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Complexity and cohomology for cut-and-projection tilings

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Abstract. We consider a subclass of tilings: the tilings obtained by cut-and-projection. Under somewhat standard assumptions, we show that the natural complexity function has polynomial growth. We compute its exponent α in terms of the ranks of certain groups which appear in the construction. We give bounds for α. These computations apply to some well-known tilings, such as the octagonal tilings, or tilings associated with billiard sequences. A link is made between the exponent of the complexity, and the fact that the cohomology of the associated tiling space is finitely generated over $\mathbb{Q}$. We show that such a link cannot be established for more general tilings, and we present a counterexample in dimension one.

1. *Introduction*

A major motivation for studying aperiodic tilings came from the physics of quasi-crystals. Even though the aperiodic Wang tilings and Penrose tilings have already been studied in the 1960s and the 1970s, respectively, most of the work on this subject was performed after the first discovery of a quasi-crystalline material in the 1980s. Such a material appeared to be very ordered, because its diffraction pattern showed clear peaks. However, it could not be periodic, as the diffraction pattern had symmetries which should not have occurred according to the classical periodic models of crystals. Before this observation, a crystalline material was seen as a set of atoms, the position of which was determined by a regular lattice. The group of transformations of $\mathbb{R}^3$ preserving the lattice determined the symmetries of the crystal. The appearance of quasi-crystals in physics created the need for new mathematical objects in order to model these materials.

The cut-and-projection method plays a major role in the description of such aperiodic yet ordered sets. The general idea is to project a 'slice' of a higher-dimensional regular lattice of $\mathbb{R}^N$ to a subspace of dimension d. The resulting point pattern should not

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be periodic, but should inherit the symmetry properties of the larger lattice. This method actually provides important examples of tilings, such as the octagonal tilings, the dodecagonal tilings, or the icosahedral tilings.

The notion of an ordered point set, or ordered tiling, is ambiguous. The definition itself of what should be an ordered tiling varies according to the author's preoccupations. A most common assumption is the notion of finite local complexity (FLC). One requires that there are only finitely many local configurations in the tiling (up to translation). This is a qualitative condition on complexity. A dynamical definition of order could be recurrence of orbits for the associated dynamical system. In combinatorial terms, this corresponds to a property of repetitivity: every finite patch of the tiling appears infinitely often, and within a prescribed range depending on the size of the patch. Another dynamical condition would be to require that the entropy of the dynamical system is zero. In terms of 'patch-counting entropy', it is equivalent to requiring that the growth of the patch-counting function (which to R associates the number of patches included in a ball of size R) is sub-exponential.

Our approach to understanding the notion of an ordered point pattern involves a complexity function, as in the entropy case, but with greater precision. The notion of complexity is well defined for bi-infinite words over a finite alphabet: the function $n \mapsto p(n)$ associated with a word w counts the number of subwords of w of length n. This definition can be extended to multi-dimensional tilings, although not in a canonical way: $p(n)$ should count (up to translation) the number of patches of size n. This definition depends on a suitable definition of 'patch of size n'. While many properties are known about the complexity of one-dimensional words (that is, for one-dimensional tilings), the situation is not as well known for multi-dimensional tilings.

In this article, we define a notion of 'patch of size n' in the framework of cut-and-projection tilings, in order to define a complexity function. We compute this function for cut-and-projection tilings with almost-canonical acceptance domain. Almost-canonical acceptance domains, as we defined here, are a generalization of the canonical case, when the projected 'slice' of the lattice is a cylinder, the base of which is the projection of the unit cube. Almost canonical acceptance domains are domains for which it is possible to define cut hyperplanes in the 'internal space' (a supplementary of the 'physical space').

We prove (Theorem 3.3) that for a cut-and-projection tiling of dimension d arising as the projection of the regular lattice $\mathbb{Z}^N$ with almost-canonical acceptance domain, the complexity $p(n)$ grows as n^α, where α is an integer which depends on the data of the cut-and-projection method. Namely, it depends on the rank of the stabilizers of the cut planes, and is computed in examples. We give bounds for α: for a tiling with no period, we have $d \leq \alpha \leq d(N-d)$ (Theorems 3.3 and 5.1). For a generic choice of parameters of the cut-and-projection method, the complexity is maximal, that is $\alpha = d(N-d)$ (see Remark 3.4).

Another notion of 'ordered structure' could be understood in terms of the topology of the tiling space. Given a tiling and its associated tiling space Ω, one can compute its Čech cohomology. A qualitative distinction can be made between tiling spaces with finitely generated rational cohomology and tiling spaces with infinitely generated rational cohomology. This distinction seems to characterize the difference between a topologically complex space and a topologically simpler space, and our intuition is that the class of tilings with finitely generated rational cohomology should have some specific interest, being a

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class of more 'ordered' tilings. Note that the cohomology groups of substitution tiling spaces are finitely generated over $\mathbb{Q}$ (see [**AP**]), as well as all of the tilings which have been used to model quasi-crystals in physics.

For cut-and-projection tilings, there is actually a link between these two notions of order: the cohomological 'complexity' and the growth of the complexity function p. We prove in §5 (Theorem 5.1) that, in the case of a d-dimensional cut-and-projection tiling with almost-canonical acceptance domain and with no periods, the following equivalence holds: the cohomology of the tiling space is finitely generated if and only if the complexity of the tiling is low, that is $p(n)$ grows as n^d. This result is adapted in the case of tilings with periods.

Such a link between high complexity and infinitely generated cohomology does not hold in a more general framework. We prove that if a word is not too complex (which means that its complexity is dominated by Cn with C a constant), then its associated tiling space has finitely generated cohomology. This result is in line with the statement above, but the converse does not hold in full generality: we define a word, the complexity of which is exponential, but such that its associated tiling space has finitely generated cohomology. In order to prove these results, we prove that the tiling space of a word is homeomorphic to the inverse limit of its Rauzy graphs. It was already known that the space was an inverse limit of graphs, thanks to the Anderson–Putnam–Gähler complexes (see [**Sad**]). However, Rauzy graphs are more suitable than Gähler complexes for understanding the combinatorics of a word. Our proof uses the now classical methods developed in [**AP**] and [**Sad**].

The questions raised in this article have interested mathematicians for some time. Answers for some of these problems are already known in specific cases. The case of complexity for one-dimensional cut-and-projection tilings, also known as cubic billiard sequences, was investigated in [**AMST**] in dimension three and then in [**Bar**] in any dimension. In these articles, an exact and explicit polynomial formula is given for the complexity function, assuming certain additional arithmetic conditions.

The simplest case of the Sturmian sequences has itself many different (and equivalent) descriptions: cut-and-projection tiling, billiard sequence in the square, word coded by the exchange of intervals or by rotations on the circle, cutting word of a square regular tiling of the plane, etc. Each of these points of view leads to different definitions of generalized Sturmian sequences, and subsequent calculations for their complexity. One can see, for example, [**HV**] for the complexity of cutting words defined on more general tilings than the square tiling.

Our point of view is of cut-and-projection tilings, and it seems that the complexity problem has not yet been solved in general in this framework. Berthé and Vuillon obtained more precise results than ours for the complexity, in specific case of $(3, 2)$, $(N, N-1)$ and to a certain extent $(N, 2)$-cut-and-projection tilings (or 'discrete planes'); see [**BV**] and [**Vui**]. Their method consists of modifying the tiling in order to obtain a $\mathbb{Z}^2$-subshift, and studying its rectangular complexity. The method used in our paper seems more similar to that used in [**AMST**] in the sense that the problem of counting patches (or words in their case) is reduced to the problem of counting connected components of a certain set (counting cells of a certain decomposition of the two torus, respectively).

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It seems also that the question of complexity was solved in the framework of tilings given by inflation and substitution. Robinson and Hansen (see [**Rob**] and [**HR**]) gave an upper bound for the complexity function for substitutive tilings in dimension d, which agrees with our computation for examples which are both substitutive and obtained by cut-and-projection. See §4.1 for the example of the octagonal tiling.

There are also other lines of approach to study complexity in more than one dimension: Lagarias and Pleasants [**LP**] have worked on the complexity of general Delone sets of $\mathbb{R}^d$. They proved results of minimal complexity. More precisely, they stated the following result: assume that the complexity function p of a Delone set $\mathcal{D}$ satisfies $p(n) \leq Cn$ for some $n \in \mathbb{N}$ and for some constant C depending (in a weak sense) on the set. Then $\mathcal{D}$ is an ideal crystal, that is, its group of periods has rank d. We prove here a result which is related to a conjecture in their article, even though we prove it for the much more specific case of cut-and-projection Delone sets: the group of periods of a cut-and-projection Delone set of $\mathbb{R}^d$ is of rank at most k, if and only if $p(n)/n^{d-k}$ is bounded from below by a positive constant. However, we do not have control on the constant involved, and so this does not yet fully confirm the conjecture for almost-canonical cut-and-projection tilings.

2. *Definitions and general properties*

2.1. *Cut-and-projection tiling spaces.* Let $\mathbb{R}^N$ be the standard Euclidean space with basis $(e_i)_{i=1}^N$, and distinguished sublattice $\mathbb{Z}^N$. In the following, this space is normed by the 1-norm, which is denoted by $\|.\|_1$, and the closed balls are denoted by $\mathcal{B}(x, r) := \{y \in \mathbb{R}^N \mid \sum_i |x_i - y_i| \leq r\}$, for $r > 0$ and $x \in \mathbb{R}^N$.

Let E be a linear subspace of $\mathbb{R}^N$, F a complementary subspace, π and π_F be the projections on E and F, respectively, associated with the decomposition $\mathbb{R}^N = E \oplus F$. We use d to denote the dimension of E, and we make the following standard assumption.

Assumption. The restriction $\pi|_{\mathbb{Z}^N}$ is one-to-one, and $\pi_F(\mathbb{Z}^N)$ is a dense subgroup of F.

Definition 2.1. We define the following objects:

- the *acceptance domain* K is a compact subset of F, which is the closure of its interior, and which we assume to be a convex polytope (the closed convex hull of a finite set);
- V the (finite) set of vertices of K;
- the *acceptance strip*, which we denote by S, is the set $K + E$;
- $\Gamma := \pi_F(\mathbb{Z}^N)$ the projection of $\mathbb{Z}^N$ on F, which we assume to be a dense subgroup of F (by Assumption 2.1).

Let C be the closed unit cube, i.e. the following convex set:

$$C = \left\{ \sum_{i=1}^{N} \lambda_i e_i \;\middle|\; 1 \leq i \leq N,\ 0 \leq \lambda_i \leq 1 \right\}.$$

In the case $K = \pi_F(C)$, the cut-and-projection method is said to have *canonical acceptance domain*. The interested reader may refer to [**ODK**] for general results about cut-and-projection tilings with canonical acceptance domain, and their relation with other construction methods.

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In a more general setting, we assume that the acceptance domain is *almost-canonical* in the following sense. From now on, we assume that the acceptance domain is almost canonical.

Definition 2.2. The acceptance domain is said to be *almost-canonical* if the following holds: K is a convex polytope, such that $(\partial K + \Gamma)$ is a countable union of hyperplanes of F. In addition, given any $(N-d-1)$-dimensional face f of K, and any vertex v of f, and H_f the linear hyperplane parallel to f, we assume that there is a neighbourhood of 0 in H_f which can be covered by finitely many translates of f of the form $(f-v)+\gamma$, with $\gamma \in \Gamma \cap H_f$.

We remark that $f - v$ has non-empty interior in H_f. Therefore, if $H_f \cap \Gamma$ is dense in H_f for all hyperplanes H_f, then the acceptance domain K is almost-canonical. We prove later that a canonical acceptance domain is almost-canonical. An almost-canonical domain is a generalization of a canonical domain which allows us to define cut planes, and such that the singular points (cf. infra) have the same structure as in the canonical case.

Definition 2.3. The set $(\mathbb{R}^N = E \oplus F, \mathbb{Z}^N, \pi, K)$ is called the *data* of a (N, d)*-cut-and-projection method.*

Remark 2.4. Sometimes, the space E (the space which will be tiled) is referred to as the *physical space*, whereas the space F (which can be seen as a parameter space) is referred to as the *internal space*.

A cut-and-projection method can allow one to construct tilings. Namely, in the case of a canonical acceptance domain, a very elegant and canonical way to do so is described in [**ODK**]. However, we do not describe cut-and-projection *tilings* but rather cut-and-projection *Delone sets*, which amounts to the same thing when it comes to computing the complexity.

A Delone set of $\mathbb{R}^d$ is a set $\mathcal{P}$ which is uniformly discrete and relatively dense. Uniform discreteness means that there exists $r > 0$ such that for all $x \in \mathcal{P}$, $\mathcal{B}(x, r) \cap \mathcal{P} = \{x\}$. Relative density means that there exists $R > 0$ such that for all $x \in \mathbb{R}^d$, the set $\mathcal{B}(x, R) \cap \mathcal{P}$ is not empty. Such a set is called a (r, R)-Delone set.

We construct Delone sets by projection on E of a subset of the lattice $\mathbb{Z}^N$ determined by the acceptance domain K. The general idea is to project the 'slice' $S \cap \mathbb{Z}^N$ on E, as in Figure 1, but we have to be a little bit careful in the case when $\mathbb{Z}^N$ intersects the border of S.

Definition 2.5. We say that $x \in \mathbb{R}^N$ is a *singular* point ($x \in \mathrm{Sing}$), if $(x + \mathbb{Z}^N) \cap \partial S \neq \varnothing$, where ∂S stands for the boundary of the acceptance strip S in E. If $x \notin \mathrm{Sing}$, we say that x is *regular* and we use the notation $x \in \mathrm{Reg}$.

Given $n \in \mathbb{N} \cup \{0\}$, we say that $x \in S$ is n-singular if $(x + \mathbb{Z}^N) \cap \mathcal{B}(x, n) \cap \partial S \neq \varnothing$. We then use the notation $x \in \mathrm{Sing}_n$. We say that $x \in S$ is n-regular if it is not n-singular, and we use the notation $x \in \mathrm{Reg}_n$.

When a point x is singular, there is some $y \in (\mathbb{Z}^N + x) \cap \partial S$, so that it is unclear whether y should be considered as inside or outside S. This could cause an ambiguity in the definition of the cut-and-projection Delone set associated with x. This is why in

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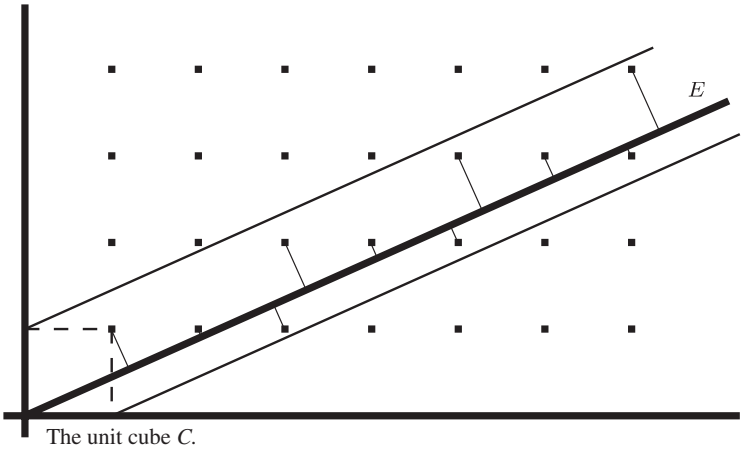


FIGURE 1. A one-dimensional cut-and-projection tiling: elements of the 'slice' $C + E$ are projected on E.

the definition below, we only consider the Delone set associated to regular points. The n-regular points are an adaptation of the regular points: they are points for which it is possible to construct without ambiguity a patch of size n in the following sense.

Definition 2.6. For all $x \in \mathrm{Reg}$, we define the following objects:

- the lifted set associated with x is $\widetilde{\mathcal{P}}(x) := (x + \mathbb{Z}^N) \cap S$;
- the Delone set associated with x is $\mathcal{P}(x) := \pi(\widetilde{\mathcal{P}}(x)) \subseteq E$.

For all $n \in \mathbb{N} \cup \{0\}$, for all $x \in \mathrm{Reg}_n$, we define:

- the lifted patch of size n associated with x as $\widetilde{A}(x, n) = (x + \mathbb{Z}^N) \cap \mathcal{B}(x, n) \cap S$ (we say that x is a *seed* of this lifted patch);
- the patch of size n of seed x as $A(x, n) := \pi(\widetilde{A}(x, n))$.

Finally, we define the pointed lifted patch $\widetilde{A}_{\mathrm{pt}}(x, n)$ as $\widetilde{A}(x, n)$ with distinguished point x. Similarly, the pointed patch $A_{\mathrm{pt}}(x, n)$ is $A(x, n)$ with distinguished point $\pi(x)$.

The $\mathcal{P}(x)$ are Delone sets of E which are known to be aperiodic and repetitive. More precisely, under our assumptions, the group of periods of $\mathcal{P}(x)$ is never a regular lattice of E. If $E \cap \mathbb{Z}^N = \{0\}$, the group of periods is trivial, and the resulting Delone sets are said to be completely aperiodic.

The $A(x, n)$ are bounded sets. One may be tempted to call $A(x, n)$ the 'patch of center x and radius n', but this could be misleading: $x \in \mathbb{R}^N$ whereas $A(x, n) \subset E$. We call x a *seed* of $A(x, n)$. Actually, the 'physical' center of $A(x, n)$ in E is $\pi(x)$: in the decomposition $x = \pi(x) + \pi_F(x)$, the term $\pi(x)$ encodes the position of $A(x, n)$ in E, and $\pi_F(x)$ encodes the shape of the patch. The question 'how does $\pi_F(x)$ encode the shape of $A(x, n)$?' is what this paper is about.

Remark that we chose $x \in S$ for the definition of n-singular points, and hence for the definition of patches of size n. This is because if n is given, and one takes x far away from E, one can have that $A(x, n)$ is empty. As we want to consider patches of size n and not patches of size at most n, we take $x \in S$ in the definition of the patches.

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If $x \in \mathrm{Reg}$, the patch $A(x, n)$ is a subpatch of $\mathcal{P}(x)$, as it can be extended: it is actually a subset. If $x \in (\mathrm{Reg}_n \setminus \mathrm{Reg})$, $A(x, n)$ is a subpatch of $\mathcal{P}(y)$ for all y (by repetitivity), that is *up to translation* by an element of E, $A(x, n)$ appears in $\mathcal{P}(y)$.

Remark 2.7. There is *a priori* a difference between patches and pointed patches. Beware that a patch of radius n could have several seeds: there could exist x, y such that $\pi(x) \neq \pi(y)$, and $A(x, n) = A(y, n)$. That is, a patch could correspond to several pointed patches. However, the number of candidates to be the seed of any patch of the form $A(x, n)$ is bounded, and this bound only depends on the data of the cut-and-projection method. This fact will be proved in Lemma 2.12.

We can now define the complexity function, the function which counts the number of different patches of a given size.

Definition 2.8. Let $\mathcal{P}(x)$ be a Delone set. The complexity function is the function $p : \mathbb{N} \to \mathbb{N}$, such that $p(n)$ is the number of subpatches of size n of $\mathcal{P}(x)$, up to translation by an element of E. We can write it as follows:

$$p(n) := \mathrm{Card}(\{A(y, n) \mid y \in \widetilde{\mathcal{P}}(x)\}/E).$$

The repetitivity ensures the following property.

PROPOSITION 2.9. *The complexity p of a given Delone set $\mathcal{P}(x)$ is equal to the complexity of any Delone set $\mathcal{P}(x')$ obtained with the same cut-and-projection data. We deduce that*

$$p(n) = \mathrm{Card}(\{A(y, n) \mid y \in \mathrm{Reg}_n\}/E).$$

Furthermore, as we count the patches only up to translation by an element of E, we have

$$p(n) = \mathrm{Card}(\{A(y, n) \mid y \in K \cap \mathrm{Reg}_n\}/E).$$

Similarly, we define the pointed complexity function, which counts the number of pointed patches up to a translation of E.

Definition 2.10. The pointed complexity function associated with the cut-and-projection method is

$$p_{\mathrm{pt}}(n) = \mathrm{Card}(\{A_{\mathrm{pt}}(y, n) \mid y \in K \cap \mathrm{Reg}_n\}).$$

The complexity function and pointed complexity function are linked by the following inequalities.

PROPOSITION 2.11. *There exists $\lambda > 0$, which only depends on the data of the cut-and-projection method, such that for all $n \in \mathbb{N}$,*

$$\lambda p_{\mathrm{pt}}(n) \leq p(n) \leq p_{\mathrm{pt}}(n).$$

We first state the following lemma, which proves the assertion of Remark 2.7.

LEMMA 2.12. *Given a cut-and-projection method, there exists $M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, for all $x \in \mathrm{Reg}_n$, the patch $A(x, n)$ has at most M seeds; that is,*

$$\mathrm{Card}\,(\{y \in \widetilde{A}(x, n) \mid A(x, n) = A(y, n)\}) \leq M.$$

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Proof. The size of a patch is given by the norm in $\mathbb{R}^N$. We want to express the size of a patch of size n in terms of the norm induced in E by the norm of $\mathbb{R}^N$. Remark that if $n \in \mathbb{N}$ and $x \in K$, then for all $y \in \widetilde{\mathcal{P}}(x)$,

$$\|\pi(y) - \pi(x)\|_1 \leq \|\pi(y) - y\|_1 + \|y - x\|_1 + \|x - \pi(x)\|_1,$$

and

$$\|y - x\|_1 \leq \|\pi(y) - y\|_1 + \|\pi(y) - \pi(x)\|_1 + \|x - \pi(x)\|_1.$$

However, $\pi(y) - y \in K$ and K is compact, so there exists an upper bound to $\|\pi(y) - y\|$ which does not depend on $y \in S$. Let us call this bound $\mu/2$. Then

$$\|y - x\|_1 - \mu \leq \|\pi(y) - \pi(x)\|_1 \leq \|y - x\|_1 + \mu.$$

So, applying this inequality with fixed x and all $y \in \widetilde{A}(x, n)$, we have

$$(\mathcal{B}(\pi(x), n - \mu) \cap \mathcal{P}(x)) \subseteq A(x, n) \subseteq (\mathcal{B}(\pi(x), n + \mu) \cap \mathcal{P}(x)),$$

where $\mathcal{B}$ denotes the ball in E.

A consequence is that, given a patch $A(x, n)$, the number of different $y \in \widetilde{A}(x, n)$ such that $A(x, n) = A(y, n)$ is bounded, and its bound does not depend on x. Indeed, if $\|x - y\| > 2\mu$, then $\mathcal{B}(\pi(x), n - \mu) \cap \mathcal{P}(x)$ is not included in $\mathcal{B}(\pi(y), n + \mu) \cap \mathcal{P}(y)$, so that we cannot have $A(x, n) = A(y, n)$. So a bound of the number of candidates for being the seed of a given patch is, for example, $\max_u (\mathcal{P}(u) \cap \mathcal{B}(\pi(u), 2\mu)) =: M$, which is finite by uniform discreteness of the Delone sets associated with the cut-and-projection data. This shows that a patch of size n can have at most M seeds. □

Proof of Proposition 2.11. The proposition is an immediate consequence of the lemma above: as a given patch $A(x, k)$ has at most M seeds, it means that such a patch corresponds to at most M pointed patches $A(x, k)$ with distinguished point $y \in A(x, k)$ (y being a seed for the patch). Therefore, we have the result with $\lambda = 1/M$. The other inequality is trivial. □

2.2. *The structure of singular points.* In the following sections of this paper, the singular points will play a crucial role to compute the complexity function. Therefore, it is necessary to understand their geometrical structure.

PROPOSITION 2.13. *We have the following properties:*

$$\text{Sing} = \partial S + \mathbb{Z}^N, \tag{1}$$

and

$$\text{Sing}_k = (\mathcal{B}(0, k) \cap \mathbb{Z}^N + \partial S) \cap S, \tag{2}$$

so that Sing_k *is an increasing sequence of subsets of* S*, the union of which is* $\text{Sing} \cap S$*. Moreover,* Sing_k *and* Sing *are invariant under the action of* E *by translation.*

Proof. It is straightforward, using the definition. □

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As the singular points are invariant by the action of E, it is enough to study their restriction to F, which is a complementary subspace of E in $\mathbb{R}^N$. In the following sections, we mostly work on F. Applying π_F to the equations above, we can see that the first condition which defines an almost-canonical acceptance domain in Definition 2.2 is equivalent to the fact that the singular points are a countable union of hyperplanes.

We state two results on singular points. The first is a qualitative result, the second is a quantitative (and a bit technical) result. The proofs will be quite straightforward in the case of almost-canonical acceptance domains, as the definition of almost-canonical was specifically designed so that these results are true. In order to prove that they hold for canonical acceptance domains as well, we prove the following proposition.

PROPOSITION 2.14. *A canonical acceptance domain is almost-canonical.*

Proof. Consider a canonical acceptance domain K. Let f be a $(N-d-1)$-dimensional face of K, and H_f the hyperplane parallel to it. Up to renaming the vectors, the face f is of the following form:

$$f = v + \left\{ \sum_{k=1}^{N-d-1} \lambda_k \pi_F(e_k) \;\middle|\; 0 \leq \lambda_k \leq 1 \right\},$$

where $v \in V$. Note that Λ is the group generated by $(\pi_F(e_i))_{i=1}^{N-d-1}$. This group is a regular lattice of H_f and a subgroup of $H_f \cap \Gamma$. Furthermore, $f - v$ it is a fundamental domain for the action of Λ. Therefore, $f + \Gamma$ is a union of hyperplanes. The same argument applies to all of the faces, and $\partial K + \Gamma$ is a union of hyperplanes.

To prove the second condition, remark once again that $f - x$ is compact for any $x \in f$, and is a fundamental domain for the action of the locally finite group Λ. Therefore, there exist only a finite number of translates of the form $f - x + \gamma$ with $\gamma \in \Lambda$ that intersect a given neighbourhood of the origin. As f is bounded, this finite number can be chosen such as not to depend on x. On the other hand, $f - x + \Lambda$ covers H_f. So a given neighbourhood of the origin, it is covered by a finite number of translates of $f - x$ of the form above. It proves that a canonical acceptance domain is almost-canonical. □

Let us now make the structure of singular points precise.

Definition 2.15. We use $H_1, \ldots, H_m$ to denote all of the distinct hyperplanes of F which are parallel to faces of K. We call these hyperplanes the *singular hyperplanes*, or *cut hyperplanes* of F. Remember that $\Gamma := \pi_F(\mathbb{Z}^N)$. We define $\Gamma^i := \mathrm{Stab}_\Gamma(H_i)$.

Remark that as a face f parallel to H_i is compact, and $f + \Gamma^i$ is an hyperplane, then Γ^i always contains a regular lattice of H_i. It implies that it is at least of rank $N - d - 1$. So the hyperplanes H_i are always directed by $N - d - 1$ vectors of Γ.

The following proposition follows quite easily from Definition 2.2. It is best understood with a picture: see Figure 2.

PROPOSITION 2.16. *The singular points of F are the union of translates of singular hyperplanes:*

$$\mathrm{Sing} \cap F = \bigcup_{i=1}^{m} (H_i + V_i) + \Gamma,$$

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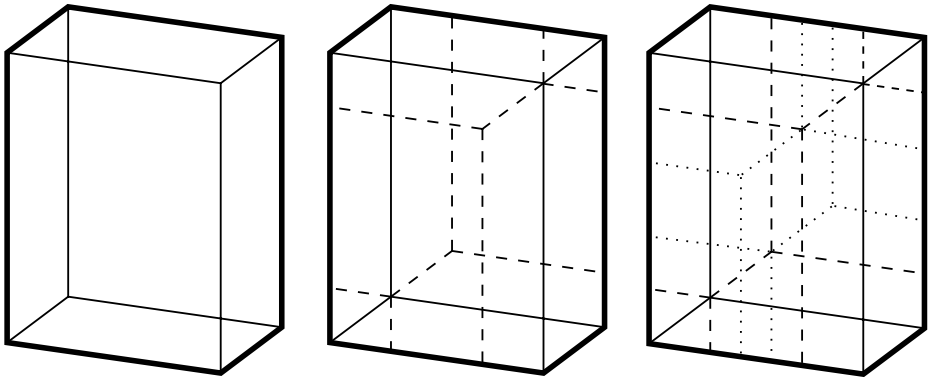


FIGURE 2. The 1-, 2- and 3-singular points of a (3, 1)-cut-and-projection tiling, seen on F (for a canonical acceptance domain).

where V_i is the subset of V consisting of the vertices of K which belong to a face parallel to H_i. In particular, we always have

$$\mathrm{Sing} \cap F \subseteq \left(\bigcup_{i=1}^{m} H_i \right) + V + \Gamma,$$

and if $V \subseteq \Gamma$ (for example, in the case of a canonical acceptance domain), we have

$$\mathrm{Sing} \cap F = \left(\bigcup_{i=1}^{m} H_i \right) + \Gamma.$$

The following technical lemma is a qualitative version of the previous proposition. It uses the second property of almost-canonical acceptance domains (cf. Definition 2.2). Once again, having in mind Figure 2 helps.

LEMMA 2.17. *We use the notation above. Then for all $1 \leq i \leq m$, there exists $\varepsilon_i > 0$ and $M_i \in \mathbb{N}$ such that for all $y \in (\pi_F(\mathcal{B}(0, n) \cap \mathbb{Z}^N) + V_i)$, the points of $(H_i + y) \cap \mathcal{B}(y, \varepsilon_i)$ are $(n + M_i)$-singular.*

Proof. First, remark that if we apply π_F to equation (2), we have

$$\mathrm{Sing}_n \cap F = \partial K + \pi_F(\mathbb{Z}^N \cap \mathcal{B}(0, n)).$$

Let $1 \leq i \leq m$. Consider $\varepsilon_i > 0$ such that, according to Definition 2.2, the ball of radius ε_i around 0 in H_i is covered by a finite number of translates of any face parallel to H_i (there are only finitely many such faces). Let $n \in \mathbb{N}$ and $y \in \pi_F(\mathcal{B}(0, n) \cap \mathbb{Z}^N) + V_i$. We write $y = v + \pi_F(z)$ with $z \in \mathbb{Z}^N$ such that $\|z\|_1 \leq n$, and $v \in V_i$. Call f the face parallel to H_i to which v belongs. However, there are a finite number of translates of $f - v$ by elements of Γ^i which cover $\mathcal{B}(0, \varepsilon_i)$. Let us call $\gamma_1, \ldots, \gamma_k$ these elements, and let M_i be

$$M_i = \max_{1 \leq j \leq k} \min\{\|w\|_1 \mid w \in \mathbb{Z}^N, \pi_F(w) = \gamma_j\}.$$

Note that M_i can be chosen such as not to depend on v (as f has only finitely many vertices). So if $x \in \mathcal{B}(y, \varepsilon_i) \cap H_i$, then $x - y \in \mathcal{B}(0, \varepsilon_i) \cap H_i$, and $x - y$ is in some $f - v + \pi_F(w)$ with $w \in (\mathbb{Z}^N \cap \mathcal{B}(0, M_i))$. This means that $x \in f + \pi_F(z + w)$ with $z + w \in \mathbb{Z}^N \cap \mathcal{B}(0, n + M_i)$; that is, x is $(n + M_i)$-singular. □

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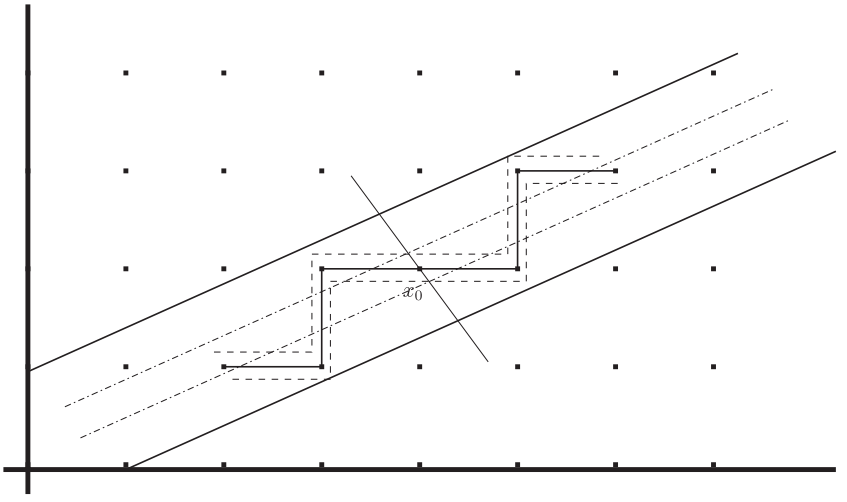


FIGURE 3. When we move x_0 (the seed of the lifted patch), we do not change its shape, until it becomes 3-singular (the patch has size 3). As long as x_0 stays in the band (the connected component of Reg_3, here pictured with dash-dotted lines), the patch $\widetilde{A}(x_0, 3)$ will not change up to translation.

We know (Proposition 2.16) that if a point $y \in \Gamma + V_i$ is n-singular, then $H_i + y$ is singular. What this lemma states is that, restricted to H_i, there exists a neighbourhood of y for the relative topology of H_i which is $(n + M)$-singular, and M does not depend on n. It states that locally, around a n-singular point, the $(n + M)$-singular points look like hyperplanes.

Therefore, we have two inclusions: the set of n-singular points is included in a union of hyperplanes, and it contains (at least locally) a union of hyperplanes.

3. *Complexity for cut-and-projection tilings*

3.1. *Statement of the theorem.* We state in this section the key proposition and the main theorem of this paper. The first proposition states that, up to a constant, counting the number of different patches of given size n is the same as counting the number of connected components of $\mathrm{Reg}_n \cap K$. It is the precise statement corresponding to the fact that when considering the patch $A(x, n)$, $\pi(x)$ encodes its position in E, while $\pi_F(x)$ encodes its shape. This is illustrated by Figure 3.

PROPOSITION 3.1. *We use p_{pt} to denote the pointed complexity function, as defined above. For all $n \in \mathbb{N} \cup \{0\}$, $c(n)$ will denote the number of connected components of* Reg_n. *Then, for all $n \in \mathbb{N}$, we have the following:*

$$p_{\mathrm{pt}}(n) = c(n).$$

The following corollary is an immediate result of the proposition above and Proposition 2.11.

COROLLARY 3.2. *We use p to denote the complexity function and c to denote the function which counts the number of connected components of* Reg_n. *Then there exists $0 < \lambda < 1$*

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such that for all $n \in \mathbb{N}$, we have

$$\lambda c(n) \leq p(n) \leq c(n).$$

We now state the main theorem of this paper, which provides a link between the complexity of cut-and-projection tilings associated with given data, and the rank of the stabilizer in Γ of the singular hyperplanes.

THEOREM 3.3. *Let $(\mathbb{R}^N = E \oplus F, \mathbb{Z}^N, \pi, K)$ be the data of a cut-and-projection method with K almost-canonical. Let p be the associated complexity function. Let $(H_i)_{i=1}^m$ be the singular hyperplanes in F, and $\Gamma := \pi_F(\mathbb{Z}^N)$. We define the following objects:*

- *for all $1 \leq i \leq m$, let $\Gamma^i := \mathrm{Stab}_\Gamma(H_i)$;*
- *let $\alpha_i + 1$ be the rank of Γ / Γ^i;*
- *let $(\alpha_i)_{i \in I}$ be $N - d$ elements amongst the $(\alpha_i)_{i=1}^m$ satisfying $\bigcap_{i \in I} H_i = \{0\}$, and such that $\alpha := \sum_{i \in I} \alpha_i$ is maximal.*

Then there exist $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$,

$$C_1 n^\alpha \leq p(n) \leq C_2 n^\alpha.$$

Remark 3.4. Generically, the rank of Γ is N. What do we mean by generically? For any choice of E which does not intersect $\mathbb{Z}^N$, the rank of Γ is N. Given $x \in \mathbb{R}^N$, the set of d-dimensional spaces which contain x is a closed set with empty interior in the Grassmannian manifold $G(N, d)$. Therefore, the set of E of dimension d which do not intersect $\mathbb{Z}^n$ is a countable intersection of dense open subsets of $G(N, d)$, and hence is generic.

Similarly, it is a generic situation that the rank of the stabilizers of H_i is minimal. In this case, this rank is equal to $N - d - 1$. So for all i, $\alpha_i = d$, and the complexity $p(n)$ grows as $n^{d(N-d)}$.

This is also the maximal complexity of a (N, d)-cut-and-projection tiling with our assumptions on K (in particular, with canonical acceptance domain). It is easily seen that $\alpha \geq N - d$. There is another less trivial lower bound for α: in Theorem 5.1, we prove that $\alpha = d$ is the minimal value of α for a cut-and-projection Delone set with no period.

Proposition 3.1 shows that the functions $p(n)$ (complexity) and $c(n)$ (the number of connected components of $K \cap \mathrm{Reg}$) have the same order of growth. The theorem is then proved by counting the number of connected components of $K \cap \mathrm{Reg}$, using a few geometric lemmas, which we state and prove in the following sections.

Proof of Proposition 3.1. Let us first show that for all $n \in \mathbb{N}$, $p_{\mathrm{pt}}(n) \leq c(n)$. Remember that $p_{\mathrm{pt}}(n)$ is the number of pointed patches $A_{\mathrm{pt}}(x, n)$ which differ up to a translation by an element of E, with $x \in K \cap \mathrm{Reg}_n$. What we have to show is that, if x and y are in the same connected component of $\mathrm{Reg}_n \cap K$, then $(A(x, n), \pi(x)) = (A(y, n), \pi(y))$. Let $x \in \mathrm{Reg}_n \cap K$. We show that the set X of all of the $y \in \mathrm{Reg}_n \cap K$ such that $A(x, n) = A(y, n)$ and $\pi(x) = \pi(y)$ is both open and closed for the relative topology on $\mathrm{Reg}_n \cap K$. First, remark that it is superfluous to check $\pi(x) = \pi(y)$, as by hypothesis, $x, y \in K$, and so $\pi(x) = \pi(y) = 0$. Let $y \in X$, and d be the distance between $\widetilde{A}(y, n)$ and ∂S, so that $\mathcal{B}(0, d/2) + \widetilde{A}(y, n)$ is included in the interior of S. Then $\mathcal{B}(y, d/2) \cap K \subseteq X$, and X

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is open. Conversely, consider $(y_k)_{k\geq 0}$, a sequence in X which converges to $y \in \mathrm{Reg}_n \cap K$. Let d be the distance between $\widetilde{A}(y, n)$ and ∂S. Then for all $y' \in \mathcal{B}(y, d/2)$, $A(y, n) = A(y', n)$. However, for k large enough, $y_k \in \mathcal{B}(y, d/2)$, so that $A(y, n) = A(y_k, n) = A(x, n)$, and $y \in X$. The set X being open and closed, it contains the connected component of Reg_n containing x, and $p_{\mathrm{pt}}(n) \leq c(n)$.

Let us now show the other inequality. Let $A_{\mathrm{pt}}(x, n)$ be a pointed patch of size n; we show that there exists exactly one connected component $\mathcal{C}$ such that if $y \in C$, $A_{\mathrm{pt}}(y, n) = A_{\mathrm{pt}}(x, n)$ up to translation. If $A_{\mathrm{pt}}(x, n) = A_{\mathrm{pt}}(y, n)$, by injectivity of $\pi|_{\mathbb{Z}^N}$, we have that $\widetilde{A}(x, n)$ and $\widetilde{A}(y, n)$ differ by an element of F. Moreover, as we consider pointed patches, this element is exactly $y - x =: v$. Then we have $\widetilde{A}(x, n) + v = \widetilde{A}(y, n)$. Both of these sets being included in S, by convexity, we have that $\widetilde{A}(x, n) + tv = \widetilde{A}(x + tv, n)$ is included in S for all $t \in [0, 1]$, so that all of the $x + tv$ are n-regular, and x is in the same connected component of Reg_n as $x + v = y$. Thus, x and y are in the same connected component of Reg_n. □

Remark 3.5. The second part of the proof above proves that the connected components of Reg_n are convex.

We leave the proof of Theorem 3.3 for the next sections, but let us give a motivation for the technical lemmas we are going to state (and prove). Our goal is to give an asymptotic estimation of $c(n)$, the number of connected components of $\mathrm{Reg}_n \cap K$. Remember that the set of regular points is a convex polytope K minus hyperplanes:

$$\mathrm{Reg} \cap K = K \setminus \left(\Gamma + \bigcup_{i=1}^{m} (H_i + V_i) \right),$$

where the V_i are finite sets. Similarly, one can prove that

$$\mathrm{Sing}_n \subseteq ((\pi_F(\mathbb{Z}^N \cap \mathcal{B}(0, n)) + V) \cap K) + \bigcup_{i=1}^{m} H_i.$$

The idea is to compare $c(n)$ with the number of connected components of K minus a finite number of hyperplanes in each direction. To do so, we have to count the number of hyperplanes actually cutting K in the formula above. For this, according to the formula above, we might need to count the number of points of $\pi_F(\mathbb{Z}^N \cap \mathcal{B}(0, n)) + V$ which fall inside of K. Actually, the set we count is not exactly this one, as we have to take into account the fact that some hyperplanes are stabilized by elements of Γ. However, the idea remains the same. Then, we need to estimate the number of connected components of a compact convex subset of $\mathbb{R}^{N-d}$ when it is cut by hyperplanes. We state two lemmas which answer these questions. Combining these two lemmas, we then prove the theorem.

3.2. *Preliminary lemmas.* We begin with an 'equidistribution-like' lemma: this lemma aims at giving an estimation of the number of points of $\pi_F(\mathcal{B}(0, n) \cap \mathbb{Z}^N) \cap K$, when n tends to infinity.

LEMMA 3.6. *Let F be a p-dimensional vector space, $\gamma_1, \ldots, \gamma_q$ be q vectors of F, which span a dense subgroup of rank q ($q > p$).*

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Then for all bounded sets U with non-empty interior, there exist $C_1, C_2 > 0$, such that for all n, we have

$$C_1 n^{q-p} \leq \mathrm{Card}\left\{ m = (m_i)_{i=1}^q \in \{1, \ldots, n\}^q \;\middle|\; \sum_{i=1}^q m_i \gamma_i \in U \right\} \leq C_2 n^{q-p}.$$

We can apply this lemma with $p = N - d$, $q = \mathrm{rk}\,\Gamma$, and where the γ_i are a system of generators of Γ chosen amongst the $(\pi_F(e_i))_{1 \leq i \leq N}$. In the special case where we make the additional assumption that $\mathbb{Z}^N \cap E = \{0\}$, we have $q = N$. In this case, this lemma states that the cardinality of $\pi_F(\widetilde{A}(0, n))$ grows as n^d. This gives the cardinality of a patch of size n. An interpretation is that the number of integer points contained in the convex set $(K + E) \cap B(0, n)$ grows as the volume of this set, which is roughly speaking the volume of $K + (E \cap B(0, n))$, that is, grows as n^d, up to constants.

Proof. Let us first remark that by precompacity of bounded sets, it is enough to check the property for a set U of the form $U = \mathcal{B}(0, \varepsilon')$ with small ε'.

Up to reordering the vectors, we assume that $\gamma_1, \ldots, \gamma_p$ define a parallelotope of maximal volume amongst all of the parallelotopes which can be formed from the $\{\gamma_i\}_{i=1}^q$. Thus, $(\gamma_i)_{i=1}^p$ form a basis of F. Let $\|.\|_2$ and $\|.\|_\infty$ be respectively the Euclidean and sup norms associated with this basis. One can check that for all $i > p$, we have $\|\gamma_i\|_\infty \leq 1$. Let Λ be the lattice of F generated by $\gamma_1, \ldots, \gamma_p$.

We prove a result of equidistribution relative to the subgroup of the torus $T := F/\Lambda$ generated by the vectors $(\gamma_i)_{i>p}$. Let $i > p$. By hypothesis, the subgroup of E generated by all of the $(\gamma_j)_{j=1}^q$ is of rank q, so that the group $\mathbb{Z}\gamma_i$ is infinite as a subgroup of T. Let us call T_i its adherence. This is a sub-torus of T, in which $\mathbb{Z}\gamma_i$ is dense. It is then classical (see, for example, [**Kör**]), that the sequences $(n\gamma_i)_{n \geq 0}$ and $(-n\gamma_i)_{n>0}$ are equidistributed in T_i. Thus, for all $\varepsilon > 0$, if $V(\varepsilon)$ is a small neighbourhood of 0 in T (which is the image on the quotient of a small ball $\mathcal{B}(0, \varepsilon)$ in F, with $\mathcal{B}(0, \varepsilon)$ being included in a fundamental domain), then $V(\varepsilon) \cap T_i$ has positive measure (for the invariant measure on T_i), and thus

$$\frac{1}{2n+1} \mathrm{Card}\{k \in \{-n, \ldots, n\} \mid k\gamma_i \in V(\varepsilon) \subseteq T\} \xrightarrow[n \to +\infty]{} l_i > 0.$$

Lifting this result, we deduce that

$$\frac{1}{2n+1} \mathrm{Card}\left\{ k = (k_j)_{j=1}^{p+1} \in \mathbb{Z}^p \times \{-n, \ldots, n\} \;\middle|\; \sum_{j=1}^p k_j \gamma_j + k_{p+1}\gamma_i \in \mathcal{B}(0, \varepsilon) \subseteq E \right\}$$
$$\xrightarrow[n \to +\infty]{} l_i > 0.$$

To simplify the notation, let us define for all $i > p$:

$$A_n^{(i)}(\varepsilon) := \left\{ k = (k_j)_{j=1}^{p+1} \in \{-n, \ldots, n\}^{p+1} \;\middle|\; \sum_{j=1}^p k_j \gamma_j + k_{p+1}\gamma_i \in \mathcal{B}(0, \varepsilon) \subseteq E \right\}.$$

Remark that, the condition on the norms (that is, $\|\gamma_i\|_\infty \leq 1$) implies that $\|n\gamma_i\|_\infty \leq |n|$ for all $n \in \mathbb{Z}$. Thus, for all $n \in \mathbb{Z}$, there exists a unique vector v such that $n\gamma_i + v$ belongs to a fundamental domain containing $\mathcal{B}(0, \varepsilon)$ of the lattice Λ, and we have $\|v\|_\infty \leq n$. Thus, we deduce the following:

$$\frac{1}{2n+1} \mathrm{Card}\, A_n^{(i)}(\varepsilon) \xrightarrow[n \to +\infty]{} l_i > 0.$$

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Remark that the same reasoning holds for all γ_i, with $i > p$.

Let us now prove the lemma itself: let $\varepsilon' > 0$, we want to find both an upper bound and a positive lower bound for the sequence $(2n+1)^{-(q-p)}$ Card $A_n(\varepsilon')$, where

$$A_n(\varepsilon') := \left\{ k = (k_j) \in \{-n, \ldots, n\}^q \;\middle|\; \sum_{j=1}^{q} k_j \gamma_j \in \mathcal{B}(0, \varepsilon') \subseteq E \right\}.$$

Let us first find bounds to the cardinality of $(2n+1)^{-(q-p)} A'_n(\varepsilon')$, where

$$A'_n(\varepsilon') := \left\{ k = (k_j)_{j=1}^q \in \{-(q-p)n, \ldots, (q-p)n\}^p \right.$$
$$\left. \times \{-n, \ldots, n\}^{q-p} \;\middle|\; \sum_{j=1}^{q} k_j \gamma_j \in \mathcal{B}(0, \varepsilon') \subseteq E \right\}.$$

Consider $k^{(i)} \in A_n^{(i)}(\varepsilon)$, for all $i \in \{p+1, \ldots, q\}$, with $\varepsilon := \varepsilon'/(q-p)$. Then to these $(k^{(i)})_{i=p+1}^q$, one can associate a $k \in A'_n(\varepsilon')$ defined by $k_j := \sum_{i=p+1}^q k_j^{(i)}$ if $j \leq p$, and $k_j := k_{p+1}^{(j)}$ if $j > p$. In addition, this application

$$\prod_{i=p+1}^{q} A_n^{(i)}(\varepsilon) \longrightarrow A'_n(\varepsilon')$$

is one-to-one. Indeed, if $(k^{(i)})_{i=p+1}^q \neq (k'^{(i)})_{i=p+1}^q$ (with respective images k and k' in $A'_n(\varepsilon')$), then there exists $p+1 \leq j \leq q$ such that $k^{(j)} \neq k'^{(j)}$. It necessarily means that $k_{p+1}^{(j)} \neq k_{p+1}^{\prime(j)}$, because if $k_{p+1}^{(j)} = k_{p+1}^{\prime(j)}$, then as ε' (and thus ε) is chosen small enough, there exists a *unique* vector $v \in \{-n, \ldots, n\}^p$ such that $k_{p+1}^{(j)} \gamma_j + v \in \mathcal{B}(0, \varepsilon)$. Thus, $k^{(j)} = (v, k_{p+1}^{(j)}) = k'^{(j)}$, which is a contradiction. Thus, $k_{p+1}^{(j)} \neq k_{p+1}^{\prime(j)}$, and so $k \neq k'$, and the application defined above is one-to-one.

As a consequence, the cardinality of $A'_n(\varepsilon')$ is larger than the product of the cardinalities of the $A_n^{(i)}(\varepsilon)$, and so

$$\liminf_{n\to\infty} \frac{1}{(2n+1)^{q-p}} \text{ Card } A'_n(\varepsilon') > 0.$$

Let us now find an upper bound to the cardinality of the $A'_n(\varepsilon')$: if ε' is chosen small enough, i.e. $\mathcal{B}(0, \varepsilon')$ is included in a fundamental domain of the lattice Λ, then

$$\text{Card}(A'_n(\varepsilon')) \leq (2n+1)^{q-p}$$

because an element of $A'_n(\varepsilon')$ is entirely determined by its $q-p$ last coordinates.

Now, the cardinality of A_n can be easily compared with the cardinality of A'_n, as we have

$$\text{Card } A'_n(\varepsilon') \leq \text{Card } A_{(q-p)n}(\varepsilon') \leq \text{Card } A'_{(q-p)n}(\varepsilon').$$

As the upper bound we want to find is polynomial in n, and Card$(A_n(\varepsilon'))$ is increasing, we have the result. □

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The next lemma gives an upper bound for the number of connected components of a polytope cut by affine hyperplanes, assuming we have an estimation of the number of hyperplanes involved. As we said, Reg_k is 'almost' K minus translates of the singular hyperplanes. Therefore, this lemma will be a key lemma in order to have an upper bound of $c(n)$.

LEMMA 3.7. *Let F be a p-dimensional real vector space. Then for all $m \in \mathbb{N}$, for all $H_1, \ldots, H_m$ distinct linear hyperplanes of F, for all $(\alpha_i)_{i=1}^m$ integers, for all finite families of integer-valued increasing sequences $(\beta^{(i)})_{i=1}^m$ such that for all i, $\beta_n^{(i)} = O(n^{\alpha_i})$, we have the following.*

If K is an open convex and bounded subset of F and $(K_n)_{n\geq 0}$ is a decreasing sequence of sets, such that $K_0 = K$ and K_n can be obtained from K by removing $\beta_n^{(i)}$ distinct translates of the H_i (for all i), then the number $c(n)$ of connected components of K_n satisfies

$$c(n) = O(n^{\alpha}),$$

where $\alpha := \sum_{i\in I} \alpha_i$, with I defined as follows:

- *$\bigcap_{i\in I} H_i$ is of minimal dimension;*
- *amongst such sets, I is of minimal cardinality (the cardinality of I is then $p - \dim \bigcap_{i=1}^m H_i$);*
- *amongst such sets, I is such that α is maximal.*

Proof. We prove this lemma by recurrence on p. If $p = 1$, then $m = 1$ as the only hyperplane of a one-dimensional space is $\{0\}$. Then K_n is a segment with β_1 points removed, i.e. a union of $\beta_1 + 1$ intervals. So its number of connected components is $\beta_1 + 1 = O(n^{\alpha_1})$, and the lemma is proved for $p = 1$.

Let us assume that the lemma is true for all $(p-1)$-dimensional vector spaces. Let F be a p-dimensional vector space. Let K, m, $(H_i)_{i=1}^m$ $(\beta^{(i)})_{i=1}^m$ and $(K_n)_{n\geq 0}$ be as in the statement of the lemma. We use $\mathcal{C}$ to denote the function which associates its number of connected components with a subset of F ($\mathcal{C}$ takes values in $\mathbb{N} \cup \{\infty\}$), so that $c(n) = \mathcal{C}(K_n)$. To fix the notation, for all i, we use $(X_n^{(i)})_{n\geq 0}$ to denote an increasing sequence of finite subsets of F, so that $\mathrm{Card}\, X_n^{(i)} = \beta_n^{(i)}$ for all i, n, and

$$K_n = K \setminus \left(\bigcup_{i=1}^{m} (X_n^{(i)} + H_i) \right).$$

We use the notation

$$c(x, k, n) := \mathcal{C}\left((H_k + x) \cap \left(K \setminus \bigcup_{i=1}^{k-1} (X_n^{(i)} + H_i) \right) \right),$$

and we remark that the connected components of K_n are convex (being the intersection of half-spaces), so that $c(x, k, n)$ is exactly the number of components intersected by the hyperplane $H_k + x$. Therefore, for all $x \in X_n^{(k)}$, we have

$$\mathcal{C}\left(K \setminus \left((x + H_k) \cup \bigcup_{i=1}^{k-1} (X_n^{(i)} + H_i) \right) \right) = \mathcal{C}\left(K \setminus \bigcup_{i=1}^{k-1} (X_n^{(i)} + H_i) \right) + c(x, k, n). \quad (3)$$

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An interpretation of the equation above is the following: if we have an open set K', made of a certain number $\mathcal{C}(K')$ of convex connected components, we want to count the number of connected components of K' minus an affine hyperplane H. The number of additional components obtained by cutting by this hyperplane is equal to the number of components of K' which are intersected by H, that is the number of components of $K' \cap H$ (by convexity of the components of K'). Equation (3) expresses this fact with a particular K', which allows us to iterate the process: if we consider that K_n is nothing more than K with a certain number of hyperplanes removed *one after another*, we can deduce $c(n)$ by successive sums and we have

$$c(n) = \sum_{k=0}^{m} \left(\sum_{x \in X_n^{(i)}} c(x, k, n) \right).$$

So if we find an upper bound of the number of $c(x, k, n)$ independently of x for all k, we should be able to find an upper bound of $c(n)$. The upper bound of $c(x, k, n)$ will be obtained by using the recurrence hypothesis.

Let $k \in \{1, \ldots, m\}$. We want to have an upper bound of $c(x, k, n)$. Consider the following set:

$$\left(K \setminus \bigcup_{i=1}^{k-1} (X_n^{(i)} + H_i) \right) \cap (H_k + x).$$

This set is obtained by considering the space $F' := H_k + x$ (which we regard as a vector space of dimension $p - 1$ by taking $\{x\}$ as the origin), and a compact convex subset, $K' := K \cap F'$ which is cut by hyperplanes directed by the $H_j \cap H_k$, where $j \neq k$. It can happen that some of these hyperplanes are not distinct, so we rename them $H'_1, \ldots, H'_s$, with $s < m$. The sequence of subsets $(K_n)_{n \geq 0}$ of K induces a sequence $(K'_n)_{n \geq 0}$ of subsets of K', obtained by removing hyperplanes directed by the H'_i. The number of hyperplanes removed at each step is given by sequences $(\gamma_n^{(i)})_{n \geq 0}$ for $1 \leq i \leq s$. Let us now find an estimation of the $(\gamma_n^{(i)})_{n \geq 0}$ to apply the recurrence hypothesis: if $H'_i = H_{i(1)} \cap H_k = \cdots = H_{i(q)} \cap H_k$, then $\gamma_n^{(i)} = O(n^{\delta_i})$, where $\delta_i := \max_j (\alpha_{i(j)})$. The fact that we required the $(H_i)_{i \in I}$ to be transverse (i.e. $\bigcap_{i \in I} H_i$ of minimal dimension) implies that if $i, j \in I$, then $H_k \cap H_i \neq H_k \cap H_j$. Therefore, the sum of the $p - 1$ largest elements amongst the $\{\delta_i\}_{i=1}^{s}$ (we call this sum δ) with suitable additional condition of transversality for the associated hyperplanes, is lower than the sum of the $p - 1$ largest elements amongst the $\{\alpha_i\}_{i=1}^{m}$ (with the same condition of transversality).

So, by recurrence, we have $c(x, k, n) = O(n^{\delta})$. This upper bound does not depend on x so if we sum the $x \in X_n^{(k)}$, we obtain that $\sum_{x \in X_n^{(i)}} c(x, k, n) = O(n^{\delta + \alpha_k})$. However, $\delta + \alpha_k \leq \alpha$. We deduce that $\sum_{x \in X_n^{(i)}} c(x, k, n) = O(n^{\alpha})$ and so, as we can have the same bound for all k, summing again, we obtain the expected result:

$$c(n) = O(n^{\alpha}). \qquad \square$$

3.3. *Proof of the theorem.* Let us now prove the Theorem 3.3. The proof falls into two parts: upper and lower bound. Actually, thanks to Proposition 3.1, the quantity we bound (above and below) is $c(n)$ rather than $p(n)$. We then use Lemmas 2.17, 3.6 and 3.7 to count the number of connected components.

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Upper bound. Remember that V is the set of vertices of K, seen as a polygon. As Γ is a free Abelian group of rank r, we identify Γ and $\mathbb{Z}^r$ by the choice of a group basis $\gamma_1, \ldots, \gamma_r$, such that the γ_i are projections of elements of the canonical basis of $\mathbb{R}^N$. If $a \in \Gamma$, and $a = (a_i)_{i=1}^r$ when decomposed in this basis, we define

$$\mathcal{B}_\Gamma(a, n) := \left\{ b = (b_i)_{i=1}^r \in \Gamma \mid \max_i |b_i - a_i| \leq n \right\}.$$

There exists a constant ρ such that $\pi_F(\mathbb{Z}^N \cap \mathcal{B}(0, 1)) \subseteq \mathcal{B}_\Gamma(0, \rho)$.

We have already mentioned that

$$\mathrm{Sing}_n \cap K = (\pi_F(\mathcal{B}(0, n) \cap \mathbb{Z}^N) + \partial K) \cap K$$

(apply π_F to equation (2)). As ∂K is included in $V + \bigcup_{i=1}^m H_i$, we have

$$\mathrm{Sing}_n \cap K \subseteq \left(\mathcal{B}_\Gamma(0, \rho n) + V + \bigcup_{i=1}^m H_i \right) \cap K.$$

Given $i \in \{1, \ldots, m\}$, we now have to find an upper bound of the number of distinct hyperplanes of the following form:

$$\mathcal{B}_\Gamma(0, \rho n) + V + H_i$$

and which actually intersect K. We call this number $\beta_n^{(i)}$. We have to determine the asymptotic behaviour of the sequence $(\beta_n^{(i)})_{n \in \mathbb{N}}$. We then apply Lemma 3.7.

Let $q_i : F \to F/H_i$ be the quotient map. It maps F to a one-dimensional $\mathbb{R}$-vector space, and it maps Γ to Γ/Γ^i, which is of rank $\alpha_i + 1$. We want to find an upper bound of the number of hyperplanes of the form $H_i + x$, with $x \in V + \mathcal{B}_\Gamma(0, \rho n)$ intersecting K. We write these hyperplanes $H_i + \gamma + v$ with $\gamma \in \mathcal{B}_\Gamma(0, \rho n)$ and $v \in V$. However, for a given $v \in V$, it amounts to counting the number of points of $q_i(\mathcal{B}_\Gamma(0, \rho n) + v)$ which belong to $q_i(K)$. Equivalently, this amounts to counting the number of points of $q_i(\mathcal{B}_\Gamma(0, \rho n))$ which belong to $q_i(K - v)$. Choosing a basis for $q_i(\Gamma) = \Gamma/\Gamma^i \simeq \mathbb{Z}^{\alpha_i + 1}$, there exists a $\rho' > 0$ such that $q_i(\mathcal{B}_\Gamma(0, \rho n)) \subseteq \mathcal{B}_{\Gamma/\Gamma^i}(\rho' n)$ at least for all n large enough. We now apply Lemma 3.6: the number of points in the set $\mathcal{B}_{\Gamma/\Gamma^i}(\rho' n)$ which fall in $q_i(K - v)$ is dominated by n^{α_i}. We do the same reasoning for all $v \in V$, we sum the inequalities, and we have

$$\beta_n^{(i)} = O(n^{\alpha_i}).$$

This reasoning holds for all $1 \leq i \leq m$, so $(\beta_n^{(i)})_{n \in \mathbb{N}}$ is dominated by n^{α_i} for all i. In addition, as we said, $\beta_n^{(i)}$ is the number of hyperplanes parallel to H_i of the form above intersecting K. Then, we apply Lemma 3.7, and we have

$$p(n) \leq c(n) = O(n^\alpha). \qquad \square$$

Lower bound. Let us then consider the set of indices I satisfying condition that $\alpha = \sum_{i \in I} \alpha_i$, and $\bigcap_{i \in I} H_i = \{0\}$. Let ε be the minimum over $i \in I$ of the ε_i. Consider an open parallelotope P of non-empty interior, of diameter lesser than ε, included in K, and the faces of which are parallel to the hyperplanes $(H_i)_{i \in I}$. Such a parallelotope does exist because all of the $(H_i)_{i \in I}$ are transverse.

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Using the same reasoning as in the upper bound part above, we can say that, for all $i \in I$ and for a given $v_i \in V_i$, the number of points in $(\pi_F(\mathcal{B}(0, n \cap \mathbb{Z}^N)) + v_i) \cap P$ modulo H_i is greater than $C_i n^{\alpha_i}$ for a certain $C_i > 0$.

This implies, by Lemma 2.17, that the $(n + M)$-singular points inside P contains at least $C_i n^{\alpha_i}$ distinct affine hyperplanes (intersected with P) directed by H_i for all $i \in I$. Thus, the number of connected components of the $(n + M)$-singular points, when restricted to P, is at least $\prod_i C_i n^{\alpha_i}$, that is at least Cn^{α}, for suitable C. So we have

$$p(n) \geq \lambda c(n) \geq \lambda C n^{\alpha}. \qquad \square$$

4. *Examples and applications*

We now apply our results to three examples: the case $d = 1$, the case $d = N - 1$, and the case of the octagonal tiling (a specific case with $d = 2$ and $N = 4$).

4.1. *The octagonal tiling.*

The octagonal tiling can be defined as a cut-and-projection tiling with $N = 4$ and $d = 2$. We use $(e_1, \ldots, e_4)$ to denote the canonical basis of $\mathbb{R}^4$.

We consider the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $X^4 + 1$, so that its invariant subspaces are two orthogonal subspaces of $\mathbb{R}^4$ of dimension two. Let us call E and $F = E^{\perp}$ these subspaces ($E \perp F$ because the matrix A is orthogonal), and π the orthogonal projection of $\mathbb{R}^4$ on E.

We compute that we can choose the following basis for F:

$$F = \text{Vect}\,((\sqrt{2}/2, 1/2, 0, -1/2), (0, 1/2, \sqrt{2}/2, 1/2)),$$

and that these vectors form an orthonormal basis of F. Thus, we can write the matrix of π_F in the canonical basis:

$$\text{Mat}_{\text{can}}\, \pi_F = \begin{pmatrix} 1/2 & \sqrt{2}/4 & 0 & -\sqrt{2}/4 \\ \sqrt{2}/4 & 1/2 & \sqrt{2}/4 & 0 \\ 0 & \sqrt{2}/4 & 1/2 & \sqrt{2}/4 \\ -\sqrt{2}/4 & 0 & \sqrt{2}/4 & 1/2 \end{pmatrix}.$$

Then the octagonal tiling is the tiling obtained from these data, with canonical acceptance domain (that is, the projection of the four-dimensional cube). Figure 4 represents the acceptance domain $K \subseteq F$ of this tiling, with the 1-singular points.

We use $f_1, \ldots, f_4$ to denote the projections of the $\{e_i\}$ on F. The four singular hyperplanes are $H_i = \text{Vect}(f_i)$. From the form of the matrix $\text{Mat}_{\text{can}}\, \pi_F$, we see that the H_i have non-trivial stabilizers:

$$\begin{aligned} \text{Stab}_{\Gamma}(H_1) &= \langle f_1, f_2 - f_4 \rangle; \\ \text{Stab}_{\Gamma}(H_2) &= \langle f_2, f_1 + f_3 \rangle; \\ \text{Stab}_{\Gamma}(H_3) &= \langle f_3, f_2 + f_4 \rangle; \\ \text{Stab}_{\Gamma}(H_4) &= \langle f_4, f_1 - f_3 \rangle. \end{aligned}$$

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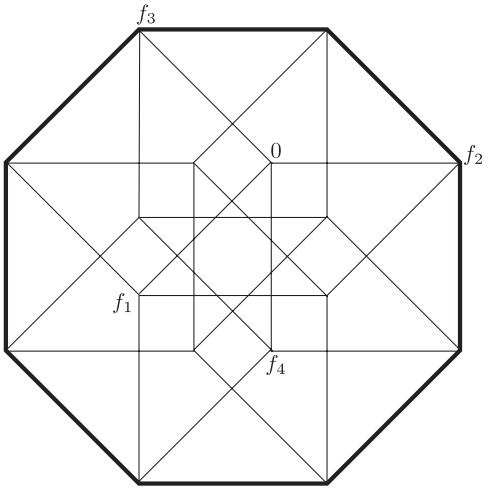


FIGURE 4. The acceptance domain of the octagonal tiling, seen on F, with the 1-singular points.

So we have for all $i \in \{1, \ldots, 4\}$, $\alpha_i = 2$, so that $\alpha = 2$, and the complexity of the octagonal tiling grows as n^2.

This computation agrees with the results of Robinson and Hansen (see [**Rob**, Theorem 7.17]): for primitive substitution tilings of dimension d, the complexity $p(n)$ is dominated by Cn^d for some constant C. This result applies to the octagonal tiling, which is known to be a substitution tiling. Our computations provide a new method for the computation of the complexity of the octagonal tiling, as well as an additional lower bound.

4.2. *Cubic billiards.* The complexity of cubic billiards has already been studied thoroughly by mathematicians as well as computer scientists. We show that our result gives the (well-known) order of growth for the complexity of a word obtained by a cubic billiard, even though we have no control on the constants involved.

The assumption that $\Gamma := \pi_F(\mathbb{Z}^N)$ is dense in F, and F is of dimension $N-1$ implies that Γ is of rank N. The singular hyperplanes are $H_i := \mathrm{Vect}(\{\pi_F(e_k)\}_{k \neq i})$. The rank of $\mathrm{Stab}_\Gamma(H_i)$ is at least $N-1$ (by definition), and cannot be N, because it would imply $\Gamma \subseteq H_i$, and Γ would not be dense in F.

So for all i, we have $\alpha_i = N-1$, and the complexity of a billiard sequence in an N-dimensional hypercube in a suitable direction (which ensures that Γ is dense in F), grows as n^{N-1}.

This result was already known, with greater precision. See [**Bar**].

4.3. *Discrete hyperplane.* In the case $d = N-1$ (the Sturmian sequence being a particular case), the space F is one-dimensional, and there exists only one singular hyperplane (the point $\{0\} =: H_0$).

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Then $\mathrm{Stab}_\Gamma(H_i) = \{0\}$, and the complexity only depends of the rank of Γ. Let us assume that $E \cap \mathbb{Z}^N$ is of rank k, that is the group of periods of the resulting Delone set is of rank k. Then Γ is of rank $N - k$, and $\alpha_i = \mathrm{rk}\ \Gamma - 1 = N - k - 1 = d - k$.

We deduce that the complexity of a Delone set of dimension d, periodic in k independent directions, obtained by cut-and-projection in a total space of dimension $N = d + 1$, grows as n^{d-k}.

In the case $k = 0$ (no period), the complexity grows as n^d. This result agrees with (and extends) the computations of the complexity for discrete planes of [**BV**] in dimension two: in their case, a $(3 - 1)$-cut-and-project tiling (hence a tiling of the plane) is coded by a two-dimensional word, and the rectangular complexity (the number of patches of size $n \times m$) is $nm + n + m$. In the case of square patches, as in our case, this is a quadratic complexity.

5. *Links between complexity and cohomology*

5.1. *Cohomology for tiling spaces.* Cohomology first originates from algebraic topology. The idea of cohomology is to associate Abelian groups (the *cohomology groups*) with a topological space X. These groups are denoted by $(H^i(X))_{i\geq 0}$. The construction is functorial (and contravariant) in the sense that given a continuous morphism $f : X \to Y$, then there is an induced map $f^* : H^i(Y) \to H^i(X)$. Cohomology groups can be defined by various means on topological spaces such as manifolds. We can mention for example the singular cohomology, the de Rham cohomology, or the Čech cohomology. For manifolds, these definitions agree. However, for spaces with less regularity, some definitions are more suited than others.

The strength of cohomology is that it is a topological invariant. Homeomorphic (or even homotopic) spaces have the same cohomology groups. Furthermore, an exact sequence machinery exists which actually allows, to some extent, these groups to be computed.

Tiling spaces are quite complicated topological spaces: they are everywhere locally the product of a Cantor set by an Euclidean ball. It appeared that the good notion of cohomology for these spaces was Čech cohomology. A remarkable feature of Čech cohomology is that it is well behaved towards inverse limits: the ith Čech cohomology group of an inverse limit of spaces is the direct limit of the ith cohomology groups of these spaces. This fact led to an actual computation of cohomology groups for a subclass of tilings including some important examples. Anderson and Putnam [**AP**] showed that substitution tiling spaces were inverse limits of simple complexes, which allowed the associated invariants to be computed.

In the framework of cut-and-projection tiling spaces, another method exists and was carried out by Forrest, Hunton and Kellendonk [**FHK**]. This method uses the fact that a group acts on the tiling space. The cohomology groups actually computed with this method are the cohomology groups of the group $\mathbb{Z}^N$ with value in a certain module of functions, and this other notion of cohomology agrees with the Čech cohomology of the topological space. Their computations provided a criterion for infinitely generated cohomology, which we give below.

5.2. *Preliminaries and statement of the theorem.* We may wonder whether complexity has a topological interpretation. Indeed, intuitively, the more complex a tiling is, the less it

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should be repetitive. As the usual distance which defines the topology of tiling spaces (see, for example, [**AP**]) is linked to the notion of recurrence of patches, complexity should have a dynamical or topological interpretation in terms of the associated tiling space. We will show that, for cut-and-projection tilings with almost-canonical acceptance domain, one has such a link: the cohomology of a cut-and-projection tiling space is finitely generated (over $\mathbb{Q}$) if and only if its complexity grows as slow as possible. More precisely, the statement is as follows.

THEOREM 5.1. *Consider a cut-and-projection method* $(\mathbb{R}^N = E \oplus F, \mathbb{Z}^N, \pi, K)$ *which satisfies our assumptions. Call* $\Gamma = \pi_F(\mathbb{Z}^N)$. *Let* p *be the associated complexity function. Then there exists a constant* $C_1 > 0$ *such that* $p(n) \geq C_1 n^\alpha$ *with* $\alpha = d - (N - \mathrm{rk}\,\Gamma)$. *The cohomology groups over* $\mathbb{Q}$ *of the associated tiling space are finitely generated if and only if the complexity is minimal, that is, there exists* C_2 *such that* $p(n) \leq C_2 n^\alpha$.

In the following sections, we show that this theorem is specific to cut-and-projection Delone sets: such an equivalence is actually false in greater generality, and we give a counterexample of one of these implications in dimension one.

Definition 5.2. Given a (N, d)-cut-and-projection method, we call Sing the set of singular points, and Sing_n the set of n-singular points. These have a structure of cell complex, and we denote the set of k-cells of Sing (for $0 \leq k \leq N - d - 1$) by $L^{(k)}$, and the set of k-cells of Sing_n by $L_n^{(k)}$. We remark that $L^{(k)}$ carries naturally a Γ-action, induced by the translation.

Definition 5.3. We call an affine space of dimension k, which is singular, and is parallel to a space of the form $A = \bigcap_{j \in J} H_j$ a *singular* k*-space*.

THEOREM 5.4. [**FHK**] *The cohomology groups associated with a cut-and-projection method are finitely generated if and only if* $L^{(0)}/\Gamma$ *is finite.*

Theorem 5.1 actually has two parts. First, it gives a lower bound for the exponent of the complexity. Then, it makes a link with cohomology. The cohomology groups are finitely generated if and only if the exponent of the complexity is minimal. Let us make a connection with our result on the minimal complexity exponent and a conjecture by Lagarias and Pleasant.

CONJECTURE. (Lagarias and Pleasant [**LP**]) *Let the dimension* d *be given. For each* $1 \leq j \leq d$, *there is a positive constant* $c_j = c_j(d, r, R)$ *such that any Delone set* $\mathcal{D}$ *in* $\mathbb{R}^d$ *with constants* (r, R) *that satisfies*

$$p(n)/n^{d+1-j} < c_j$$

for all n *large enough has at least* j *linearly independent periods.*

What Theorem 5.1 proves is that *for a cut-and-projection Delone set*, if $p(n)/n^{d+1-j}$ is bounded, then $d + 1 - j \geq d + \mathrm{rk}\,\Gamma - N$ which means $\mathrm{rk}\,\Gamma \leq N - j + 1$, which means that the Delone set has at least $j - 1$ independent periods. The Delone set has at least j independent periods if and only if the quantity $p(n)/n^{d+1-j}$ tends to zero. Our method gives a rather asymptotic result, and we do not have control on the constants involved.

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Therefore, even if we do not fully prove the conjecture in the case of cut-and-projection Delone sets, the theorems in this article seem to confirm the result. We do not know whether our methods can be adapted to prove the conjecture.

5.3. *Proof of the theorem.* Let us now prove Theorem 5.1. First, we make the following remark.

Remark 5.5. An element x is in $L^{(0)}$ if and only if there exist $N-d$ singular hyperplanes $(H_j)_{j\in J}$ the intersection of which is $\{0\}$, elements $(\gamma_j)_{j\in J}$ of Γ, and elements $v_j \in V_j$, such that

$$x = \bigcap_{j\in J} (H_j + \gamma_j + v_j).$$

Consider a (N, d)-cut-and-projection method, with complexity function $p(n)$. We proved (Theorem 3.3) that $p(n)$ grows as n^α. We want to prove that $\alpha \geq d + \operatorname{rk}\Gamma - N$, with equality if and only if $L^{(0)}/\Gamma$ is finite.

Notation. Recall that Γ^j is the stabilizer in Γ of the singular hyperplane H_j. For $J \subset \{1 \dots m\}$, we use Γ^J to denote the stabilizer of the intersection $\bigcap_{j\in J} H_j$, and $a_J = \operatorname{rk}\Gamma^J$. We use the convention $\Gamma^\emptyset = \Gamma$.

Proof of Theorem 5.1. We use the notation of Theorem 3.3: $\alpha = \sum_{i\in I} \alpha_i$, where $\alpha_i = \operatorname{rk}(\Gamma/\Gamma^i) - 1 = \operatorname{rk}\Gamma - a_i - 1$, and the α_i, $i \in I$ are chosen to be the $N-d$ greatest amongst the α_j, $1 \leq j \leq m$, and such that $\bigcap_{i\in I} H_i = \{0\}$. Therefore, we have

$$\begin{aligned} \alpha &= (N-d)\operatorname{rk}\Gamma - (N-d) - \sum_{i\in I} a_i \\ &= (d + \operatorname{rk}\Gamma - N) + \left[(N-d-1)\operatorname{rk}\Gamma - \sum_{i\in I} a_i\right] \end{aligned} \tag{4}$$

and the a_i, $i \in I$, are chosen to be the $N-d$ *smallest* amongst the a_j, $1 \leq j \leq m$ (with the transversality condition).

Given two sets of indices J, and J' in $\{0, \dots, m\}$, consider the following application:

$$\begin{array}{rccc} \varphi_{J,J'}: & \Gamma/\Gamma^{J\cup J'} & \longrightarrow & \Gamma/\Gamma^J \times \Gamma/\Gamma^{J'} \\ & [x]_{J\cup J'} & \longmapsto & ([x]_J, [x]_{J'}) \end{array}$$

where $[x]_J$ stands for the class of x in Γ modulo Γ^J. One checks easily that this application is well-defined and one-to-one. So the rank of the target group is equal or greater than the rank of the source group (we remark that both of these groups are free Abelian). Therefore, the following condition on the ranks holds:

$$\operatorname{rk}\Gamma - a_{J\cup J'} \leq \operatorname{rk}\Gamma - a_J + \operatorname{rk}\Gamma - a_{J'},$$

which gives

$$a_J + a_{J'} - a_{J\cup J'} \leq \operatorname{rk}\Gamma. \tag{5}$$

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Let us iterate this inequality on equation (4) in order to deduce the first part of the theorem. We label $i(1), \ldots, i(N-d)$ the elements of I. Then we write

$$\begin{aligned} a_{i(1)} + a_{i(2)} &\leq a_{\{i(1),i(2)\}} + \operatorname{rk}\Gamma; \\ a_{i(1)} + a_{i(2)} + a_{i(3)} &\leq a_{\{i(1),i(2)\}} + a_{i(3)} + \operatorname{rk}\Gamma \\ &\leq a_{\{i(1),i(2),i(3)\}} + 2\operatorname{rk}\Gamma; \\ &\vdots \\ \sum_{i\in I} a_i &\leq (N-d-1)\operatorname{rk}\Gamma + a_I \\ &= (N-d-1)\operatorname{rk}\Gamma. \end{aligned} \tag{6}$$

We deduce that $[(N-d-1)\operatorname{rk}\Gamma - \sum_{i\in I} a_i]$ is positive, which yields

$$\alpha \geq d + \operatorname{rk}\Gamma - N.$$

This proves the first part of Theorem 5.1.

Let us now prove that $\alpha = d - N + \operatorname{rk}\Gamma$ if and only if $L^{(0)}/\Gamma$ is finite. First, we prove that the equality implies finiteness of $L^{(0)}/\Gamma$, by a decreasing induction. Assume that $\alpha = d + \operatorname{rk}\Gamma - N$, and consider the following recurrence hypothesis: for a given $k < N-d$ and every space of the form $A = \bigcap_{j\in J} H_j$ of dimension k (Card $J = N-d-k$), then the set of singular k-spaces (recall Definition 5.3) which are directed by A is the union of only finitely many Γ-orbits of translates of A.

The assertion above is obviously true for $k = N-d-1$, since the set of singular $(N-d-1)$-spaces directed by H_i is included in $H_i + V + \Gamma$ for all i, and V is finite.

Let us assume that the recurrence hypothesis is true for a given $k+1$, and let us prove that the set of singular k-spaces directed by a given space of dimension k is the union of finitely many Γ-orbits. Let $A = \bigcap_{i\in J} H_i$ be a vector subspace of dimension k, with Card $J = N-d-k$. We choose $j \in J$, and define $J' := J \setminus j$, and $A' = \bigcap_{j\in J'} H_j$. Then consider the map $\varphi_{J',j} := \varphi_{J',\{j\}}$ as defined above.

We deduce from the iterative reasoning of equation (6) that for all the sets I' of $N-d$ elements such that $\bigcap_{i\in I'} H_i = \{0\}$, the following holds:

$$(N-d-1)\operatorname{rk}\Gamma - \sum_{i\in I'} a_i \geq 0$$

(equation (6) is not specific to the choice of I). Furthermore, we have assumed that $\alpha = d + \operatorname{rk}\Gamma - N$. Therefore, for the specific choice $I' = I$, the inequality above is an equality. However, the choice of I is the choice for which the a_i are *minimal*. Therefore, the above expression is positive, and its maximum is zero. So for all choices of I' such that $\bigcap_{i\in I'} H_i = \{0\}$:

$$(N-d-1)\operatorname{rk}\Gamma - \sum_{i\in I'} a_i = 0. \tag{7}$$

We claim that the following equality holds, which means inequality (5) is in fact an equality:

$$a_{J'} + a_j = a_J + \operatorname{rk}\Gamma, \tag{8}$$

otherwise if we had $a_{J'} + a_j < a_J + \operatorname{rk}\Gamma$, then we could choose I' to be a set of $N-d$ elements of $\{1, \ldots, m\}$, containing J. Then, applying the inequalities of (6) to I', we

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would obtain a strict inequality at some step, and so we would obtain

$$(N - d - 1)\,\mathrm{rk}\,\Gamma - \sum_{i \in I'} a_i > 0,$$

in contradiction to equation (7) above.

Consider the map $\varphi_{J',j} : \Gamma/\Gamma^J \to (\Gamma/\Gamma^{J'}) \times (\Gamma/\Gamma^j)$: it is one-to-one. Furthermore, the source group has rank $\mathrm{rk}\,\Gamma - a_J$ and the target group has rank $2\,\mathrm{rk}\,\Gamma - a_{J'} - a_j$. Those have the same rank, by equation (8). It does not necessarily mean that $\varphi_{J',j}$ is onto, but its image is of finite index, so its cokernel $G := (\Gamma/\Gamma^{J'}) \times (\Gamma/\Gamma^j)$ is finite.

We consider the set of k-dimensional affine spaces of the following form:

$$(A' + \gamma_1) \cap (H_j + \gamma_2)$$

with $\gamma_1, \gamma_2 \in \Gamma$. Owing to stabilization of the spaces involved, it is enough to consider $\gamma_1 \in \Gamma^{J'}$ and $\gamma_2 \in \Gamma^j$. Then the set of translates of singular k-spaces of the form above is in correspondence with $(\Gamma/\Gamma^{J'}) \times (\Gamma/\Gamma^j)$. Quotienting by the action of Γ (or by the image of $\varphi_{J',j}$, which amounts to the same thing), we have that the set of Γ-orbits of k-dimensional spaces of the form above is finite.

In general, the singular k-spaces parallel to A are of the form

$$(A' + \gamma_1 + v_1) \cap (H_j + \gamma_2 + v_2),$$

where A' is chosen amongst the (finite) set of $(k+1)$-dimensional vector spaces containing A and of the form $\bigcap_i H_i$, j is chosen amongst the hyperplanes such that $H_j \cap A' = A$, $\gamma_1 \in \Gamma/\Gamma^{J'}$, $\gamma_2 \in \Gamma/\Gamma^j$, and v_2 is chosen in V (the set of vertices of the acceptance domain), which is finite. In addition, by the recurrence hypothesis, there are only finitely many Γ-orbits of translates of singular $k+1$-spaces directed by A', so v_1 can be chosen in a finite set. Thus, as all choices we made were finite, the number of Γ-orbits of singular k-spaces directed by A is finite. We can say this for every singular k-space A, so the recurrence hypothesis holds for k.

We conclude, using induction down to $k = 0$, that $L^{(0)}$ is a union of finitely many Γ-orbits.

Conversely, assume that $\alpha > d - N + \mathrm{rk}\,\Gamma$. This means that there is at least one inequality in equations (6) which is not an equality. This means that there is $J' \subseteq \{1, \ldots, m\}$ and $j \in \{1, \ldots, m\}$ such that the $(H_i)_{i \in J \cup \{j\}}$ are transverse. We note, as above, $J = J' \cup \{j\}$. Then,

$$a_{J'} + a_j < a_J + \mathrm{rk}\,\Gamma.$$

As we said above, this proves that the set of singular k-spaces of the form:

$$\left(\bigcap_{i \in J'} H_i + \gamma_1 \right) \cap (H_j + \gamma_2)$$

is the union of infinitely many Γ-orbits of translates of $\bigcap_{j \in J} H_j$. Therefore, taking the intersection of these spaces with any complementary singular subspace, we deduce that $L^{(0)}$ is made of infinitely many Γ-orbits. □

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6. *The case of dimension one*

In dimension one, for general tilings (not necessarily obtained by cut-and-projection), the links between cohomology and complexity are quite different.

To a tiling of $\mathbb{R}$, one can associate a bi-infinite word w, which corresponds to the sequence of its tiles. In this section $p(n)$ stands for the number of subwords of length n of a given word w. Beware that before this section, $p(n)$ in the one-dimensional case was defined as the number of words of length $2n$. These two definitions of course give the same order of growth, only the constants are changed.

6.1. *Definitions and construction.*

Let us fix the notation we use in the following sections. We consider a finite alphabet $\mathcal{A}$. Given $w \in \mathcal{A}^{\mathbb{Z}}$ a bi-infinite word over $\mathcal{A}$, we write

$$w = \ldots w_{-2}w_{-1}.w_0w_1w_2 \ldots .$$

The space $\mathcal{A}^{\mathbb{Z}}$ is a topological space in a natural way (it is a Cantor set). The group $\mathbb{Z}$ acts naturally on $\mathcal{A}^{\mathbb{Z}}$ by the *shift* homeomorphism. This homeomorphism, τ, is defined as follows:

$$\tau(\ldots w_{-2}w_{-1}.w_0w_1 \ldots) = (\ldots w_{-1}w_0.w_1w_2 \ldots).$$

Definition 6.1. We call the closure in $\mathcal{A}^{\mathbb{Z}}$ of the set

$$\Theta_w := \{\tau^n(w) \mid n \in \mathbb{Z}\},$$

together with the shift application τ the *subshift* of $\mathcal{A}^{\mathbb{Z}}$ generated by w.

Such a subshift is totally disconnected, compact, and τ-stable. We now define the suspension of a subshift Θ_w.

Definition 6.2. If (Θ_w, τ) is the subshift of $\mathcal{A}^{\mathbb{Z}}$ associated with w, we define its *suspension* (Ω_w, φ) by

$$\Omega_w = \Theta_w \times \mathbb{R}/\sim,$$

where $\sim$ is the equivalence relation generated by the relations $(x, t) \sim (\tau(x), t - 1)$ for all $x \in \Theta_w$ and all $t \in \mathbb{R}$. The map φ is an action of $\mathbb{R}$ defined by

$$\varphi_s([(x, t)]) := [(x, t + s)].$$

With these definitions, (Ω_w, φ) is a $\mathbb{R}$-dynamical system.

Remark 6.3. The construction of the suspension allows us to make the link between words and tilings of $\mathbb{R}$. An element $[(x, t)]$ of Ω_w can be seen as a tiling of $\mathbb{R}$ by collared tiles, which would all be of length one, the set of colours being $\mathcal{A}$. The tiling follows then the pattern of the word x and the parameter t indicates the position of the tiling. There is a natural action of $\mathbb{R}$ by translation on tilings: this action corresponds to φ on the suspension space.

Again, Ω_w has a natural topology, which we now define.

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Definition 6.4. The following subsets of Ω_w define a basis of open sets for the topology:

$$U(V, I) = \{[(x, t)] \mid x \in V,\ t \in I\},$$

with V an open subset of Θ_w and I an open interval of $\mathbb{R}$.

The following proposition is classical, and we state it without a proof.

PROPOSITION 6.5. *With this topology, Ω_w is compact and connected.*

Now, let us define the Rauzy graphs associated with w.

Definition 6.6. For all $n \in \mathbb{N}$, define $F_n(w)$ the set of factors of w, of size n, that is the set of all finite subwords of w of length n. The nth Rauzy graph R_n associated with w is an oriented graph defined by:

- a set of vertices $V_n = F_n(w)$;
- a set of oriented edges E_n (there is an edge from $a_1 \dots a_n \in V_n$ to $b_1 \dots b_n \in V_n$ if $a_i = b_{i-1}$ for $2 \leq i \leq n$, and if the word $a_1 \dots a_n b_n$ is a factor of w).

We use ${}_a f_b$ to denote the oriented edge from af to fb where $a, b \in \mathcal{A}$ and $f \in F_{n-1}$, and such that $af,\ fb \in V_n$. Such an edge exists if and only if $afb \in F_{n+1}$.

Rauzy graphs are very well suited to the study of combinatorial properties on words. Namely, the vertices with more than one outgoing edge correspond exactly to the factors which can be extended to the right in more than one way (such factors are sometimes called 'special factors'). As all factors of w can be extended to the right or to the left, the Rauzy graphs R_n are connected by oriented paths (or strongly connected).

Remark that these graphs are combinatorial objects. However, we can (and will) regard them as metric spaces as well: a graph is the disjoint union indexed over the edges of copies of the interval [0, 1], quotiented by the adjacency condition (gluing the vertices together).

6.2. *Inverse limits of Rauzy graphs.* We first prove that one-dimensional tiling spaces are inverse limits of Rauzy graphs. It is already known that they are inverse limits of graphs, namely the Gähler complexes. Even though Rauzy graphs and Gähler complexes are constructed in a very similar way, they can be different. Rauzy graphs are best suited to capture combinatorial information on the word, such as complexity or the number of 'bi-extendable' words (or special factors). Therefore, they are a convenient technical tool if we want to make a link between complexity and cohomology.

In order to build inverse limits, we need projection maps $R_{n+1} \to R_n$. The simple fact that a word of length $n+1$ can be shortened to a word of length n will allow us to define maps $\gamma_n : R_{n+1} \to R_n$. In this construction, one could shorten the word from the left or from the right. We make the following choice.

Definition 6.7. Define $\gamma_n : R_{n+1} \to R_n$ in the following way. If n is even, γ_n is defined by:

- $\gamma_n(a := a_1 \dots a_{n+1}) = a_1 \dots a_n$ if $a \in S_{n+1}$ (remove a letter on the right);
- if an edge $e \in E_n$ goes from a to b, then there exists in R_n an edge from $\gamma_n(a)$ to $\gamma_n(b)$; the application γ_n then maps e on that edge, and its restriction on e is the identity map on the segment [0, 1].

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If n is odd, γ_n is defined in a similar way by:

- $\gamma_n(a := (a_1 \dots a_{n+1})) = a_2 \dots a_{n+1}$, if $a \in S_{n+1}$ (remove a letter on the left);
- the definition of γ_n on the edges is exactly the same.

PROPOSITION 6.8. *The maps γ_n are continuous.*

Proof. The proof is straightforward. □

We can now consider inverse limits of the Rauzy graphs, with respect to the maps γ_n.

Definition 6.9. The inverse (or projective) limit of the complex $(R_n, \gamma_n)_{n\in\mathbb{N}}$ is the following subset of $\prod_{n\in\mathbb{N}} R_n$:

$$\varprojlim (R_n, \gamma_n)_{n\in\mathbb{N}} := \left\{ (x_1, x_2, \dots) \in \prod_{n\in\mathbb{N}} R_n \;\middle|\; \forall n \in \mathbb{N},\ \gamma_n(x_{n+1}) = x_n \right\}.$$

This inverse limit is a topological space with the relative topology induced by the product topology of $\prod_{n\in\mathbb{N}} R_n$. It is a compact set, being a closed subset of a compact set.

THEOREM 6.10. *Let w be a bi-infinite word over the finite alphabet $\mathcal{A}$. Let Θ_w be its associated subshift of $\mathcal{A}^{\mathbb{Z}}$. Use Ω_w to denote the suspension of (Θ_w, τ). We then have a homeomorphism:*

$$\Omega_w \simeq \varprojlim (R_n, \gamma_n)_{n\in\mathbb{N}}.$$

For the proof, we use the now classical methods developed in [**AP**].

Proof. Let us build a homeomorphism between $X := \varprojlim_n R_n$ and $\Omega := \Omega_w$. For all $n \in \mathbb{N}$, we define an application $\psi_n : \Omega \to R_n$. This allows us to define an application $\psi : \Omega \to X$, which we prove, is a homeomorphism. Let $P \in \Omega$. Then $P = (m, t)$ with $m \in \Theta_w$, and t can be chosen in $[0, 1[$. Note $m = (m_i)_{i\in\mathbb{Z}}$. Define $\psi_n(P)$ as follows.

- If n is even $(n = 2p)$, consider the vertices s_1 and s_2 of R_n defined by the following words:
$$s_1 := m_{-p} \dots m_{p-1};$$
$$s_2 := m_{-p+1} \dots m_p.$$
There exists an oriented edge isometric to $[0, 1]$ from s_1 to s_2. Then $\psi_n(P)$ is defined to be the point in position $t \in [0, 1]$ in this edge.
- If n is odd, $(n = 2p + 1)$, consider the vertices s_1 and s_2 of R_n defined by the following words:
$$s_1 := m_{-p-1} \dots m_{p-1};$$
$$s_2 := m_{-p} \dots m_p.$$
Here $\psi_n(P)$ is defined to be the point in position $t \in [0, 1]$ in the edge from s_1 to s_2.

A direct computation shows that the ψ_n are compatible with the maps γ_n, that is, for all $n \in \mathbb{N}$:

$$\psi_n = \gamma_{n+1} \circ \psi_{n+1}.$$

Then, owing to this compatibility relations, the application ψ_n defines a map $\psi : \Omega \to X$. Let us check that ψ is one-to-one, onto, and continuous. By compactness of Ω, it will be a homeomorphism.

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Let $P, P' \in \Omega$. We write $P = (m, t)$ and $P' = (m', t')$, with $t, t' \in [0, 1[$. We assume that $P \neq P'$. If $t \neq t'$, it is clear that $\psi_n(P) \neq \psi_n(P')$ for all n, and hence $\psi(P) \neq \psi(P')$. If $t = t'$, then $m \neq m'$, and hence there is some p such that

$$m_{-p-1} \dots m_{p-1} \neq m'_{-p-1} \dots m'_{p-1}.$$

Hence, $\psi_{2p+1}(P)$ and $\psi_{2p+1}(P')$ do not belong to the same edge of R_{2p+1}, and so they cannot be equal. In all cases, if $P \neq P'$, then $\psi(P) \neq \psi(P')$, and ψ is one-to-one.

Let us prove that ψ is onto. Given $(x_n)_{n\in\mathbb{N}}$ an element of X, let us construct a preimage in Ω. By construction, there exists at most one edge from one given vertex of a Rauzy graph to another. Therefore, the element $x_n \in R_n$ is entirely determined by two vertices s_n (starting point), s'_n (ending point), and an element $t_n \in [0, 1[$ (position on the edge). It is obvious that $t_n =: t$ does not depend on n (because of the compatibility conditions). From the sequence of the s_n, we construct a bi-infinite word: by the compatibility condition, each s_n extends s_{n-1}. We define m to be the bi-infinite word obtained by increasing union of these finite words. The factor $s_n = s_n^{(1)} \dots s_n^{(n)}$ will be found in the following position (say, for n even):

$$m = \dots s_n^{(1)} \dots s_n^{(n/2)} . s_n^{(n/2+1)} \dots s_n^{(n)} \dots$$

where the dot lies just before the zeroth letter of m. Then we check that $(m, t) \in \Omega$ is a preimage for $(x_n)_{n\in\mathbb{N}}$.

Finally, let us prove that ψ is continuous. Let $P \in \Omega$, and V be an open neighbourhood around $x := \psi(P)$. Then V is of the form:

$$V = V_1 \times \dots \times V_n \times \prod_{i>n} R_i,$$

where the V_i are neighbourhoods of $x_i := \psi_i(P)$ in R_i. Let us prove that $\psi^{-1}(V)$ contains an open set containing P. There are two cases to consider.

- If $P = [(m, t)]$ with $t \in]0, 1[$, then we can restrict all of the V_i such that for all $i \leq n$, V_i is an open neighbourhood of x_i which is entirely included in an edge of R_n. Furthermore, if, as above, we use the notation $x_i = (s_i, s'_i, t)$, the union of s_n and s'_n define a finite word of length $n + 1$ centred in zero, which, in turn, defines a clopen set of Θ_w (that is a set which is both open and closed). We call W this clopen set. Finally, as the V_i are all included on edges isometric to $[0, 1]$, each of these neighbourhoods define an interval I_i of $]0, 1[$. We call I their (finite) intersection, which is an open interval containing t. Then $U(W, I)$ is an open set of Ω included in $\psi^{-1}(V)$, and containing P.
- If $P = [(m, 0)]$, the construction is about the same. We construct a finite subword of m, which defines a neighbourhood W of Θ_w, and such that $\psi([(W, 0)]) \subseteq V$. Then, we remark that the V_i define (up to isometry), a certain number of neighbourhoods of 1 in $[0, 1]$ which correspond to the incoming edges, and a certain number of neighbourhoods of 0 in $[0, 1]$ which correspond to the outgoing edges. We call I_k all of the neighbourhoods of 1 (when $1 \leq i \leq n$), and J_k the neighbourhoods of 0. Now, define I as the intersection of all of the I_k, and J the intersection of all of the J_k. Finally, call $I' = (J - 1) \cup I$, which is an open neighbourhood of 0 in $\mathbb{R}$. Then $U(W, I')$ is an open set of Ω included in $\psi^{-1}(V)$, and containing P.

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We have proved that ψ is continuous. Hence, by compactness of Ω_w, it is an homeomorphism, and the theorem is proved. □

6.3. *Preliminaries.* We proved (Theorem 6.10) that tilings spaces can be seen as inverse limits of some nice topological spaces, namely the Rauzy graphs, under the maps $(\gamma_n)_{n\in\mathbb{N}}$.

The number of vertices of the nth Rauzy graph R_n is equal to the complexity $p(n)$, and its number of edges is related to the function $p(n+1)-p(n)$. This makes it suited to study complexity. In comparison, the number of edges of G_n, the nth Gähler complex equals $p(n)$, but we have no real control on the number of vertices: this makes it less suited for our purpose.

Let us actually make the link between complexity and the combinatoric properties of the Rauzy graphs associated with a word. As in the sections above, w is a given bi-infinite word over a finite alphabet $\mathcal{A}$. Remember that $p(n)$ counts the number of different factors of n letters which appear in w.

Definition 6.11. We define $s(n) := p(n+1) - p(n)$.

Remark that a factor of w of length $n+1$ can be obtained from a factor of length n by extending it on the right by a suitable letter. Any factor of F_{n+1} can be obtained in this way. This simple fact gives the following proposition.

PROPOSITION 6.12. *We have*

$$s(n) = \sum_{w\in F_n} [\textit{number of ways to extend } w \textit{ on the right } - 1].$$

So the function s counts the number of 'exceptional' ways to extend words on a given side, and in a way that the extension is still a subword of w.

Remark 6.13. Over a two-letter alphabet, $s(n)$ simply counts the number of 'special factors' of length n of w: all subwords of w of a given length can be extended in at least one way. The special factors are those which can be extended in more than one way.

Let us now introduce a notation which will make the formulation of the notion of branching point easier.

Notation. Let G be an oriented graph.

- If v is a vertex of G, we use $d^+(v)$ to denote the number of outgoing edges starting from v. We define $d^-(v)$ similarly as the number of incoming edges.
- If e is an edge of G, we call e^+ the vertex towards which e is directed (so that e is an incoming edge with respect to the vertex e^+).

We now make explicit the link between $s(n)$ and the cohomology of the nth Rauzy graph R_n.

PROPOSITION 6.14. *The rank of the cohomology $H^1(R_n;\mathbb{Z})$ of the nth Rauzy graph equals $s(n)+1$.*

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Proof. Let $n \in \mathbb{N}$. We call V the set of vertices of R_n and E the set of its edges. By construction Card $V = p(n)$. Every factor of w of length $n+1$ can be obtained by extending by one letter to the right a word of length n. That is, by choosing a vertex $v \in V$ and an outgoing edge from v. Plus, every such choice of vertex and edge provides a different subword of w (by construction of the Rauzy graphs). Therefore, we can write

$$p(n+1) = \sum_{v \in V} d^+(v) = \text{Card } E.$$

Furthermore, it is classical for connected graphs that

$$\text{rk } H^1(R_n; \mathbb{Z}) = \text{Card } E - \text{Card } V + 1.$$

We deduce that

$$\text{rk } H^1(R_n; \mathbb{Z}) = p(n+1) - p(n) + 1 = s(n) + 1. \qquad \square$$

We now state a key lemma which links cohomology in the Rauzy graphs and in their inverse limit.

LEMMA 6.15. *We assume that there exists $M \in \mathbb{N}$ such that the rank of the cohomology of infinitely many Rauzy graphs of the word w is bounded by M. Then the Čech cohomology over $\mathbb{Q}$ of the suspension space of w is generated by at most M elements.*

Proof. We know that the suspension space of w is homeomorphic to the inverse limit of the Rauzy graphs. This implies that the Čech cohomology of the suspension space is the direct limit of the cohomology of the graphs. Let $x_1, \ldots, x_{M+1}$ be $M+1$ elements in the cohomology of the limit. We call $\hat{x}_1, \ldots, \hat{x}_{M+1}$ lifts of the x_i, so that $\hat{x}_i$ is an element of the cohomology of the complex $R_{\varphi(i)}$. Let N be an upper bound for all of the $\varphi(i)$. We can assume that N is chosen in such a way that the cohomology of R_N is generated by less than M elements. We call $y_1, \ldots, y_{M+1}$ the images of the $\hat{x}_i$ in the cohomology of the graph R_N. The rational cohomology of R_N is a vector space of dimension at most M. Therefore, there exists a linear relation between the $(y_i)_{i=1}^{M+1}$. Taking the image of this linear relation in the rational cohomology of the suspension space, we deduce that the x_i do not form a free family. $\square$

6.4. *Simple words have finitely generated cohomology.* We now prove that if a word is not too complex (that is, if its complexity function $p(n)$ is bounded by Cn with C a constant), then the cohomology of its associated subshift is rationally finitely generated.

LEMMA 6.16. *Let w be a word over a finite alphabet, and p the corresponding complexity function. We assume that there exists a constant C such that for all n,*

$$p(n) \leq Cn.$$

Then the function s does not tend to infinity.

Actually, a much stronger result was proved by Cassaigne: the function s is bounded (under some assumptions on the word, see [**Cas**] for this result). However, we just need this weaker result to prove Proposition 6.17.

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Proof. It is enough to remark that $p(n+1) = p(n) + s(n)$, so that $p(n) = \sum_{k=0}^{n-1} s(k)$. Assume that $s(k)$ is greater than $M+1$ for all k large enough. Then $p(n)$ is greater than Mn for all n large enough. If $s(k)$ tends to infinity, M can be chosen as large as one wants, which implies that p is not dominated by a linear function. □

With this, Theorem 6.10, and the results of §6.3, we can prove the following proposition.

PROPOSITION 6.17. *Let w be a bi-infinite word over a finite alphabet. Let p be the complexity function associated with w. We assume that $p(n) \leq Cn$ for all n large enough. Then the first rational cohomology group of the associated suspension space is finitely generated.*

Proof. By Theorem 6.10, the suspension space of w is the inverse limit of its Rauzy graphs. This implies that its first cohomology group is the direct limit of the first cohomology groups of the complexes. However, $s(n)$ does not tend to infinity, by Lemma 6.16. Therefore, by Proposition 6.14, there exists an $M \in \mathbb{N}$ such that the first cohomology group of infinitely many Rauzy graphs is generated by less than M elements. Finally, Lemma 6.15 allows us to conclude that the first cohomology group of the suspension space associated with w is finitely generated (over $\mathbb{Q}$). □

6.5. *Complex tilings can have an homologically simple tiling space.*

Notation. Let f be a finite word over the finite alphabet $\mathcal{A}$. We use $|f|$ to denote the length of f.

We now want to give an example of a minimal, recurrent word, which has an exponential complexity, and such that its associated tiling space has finitely generated cohomology over $\mathbb{Q}$. The idea is to find a word of high complexity, but such that infinitely many of its Rauzy graphs are homotopic to a wedge of two circles, and conclude thanks to Lemma 6.15.

This example was suggested to me by Julien Cassaigne, during a discussion we had when he came to Lyon in 2007.

Let $\{a, b\}$ be an alphabet of two letters. We choose a finite word in which all factors of length two over $\{a, b\}$ appear, except aa. We call f_1 this word. We define $u_1 = aabf_1baa$ and $v_1 = abbf_1baa$.

Assuming we constructed an f_k, v_k and u_k of same length, let us construct f_{k+1}, v_{k+1} and u_{k+1} by induction. We define a word f_{k+1} over the alphabet $\{0, 1\}$ in the following way: f_{k+1} is a finite word where all factors of length $|u_k|$ without 00 over the alphabet $\{0, 1\}$ appear. Then, apply the following substitution on f_{k+1}:

$$0 \mapsto u_k;$$
$$1 \mapsto v_k.$$

We still call f_{k+1} the word we obtained after substitution. It is now a word over $\{a, b\}$. Then, we define

$$u_{k+1} = u_k u_k v_k f_k v_k u_k u_k;$$
$$v_{k+1} = u_k v_k v_k f_k v_k u_k u_k.$$

Then, u_{k+1} and v_{k+1} have the same length.

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For all k, the word u_k appears in f_k (at various places), and so it appears in the middle of u_{k+1}. By making choices of the position of u_k in f_k for all k, we can form a sequence of finite words over $\{a, b\}$, increasing on both ends. Such a sequence is not unique, as u_k appears many times in f_k, and so many choices can be made. We can take a limit to obtain a bi-infinite word over $\{a, b\}$. We choose a word given by this method, and we call it w.

As an aside, we remark that one can prove that w is uniformly repetitive, that is, for all finite factors f of w, there exists a $N \in \mathbb{N}$ such that an occurrence of f appears in every subword of length N of w.

PROPOSITION 6.18. *Let p be the complexity function of w. Then $p(n)$ is not dominated by any polynomial function in n.*

Proof. Let p' be the complexity function of the subshift of $\{a, b\}$ without aa. This subshifts contains $\{bab, bbb\}^{\mathbb{Z}}$, so we have $p'(3n) \geq 2^n$, which means that p' is exponential.

Let us now compare p and p'. By definition, $p(|u_n|^2) \geq p'(|u_n|) \geq 2^{|u_n|/3}$. Therefore, for some values of $N \in \mathbb{N}$, $p(N) \geq 2^{\sqrt{N}/3}$. It proves that infinitely often, p is greater than any polynomial function. □

Now, let us prove that the Rauzy graphs of w are infinitely often homotopic to a wedge of two circles.

LEMMA 6.19. *For infinitely many $N \in \mathbb{N}$, there is only one right special factor of length N in w (i.e. a factor which can be extended in more than one way on the right).*

Proof. Remark that w has a very specific structure: one can uniquely rewrite w as a word over the alphabet $\{u_1, v_1\}$. To do so, mark each occurrence of $aaaa$ or $baaab$, and cut it by adding a dot: $aaaa \to aa.aa$ and $baaab \to baa.ab$. We then see w as a concatenation of blocks, these blocks being precisely u_1 and v_1. By this method, we can rewrite w as a word over the alphabet $\{u_1, v_1\}$ (seen as symbols rather than words).

Similarly, we can rewrite w uniquely as a word over $\{u_k, v_k\}$ for all k. We denote this rewriting by $\varphi_k(w)$.

Consider now $N = |u_{k+1}| + 4\,|u_k|$. A factor f of w is right-special if it can be extended in two ways on the right. Given f of length N, it can be rewritten as a word over the symbols $\{u_k, v_k\}$, apart perhaps from a prefix and a suffix, each of which is of length at most $2\,|u_k|$ (the length is taken over $\{a, b\}$). These prefix and suffix can possibly not be recognized, which means it cannot be determined whether they belong to u_k or v_k. Still, there is in f at least one occurrence of one of the following patterns: $(u_k)^4$ or $v_k(u_k)^3v_k$. The occurrence of either one of these patterns allows to recognize that f is a subfactor of xu_{k+1} or xv_{k+1}, respectively, where $x \in \{u_{k+1}, v_{k+1}\}$. Therefore, in most cases, it allows us to predict how f can be extended on the right, and the extension is unique. We know that u_i and v_i begin by u_{i-1} for all i, and u_1, v_1 begin by a. So the only case when the extension is not unique is when f contains $xu_ku_{k-1}\ldots u_1a$ as a right factor, with $x \in \{u_{k+1}, v_{k+1}\}$.

- If the next letter is a, then the aa at the end of fa will determine that the unique extension of fa is $x_{k+1}u_ku_{k-1}\ldots u_1u_1$. This will imply (by induction), that the unique extension of fa is $x_{k+1}u_{k+1}$.
- Similarly, if the next letter is b, the unique extension of fb will be $x_{k+1}v_{k+1}$.

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We have to prove that a factor of length N which ends by $x_{k+1}u_k u_{k-1} \dots u_1 a$ is unique: indeed, looking at the form of v_{k+1} and u_{k+1}, the finite word f is a factor of $v_k(u_k)^2 x_{k+1}u_k \dots u_1 a$, which is of length greater than N.

Therefore, there exists exactly one right-special word for this value of N. □

We remarked that in the case of a two-letter alphabet, the number of right-special words was equal to $s(n)$. Therefore, the lemma proves (thanks to Proposition 6.14) that infinitely often, the Rauzy graph of w has an homology of rank two. This means that it is homotopic to a wedge of two circles. Application of Proposition 6.14 and Lemma 6.15 yields the following result.

PROPOSITION 6.20. *The suspension space associated with the word w has finitely generated cohomology over $\mathbb{Q}$.*

This last example proves that cut-and-projection tiling spaces, unlike other tiling spaces, have some kind of rigidity which links complexity and cohomology.

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