big(\textsc{Sample}\mathcal{M}[\xi_n; n]\big) + \mathbb{E}_n(\textsc{PakBijection}),$$

where $\mathbb{E}_n(\textsc{PakBijection})$ is the expected running time of Algorithm 1 (Pak's bijection) for a multiset of size n taken uniformly at random. Lemma 5.3 ensures that $\Lambda(\textsc{Sample}\mathcal{M}[\xi_n; n])$ is $O(n^{4/3})$; by Lemma 6.6, $\mathbb{E}_n(\textsc{PakBijection}$ is $O(n(\ln n)^3)$. Thus $\Lambda(\textsc{SamplePartitions}[n])$ is $O(n^{4/3})$.

Now consider the approximate-size sampler. By definition, we have

$$\Lambda\big(\textsc{SamplePartitions}[n, \varepsilon]\big) = \Lambda\big(\textsc{Sample}\mathcal{M}[\xi_n; n, \varepsilon]\big) + \mathbb{E}_{n,\varepsilon}(\textsc{PakBijection}),$$

where $\mathbb{E}_{n,\varepsilon}(\textsc{PakBijection})$ is the expected running time of Pak's bijection for a multiset drawn from $\textsc{Sample}\mathcal{M}(\xi_n; n, \varepsilon)$. By Lemma 5.3, $\Lambda(\textsc{Sample}\mathcal{M}(\xi_n; n, \varepsilon))$ is $O(n^{2/3})$. Since the size of an object output by $\textsc{Sample}\mathcal{M}(\xi_n; n, \varepsilon)$ is at most $2n$ (because $\varepsilon \in (0, 1)$), Lemma 6.6 ensures that $\mathbb{E}_{n,\varepsilon}(\textsc{PakBijection})$ is $O(n(\ln n)^3)$. Consequently $\Lambda(\textsc{SamplePartitions}[n, \epsilon])$ is $O(n(\ln n)^3)$. □

6.6. Complexity of the samplers for $(a \times b)$-boxed plane partitions

By definition (see Algorithm 5), the Boltzmann sampler $\Gamma M_{a,b}(x)$ for $(a \times b)$-boxed multisets just consists of ab calls to geometric laws. Giving unit cost to a call to a geometric law, we have

$$\Lambda M_{a,b}(x) = ab.$$

Next, recall that

$$M_{a,b}(x) = \prod_{\substack{0 \leqslant i < a \\ 0 \leqslant j < b}} \frac{x^{i+j+1}}{1 - x^{i+j+1}} \underset{x \to 1^-}{\sim} \frac{c}{(1-x)^{ab}}, \quad \text{with } c = \prod_{\substack{0 \leqslant i < a \\ 0 \leqslant j < b}} \frac{1}{i+j+1}.$$

For $\alpha > 0$, a class $\mathcal{C}$ is called α-singular at $x = 1$ if $C(x) \sim c/(1-x)^\alpha$ for some constant $c > 0$ (the $\sim$ holding in a complex neighbourhood of 1), and if 1 is the only singularity of $C(x)$ in a disk of the form $\{z \in \mathbb{C} \text{ s.t. } |z| < 1 + \delta\}$ for some $\delta > 0$. Note that $\mathcal{M}_{a,b}$ is α-singular for $\alpha = ab$. In [5] it is shown that if $\mathcal{C}$ is α-singular at $x = 1$, then for $n \geqslant 1$ the probability π_n of being of size n under the Boltzmann model at $x_n := 1 - \alpha/n$ satisfies

$$\pi_n \underset{n \to \infty}{\sim} \frac{\phi(\alpha)}{n}, \quad \text{where } \phi(\alpha) = \frac{(\alpha/e)^\alpha}{\Gamma(\alpha)}. \tag{6.9}$$

Moreover, for fixed $\varepsilon \in (0, 1)$, the probability $\pi_{n,\varepsilon}$ of being in the size domain $[n(1-\varepsilon), n(1+\varepsilon)]$ under the Boltzmann model at x_n satisfies

$$\pi_{n,\varepsilon} \underset{n \to \infty}{\to} \Phi(\alpha, \varepsilon), \quad \text{where } \Phi(\alpha, \varepsilon) = \frac{(\alpha/e)^\alpha}{\Gamma(\alpha)} \int_{-\varepsilon}^{\varepsilon} (1+s)^{\alpha-1} e^{-\alpha s} ds. \tag{6.10}$$

Since the expected running time of a rejection sampler is the expected running time of the sampler divided by the probability of success at each attempt, the expected running

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times of the targeted samplers for $\mathcal{M}_{a,b}$ satisfy asymptotically

$$\Lambda\big(\textsc{Sample}\mathcal{M}_{a,b}[\xi_n^{a,b};n]\big) \underset{n\to\infty}{\sim} \frac{ab}{\phi(ab)}n, \quad \Lambda\big(\textsc{Sample}\mathcal{M}_{a,b}[\xi_n^{a,b};n,\varepsilon]\big) \underset{n\to\infty}{\to} \frac{ab}{\Phi(ab,\varepsilon)}.$$

The second step of the targeted samplers for $(a\times b)$-boxed plane partitions is Algorithm 1 (Pak's bijection). When $x\to 1^-$, all entries of the rectangle $R_{a,b} := [0..a-1]\times[0..b-1]$ in the diagram of $\mu \leftarrow \Gamma M_{a,b}(x)$ are non-zero with high probability, hence the bounding rectangle of μ is $R_{a,b}$ with high probability. As a consequence, the complexity of Algorithm 1 applied to μ is with high probability the quantity $\psi(a,b)$ defined in (6.7). Hence

$$\Lambda\big(\textsc{SamplePartitions}_{a,b}[n]\big) \underset{n\to\infty}{\sim} \frac{ab}{\phi(ab)}n + \psi(a,b) \underset{n\to\infty}{\sim} \frac{ab}{\phi(ab)}n,$$

$$\Lambda\big(\textsc{SamplePartitions}_{a,b}[n,\varepsilon]\big) \underset{n\to\infty}{\to} \frac{ab}{\Phi(ab,\varepsilon)} + \psi(a,b),$$

which concludes the proof of Theorem 5.6. □

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