East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.010116.220516a

Vol. **6**, No. 3, pp. 314-336
August 2016

Distribution of Discrete Time Delta-Hedging Error via a Recursive Relation

Minseok Park[1,*], Kyungsub Lee[2] and Geon Ho Choe[1]

[1] *Department of Mathematical Sciences, KAIST, Daejeon, Republic of Korea.*
[2] *Department of Statistics, Yeungnam University, Gyeongsan, Republic of Korea.*

Received 1 January 2016; Accepted (in revised version) 22 May 2016.

Abstract. We introduce a new method to compute the approximate distribution of the Delta-hedging error for a path-dependent option, and calculate its value over various strike prices via a recursive relation and numerical integration. Including geometric Brownian motion and Merton's jump diffusion model, we obtain the approximate distribution of the Delta-hedging error by differentiating its price with respect to the strike price. The distribution from Monte Carlo simulation is compared with that obtained by our method.

AMS subject classifications: 91G20,91G60

Key words: Delta-hedging errors, profit and loss distribution, discrete trading, jump-diffusion model, transaction cost.

1. Introduction

In 1973 Black, Scholes & Merton [3, 10, 11] derived their famous equation for the price of a financial derivative. By calculating the price of replicating a portfolio under several assumptions — including continuous trading, no transaction cost, perfect dynamic hedging and stock price movement with geometric Brownian motion — an option trader can eliminate the randomness of the option price. Thus a trader can continuously replicate the option by adjusting a portfolio with an appropriate amount of stock and risk-free assets to remove its Delta-risk (the randomness caused by changes in the stock price), and this strategy is called 'Delta-hedging'.

However, historical data and many empirical results show that the Black–Scholes framework is imperfect and needs adjustment. For example, in practice there are transaction costs, we are not allowed to trade continuously, model parameters are unknown and seem to vary, and stock price movements are discontinuous. Thus perfect replication of a financial derivative is impossible, and there are errors caused by incomplete replication. Delta-hedging error, and the profit and loss (P&L) from imperfect Delta-hedging, has been studied

*Corresponding author. *Email addresses:* minseok.park@kaist.ac.kr (M. Park), ksublee@yu.ac.kr (K. Lee), choe@euclid.kaist.ac.kr (G. H. Choe)

 ©2016 Global-Science Press

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

extensively by many researchers under various conditions — e.g. unknown model parameters or discontinuous trading. Several authors [2,7,8] showed that the discrete time Delta-hedging error is proportional to $1/\sqrt{N}$, where N is the hedging frequency with equally distributed time intervals, and proposed a weak convergence of the Delta-hedging error. A more general convergence of the Delta-hedging error was considered by Tankov [15], and Bakshi & Kapadia [1] claimed that under Heston's stochastic volatility model the Delta-hedging error is driven by the option Vega and volatility risk premium. Cont *et al.* [6] and Mina [12] claim it is possible to hedge the Delta-hedging error with risk-free assets and additional options with the same underlying asset. Recently, Sepp [14] derived an approximated distribution of the Delta-hedging error using a characteristic function, which he found is approximated by the chi-squared distribution.

Option replicating, pricing and hedging with a transaction cost have been studied [4,9, 13,16]. Boyle &Vorst [4] discussed option replication in a discrete-time framework with a transaction cost under a binomial option pricing model. Leland [9] claims that the volatility should change when the option price and its Greek parameters are calculated, if there is a transaction cost in the Black–Scholes framework. He also suggests a modified replicating strategy and Delta-hedging error under a discrete time framework with a transaction cost, and Toft [16] extended this to a derivation of a closed form for the expected Delta-hedging error.

We propose a new method to estimate the distribution of the Delta-hedging error with discrete time trading and transaction cost. In Section 2, we summarise the Black–Scholes framework with geometric Brownian motion and Merton's jump diffusion model, and state some basic facts about Delta-hedging strategy. In Section 3, we derive a recursive relation for option pricing with underlying P&L Delta-hedging error, and obtain its distribution by differentiating the option price twice with respect to the strike price. Numerical simulations presented in Section 4 are compared with distributions obtained by the Monte Carlo method, and our conclusions follow in Section 5.

2. Preliminaries

2.1. Black–Scholes framework

We assume that there exists a risk-neutral measure $\mathbb{Q}$ such that the discounted asset process $e^{-rt}S_t$ is $\mathbb{Q}$-martingale, where r is a risk-free rate. We say that the stock price process follows geometric Brownian motion (GBM) if S_t satisfies the stochastic differential equation (SDE)

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t \,, \tag{2.1}$$

where μ denotes the drift rate and σ is the volatility. From Ito's lemma, Eq. (2.1) may be rewritten

$$\mathrm{d}(\log S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W_t \,,$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

such that under some conditions the price $V = V(t, S_t)$ of a financial derivative satisfies

$$rV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} . \tag{2.2}$$

Conventionally, Greek letters are used to denote some of the partial derivatives of option prices — e.g.

$$\Delta = \frac{\partial V}{\partial S}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2}, \quad \rho = \frac{\partial V}{\partial r}, \quad \Theta = \frac{\partial V}{\partial t}, \quad \text{Vega} = \frac{\partial V}{\partial \sigma} ;$$

and consequently, Eq. (2.2) can be rewritten as

$$rV = \Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta . \tag{2.3}$$

Specifically, with the boundary condition $C(T, S_T) = \max(S_T - K, 0)$ where K is the strike price and the maturity is T, a European call option price at time t is

$$C(t,S) = S_t N(d_1) - K\mathrm{e}^{-r(T-t)} N(d_2) ,$$

where

$$d_1 = \frac{\log S_t/K + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} ,$$
$$d_2 = d_1 - \sigma\sqrt{T-t} ,$$

and N is the cumulative distribution function of the standard normal distribution. We can then readily calculate the Greek parameters as defined above for a European call option.

2.2. Jump-diffusion model

In Merton's jump diffusion model (JDM), the stock price S_t satisfies the SDE

$$\mathrm{d}S_t = (\mu - \lambda\beta)S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t + (\mathrm{e}^J - 1)S_t \mathrm{d}N_t , \tag{2.4}$$

where μ is the drift rate, σ the volatility conditional on no arrivals of some important event, W_t defines a Brownian motion, and N_t is a Poisson process with deterministic intensity λ and $\beta = \mathbb{E}[\mathrm{e}^J - 1]$. In this model, the jump size of the logarithmic return J follows a normal distribution with mean ν and variance δ^2 and the N_t, J, W_t are independent. It is easy to see that

$$\beta = \mathbb{E}[\mathrm{e}^J - 1] = \mathrm{e}^{\nu + \frac{1}{2}\delta^2} - 1 ,$$

and that Eq. (2.4) can be rewritten as

$$\frac{\mathrm{d}S_t}{S_t} = \begin{cases} (\mu - \lambda\beta)\mathrm{d}t + \sigma \mathrm{d}W_t & \text{if the Poisson event does not occur} , \\ (\mu - \lambda\beta)\mathrm{d}t + \sigma \mathrm{d}W_t + (\mathrm{e}^J - 1) & \text{if the Poisson event occurs} . \end{cases}$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

Moreover, from Ito's lemma Eq. (2.4) may be written

$$\mathrm{d}\log S_t = \left(\mu - \frac{1}{2}\sigma^2 - \lambda\beta\right)\mathrm{d}t + \sigma\mathrm{d}W_t + J\mathrm{d}N(t)\,,$$

so that

$$\log\frac{S_t}{S_0} = \left(\mu - \frac{1}{2}\sigma^2 - \lambda\beta\right)t + \sigma W_t + \sum_{k=1}^{N_t} J_k\,. \tag{2.5}$$

The mean and variance of $\log(S_t/S_0)$ are

$$\begin{aligned}\text{mean} &= \left(\mu - \frac{1}{2}\sigma^2 - \lambda\beta + \lambda\nu\right)t,\\ \text{variance} &= \left(\sigma^2 + \lambda(\nu^2 + \delta^2)\right)t,\end{aligned}$$

respectively, and the transition probability of $\log(S_t/S_0)$ obtained by an inverse Fourier transform has the characteristic function described in Appendix A.

However, under a risk-neutral measure $\mathbb{Q}$ we can consider the drift rate of the JDM as risk-free, so that the discounted stock price $\mathrm{e}^{-rt}S_t$ is a $\mathbb{Q}$-martingale — i.e. $\mu = r$ and

$$\frac{\mathrm{d}S_t}{S_t} = (r - \lambda\beta)\mathrm{d}t + \sigma\mathrm{d}X_t + (\mathrm{e}^J - 1)\mathrm{d}N_t\,,$$

where X_t is a $\mathbb{Q}$-Brownian motion. Further, under the JDM, the option price $V(t,S_t)$ satisfies Merton's differential equation

$$rV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r - \lambda\beta)S\frac{\partial V}{\partial S} + \lambda\mathbb{E}[V(t,Se^J) - V(t,S)]\,, \tag{2.6}$$

so $V(t,S_t)$ is given by the weighted sum of solution of the Black–Scholes equation — i.e.

$$V(t,S_t) = \sum_{k=0}^{\infty}\frac{\mathrm{e}^{-\lambda'\tau}(\lambda'\tau)^k}{k!}V^{BS}(t,S_t,\tau,r_k,\sigma_k)\,, \tag{2.7}$$

where

$$\begin{aligned}\tau &= T - t\,,\\ \lambda' &= \lambda(1+\beta)\,,\\ r_k &= r - \lambda\beta + \frac{k(\nu + \frac{1}{2}\delta^2)}{\tau}\,,\\ \sigma_k^2 &= \sigma^2 + \frac{k\delta^2}{\tau}\,,\end{aligned}$$

and $V^{BS}(t,S_t,\tau,r_k,\sigma_k)$ is an option price under the Black–Scholes framework with the risk-free interest rate r_k and volatility σ_k. For a detailed proof, see Appendix B.

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

Similarly, we obtain the Greek parameters of an option under the JDM as a weighted sum of the Greek parameters under the Black–Scholes framework with adjustments. For example, for an option under the JDM we have

$$\Gamma^{JDM} = \sum_{k=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^k}{k!} \Gamma^{BS}(t, S_t, \tau, r_k, \sigma_k) ,$$

where $\Gamma^{BS}(t, S_t, \tau, r_k, \sigma_k)$ is the option Gamma under the Black–Scholes framework.

2.3. Delta-hedging error

Consider an option trader who sells a European option V and hedges its δ. Let $V_t = V(t, S_t)$ be a option price at time t with stock price S_t, and Π_t be a Delta-hedging portfolio at time t such that

$$\Pi_t = \Delta_t S_t - V_t ,$$

where $\Delta_t = \partial V(t, S_t)/\partial S$ and thus the δ of this portfolio is zero — i.e., $\partial \Pi/\partial S = 0$. Now suppose that $\Delta_t = \Delta(t, S_t)$ is fixed on the time interval $[t, t + \delta t)$ and adjust its δ at time $t + \delta t$. The Delta-hedging error on time interval $[t, t + \delta t)$ then becomes

$$\begin{aligned} \Pi_{t+\delta t-} - \Pi_t &= \Delta_t S_{t+\delta t} - V_{t+\delta t} - (1 + r\delta t)(\Delta_t S_t - V_t) \\ &= \Delta_t \delta S_t - \delta V_t - r\delta t(\Delta_t S_t - V_t) , \end{aligned} \tag{2.8}$$

where $\delta S_t = S_{t+\delta t} - S_t$ and $\delta V_t = V_{t+\delta t} - V_t$. By Ito's lemma, we can approximate δV by

$$\begin{aligned} \delta V_t =& V_{t+\delta t} - V_t \approx \frac{\partial V}{\partial t}(t, S_t)\delta t + \frac{\partial V}{\partial S}(t, S_t)\delta S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)(\delta S)^2 \\ =& \Theta_t \delta t + \Delta_t \delta S_t + \frac{1}{2}\Gamma_t (\delta S_t)^2 , \end{aligned} \tag{2.9}$$

where $\Theta_t = \partial V(t, S_t)/\partial t$ and $\Gamma_t = \partial^2 V(t, S_t)/\partial S^2$. It is notable that Eq. (2.3) implies

$$\Theta_t = rV_t - \frac{1}{2}\sigma^2 S_t^2 \Gamma_t - rS_t\Delta_t . \tag{2.10}$$

Combining Eqs. (2.8), (2.9) and (2.10), we therefore obtain

$$\begin{aligned} \Pi_{t+\delta t-} - \Pi_t &= \Delta_t \delta S_t - \delta V_t - r\delta t(\Delta_t S_t - V_t) \\ &\approx \Delta_t \delta S_t - \left(\Theta_t \delta t + \Delta_t \delta S_t + \frac{1}{2}\Gamma_t (\delta S_t)^2\right) - r\delta t(\Delta_t S_t - V_t) \\ &= -\Theta_t \delta t - \frac{1}{2}\Gamma_t (\delta S_t)^2 - r\delta t(\Delta_t S_t - V_t) \\ &= \left(\frac{1}{2}\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t - rV_t\right)\delta t - \frac{1}{2}\Gamma_t (\delta S_t)^2 - r\delta t(\Delta_t S_t - V_t) \\ &= \frac{1}{2}\sigma^2 S_t^2 \Gamma_t \delta t - \frac{1}{2}\Gamma_t (\delta S_t)^2 , \end{aligned}$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

so we may write

$$\Pi_{t+\delta t-} - \Pi_t \approx \frac{1}{2} S_t^2 \Gamma_t \left\{ \sigma^2 \delta t - \left(\frac{\delta S_t}{S_t} \right)^2 \right\}. \tag{2.11}$$

Thus even if the stock price follows a GBM and the option price satisfies the Black–Scholes equation, risk remains since the Delta-hedging portfolio is rebalanced discretely. Since Γ is always positive for a European option, the option seller who hedges its Δ discretely will lose money if the realised volatility is larger than the volatility used for calculating the Δ.

2.4. Transaction cost

In a real market, important differences from the Black–Scholes framework are that there is not continuous trading and there is a transaction cost. If there were no transaction cost, we could reduce the risk from the discrete Delta-hedge by increasing the frequency of portfolio adjustment. However, even if the trading is assumed to be continuous, many portfolio adjustments generate large losses due to the nonzero transaction cost as follows.

Consider the time intervals $0 = t_0 < t_1 < \cdots < t_N = T$. For convenience, let S_n, Γ_n and Δ_n denote S_{t_n}, Γ_{t_n} and Δ_{t_n}, respectively. Suppose that the portfolio at time t_n is $\Delta_n S_n - V(t_n, S_n)$ and we rebalance this portfolio to $\Delta_{n+1} S_{n+1} - V(t_{n+1}, S_{n+1})$ at t_{n+1} — i.e. the holding assets number Δ_n at time t_n and Δ_{n+1} at time t_{n+1}. Let us assume that transaction cost is proportional to $|\Delta_n - \Delta_{n+1}|$ and $S(n+1)$ — i.e.

$$\text{transaction cost on } [t_n, t_{n+1}] = \alpha S_{n+1} |\Delta_n - \Delta_{n+1}|$$

for some constant α. Since $\Delta = \partial V / \partial S$, we have

$$\begin{aligned} \alpha S_{n+1} |\Delta_n - \Delta_{n+1}| &\sim \alpha S_{n+1} |\Gamma_n \delta S_n| \\ &= \alpha S_n^2 \Gamma_n \left| \left(\frac{S_{n+1}}{S_n} - 1 \right) \frac{S_{n+1}}{S_n} \right| \\ &= \alpha S_n^2 \Gamma_n \frac{|\delta S_n|}{S_n} \left(1 + \frac{\delta S_n}{S_n} \right). \end{aligned} \tag{2.12}$$

(It is notable that transaction cost is also driven by the option Γ and the realised variance.)

3. Profit and Loss Distribution Estimation

Even if we may reduce our risk from the Delta-hedging error by holding some derivatives on variance, we cannot determine whether we should bet on higher volatility or lower volatility than expected, so we need to predict the Delta-hedging error. Many researchers have studied the distribution of the discrete Delta-hedging error with various methods under various assumptions, and it is well known that the distribution corresponds to the second derivative if the price of European call option with respect to the strike price. In this section, we view the Delta-hedging error as a path-dependent contingent claim, and find the European call option price with various strike prices to estimate its distribution.

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

3.1. Setting

We consider an option trader who sells a European style option V with maturity T, and suppose the trader tries to hedge its δ discretely at times $0 = t_0 < t_1 < \cdots < t_N = T$. Suppose that the interest rate $r(t)$, drift rate $\mu(t)$ and volatility $\sigma(t)$ are known deterministic functions of t, and that the stock price process S_t follows a GBM

$$\mathrm{d}S_t = \mu(t)S_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}W_t \ .$$

Since the option trader changes the portfolio at each t_n for $n = 0, 1, \cdots, N$ such that its Δ is zero, we can assume that the number of assets in the portfolio is constant from time t_n and $t_{n+1}-$ and changes at t_{n+1} for all $n = 0, 1, \cdots, N-1$.

In the remainder of this article, we adopt the notation

$$\begin{aligned} r(s:t) &:= \int_s^t r(t)\mathrm{d}t \ , \\ \mu(s:t) &:= \int_s^t \mu(t)\mathrm{d}t \ , \\ \sigma^2(s:t) &:= \int_s^t \sigma^2(t)\mathrm{d}t \ . \end{aligned}$$

Moreover, we let Y_n denote a Delta-hedging error over the time interval $[t_{n-1}, t_n]$, and define

$$Y = \sum_{n=1}^{N} Y_n \mathrm{e}^{r(t_n:T)}$$

as the total Delta-hedging error on time interval $[t_0, t_N] = [0, T]$.

Our goal in this section is to find $\mathbb{E}[(Y-\kappa)^+]$ for various values of κ under the Black–Scholes framework and Merton's jump diffusion model. For this purpose, we define a function

$$\begin{aligned} g_N^{(n)}(S_0, S_1, \cdots, S_n, K, \kappa) &= \mathbb{E}\left[\left(\sum_{n=1}^{N} Y_n \mathrm{e}^{r(t_n:T)} - \kappa\right)^+ \middle| \mathscr{F}_n\right] \\ &= \mathbb{E}[(Y-\kappa)^+ | \mathscr{F}_n] \end{aligned}$$

denoting the expectation of $(Y-\kappa)^+$ conditional on $\mathscr{F}_n$ (which means $S_0, S_1, \cdots, S_n$ are given), where Y is the total Delta-hedging error on the time interval $[t_0, t_N]$. For example, $g_3^{(1)}$ is a call option price of Y at time t_1, where Y is a total Delta-hedging error on the time interval $[t_0, t_3]$ with 3 time steps.

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

3.2. Recursive relation under the GBM

The Delta-hedging error between time t_n and t_{n+1} is

$$\begin{aligned}\Pi_{t_{n+1}-}-\Pi_{t_n} &\approx \frac{1}{2}S_n^2\Gamma_n\left\{\sigma^2(t_n:t_{n+1})-\left(\frac{S_{n+1}-S_n}{S_n}\right)^2\right\}\\ &= \frac{1}{2}S_n^2\Gamma_n\left\{\sigma^2(t_n:t_{n+1})-\left(\frac{S_{n+1}}{S_n}-1\right)^2\right\},\end{aligned} \tag{3.1}$$

and the error caused by discrete Delta-hedging consists of two parts — i.e.

$$\text{Delta-hedging error} = \text{Error from portfolio} - \text{Transaction cost}.$$

Thus from Eqs. (3.1) and (2.12), the Delta-hedging error Y_n over the time interval $[t_{n-1}, t_n]$ under the geometric Brownian motion model is

$$\begin{aligned}Y_n \approx& \frac{1}{2}S_{n-1}^2\Gamma_{n-1}\left\{\sigma^2(t_{n-1}:t_n)-\left(\frac{S_n}{S_{n-1}}-1\right)^2\right\}-\alpha S_{n-1}^2\Gamma_{n-1}\left|\left(\frac{S_n}{S_{n-1}}-1\right)\frac{S_n}{S_{n-1}}\right|\\ =& \frac{1}{2}S_{n-1}^2\Gamma_{n-1}\left\{\sigma^2(t_{n-1}:t_n)-\left(\frac{S_n}{S_{n-1}}-1\right)^2-2\alpha\left|\frac{S_n}{S_{n-1}}-1\right|\frac{S_n}{S_{n-1}}\right\},\end{aligned}$$

where $\sigma^2(t_{n-1}:t_n)$ is the expected variance and $(S_n/S_{n-1}-1)^2$ is the realised variance on $[t_{n-1}, t_n]$.

For simplicity, let $s_n = S_n/S_{n-1}$ such that s_n follows a log-normal distribution with initial value 1 and

$$Y_n = \frac{1}{2}S_{n-1}^2\Gamma_{n-1}\left\{\sigma^2(t_{n-1}:t_n)-(s_n-1)^2-2\alpha|(s_n-1)s_n|\right\}.$$

We can now derive a recursive relation to find the call option price with an underlying Y, and thus obtain the distribution of the Delta-hedging error with the stock price under the geometric Brownian motion model (GBM).

Theorem 3.1. *We have the recursive relation*

$$g_{N-n}^{(0)}(S_n, K, \kappa_n) = \mathbb{E}\left[g_{N-n-1}^{(0)}(S_{n+1}, K, \kappa_{n+1})|\mathscr{F}_n\right] \tag{3.2}$$

for $n = 0, 1, \cdots, N-2$ *and*

$$g_1^{(0)}(S_{N-1}, K, \kappa) = \mathbb{E}\left[(Y_N-\kappa)^+\,\middle|\,\mathscr{F}_{N-1}\right]$$

for $n = N-1$, *where*

$$\begin{aligned}Y_n &= \frac{1}{2}S_{n-1}^2\Gamma_{n-1}\left\{\sigma^2(t_{n-1}:t_n)-\left(\frac{S_n}{S_{n-1}}-1\right)^2-2\alpha\left|\frac{S_n}{S_{n-1}}-1\right|\frac{S_n}{S_{n-1}}\right\},\\ \kappa_n &= \kappa-\sum_{i=1}^{n}Y_i e^{r(t_i:T)},\end{aligned}$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

and

$$f^{DH}(\kappa) = \frac{\partial^2 g_N^{(0)}(S_0, K, \kappa)}{\partial \kappa^2}$$

approximates the probability distribution of the Delta-hedging error in the Black–Scholes framework under GBM.

Proof. See Appendix C.

For simplicity, we may assume all of the parameters are constant and that the t_n's are equally spaced in $[0, T]$, to render a simpler form of Theorem 3.1.

Lemma 3.1. *Let r be a constant risk-free interest rate and assume the stock price follows a GBM with constant drift rate μ and constant volatility σ, and that t_n's are equally spaced in $[0, T]$ with $\delta t = T/N$ — i.e. $t_n = n\delta t$. Then*

$$g_{N-n}^{(0)}(S_n, K, \kappa) = \mathbb{E}\left[g_{N-n-1}^{(0)}\left(S_n X, K, \kappa - y_n(X) e^{r(T-t_{n+1})}\right)\right] \tag{3.3}$$

for $n = 0, 1, \cdots, N-2$ where

$$y_n(X) = \frac{1}{2} S_n^2 \Gamma_n(S_n, K)\left\{\sigma^2 \delta t - (X-1)^2 - 2\alpha |X-1| x\right\}$$

and

$$g_1^{(0)}(S_{N-1}, K, \kappa) = \mathbb{E}\left[\left(\frac{1}{2} S_{N-1}^2 \Gamma_{N-1}\left\{\sigma^2 \delta t - (X-1)^2 - 2\alpha |X-1| X\right\} - \kappa\right)^+\right]$$

for $n = N-1$, where $\log X \sim N((\mu - \frac{1}{2}\sigma^2)\delta t, \sigma^2 \delta t)$.

In order to calculate the expectation in Eqs. (3.2) and (3.3), we need to find the values of g with sufficiently wide ranges of x and κ at each time step. However,

$$g_{N-n}^{(0)}(S_n, K, \kappa_n) = S_n g_{N-n}^{(0)}\left(1, \frac{K}{S_n}, \frac{\kappa_n}{S_n}\right)$$

for any n, and using this property we proceed to find $g_N^{(0)}(1, K, \kappa)$ for various values of K and κ to allow a much simpler numerical calculation.

Lemma 3.2. *Let $\Gamma_n(1, K/S_n)$ be the Gamma of a European option with initial stock price 1 and strike price K/S_n at time t_n. Under the conditions in Lemma 3.1,*

$$g_{N-n}^{(0)}(1, K, \kappa) = \mathbb{E}\left[X g_{N-n-1}^{(0)}\left(1, \frac{K}{X}, \frac{\kappa - y_n(X) e^{r(T-t_{n+1})}}{X}\right)\right] \tag{3.4}$$

for $n = 0, 1, \cdots, N-2$, where

$$y_n(X) = \frac{1}{2}\Gamma_n\left(1, \frac{K}{S_n}\right)\left\{\sigma^2 \delta t - (X-1)^2 - 2\alpha |X-1| X\right\}.$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

For $n = N-1$,

$$g_1^{(0)}(1,K,\kappa) = \mathbb{E}\left[\left(\frac{1}{2}\Gamma_{N-1}(1,K)\left\{\sigma^2\delta t - (X-1)^2 - 2\alpha|X-1|X\right\} - \kappa\right)^+\right],$$

where $\log X \sim N((\mu - \frac{1}{2}\sigma^2)\delta t, \sigma^2\delta t)$.

In Lemmas 3.1 and 3.2, it is notable that

$$f^{DH}(\kappa) = \frac{\partial^2 g_N^{(0)}(S_0,K,\kappa)}{\partial\kappa^2}$$

approximates the probability distribution of the Delta-hedging error under the Black–Scholes framework with the GBM, where the seller of a European option with strike price K hedges its Delta for $t_0, t_1, \cdots, t_N$ where $t_n = n\delta t$.

3.3. Recursive relation under the JDM

Let us now estimate the distribution of Delta-hedging error under Merton's jump diffusion model. First, we find the jump diffusion model (JDM) version of Eq. (2.11), noting that Merton's differential equation (2.6) can be rewritten

$$\Theta_t = rV_t - \frac{1}{2}\sigma^2 S_t^2\Gamma_t - (r-\lambda\beta)S_t\Delta_t - \lambda\mathbb{E}\left[V(t,S_t e^J) - V(t,S_t)\right].$$

Using a similar argument as for Eq. (2.8) before, we have

$$\begin{aligned}
\Pi_{t+\delta t-} - \Pi_t =& \Delta_t\delta S_t - \delta V_t - r\delta t(\Delta_t S_t - V_t)\\
\approx& \Delta_t\delta S_t - (\Theta_t\delta t + \Delta_t\delta S_t + \frac{1}{2}\Gamma_t(\delta S_t)^2) - r\delta t(\Delta_t S_t - V_t)\\
=& -\Theta_t\delta t - \frac{1}{2}\Gamma_t(\delta S_t)^2 - r\delta t(\Delta_t S_t - V_t)\\
=& \left(\frac{1}{2}\sigma^2 S_t^2\Gamma_t + (r-\lambda\beta)S_t\Delta_t + \lambda\mathbb{E}[V(t,S_t e^J) - V(t,S_t)] - rV_t\right)\delta t\\
& - \frac{1}{2}\Gamma_t(\delta S_t)^2 - r\delta t(\Delta_t S_t - V_t)\\
=& \frac{1}{2}\sigma^2 S_t^2\Gamma_t\delta t - \frac{1}{2}\Gamma_t(\delta S_t)^2 + \left(\mathbb{E}[V(t,S_t e^J) - V(t,S_t)] - \beta S_t\Delta_t\right)\lambda\delta t.
\end{aligned}$$

With $\beta = \mathbb{E}[e^J - 1]$, we obtain

$$\begin{aligned}
\mathbb{E}[V(t,S_t e^J) - V(t,S_t)] - \beta S_t\Delta_t \approx& \mathbb{E}\left[\frac{\partial V}{\partial S}(S_t e^J - S_t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(S_t e^J - S_t)^2\right] - \beta S_t\Delta_t\\
=& \mathbb{E}\left[\Delta_t S_t(e^J - 1) + \frac{1}{2}\Gamma_t S_t^2(e^J - 1)^2\right] - \beta S_t\Delta_t\\
=& \Delta_t S_t\mathbb{E}[e^J - 1] + \frac{1}{2}\Gamma_t S_t^2\mathbb{E}[(e^J - 1)^2] - \beta S_t\Delta_t\\
=& \frac{1}{2}S_t^2\Gamma_t\mathbb{E}[(e^J - 1)^2].
\end{aligned}$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

Thus we can approximate the Delta-hedging error over $[t, t+\delta t)$ by

$$\Pi_{t+\delta t-} - \Pi_t = \frac{1}{2} S_t^2 \Gamma_t \left\{ \left(\lambda \mathbb{E}[(e^J - 1)^2] + \sigma^2 \right) \delta t - \left(\frac{\delta S_t}{S_t} \right)^2 \right\}, \tag{3.5}$$

where $\mathbb{E}[(e^J-1)^2] = \mathbb{E}[e^{2J} - 2e^J + 1] = e^{2\nu+2\delta^2} - 2(\beta+1) + 1 = e^{2\nu+2\delta^2} - 2e^{\nu+\frac{1}{2}\delta^2} + 1$. In passing, we note $(\lambda\mathbb{E}[(e^J-1)^2] + \sigma^2)\delta t$ is the expected variance and $(S_n/S_{n-1} - 1)^2$ is the realised variance on $[t_{n-1}, t_n]$.

Since the transaction cost part of the Delta-hedging error is independent of the model, we can estimate the distribution of the Delta-hedging error under the JDM by replacing the portfolio error term given in Eq. (2.11) with the result in Eq. (3.5).

Theorem 3.2. *Let*

$$g_N^{(n)}(S_0, S_1, \cdots, S_n, K, \kappa) = \mathbb{E}[(Y-\kappa)^+ | \mathscr{F}_n]$$

where

$$\begin{aligned} Y_n = \frac{1}{2} S_{n-1}^2 \Gamma_{n-1} \Bigg\{ & \lambda \mathbb{E}[(e^J-1)^2](t_n - t_{n-1}) + \sigma^2 (t_{n-1} : t_n) - \left(\frac{S_n}{S_{n-1}} - 1 \right)^2 \\ & - 2\alpha \left| \frac{S_n}{S_{n-1}} - 1 \right| \frac{S_n}{S_{n-1}} \Bigg\}, \\ \kappa_n = \kappa - \sum_{i=1}^{n} & Y_i e^{r(t_i:T)}. \end{aligned}$$

Then the following recursive relation holds:

$$g_{N-n}^{(0)}(S_n, K, \kappa_n) = \mathbb{E}\left[g_{N-n-1}^{(0)}(S_{n+1}, K, \kappa_{n+1}) | \mathscr{F}_n \right]$$

for $n = 0, 1, \cdots, N-2$, *and*

$$g_1^{(0)}(S_{N-1}, K, \kappa) = \mathbb{E}\left[(Y_N - \kappa)^+ \,\middle|\, \mathscr{F}_{N-1} \right].$$

Under the JDM,

$$f^{DH}(\kappa) = \frac{\partial^2 g_n^{(0)}(S_0, K, \kappa)}{\partial \kappa^2}$$

approximates the probability distribution of the Delta-hedging.

Proof. We proceed as in the proof of Theorem 3.1.

As in the GBM case, we obtain a simpler form of Theorem 3.1 with constant parameters and re-hedging on equally distributed time intervals over $[0, T]$.

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

Lemma 3.3. *Let r be a constant risk-free rate and assume the stock price follows the JDM satisfying Eq.* (2.4) *where all parameters are constant. Also assume that the t_n's are equally spaced over $[0,T]$ with $\delta t = T/N$ — i.e. $t_n = n\delta t$. Then*

$$g^{(0)}_{N-n}(1,K,\kappa) = \mathbb{E}\left[X\, g^{(0)}_{N-n-1}\left(1, \frac{K}{X}, \frac{\kappa - y_n(X)e^{r(T-t_{n+1})}}{X}\right)\right]$$

for $n = 0,1,\cdots,N-2$, where

$$X = \exp\left(\left(\mu - \frac{1}{2}\sigma^2 - \lambda\beta\right)\delta t + \sigma W_{\delta t} + \sum_{k=1}^{P_{\delta t}} J\right),$$
$$y_n(X) = \frac{1}{2}\Gamma_n\left(1, \frac{K}{S_n}\right)h(X),$$
$$h(X) = \left(\lambda\mathbb{E}[(e^J-1)^2] + \sigma^2\right)\delta t - (X-1)^2 - 2\alpha\,|X-1|X,$$

and $P_{\delta t}$ follows a Poisson distribution with parameter $\lambda\delta t$. For $n = N-1$,

$$g^{(0)}_1(1,K,\kappa) = \mathbb{E}\left[\left(\frac{1}{2}\Gamma_{N-1}(1,K)h(X) - \kappa\right)^+\right].$$

It is notable that

$$f^{DH}(\kappa) = \frac{\partial^2 g^{(0)}_N(S_0,K,\kappa)}{\partial\kappa^2}$$

approximates the probability distribution of the Delta-hedging error under the JDM.

In practice, there is no analytic form of the probability density function for X, so we take the inverse Fourier transform of its characteristic function numerically to simulate the random variable in the jump diffusion model.

4. Numerical Simulations

For simplicity, in our numerical simulations we took $\delta t = T/N$ and $t_n = n \times \delta t$ for $n = 0,1,\cdots,N$, so the t_i's produce an equally distributed partition over $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_N = T$, and all parameters to be constants (scaled per annum) over the time period $[0,T]$. In the Monte Carlo simulation undertaken for comparison, we computed the Delta-hedging error at time t_n as

$$\begin{aligned}\delta\Pi^{MC}_n =& \text{Error from portfolio} - \text{transaction cost}\\ =& \Delta_t\delta S_t - \delta V_t - r\delta t(\Delta_t S_t - V_t) - \alpha S_n|\Delta_n - \Delta_{n-1}|\,,\end{aligned}$$

and the total Delta-hedging error

$$\sum_{n=1}^{N}\delta\Pi^{MC}_n e^{r(T-n)}.$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

Table 1: Statistics and 95% VaR of distribution of Delta-hedging error under GBM obtained by Monte Carlo simulation and the recursive relation in Theorem 3.1 with $\delta t = 1/50$.

	MC simulation(%)	Recursive relation(%)
Mean	0.0248	1.1556
Variance	501.7	470.0
95% VaR	-36.00	-34.95

Table 2: Statistics and 95% VaR of distribution of Delta-hedging error under GBM obtained by Monte Carlo simulation and the recursive relation in Theorem 3.1 with $\delta t = 1/150$.

	MC simulation(%)	Recursive relation(%)
Mean	-0.0045	0.3966
Variance	172.8	166.0
95% VaR	-22.00	-22.13

Table 3: Statistics and 95% VaR of distribution of Delta-hedging error under GBM obtained by Monte Carlo simulation and the recursive relation in Theorem 3.1 with $\delta t = 1/252$.

	MC simulation(%)	Recursive relation(%)
Mean	-0.0091	0.2374
Variance	103.7	100.6
95% VaR	-18.00	-17.47

Table 4: Statistics and 95% VaR of distribution of Delta-hedging error under GBM with transaction cost coefficient $\alpha = 0.001$ obtained by Monte Carlo simulation and the recursive relation in Theorem 3.1 with $\delta t = 1/50$.

	MC simulation(%)	Recursive relation(%)
Mean	-5.6607	-3.7159
Variance	525.0	493.4
95% VaR	-46.00	-43.10

4.1. Geometric Brownian motion

We supposed the stock price follows a GBM with $r = \mu = 0$, $\sigma = 0.2$ and considered a trader who sells a European call option with a strike price 110% of the initial purchase price with maturity $T = 1$. In the following figures and tables, we present the distributions of the Delta-hedging errors with hedging frequencies $N = 50$ (weekly), $N = 150$ (three times a week) and $N = 252$ (daily), with a transaction cost in each case.

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

Table 5: Statistics and 95% VaR of distribution of Delta-hedging error under GBM with transaction cost coefficient $\alpha = 0.001$ obtained by Monte Carlo simulation and the recursive relation in Theorem 3.1 with $\delta t = 1/150$.

	MC simulation(%)	Recursive relation(%)
Mean	-9.2567	-8.0479
Variance	195.9	193.8
95% VaR	-36.00	-33.78

Table 6: Statistics and 95% VaR of distribution of Delta-hedging error under GBM with transaction cost coefficient $\alpha = 0.001$ obtained by Monte Carlo simulation and the recursive relation in Theorem 3.1 with $\delta t = 1/252$.

	MC simulation(%)	Recursive relation(%)
Mean	-5.6607	-3.7159
Variance	525.0	493.4
95% VaR	-48.00	-43.10

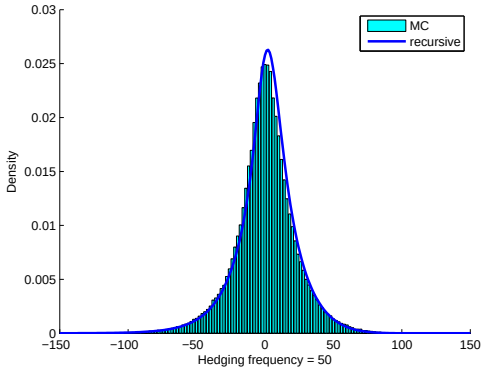


Figure 1: Distribution of Delta-hedging error.

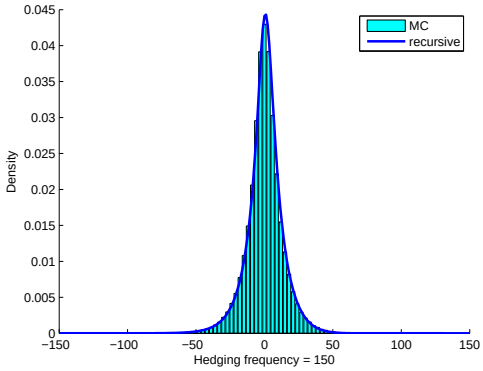


Figure 2: Distribution of Delta-hedging error.

4.2. Jump-diffusion model

The variance of the log return under the jump-diffusion model (JDM) over the time interval $[0, t]$ is given by

$$(\sigma_{\text{JDM}}^2 + \lambda(\nu^2 + \delta^2))t \,,$$

so we chose a different volatility value to make the JDM's realised variance equal to the realised variance of the GBM in subsection 4.1 — viz.

$$\sigma_{\text{JDM}} = \sqrt{\sigma_{BSM}^2 - \lambda(\nu^2 + \delta^2)} \,,$$

with the stock price following the JDM where $r = \mu = 0$, $\sigma = 0.1414$, $\lambda = 50$, $\nu = 0$ and $\delta = 0.02$. We again considered a trader who sells a European call option with the strike price 110% of the initial price and maturity at $T = 1$. The figures and tables present

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

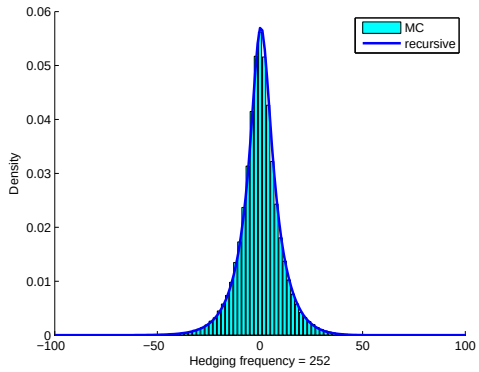


Figure 3: Distribution of Delta-hedging error.

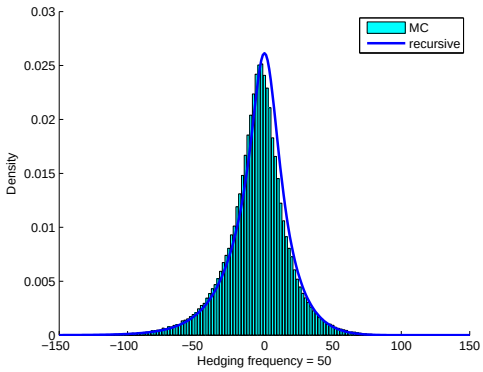


Figure 4: Distribution of Delta-hedging error.

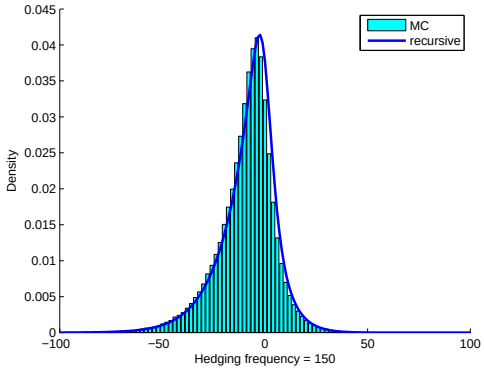


Figure 5: Distribution of Delta-hedging error

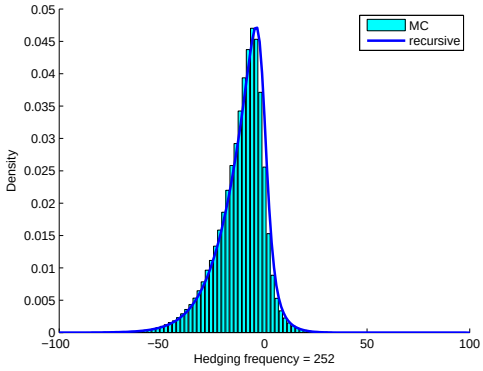


Figure 6: Distribution of Delta-hedging error.

Table 7: Statistics and 95% VaR of distribution of Delta-hedging error under JDM obtained by Monte Carlo simulation and the recursive relation in Theorem 3.2 with $\delta t = 1/50$.

	MC simulation(%)	Recursive relation(%)
Mean	-0.0057	1.5930
Variance	681.8	641.5
95% VaR	-45.00	-41.96

distributions of the Delta-hedging error with hedging frequencies $N = 50$ (weekly), $N = 150$ (three times a week), $N = 252$ (daily) and a transaction cost in each case.

In brief, we find that our method works well for the GBM and JDM whether or not there is a transaction cost, and the variance of the profit and loss (P&L) reduces as the hedging frequency increases. However, we must also note that a nonzero transaction cost may produce a loss if the trade is too frequent.

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

Table 8: Statistics and 95% VaR of distribution of Delta-hedging error under JDM obtained by Monte Carlo simulation and the recursive relation in Theorem 3.2 with $\delta t = 1/150$.

	MC simulation(%)	Recursive relation(%)
Mean	0.0384	0.8491
Variance	355.7	354.4
95% VaR	-35.00	-31.47

Table 9: Statistics and 95% VaR of distribution of Delta-hedging error under JDM obtained by Monte Carlo simulation and the recursive relation in Theorem 3.2 with $\delta t = 1/252$.

	MC simulation(%)	Recursive relation(%)
Mean	-0.0360	0.6969
Variance	293.2	292.9
95% VaR	-30.00	-27.97

Table 10: Statistics and 95% VaR of distribution of Delta-hedging error under JDM with transaction cost coefficient $\alpha = 0.001$ obtained by Monte Carlo simulation and the recursive relation in Theorem 3.2 with $\delta t = 1/50$.

	MC simulation(%)	Recursive relation(%)
Mean	-5.6099	-3.1434
Variance	707.1	671.6
95% VaR	-55.00	-48.96

Table 11: Statistics and 95% VaR of distribution of Delta-hedging error under JDM with transaction cost coefficient $\alpha = 0.001$ obtained by Monte Carlo simulation and the recursive relation in Theorem 3.2 with $\delta t = 1/150$.

	MC simulation(%)	Recursive relation(%)
Mean	-8.6607	-7.0002
Variance	385.5	382.4
95% VaR	-45.00	-43.13

Table 12: Statistics and 95% VaR of distribution of Delta-hedging error under JDM with transaction cost coefficient $\alpha = 0.001$ obtained by Monte Carlo simulation and the recursive relation in Theorem 3.2 with $\delta t = 1/252$.

	MC simulation(%)	Recursive relation(%)
Mean	-10.7197	-9.1976
Variance	325.2	330.2
95% VaR	-45.00	-43.13

5. Conclusion

The connection between Delta-hedging error and path-dependent option pricing has been analysed. The hedging error and transaction cost are proportional to the option

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

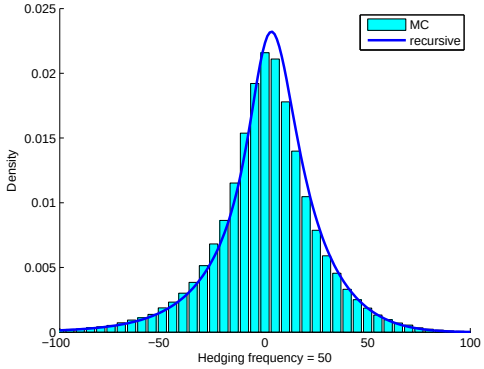


Figure 7: Distribution of Delta-hedging error.

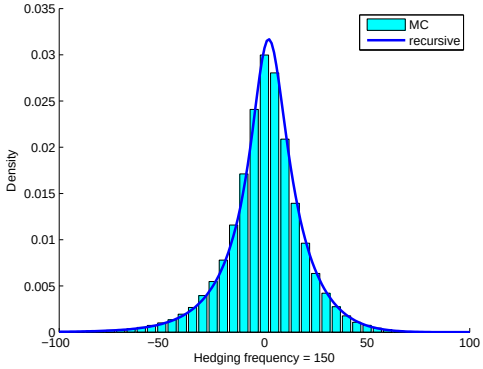


Figure 8: Distribution of Delta-hedging error.

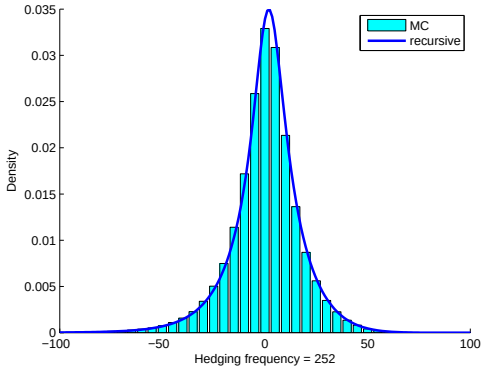


Figure 9: Distribution of Delta-hedging error

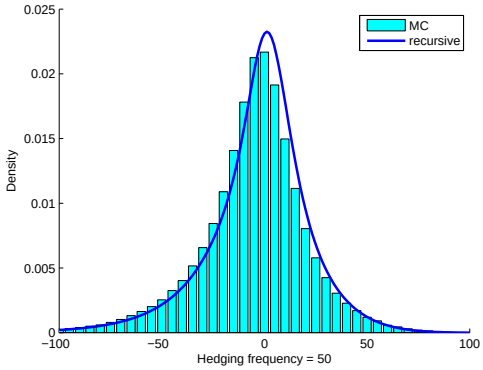


Figure 10: Distribution of Delta-hedging error.

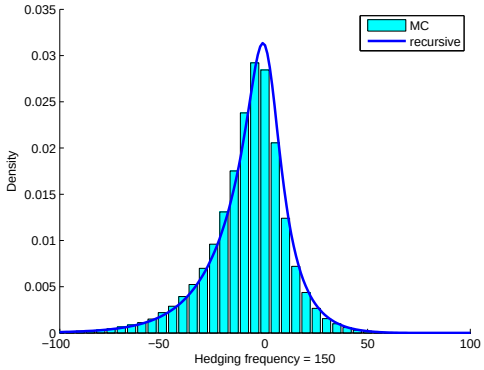


Figure 11: Distribution of Delta-hedging error.

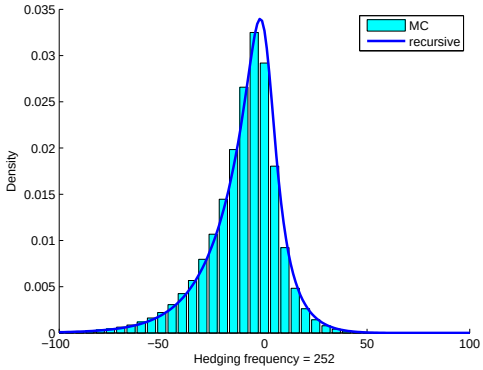


Figure 12: Distribution of Delta-hedging error.

Gamma, which therefore plays an important role in Delta-hedging strategy, Our method differs from previous approaches to Delta-hedging error, because the Delta-hedging error and option pricing are interconnected, whereas others have tried to find the distribution

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

of the Delta-hedging error directly. The presentation here involved geometric Brownian motion and Merton's jump diffusion model, but may be applicable to other models. Our method is also applicable to a diffusion process with time-dependent parameters where the approximate value of the option Gamma is calculated.

A. Characteristic Function of the JDM

Consider the JDM where $s_t = \log(S_t/S_0)$. Then from Eq. (2.5),

$$s_t = \left(\mu - \frac{1}{2}\sigma^2 - \lambda\beta\right)t + \sigma W_t + \sum_{k=1}^{N_t} J_k \ = \text{Normal part} + \text{Jump part} = X_t + Y_t$$

where X_t is a normal distribution with mean $(\mu - \frac{1}{2}\sigma^2 - \lambda\beta)t$ and variance σ^2, and $Y_t = \sum_{k=1}^{N_t} J_k$ is a compound Poisson process. Since W_t, X_t and J_k are assumed independent,

$$\mathbb{E}[e^{iws_t}] = \mathbb{E}[e^{iw(X_t+Y_t)}] = \mathbb{E}[e^{iwX_t}]\mathbb{E}[e^{iwY_t}]$$

or

$$\phi_{s_t}(w) = \phi_{X_t}(w)\phi_{Y_t}(w)\,,$$

where $\phi_{s_t}(w)$, $\phi_{X_t}(w)$ and $\phi_{Y_t}(w)$ are characteristic functions of s_t, X_t and Y_t, respectively. It is well known that the characteristic functions of the normal distribution and compound Poisson process are

$$\phi_{X_t}(w) = \exp\left\{i\left(\mu - \frac{1}{2}\sigma^2 - \lambda\beta\right)tw - \frac{\sigma^2 t}{2}w^2\right\}\,,$$
$$\phi_{Y_t}(w) = \exp\{\lambda(\phi_J(w) - 1)\}\,,$$

where $\phi_J(w)$ is a characteristic function of single jump size J — i.e.

$$\phi_J(w) = \exp\left(i\nu w - \frac{\delta^2}{2}w^2\right).$$

Thus the characteristic function of s_t is given by

$$\phi_{s_t}(w) = \exp\left[i\left(\mu - \frac{1}{2}\sigma^2 - \lambda\beta\right)tw - \frac{\sigma^2 t}{2}w^2 + \lambda\left\{\exp\left(i\nu w - \frac{\delta^2}{2}w^2\right) - 1\right\}\right].$$

B. Option Price and Greek Parameters for the JDM

For simplicity, let $\mathbb{E}_t[F]$ denote the conditional expectation given $\mathscr{F}_t$ — i.e. assume

$$\mathbb{E}_t[F] = \mathbb{E}[F|\mathscr{F}_t]\,,$$

where F is an appropriate contingent claim. By the martingale method with a risk-neutral measure $\mathbb{Q}$, we can readily obtain the price of a derivative under the JDM when its price

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

under the Black–Scholes framework is known. For a risk-neutral measure, it is notable that the discounted stock price $e^{-rt}S_t$ with risk-free rate r should be a martingale, and we assume that the stock price satisfies

$$\mathrm{d}(\log S_t) = \left(r - \frac{1}{2}\sigma^2 - \lambda\beta\right)\mathrm{d}t + \sigma \mathrm{d}X_t + J\mathrm{d}N(t)$$

where X_t is a $\mathbb{Q}$-Brownian motion. Note that

$$\begin{aligned} V(t,S_t) &= \mathrm{e}^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[F(S_T)] \\ &= \mathrm{e}^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left[F\left(S_t \exp\left\{\left(r-\lambda\beta-\frac{\sigma^2}{2}\right)\tau + \sigma X_\tau + \sum_{i=1}^{N_\tau} J_i\right\}\right)\right] \\ &= \mathrm{e}^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[F\left(S_t \exp\left\{\left(r-\lambda\beta-\frac{\sigma^2}{2}\right)\tau + \sigma X_\tau + \sum_{i=1}^{N_\tau} J_i\right\}\right)\middle| N_\tau = k\right]\right] \\ &= \mathrm{e}^{-r\tau}\sum_{k=0}^{\infty}\frac{\mathrm{e}^{-\lambda\tau}(\lambda\tau)^k}{k!}\mathbb{E}_t^{\mathbb{Q}}\left[F\left(S_t \exp\left\{\left(r-\lambda\beta-\frac{\sigma^2}{2}\right)\tau + \sigma X_\tau + \sum_{i=1}^{k} J_i\right\}\right)\right]. \end{aligned} \tag{B.1}$$

Since the J_i are independent and identically distributed (i.i.d.) with $N(\nu,\delta^2)$ and X_τ is a $\mathbb{Q}$-Brownian motion,

$$\begin{aligned} \left(r-\lambda\beta-\frac{\sigma^2}{2}\right)\tau + \sigma X_\tau + \sum_{i=1}^{k} J_i &= \left(r-\lambda\beta-\frac{\sigma^2}{2}\right)\tau + N(0,\sigma^2\tau) + N(k\nu, k\delta^2) \\ &= \left(r-\lambda\beta-\frac{\sigma^2}{2}+\frac{k\nu}{\tau}\right)\tau + N\left(0,\left(\sigma^2+\frac{k\delta^2}{\tau}\right)\tau\right) \\ &= \left(r-\lambda\beta-\frac{\sigma^2+\frac{k\delta^2}{\tau}}{2}+\frac{\frac{k\delta^2}{\tau}}{2}+\frac{k\nu}{\tau}\right)\tau + \sigma_k X_\tau \\ &= \left(r-\lambda\beta+\frac{k(\nu+\frac{1}{2}\delta^2)}{\tau}-\frac{\sigma_k^2}{2}\right)\tau + \sigma_k X_\tau \\ &= \left(r_k-\frac{\sigma_k^2}{2}\right)\tau + \sigma_k X_\tau\,, \end{aligned}$$

where $\sigma_k^2 = \sigma^2 + k\delta^2/\tau$ and $r_k = r - \lambda\beta + k(\nu+\frac{1}{2}\delta^2)/\tau$.

Thus under the condition $N_\tau = k$, Merton's jump diffusion model becomes

$$\begin{aligned} S_T &= S_t \exp\left\{\left(r-\lambda\beta-\frac{\sigma^2}{2}\right)\tau + \sigma X_\tau + \sum_{i=1}^{k} J_i\right\} \\ &= S_t \exp\left\{\left(r_k-\frac{\sigma_k^2}{2}\right)\tau + \sigma_k X_\tau\right\}, \end{aligned} \tag{B.2}$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

so we may regard S_T as a GBM with drift rate r_k and volatility σ_k.

The Black–Scholes option price can also be derived by the martingale method:

$$\begin{aligned} V^{BS}(t,S_t,\tau,r,\sigma) &= \mathrm{e}^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}[F(S_T)] \\ &= \mathrm{e}^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left[F\left(S_t\exp\left\{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma X_\tau\right\}\right)\right]. \end{aligned} \tag{B.3}$$

Thus from Eqs. (B.1), (B.2) and (B.3) we obtain

$$\begin{aligned} V(t,S_t) &= \mathrm{e}^{-r\tau}\sum_{k=0}^{\infty}\frac{\mathrm{e}^{-\lambda\tau}(\lambda\tau)^k}{k!}\mathbb{E}_t^{\mathbb{Q}}\left[F\left(S_t\exp\left\{\left(r_k-\frac{\sigma_k^2}{2}\right)\tau+\sigma_k X_\tau\right\}\right)\right] \\ &= \mathrm{e}^{-r\tau}\sum_{k=0}^{\infty}\frac{\mathrm{e}^{-\lambda\tau}(\lambda\tau)^k}{k!}\mathrm{e}^{r_k\tau}\mathrm{e}^{-r_k\tau}\mathbb{E}_t^{\mathbb{Q}}\left[F\left(S_t\exp\left\{\left(r_k-\frac{\sigma_k^2}{2}\right)\tau+\sigma_k X_\tau\right\}\right)\right] \\ &= \sum_{k=0}^{\infty}\frac{\mathrm{e}^{-(\lambda+r-r_k)\tau}(\lambda\tau)^k}{k!}V^{BS}(t,S_t,\tau,r_k,\sigma_k)\,, \end{aligned} \tag{B.4}$$

and also note that

$$\beta = \mathbb{E}[\mathrm{e}^J-1] = \mathrm{e}^{\nu+\frac{1}{2}\delta^2}-1$$

and

$$\begin{aligned} \mathrm{e}^{-(\lambda+r-r_k)\tau}(\lambda\tau)^k &= \exp\left\{-(\lambda+\lambda\beta)\tau+k\left(\nu+\frac{1}{2}\delta^2\right)\right\}(\lambda\tau)^k \\ &= \exp\left(-\lambda'\tau\right)(\beta+1)^k(\lambda\tau)^k \\ &= \exp\left(-\lambda'\tau\right)(\lambda'\tau)^k. \end{aligned} \tag{B.5}$$

From (B.4) and (B.5) we then obtain Eq. (2.7).

C. Proof of Theorem 3.1

From the tower property of conditional expectation we have

$$g_N^{(n)}(S_0,\cdots,S_n,K,\kappa) = \mathbb{E}\left[g_N^{(n+1)}(S_0,\cdots,S_n,S_{n+1},K,\kappa)|\mathscr{F}_n\right]$$

and

$$g_N^{(n)}(S_0,\cdots,S_n,K,\kappa) = \int g_N^{(n+1)}(S_0,\cdots,S_n,S_{n+1},K,\kappa)\,f(S_{n+1};S_n)\mathrm{d}S_{n+1}\,, \tag{C.1}$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

where $f(S_{n+1};S_n)$ is the transition probability function for the stock price from S_n to S_{n+1}. Now

$$\begin{aligned} g_N^{(n)}(S_0,\cdots,S_n,K,\kappa) &= \mathbb{E}[(Y-\kappa)^+|\mathscr{F}_n] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{N} Y_i \mathrm{e}^{r(t_i:T)} - \kappa\right)^+ \middle| \mathscr{F}_n\right] \\ &= \mathbb{E}\left[\left\{\sum_{i=n+1}^{N} Y_i \mathrm{e}^{r(t_i:T)} - \left(\kappa - \sum_{i=1}^{n} Y_i \mathrm{e}^{r(t_i:T)}\right)\right\}^+ \middle| \mathscr{F}_n\right] \\ &= g_{N-n}^{(0)}(S_n,K,\kappa_n)\,, \end{aligned} \tag{C.2}$$

hence from Eqs. (C.1) and (C.2) we have

$$\begin{aligned} g_{N-n}^{(0)}(S_n,K,\kappa_n) &= \int g_{N-n-1}^{(0)}(S_{n+1},K,\kappa_{n+1}) f(S_{n+1};S_n)\mathrm{d}S_{n+1} \\ &= \int g_{N-n-1}^{(0)}(S_{n+1},K,\kappa_n - y_{n+1}\mathrm{e}^{r(t_{n+1}:T)}) f(S_{n+1};S_n)\mathrm{d}S_{n+1} \end{aligned}$$

for $n=0,1,\cdots,N-2$ and

$$\begin{aligned} &g_1^{(0)}(S_{N-1},K,\kappa) \\ &= \int_0^\infty \left[\frac{1}{2}\Gamma_{N-1}(S_{N-1},K)\left\{\sigma^2(t_0:t_1) - \left(\frac{S_N}{S_{N-1}} - 1\right)^2\right\} - \kappa\right]^+ f(S_N;S_{N-1})\mathrm{d}x \end{aligned}$$

for $n=N-1$.

D. Proof of Lemma 3.1

Since

$$\kappa_n = \kappa - \sum_{i=1}^{n} Y_i \mathrm{e}^{r(t_i:T)} = \kappa - \sum_{i=1}^{n} Y_i \mathrm{e}^{r(T-t_i)},$$

we have

$$\kappa_{n+1} = \kappa_n - Y_{n+1}\mathrm{e}^{r(t_{n+1}:T)}.$$

From Theorem 3.1,

$$\begin{aligned} g_{N-n}^{(0)}(S_n,K,\kappa_n) &= \mathbb{E}\left[g_{N-n-1}^{(0)}(S_{n+1},K,\kappa_{n+1})|\mathscr{F}_n\right] \\ &= \int_0^\infty g_{N-n-1}^{(0)}(S_{n+1},K,\kappa_{n+1}) f(S_{n+1}:S_n)\mathrm{d}S_{n+1} \\ &= \int_0^\infty g_{N-n-1}^{(0)}(S_n x,K,\kappa_{n+1}) f(x)\mathrm{d}x\,, \end{aligned}$$

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

where $X = S_{n+1}/S_n$, for $n = 0, 1, \cdots, N-2$. We note that $\log X \sim N((\mu - \frac{1}{2}\sigma^2)\delta t, \sigma^2 \delta t)$. Similarly, for $n = N-1$ we have

$$\begin{aligned} g_1^{(0)}(S_{N-1}, K, \kappa) &= \mathbb{E}\left[(Y_N - \kappa)^+ \,\middle|\, \mathscr{F}_{N-1} \right] \\ &= \int_0^\infty (Y_N - \kappa)^+ f(S_N : S_{N-1}) \mathrm{d}S_N \\ &= \int_0^\infty \left(\frac{1}{2} S_{N-1}^2 \Gamma_{N-1} \left\{ \sigma^2 \delta t - (x-1)^2 - 2\alpha |x-1| \, x \right\} - \kappa \right)^+ f(x) \mathrm{d}x \,, \end{aligned}$$

where $X = S_N / S_{N-1}$.

E. Proof of Lemma 3.2

From Lemma 3.1,

$$\begin{aligned} g_{N-n}^{(0)}(S_n, K, \kappa_n) &= \int_0^\infty g_{N-n-1}^{(0)}(S_n x, K, \kappa_n - y_n(x) \mathrm{e}^{r(T-t_{n+1})}) f(x) \mathrm{d}x \\ &= \int_0^\infty S_n x \, g_{N-n-1}^{(0)}\left(1, \frac{K}{S_n x}, \frac{\kappa_n - y_n(x) \mathrm{e}^{r(T-t_{n+1})}}{S_n x} \right) f(x) \mathrm{d}x \,, \end{aligned}$$

so with $S_n = 1$, we obtain the desired result.

Acknowledgments

The authors wish to thank Professor Roger Hosking and an anonymous referee for many valuable suggestions. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (NRF-2015R1A2A2A01006176).

References

[1] G. Bakshi and N. Kapadia, *Delta-hedged gains and the negative market volatility risk premium*, Rev. Financ. Stud. **16**, 527-566 (2003).
[2] D. Bertsimas, L. Kogan and A.W. Lo, *When is time continuous*, J. Financ. Econ. **55**, 173-204 (2000).
[3] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, J. Polit. Econ. **81**, 637-659 (1973).
[4] P. Boyle and T. Vorst, *Option replication in discrete time with transaction costs*, J. Financ. **47**, 271-293 (1992).
[5] P. Carr and L. Wu, *Static hedging of standard options*, J. Financ. Econom. **12**, 3-46 (2014).
[6] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall, London (2004).

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press

[7] E. Derman, *When you cannot hedge continuously. The corrections of Black–Scholes*, Risk Books **1**, 82-85 (1999).
[8] T. Hayashi and P.A. Myklana, *Evaluating hedging errors: An asymptotic approach*, Math. Financ. **15**, 309-343 (2005).
[9] H.E. Leland, *Option pricing and replication with transaction costs*, J. Financ. **5**, 1283-1301 (1985).
[10] R. Merton, *Theory of rational option pricing*, Bell J. Econ. Manag. Sci. **4**, 141-183 (1973).
[11] R. Merton, *Option pricing when underlying stock returns are discontinuous*, J. Financ. Econ. **3**, 125-144 (1976).
[12] K.F. Mina, G.H.L. Cheang and C. Chiarella, *Approximate hedging of options under jump-diffusion processes*, Int. J. Theor. Appl. Finan. **18**, 1550024-1-1550024-26 (2015).
[13] A. Primbs and Y. Yamada, *A new computational tool for analysing dynamic hedging under transaction costs*, Quant. Financ. **4**, 405-413 (2008).
[14] A. Sepp, *An approximate distribution of delta-hedging errors in a jump-diffusion model with discrete trading and transaction costs*, Quant. Financ. **12**, 1119-1141 (2012).
[15] P. Tankov and E. Voltchkova, *Asymptotic analysis of hedginig errors in models with jumps*, Stoch. Process. Their Appl. **119**, 2004-2027 (2009).
[16] B.K. Toft, *On the mean-variance tradeoff in option replication with transactions costs*, J. Financ. Quant. Anal. **2**, 233-263 (1996).

https://doi.org/10.4208/eajam.010116.220516a Published online by Cambridge University Press