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Reduction of modules

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The notion of a reduction of an ideal I of a noetherian ring R and the related notions of the integral closure I^* of I in R and the integral dependence of an element x of R on I have proved useful in many situations in commutative algebra. The purpose of this paper is to extend these notions to modules. To motivate the definitions which follow, it will be useful to consider these notions for ideals before extending them to modules.

To introduce the concepts given above, we first associate with an ideal I of R a graded ring $S(I)$ which is defined to be the sub-ring of $R[t]$, where t is an indeterminate over R, consisting of all polynomials $\sum a_r t^r$ with a_r in I^r (with the usual convention that $I^0 = R$). Then an element x of R is integrally dependent on I if xt is integrally dependent on the ring $S(I)$. The integral closure of $S(I)$ in $R[t]$ is a graded ring $S^*(I)$ and the ideal I^* is defined to be the set of elements x of R such that xt belongs to $S^*(I)$ (and more generally, $x \in (I^r)^*$ if and only if $xt^r \in S^*(I)$). Finally, an ideal J of R contained in I is a reduction of I if and only if $S(I)$ is a finite integral extension of $S(J)$. Alternatively, we have the equivalent formulation that the ideal Jt. $S(I)$ is irrelevant, that is, contains all elements of sufficiently high degree.

The notions of integral dependence on an ideal and of reductions of an ideal properly belong to the theory of integral domains. This is a consequence of the following theorem, which is easy to prove.

Let R be a noetherian ring, and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of R. Let R_i be the ring $R/\mathfrak{p}_i$. Let a suffix i applied to an element or ideal of R denote its image in R_i. Then

(i) *an element x of R is integrally dependent on an ideal I if and only if x_i is integrally dependent on I_i for $i = 1, \ldots, r$;*

(ii) *an ideal J or R contained in I is a reduction of I if and only if J_i is a reduction of I_i for $i = 1, \ldots, r$.*

For ease of exposition, we now impose the restriction that R be a domain, with field of fractions K. In this situation we have an alternative approach to the notions above. Let v range over all discrete valuations on K such that $v(x) \geqslant 0$ for all x in R, and let O_v denote its valuation ring. Then x in R is integrally dependent on I if and only if $x \in IO_v$ for all v. Further, an ideal J contained in I is a reduction of I if and only if $JO_v = IO_v$ for all v. It is sufficient to prove the first of these statements. If x is integrally dependent on I, then xt is integrally dependent on $S(I)$, and hence on $O_v S(I) = S(IO_v)$. But the latter is isomorphic to $O_v[X]$, where X is an indeterminate, and so is integrally closed. Hence $x \in IO_v$. Conversely, suppose that x is not integrally dependent on I. Then there exists a discrete valuation w on the field of fractions $K(t)$ of $S(I)$ which is $\geqslant 0$ on $S(I)$, such that $w(xt) < 0$. Then, if v is

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the restriction of w to K, x does not belong to IO_v. It is this definition that we will extend to modules below.

It is not necessary to consider all discrete valuations v in the last two paragraphs. There is a finite set $\Sigma(I)$ of valuations v depending only on I such that $x \in (I^n)^*$ if and only if $x \in I^n O_v$ for all v in $\Sigma(I)$. Strictly speaking, the set $\Sigma(I)$ depends on the sequence of ideals I^n rather than the individual ideal I. For a proof of the existence of $\Sigma(I)$, the reader is referred to [**6**], chapter 11, where the extension of the result to the case where R is not a domain is also given.

To complete this short account of the theory of reductions of ideals, we now turn to the notion of the analytic spread of an ideal I at a prime ideal $\mathfrak{p}$ of R. Let Q denote the local ring $R_{\mathfrak{p}}$, let $\mathfrak{m}$ denote its maximal ideal and let k denote the field $Q/\mathfrak{m}$. Then the ring $F(I, \mathfrak{m}) = S(IQ)/\mathfrak{m}S(IQ)$ is a finitely generated graded extension of k, and so has a dimension, which we define to be the maximum of the transcendence degrees of the fields of fractions of $F(I, \mathfrak{m})/\mathfrak{P}$, where $\mathfrak{P}$ ranges over the minimal prime ideals of $F(I, \mathfrak{m})$. This is the analytic spread of I, and is denoted by $s(I, \mathfrak{p})$. Note that this definition does not require the restriction to domains, and, if $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal prime ideals of R contained in $\mathfrak{p}$, then

$$s(I, \mathfrak{p}) = \max\,(s(I + \mathfrak{p}_i/\mathfrak{p}_i, \mathfrak{p}/\mathfrak{p}_i)).$$

If we impose the mild restriction that k is an infinite field, then $s(I, \mathfrak{p})$ is equal to the minimal number of elements which can generate a reduction of IQ. Further we have the following inequalities for $s(I, \mathfrak{p})$:

$$\operatorname{ht}\mathfrak{p} \geqslant s(I, \mathfrak{p}) \geqslant \operatorname{alt} IQ$$

where $\operatorname{alt} IQ$ is the maximum of the heights of the minimal prime ideals of IQ.

We also note that, if $\mathfrak{p}'$ is a prime ideal contained in $\mathfrak{p}$, then

$$s(I, \mathfrak{p}') \leqslant s(I, \mathfrak{p}).$$

For further results on analytic spread we require a restriction on R. We recall first that a local ring Q is said to be quasi-unmixed if, denoting the completion of Q by Q'', for every minimal prime ideal $\mathfrak{p}$ of Q'', $\dim(Q''/\mathfrak{p}) = \dim Q$. There are three consequences of the condition which will be important in this paper:

(a) if $\mathfrak{p}$ is any prime ideal of Q, then

$$\operatorname{ht}\mathfrak{p} + \dim(Q/\mathfrak{p}) = \dim Q;$$

(b) if I is an ideal of Q of height r and generated by r element, then, for all n, the ideals $(I^n)^*$ are unmixed;

(c) if $(Q, \mathfrak{m}, k, d)$ is a quasi-unmixed domain, with field of fractions K, and $(Q', \mathfrak{m}', k', d')$ is a localization of a finitely generated extension of $(Q, \mathfrak{m}, k, d)$, with field of fractions E, such that $\mathfrak{m}'$ meets Q in $\mathfrak{m}$, then

$$d + \text{trans. deg}_K E = d' + \text{trans. deg}_k k'.$$

(Note that if we drop the assumption that Q is quasi-unmixed, then (c) remains true if we replace '$=$' by '$\geqslant$'. This is the altitude inequality, and the quasi-unmixed case is referred to as the altitude equality.)

We now introduce the definition that a noetherian ring R is LQU (or locally quasi-unmixed) if $R_{\mathfrak{p}}$ is quasi-unmixed for all prime ideals $\mathfrak{p}$ of R. It is sufficient to

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consider only maximal ideals of R. Note that the class of LQU-rings is closed under the operation of taking rings of fractions, that if R is LQU, then so is R/I, providing that the minimal prime ideals of I all have the same height, and finally, if R is LQU, then so is $R[X]$, where X is an indeterminate. For an account of LQU-rings, a convenient reference is [**6**] (see the various references to 'Locally quasi-unmixed' in the index to [**6**]).

In LQU-rings we have a result connecting analytic spread and the set of valuations $\Sigma(I)$. We recall that, if v is a valuation on the field of fractions of a domain R such that $v(x) \geqslant 0$ on R, then the centre of v on R is the prime ideal $\mathfrak{p}$ consisting of all x such that $v(x) > 0$. Then we recall the following result, due to Macadam (see [**6**], proposition 4·1 and the second sentence of chapter XI).

If R is an LQU*-domain, and I is an ideal of R, then $\mathfrak{p}$ is the centre of a valuation v in $\Sigma(I)$ if and only if*

$$s(I, \mathfrak{p}) = \operatorname{ht} \mathfrak{p}.$$

We now come to the objective of this paper. This is to extend the ideas of integral dependence on, integral closure of, and reductions of an ideal I of a noetherian ring R to corresponding notions for finitely generated modules M over R. We find it most convenient to do this in the case where R is a domain and M is torsion-free, and most of the results presented below will be stated for this case, although we will indicate the basis of the extension to the general case in the body of the paper.

We now suppose that R is a noetherian domain with field of fractions K and that M is a finitely generated torsion-free R-module. Let V denote the vector space $K \otimes_R M$, and identify M with its image in V. Then an element x of V is said to be integrally dependent on M if, for all discrete valuations v on K such that $v(z) \geqslant 0$ for z in R, $x \in O_v M$. We denote the set of all elements x of V integrally dependent on M by M^*. Then M^* is an R-module, but, in general, is not finitely generated. Next, suppose that L is a finitely generated torsion-free R-module, containing M, and let $W = K \otimes_R L$, so that V is a sub-space of W. Then we define the integral closure of M in L to be the intersection of L with M^*. Finally, M will be said to be a reduction of L if L is contained in M^*, or what is the same thing, if $O_v L = O_v M$ for all v.

At this point, we mention a result which plays only a minor role in this paper. Let the vector space V above have dimension r over K, so that M has rank r. Let $E(V) = E_0(V) + \ldots + E_r(V)$ be the exterior algebra of V and let $E(M)$ denote the image of the exterior algebra of M in $E(V)$ under the natural homomorphism of the one into the other. Suppose, further, that W above is equal to V, i.e. that L also has rank r. Then $E_r(L)$, $E_r(M)$ are respectively isomorphic to ideals $I(L)$, $I(M)$ such that $I(L)$ contains $I(M)$, and M is a reduction of L if and only if $I(M)$ is a reduction of $I(L)$.

We now turn to the initial definitions above, those depending on the ring $S(I)$, and consider the analogue of the ring $S(I)$. It should be emphasized that the definition of this ring depends on the embedding of I in R. The definition of $S(M)$ we shall use is adapted to the case where R is a domain and M is torsion-free, and runs as follows. We define $S(M)$ to be the image of the symmetric algebra $\operatorname{Sym}_R M$ of M in the symmetric algebra $\operatorname{Sym}_K V$ of the vector space V. Note that if L is a finitely generated torsion-free module containing M then $S(M)$ is a graded sub-ring of $S(L)$, both being domains. $S(M)$ is a graded domain with $S_0(M) = R$, $S_1(M) = M$, and $S_r(M)$ isomorphic to the rth symmetric power of M modulo its torsion sub-module. We can use the ring

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$S(M)$ to define the concepts of reduction of a module, integral dependence on a module and the integral closure of a module in an over-module, at least in the case where the modules are restricted to be torsion-free. We can also define analogues of the set of valuations $\Sigma(I)$ (these being sets of valuations depending on pairs of modules M, L, where L contains M), and finally, following Ooishi [9], of the analytic spread of a module. However, with the exception of the definition of reduction, the role played by $S(M)$ is adequately played by certain quotient rings of $S(M)$, which we term test domains. A test domain T for M is a graded R-algebra, with $T_0 = R$, $T_1 = M$, which is generated by its elements of degree 1, and which satisfies the condition that, for all discrete valuations v on K such that $v(x) \geqslant 0$ on R, the ring $O_v T$ is integrally closed in its field of fractions. If $V = K \otimes_R M$, we denote by $T(V)$ the domain $K \otimes_R T(M)$, and, more generally, if L is any sub-module of V, by $T(L)$ the sub-ring of $T(V)$ generated by the elements of L. We now have the following results.

(A) *An element x of $T(V)_n$ is integrally dependent on $T(M)_n$ if and only if it is integrally dependent on $T(M)$.*

(B) *If L is a finitely generated sub-module of V containing M, then there exists a finite set $\Sigma(T, L, M)$ of discrete valuations v on K satisfying $v(x) \geqslant 0$ on R with the property that, for all n, an element x of $T_n(L)$ is integrally dependent on $T_n(M)$ if and only if x belongs to $O_v T_n(M)$ for all v in $\Sigma(T, L, M)$. Further, if L' is a sub-module of L containing M, then $\Sigma(T, L', M)$ is contained in $\Sigma(T, L, M)$. Finally, the centres of the valuations in $\Sigma(T, L, M)$ all contain the annihilator $M : L$ of the module L/M.*

(C) *$S(M)$ is a test domain for M and has the further property that if N is a sub-module of M, N is a reduction of M if and only if $S(M)$ is a finite $S(N)$-module (or alternatively, N generates an irrelevant ideal of $S(M)$).*

Now we turn to the notion of analytic spread. We first take R to be a local ring Q with maximal ideal $\mathfrak{m}$ and residue field $k = Q/\mathfrak{m}$. If T is a graded Q-algebra, then Ooishi [9] defined the analytic spread $s(T)$ of T to be $\dim_k(T/\mathfrak{m}T)$, where dimension means Krull dimension. In this paper we define $\dim_k(T/\mathfrak{m}T)$ to mean the maximum of the transcendence degrees over k of the fields of fractions of the domains $T/\mathfrak{P}$, where $\mathfrak{P}$ ranges over the minimal prime ideals associated with $\mathfrak{m}T$. This is clearly the same as Ooishi's definition. We are concerned with the case when T is a domain, and we define $\dim_K T$ to be the transcendence degree of the field of fractions of T over K. Then we have the inequality

$$\dim_K T \leqslant s(T) \leqslant \dim_K T + \dim Q - 1$$

proved in [9] as proposition 3 (a proof is also given below, together with a more detailed discussion of the relationship between Ooishi's paper and the approach adopted here). If T is an R-algebra, we define $s(\mathfrak{p}, T)$ to be $s(T_{\mathfrak{p}})$, and if T is a test domain for an R-module M, we will write $s(\mathfrak{p}, T, M)$ in place of $s(\mathfrak{p}, T)$. Finally, if $T = S$, we will write simply $s(\mathfrak{p}, M)$.

We note first that, if R is a local ring Q with infinite residue field k, and M is a torsion-free Q-module, then the minimal number of generators of a reduction of M is $s(\mathfrak{m}, M)$. However, the main use we will make of analytic spread arises from a generalization of the result of Macadam already referred to. This requires the restriction that R be an LQU-domain. Let L, M be torsion-free R-modules with L containing M and contained in $K \otimes_R M$. Let T be a test domain for M. Then a prime

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ideal $\mathfrak{p}$ of R is the centre of a valuation in $\Sigma(T, L, M)$ if and only if $\mathfrak{p}$ contains Ann (L/M) and

$$s(\mathfrak{p}, T, M) = \dim_K T + \operatorname{ht} \mathfrak{p} - 1.$$

This implies that, if $s(\mathfrak{p}, T, M) < \dim_K T + \operatorname{ht}(\text{Ann}\,(L/M)) - 1$ for all prime ideals containing Ann (L/M), then M is a reduction of L.

This is used in the last section of the paper to prove the following result on Koszul Complexes. Let R be an LQU-domain, and let $\mathbf{a} = (a_1, \ldots, a_n)$ be a set of elements of R generating a proper ideal of height h. Let $\mathbf{K}$ be the Koszul Complex of R with respect to $\mathbf{a}$. Then if Z_r, B_r are, respectively, the modules of r-cycles and r-boundaries, and $r > n-h$, then B_r is a reduction of Z_r. This theorem is then recast to prove a result on Grassmann domains. We suppose that $G = G_{n,r}$ is the ring generated over R by the $r \times r$ minors of an $r \times n$ matrix $\mathbf{X}$ whose entries $X(i,j)$ are indeterminates over R. Let $x(1), \ldots, x(n)$ be elements of R generating a proper ideal I of height h. We then define the ideal B of G to be the ideal generated by the $(r+1) \times (r+1)$ minors of the matrix derived by adding the row $x(1), \ldots, x(n)$ to $\mathbf{X}$ as an $(r+1)$st row. We also define a homomorphism of $G_{n,r}$ into $G_{n,r-1}$ by defining the image of a minor of $\mathbf{X}$ to be the corresponding minor of the matrix derived from $\mathbf{X}$ by replacing the last row by $x(1), \ldots, x(n)$ and write C for the kernel of this homomorphism. Then our result is that, if $r > n-h$, C is the integral closure of B.

1. *Integral dependence on a sub-module*

In what follows R will denote a noetherian ring. Suppose that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are the minimal prime ideals of R. Let R_j be the localization of R at $\mathfrak{p}_j$, and K_j be the residue field of R_j, that is, the field of fractions of $R/\mathfrak{p}_j$. Let L, M be finitely generated R-modules, M being a sub-module of L. Then $R_j \otimes_R M$ is a sub-module of $R_j \otimes_R L$. We denote $K_j \otimes_R L$ by W_j and the image of $K_j \otimes_R M$ in W_j by V_j, and by x_j, L_j, M_j the images of x, L, M under the maps $L \to W_j$, $M \to V_j$ given by $x \to 1 \otimes_R x$, so that M_j is a sub-module of L_j for each j. We now come to the following definition:

Definition. An element x of L is said to be integrally dependent on M if, for all j and for all discrete, integer-valued valuations v on K_j such that $v(z) \geqslant 0$ on R_j, x_j belongs to $O_v M_j$.

From the form of this definition, it is clear that the general definition reduces to the case where R is a domain, and the modules L, M are torsion-free. For ease of exposition, we will therefore assume that R is a domain and that all modules considered are torsion-free in what follows, except when the contrary is indicated. We can therefore drop the suffix j. We further identify L, M with their images in W, V and consider V as a sub-space of W. It should be clear that any element x of L which is integrally dependent on M is contained in V. By taking $L = M + Rx$, where x is an element of V, we can give a meaning to the statement that an element x of V is integrally dependent on M, and this we will do below. The set of elements of V integrally dependent on M forms an R-module M^* which, however, is not, in general, finitely generated.

LEMMA 1·1. *If $M \neq 0$, M^* is finitely generated if and only if the integral closure R^* of R in K is a finite R-module.*

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If $x \neq 0$ belongs to M, then M^* contains the sub-module R^*x isomorphic to R^* as an R-module, so that if M^* is a finite R-module, so is R^*.

Conversely, suppose that R^* is a finite R-module. Let $M\dagger$ denote $\mathrm{Hom}_R(M, R^*)$, and $V\dagger$ denote $\mathrm{Hom}_K(V, K)$, so that, identifying $V\dagger\dagger$ with V, we can identify $M\dagger\dagger$ with a finitely generated R-sub-module of V. Now let μ belong to $M\dagger$. Then μ has a unique extension to an element of $V\dagger$, which we also denote by μ. Let x belong to M^*. Then if O_v is the valuation ring of a discrete valuation v on K satisfying $v(z) \geqslant 0$ on R, $\mu(x) \in O_v R^* = O_v$. But the intersection of the rings O_v is R^*. Hence $\mu(x)$ belongs to R^*, and x belongs to $M\dagger\dagger$, i.e. M^* is a sub-module of $M\dagger\dagger$ and so is finitely generated.

Definition. If N is a sub-module of M, N is said to be a reduction of M if every element of M is integrally dependent on N. Note that this is equivalent to the statement that, for all discrete valuations v on K such that $v(z) \geqslant 0$ on R, $O_v M = O_v N$.

The following is an immediate consequence of the definition.

If M is a sub-module of L and N is a sub-module of M, then N is a reduction of L if and only if M is a reduction of L and N is a reduction of M.

We now consider alternative characterizations of integral dependence on a module or reductions of a module. These characterizations depend on graded rings associated with a module. The first such ring we will consider is a modification of the exterior algebra of a module. Let M be a finitely generated R-module and let N be a sub-module of M. We now define $E(M)$, $E(N)$ to be the canonical images of the exterior algebras of M, N over R in the exterior algebra $E(V)$ of V over K. If r is the rank of M, that is, the dimension of V, then the component $E^i(V)$ of $E(V)$ of degree r is 0 if $i > r$, and $E^r(V)$ is of dimension 1. It follows that $E^r(M)$ is a torsion-free R-module of rank 1, and so is isomorphic as an R-module to an ideal I of R, and further, as a sub-module, $E^r(N)$ is mapped isomorphically by the isomorphism onto an ideal J contained in I. We now have the following result.

THEOREM 1·2. *N is a reduction of M if and only if J is a reduction of I.*

First suppose that N is a reduction of M. Then, with the usual choice of v, we have $O_v M = O_v N$, and hence $O_v I = O_v J$, implying that J is a reduction of I.

Now suppose that N is not a reduction of M, then we can choose v so that $O_v N \neq O_v M$. As finitely generated torsion-free modules over a DVR, $O_v M$, $O_v N$ are both free modules over O_v. Now if N has smaller rank than M, $E^r(N) = 0$ whereas $E^r(M) \neq 0$, implying that $J = 0$ is not a reduction of I. Hence we may suppose that M, N have the same rank. In this case we may choose a basis $u_1, \ldots, u_r$ of $O_v M$ such that N has a basis $p^{m(1)}u_1, \ldots, p^{m(r)}u_r$ where p is a generator of the maximal ideal of O_v and, since $O_v N \neq O_v M$, at least one of the integers $m(j) > 0$. Hence $JO_v = p^m IO_v$, where $m = \Sigma m(j) > 0$. It follows that J is not a reduction of I.

The natural next step is to use an adaptation of symmetric algebras. However, it is convenient at this point to introduce a wider class of graded rings which serve almost as well.

Definition. Let M be a torsion-free R-module. A graded R-algebra T is called a test domain for M if it satisfies the following conditions.

(i) It is a finitely generated domain, being generated by elements of the R-module T_1 of elements of degree 1;

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(ii) $T_0 = R$, $T_1 = M$;

(iii) with the usual choice of v, $O_v T$ is integrally closed in its field of fractions.

The use of a test domain gives us information, not merely about integral dependence on the single module M, but on the modules of the sequence of modules T_n, as the following theorem shows.

THEOREM 1·3. *Let T be a test domain for M and let V_n denote the vector space KT_n. Then, if T^* is the integral closure of T in its field of fractions, and therefore a graded ring, the module T_n^* of elements of T^* which are homogeneous of degree n coincides with the module of elements of V_n integrally dependent on T_n.*

For a proof of the fact that T^* is graded, see ([**12**], volume 2, theorem 11 on p. 157). Since $O_v T$ is integrally closed for all v, it follows that T^* is contained in $O_v T$ for all v and hence an element x of T_n^* is contained in $O_v T_n$ for all v, i.e. is integrally dependent on T_n. Conversely, suppose that x is an element of V_n not integrally dependent on T. Then there exists a discrete valuation w on the field of fractions of T, $\geqslant 0$ on T, satisfying $w(x) < 0$. Let v be the restriction of w to K. Then v takes non-zero values, since otherwise $w(x) \geqslant 0$. Then $O_v T_n$ is contained in O_w and so does not contain x. Hence x is not integrally dependent on T_n.

The following lemma, derived from Hochster [**3**], p. 53, lemma 3·14, is useful in checking that a graded domain is a test domain.

LEMMA 1·4. *The following conditions are sufficient for $O_v T$ to be integrally closed in its field of fractions*:

(i) *KT is integrally closed in its field of fractions*;

(ii) *if p is the generator of the maximal ideal of O_v, then $pO_v T$ is prime.*

Clearly we can take $R = O_v$ and replace T by $O_v T$. Suppose x is integrally dependent on T. Then, by (i), $x \in KT$. Hence we can find r such that $p^r x \in T$. Choose r minimal. Then we must show that $r \leqslant 0$. Suppose $r > 0$. Then $p^r x$ does not belong to pT. But x is integrally dependent on T. Hence $p^r x \in (p^r T)^*$ and hence to $(pT)^*$ which is pT since the latter is prime. Hence we have derived a contradiction, and $r \leqslant 0$.

Now we turn to the 'generic' test domain for M. This is the image of the symmetric algebra $\mathrm{Sym}_R M$ of M in the symmetric algebra $\mathrm{Sym}_K V$ of $V = KM$ over K. It can also be described as $\mathrm{Sym}_R M/\mathfrak{P}$, where $\mathfrak{P}$ is the unique minimal prime ideal of $\mathrm{Sym}_R M$ meeting R in the zero ideal. We leave the details to the reader. We denote this ring by $S(M)$. Note that if N is a sub-module of M, then $S(N)$ is a sub-ring of $S(M)$. We now prove that $S(M)$ is a test domain for M and a little more, effectively the result (C) of the introduction.

THEOREM 1·5. *Let M be a finitely generated R-module, and let N be a sub-module of M such that $KN = KM = V$. Then*

(i) *$S(M)$ is a test domain for M,*

(ii) *N is a reduction of M if and only if $NS(M)$ is an irrelevant ideal of $S(M)$.*

(i) We only have to verify that $O_v S(M)$ is integrally closed in its field of fractions, but this is immediate, since $O_v S(M)$ is isomorphic to $O_v[X_1, \ldots, X_m]$ where $X_1, \ldots, X_m$ are indeterminates and m is the rank of M.

(ii) First, $S(M)$ is a finitely generated extension of $S(N)$ generated by the elements of M, and since the latter is a test domain for N, the elements of M are integrally

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dependent on N if and only if $S(M)$ is a finite $S(N)$-module. Hence N is a reduction of M if and only if $NS(M)$ is irrelevant.

Theorem 1·3 contains (A) of the introduction, and Theorem 1·5 contains (C). To prove (B) requires the Mori–Nagata theorem, which we now state. For a proof we refer to Nagata's book [7], p. 118 (33·10).

THE MORI–NAGATA THEOREM. *Let R be a noether domain, field of fractions K. Let R^* denote the integral closure of R in K. Then the following is true.*

(i) *R^* is a Krull domain, that is,*

(a) *if $\mathfrak{P}$ is a height 1 prime ideal of R^*, $R^*_{\mathfrak{P}}$ is a discrete valuation ring,*

(b) *if $\mathrm{Kr}(R)$ denotes the set of valuations determined by the height 1 prime ideals of R^*, then (1) if $x \neq 0$ is an element of K and v ranges over $\mathrm{Kr}(R)$, $v(x) = 0$ for all save a finite set of v, and (2) x belongs to R^* if and only if $v(x) \geqslant 0$ for all v in $\mathrm{Kr}(R)$,*

(ii) *if $\mathfrak{P}$ is a prime ideal of R, there exist only a finite number of prime ideals $\mathfrak{P}$ of R^* meeting R in $\mathfrak{p}$, and, if $\mathfrak{P}$ is such a prime ideal, then the field of fractions of $R^*/\mathfrak{P}$ is a finite extension of the field of fractions of $R/\mathfrak{p}$.*

We refer to the valuations contained in $\mathrm{Kr}(R)$ as the Krull valuations of R.

We also require some further results concerning the sets of valuations $\mathrm{Kr}(R)$ which we give below. For proofs of these results we refer the reader to [1], chapter 7, section 1.

(1) If S is a multiplicatively closed sub-set of R not containing 0, then $\mathrm{Kr}(R_S)$ consists of those valuations v in $\mathrm{Kr}(R)$ such that $v(s) = 0$ for all $s \in S$.

(2) If $X_1, \ldots, X_n$ are indeterminates over R and $T = R[X_1, \ldots, X_n]$, then $\mathrm{Kr}(T)$ consists of the trivial extensions of the valuations v of $\mathrm{Kr}(R)$ defined by $v(\Sigma a_\mu, \mu) = \mathrm{Min}\, v(a_\mu)$, where μ denotes a monomial in $X_1, \ldots, X_n$ and $a_\mu \in R$, together with the Krull valuations of $K[X_1, \ldots, X_n]$.

(3) If T is a finite integral extension of R, so that the field of fractions E of T is a finite extension of K, then $\mathrm{Kr}(T)$ consists of the extensions of the Krull valuations of R to E. Each v in $\mathrm{Kr}(R)$ has at least one and only a finite number of such extensions, and the residue field of such an extension is a finite algebraic extension of that of v.

THEOREM 1·6. *Let R be a noetherian domain, field of fractions K, and let $T = R[x_1, \ldots, x_n]$ be a finitely generated domain over R with field of fractions E. Then*

(i) *all save a finite number of Krull valuations of T either have trivial restriction to K or are extensions of Krull valuations of R,*

(ii) *all save a finite number of Krull valuations of R are restrictions of Krull valuations of T.*

We may suppose that $x_1, \ldots, x_r$ are algebraically independent and that E is algebraic over $K(x_1, \ldots, x_r)$. Then (2) above implies that (i) and (ii) hold if we replace T by $T' = R[x_1, \ldots, x_r]$, and we can drop the phrase 'all save a finite number'. Now let a be a non-zero element of T' such that $ax_{r+1}, \ldots, ax_n$ are all integral over T'. There certainly exist such elements, and, together with 0, they form an ideal I' of T'. Let $T'' = T'[ax_{r+1}, \ldots, ax_n]$. Then (3) above implies that every Krull valuation of T'' is the extension of a Krull valuation of T' and therefore either has trivial restriction to K or is an extension of a Krull valuation of R, and further every Krull valuation of R has an extension to a Krull valuation of T''. Finally, $T''_S = T_S$ where S is the set of powers of a. Hence every Krull valuation w of T'', with the exception of the finite number satisfying $w(a) > 0$, extends to a Krull valuation of T''_S and hence, by

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restriction, to a Krull valuation of T, and every Krull valuation u of T with the exception of the finite number satisfying $u(a) > 0$, is a Krull valuation of T_S and hence of T''. We can now descend to R and obtain (i), (ii) above.

We now apply the above theorem to prove the result (B) of the introduction.

THEOREM 1·7. *Let R be a noetherian domain field of fractions K, and let M be a finitely generated R-module such that $KM = V$. Let L be another finitely generated R-module containing M and contained in V. Let T be a test domain for M. Then there exists a finite set $\Sigma(T, L, M)$ of discrete valuations v on K such that $v(x) \geqslant 0$ on R with the property that, for all n, an element x of $T_n(L)$ is integrally dependent on $T_n(M)$ if and only if $x \in O_v T_n(M)$ for all v in $\Sigma(T, L, M)$. Further, if L' is a sub-module of L containing M, $\Sigma(T, L', M)$ is contained in $\Sigma(T, L, M)$. Finally, the centres of the valuations of the set $\Sigma(T, L, M)$ contain the annihilator $M:L$ of L/M.*

By Theorem 1·3, an element x of $T_n(V)$ is integrally dependent on $T_n(M)$ if and only if it is integrally dependent on T. We now consider the two graded rings T and $T(L)$, the latter being finitely generated over the former, and both having the same field of fractions. Now an element x of $T_n(L)$ is integrally dependent on T if and only if it belongs to O_w for all Krull valuations w of T. But, with a finite number of exceptions, a Krull valuation of T is also a Krull valuation of $T(L)$. We take $\Sigma(T, L, M)$ to be the restrictions to K of these exceptional valuations. These restrictions are all proper, since those having trivial restriction to K are Krull valuations of $T(V)$ which is a ring of fractions of both T and $T(L)$ and hence are Krull valuations of both rings. Now suppose that v is a valuation in $\Sigma(T, L, M)$ and that w is a Krull valuation of T of which it is a restriction. Then if x belongs to $O_v T_n(M)$, x belongs to O_w. Hence if x belongs to $O_v T_n(M)$ for all v in $\Sigma(T, L, M)$ and to $T_n(L)$, it follows that x is integrally dependent on T and hence on $T_n(M)$. The 'only if' part of the first statement is immediate.

Now suppose that v is a valuation of $\Sigma(T, L', M)$. Then v is the restriction to K of a Krull valuation w of T which is not a Krull valuation of $T(L')$. Now if O_w contained $T(L')$, it would contain $T(L')^*$. Let $\mathfrak{p}$, $\mathfrak{p}'$ be the centres of w on T^* and $T(L')^*$. Then $T^*_{\mathfrak{p}} = O_w$ and so is a maximal sub-ring of the field of fractions of T (or $T(L')$). Hence $T(L')^*_{\mathfrak{p}'} = O_w$, and w is a Krull valuation of $T(L')$, contrary to hypothesis. Hence O_w does not contain $T(L')$ and therefore does not contain $T(L)$. Therefore w is not a Krull valuation of $T(L)$, and hence v belongs to $\Sigma(T, L, M)$.

Suppose v is a valuation in the set $\Sigma(T, L, M)$. Then v is the restriction of a Krull valuation of T which is not a Krull valuation of $T(L)$, that is, O_w does not contain $T(L)$ and hence L. This implies that $O_v M$ does not contain L and hence that O_v does not contain a^{-1} for any a in $M:L$, i.e. that $M:L$ is contained in the centre of v.

It should be emphasized that the set $\Sigma(T, L, M)$ depends on T and hence on the sequence of pairs of modules $T_n(L)$, $T_n(M)$ rather than merely on L, M. Hence, if we are only interested in determining the integral closure of M in L, a smaller set may suffice, but it seems difficult to characterize such a set.

We now turn to the case where T is taken to be the generic test domain $S = S(M)$. We first note that, for any T, the set $\Sigma(T, L, M)$ contains valuations of two types, those which are Krull valuations of R and those which are not. We now show that, if $T = S$, the valuations of the second type all belong to a finite set of valuations which does not depend on L. We first require a definition.

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DEFINITION. Let M be a finitely generated torsion-free R-module. Then we define the ideal $J(M)$ to be the union of the ideals $F:M$, where F ranges over all free sub-modules of M of the same rank as M.

THEOREM 1·8. (i) *Let M be a finitely generated torsion-free R-module. Then a discrete valuation v on K, $\geqslant 0$ on R, which is a restriction of a Krull valuation w of $S(M)$, but which is not itself a Krull valuation of R has centre containing $J(M)$, and so such valuations form a finite set $\Omega(M)$.*

(ii) *An element x of $V = KM$ is integrally dependent on M if and only if it belongs to $M\dagger\dagger$ and satisfies $x \in O_v M$ for all v in $\Omega(M)$, where*

$$M\dagger\dagger = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R^*), R^*).$$

(i) Let v have centre $\mathfrak{p}$ on R. Then, if there is an element a of $J(M)$ not in $\mathfrak{p}$, there is an element a of $F:M$ not in $\mathfrak{p}$ for some F, and hence $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module. It follows that $S(M_{\mathfrak{p}})$ is isomorphic to $R_{\mathfrak{p}}[X_1, \ldots, X_r]$ where $X_1, \ldots, X_r$ are indeterminates and r is the rank of M. Since w is a Krull valuation of $S(M_{\mathfrak{p}})$, it follows that v must be a Krull valuation of R and we have derived a contradiction. Since $J(M) \neq 0$, the number of such w is finite and hence the number of such v is finite.

(ii) A necessary and sufficient condition for x to be integrally dependent on M is that x is integrally dependent on $S(M)$, that is, if and only if $x \in O_w$ for all Krull valuations w of $S(M)$. This will be the case if $x \in O_v M$ for all Krull valuations v of R and, in addition, $x \in O_v M$ for all valuations v of $\Omega(M)$, and these conditions are necessary for x to be integrally dependent on M. Now consider the condition that x belongs to $O_v M$ for all Krull valuations v of R. This implies that, if $\alpha(z)$ is a K-linear function on V with the property that $\alpha(z)$ belongs to R^* if z belongs to M, then $\alpha(x)$ belongs to O_v for all Krull valuations v of R, and hence that $\alpha(x)$ belongs to R^*, that is, x belongs to $M\dagger\dagger = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R^*), R^*)$. Conversely, if x is an element of $M\dagger\dagger$, $\alpha(x)$ belongs to O_v for all Krull valuations v. Hence x belongs to the double dual of the O_v-module $O_v M$. But as this is a free O_v-module, it is equal to its double dual. Hence the intersection of $O_v M$ for all Krull valuations v of R is equal to $M\dagger\dagger$ and the result follows.

2. *The analytic spread of a module*

Let Q be a local noetherian ring, with maximal ideal $\mathfrak{m}$ and residue field $k = Q/\mathfrak{m}$, and let T be a finitely generated graded Q-algebra, generated by the elements of degree 1. Then, Ooishi [9] has defined the analytic spread of T to be the Krull dimension of the k-algebra $T/\mathfrak{m}T$. If P ranges over the minimal prime ideals associated with $\mathfrak{m}T$, this is clearly equal to the maximum of the Krull dimensions of the rings T/P, and this, in turn is equal to the maximum of the transcendence degrees of the fields of fractions of the domains T/P over k. It is this form of the definition that is the most convenient for our purposes, and in the special case where Q, T are domains and $T_0 = Q$, this is the definition that we shall adopt. However, in the general case we will modify the definition given by Ooishi as follows. Let S be the multiplicatively closed subset of elements of Q which belong to no minimal prime ideal of Q and let T' be the image of T under the natural map of T into T_S. Then we define the spread of T to be the analytic spread of T' according to Ooishi's definition. Note that the two definitions coincide if all minimal prime ideals of T meet

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Q in a minimal prime ideal, but if this is not the case, the two definitions may differ. For example, suppose that Q is a domain, that M is a torsion free Q-module, and that $T = \mathrm{Sym}_Q M$. According to Ooishi's definition, the analytic spread of T is then $\mu = \mu(M)$, where $\mu(M)$ is the minimal number of generators of M, since $T/\mathfrak{m}T$ is the symmetric algebra of $M/\mathfrak{m}M$ over k and so is isomorphic to $k[X_1, \ldots, X_\mu]$, where $X_1, \ldots, X_\mu$ are indeterminates over k. In general, however, the definition given here is less. In what follows we will only consider the case where Q is a domain and T is also a domain with $T_0 = Q$, and in this case the two definitions will coincide.

We now take R to be a noetherian domain, with field of fractions K, M to be a finitely generated torsion-free R-module and T to be a test domain for M. Let $\mathfrak{p}$ be a non-zero prime ideal of R. Then we denote the analytic spread of $T_{\mathfrak{p}}$ with respect to $Q = R_{\mathfrak{p}}$ by $s(T, M, \mathfrak{p})$. If T is the generic test domain $S = S_R(M)$ of M then we write $s(M, \mathfrak{p})$ for $s(S, M, \mathfrak{p})$. We also write $\dim T$ for the transcendence degree of the field of fractions of T over K. Note that if $T = S$, then $\dim S$ is simply the rank of M. The following lemma is, apart from notation, proposition 3 of [9]. Note that we do not need T to be a test domain, but only that T be a finitely generated graded R-algebra which is a domain and satisfies $T_0 = R$ and $T_1 = M$.

LEMMA 2·1. $\qquad \dim T \leqslant s(T, M, \mathfrak{p}) \leqslant \dim T + \mathrm{ht}\,\mathfrak{p} - 1.$

We can replace T by $T_{\mathfrak{p}}$ and R by Q. Now $s(T, M, \mathfrak{p})$ is the maximum of the transcendence degrees of the fields of fractions of $T/\mathfrak{P}$, where $\mathfrak{P}$ ranges over the minimal prime ideals of $\mathfrak{m}T$ over k where $\mathfrak{m}$ is the maximal ideal of Q and $k = Q/\mathfrak{m}$.

To prove the first inequality we use the theory of Hilbert functions. Consider KT. The dimensions of KT_n over K, that is the rank of T_n, is, for large n, equal to a polynomial in n of degree $\dim T - 1$. Also the dimension of $T_n/\mathfrak{m}T_n$ over k, or the minimum number of generators of T_n is for large n equal to a polynomial in n of degree $s(T, M, \mathfrak{p}) - 1$. Since the rank of T_n is at most equal to the minimum number of generators of T_n, the first inequality follows.

To prove the second inequality we use the altitude inequality. This gives

$$\mathrm{Kr.dim}\, Q + \dim T \geqslant \mathrm{trans.deg}_k (T/\mathfrak{P}) + \mathrm{ht}\,\mathfrak{P}$$

for each minimal prime ideal $\mathfrak{P}$ of $\mathfrak{m}T$. Since $\mathrm{Kr.dim}\, Q = \mathrm{ht}\,\mathfrak{p}$, and $\mathrm{ht}\,\mathfrak{P} \geqslant 1$ the inequality follows.

COROLLARY. *Let $\mathfrak{p}'$ be a prime ideal of R contained in $\mathfrak{p}$. Then*

$$s(T, M, \mathfrak{p}') \leqslant s(T, M, \mathfrak{p}).$$

Let k' be the residue field of $R_{\mathfrak{p}}$, and replace R, M, T by $R' = R/\mathfrak{p}'$, and the images M', T' of M, T in $k' \otimes_R M$ and $k' \otimes_R T$ respectively. Then the corollary follows from the first inequality applied to R', M', T' and the fact that T' need not be a test domain.

LEMMA 2·2. *Let Q be a local domain with maximal ideal $\mathfrak{m}$, infinite residue field $k = Q/\mathfrak{m}$, and let M be a finitely generated torsion-free Q-module of rank r. Then, if $s = s(M, \mathfrak{m})$, M has a reduction generated by s elements and no reduction generated by $< s$ elements. Further, if N is a reduction generated by s elements, a minimal basis of N forms part of a minimal basis of M.*

Let $u_1, \ldots, u_n$ be a minimal basis of M and let $x_1, \ldots, x_n$ be the images of $u_1, \ldots, u_n$ in $M/\mathfrak{m}M$ so that $x_1, \ldots, x_n$ is a basis of the vector space $M/\mathfrak{m}M$ over k. Now consider

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the generic test domain $S = S(M)$ of M, and let $S' = S/\mathfrak{m}S$. Then S' is a graded k-algebra and is generated by $x_1, \ldots, x_n$. Now by the definition of analytic spread and the Noether normalization theorem, we can suppose that $u_1, \ldots, u_n$ are so chosen that $x_1, \ldots, x_s$ are algebraically independent over k and that S' is a finite $k[x_1, \ldots, x_s]$-module. Alternatively expressed, this means that $x_1, \ldots, x_s$ generate an irrelevant ideal of S'. This implies that, for q large enough,

$$S'_{q+1}(M) = (x_1, \ldots, x_s) S'_q(M)$$

or

$$S_{q+1}(M) = (u_1, \ldots, u_s) S_q(M) + \mathfrak{m}S_{q+1}(M)$$

and, by Nakayama's lemma, that

$$S_{q+1}(M) = (u_1, \ldots, u_s) S_q(M).$$

But this implies that the ideal $(u_1, \ldots, u_s)$ of $S(M)$ is irrelevant, and hence that $u_1, \ldots, u_s$ generate a reduction of M. Conversely, suppose N is a reduction of M and is generated by s' elements. Then the images of these elements in $M/\mathfrak{m}M$ generate an irrelevant ideal of S'. But such an ideal cannot be generated by $< s$ elements. Hence $s' \geqslant s$ and if $s' = s$, these generators are linearly independent modulo $\mathfrak{m}M$ and so form part of a minimal basis of M.

For the remainder of this section we impose the restriction on R that it be an LQU-domain. We refer back to the introduction for the definition of such domains and for the properties (a), (b), (c) of quasi-unmixed local rings which we shall assume without proof. We require one further property of LQU-domains which we will now prove.

LEMMA 2·3. *If R is an* LQU*-domain, then a prime ideal $\mathfrak{p}$ of R is the centre of a Krull valuation of R if and only if it has height* 1.

By localization at $\mathfrak{p}$, we may assume that R is local with maximal ideal $\mathfrak{p}$. If $\mathfrak{p}$ has height 1, then every Krull valuation of R has centre $\mathfrak{p}$. Now assume that $\mathfrak{p}$ has height > 1. If $\mathfrak{p}$ is the centre of a Krull valuation of R, then it is the intersection of a height 1 prime ideal $\mathfrak{p}^*$ of R^* with R. Let x, z be any non-zero elements of $\mathfrak{p}$. Then the ideals $(xzR)^*$ are all unmixed of height 1, and so do not have $\mathfrak{p}$ as an associated prime. Divide the set of Krull valuations v of R for which $v(x) > 0$ into two classes, $v_1, \ldots, v_k$ being those which have centre $\mathfrak{p}$, and $v_{k+1}, \ldots, v_m$ being those which do not. Then we can choose y, z in R such that y/z belongs to R^* and satisfies $v_i(y/z) = 0$ if $i \leqslant k$ and $v_i(y/z) > v_i(x)$ if $i > k$. It follows that, if $i \leqslant k$, $v_i(y) < v_i(xz)$, so that y does not belong to $(zxR)^*$, while if $i > k$, $v_i(y) > v_i(xz)$. Now, since $v_i(a) > 0$ for all a in $\mathfrak{p}$ if $i \leqslant k$, it follows that if n is large enough and a belongs to $\mathfrak{p}^n$, $v_i(a) \geqslant v_i(x)$ for $i = 1, \ldots, k$. It now follows that if v is a Krull valuation of R such that $v(zx) > 0$, then $v(ay) > v(zx)$ and hence that $\mathfrak{p}$ is contained in an associated prime of $(zxR)^*$ and so has height $\leqslant 1$, contradicting assumption.

Our next theorem is a variant of a theorem of Macadam's (see [6], proposition 4·1 on p. 26).

THEOREM 2·4. *Let R be an* LQU*-domain, and let T be a finitely generated graded domain finitely generated over R. Let the field of fractions of T have transcendence degree r over that of R. Then, if $\mathfrak{p}$ is a prime ideal of R, there exists a Krull valuation of T having centre $\mathfrak{p}$ on R if and only if*

$$s(T, \mathfrak{p}) = \operatorname{ht} \mathfrak{p} + r - 1.$$

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We may assume R local with maximal ideal $\mathfrak{p}$. By the last lemma, the condition that $\mathfrak{p}$ is the centre of a Krull valuation of T is equivalent to the condition that it is the intersection of a height 1 prime ideal of T with R (since T is also an LQU-domain). Suppose that $\mathfrak{P}$ is a prime ideal of T meeting R in $\mathfrak{p}$. Then, if s is the transcendence degree of $T/\mathfrak{P}$ over $k = R/\mathfrak{p}$,

$$s + \operatorname{ht}\mathfrak{P} = \operatorname{ht}\mathfrak{p} + r,$$

by the altitude equality. This shows that $s \leqslant \operatorname{ht}\mathfrak{p} + r - 1$, with equality if and only if $\operatorname{ht}\mathfrak{P} = 1$. Hence, since $s(T, \mathfrak{p})$ is the maximum of s as $\mathfrak{P}$ ranges over the minimal prime ideals of $\mathfrak{p}S$, it follows that there is a height 1 prime ideal of S meeting R in $\mathfrak{p}$ if and only if

$$s(T, \mathfrak{p}) = \operatorname{ht}\mathfrak{p} + r - 1.$$

We now translate this theorem into a theorem giving a sufficient condition for a module M to be a reduction of a module L containing it.

THEOREM 2·5. *Let R be an* LQU*-domain with field of fractions K, and let L, M be finitely generated torsion-free R-modules with M a sub-module of L and $KL = KM$. Let T be a test domain for M, and let $r = \dim T$. Then if, for all maximal ideals $\mathfrak{m}$ of R containing the annihilator $M:L$ of L/M,*

$$s(T, M, \mathfrak{m}) < \operatorname{ht}(M:L) + r - 1,$$

M is a reduction of L.

By the corollary to 2·1, we can replace the left-hand side of the inequality by $s(T, M, \mathfrak{p})$, where $\mathfrak{p}$ is any prime ideal of R containing $M:L$, and we can replace $\operatorname{ht}(M:L)$ on the right-hand side by $\operatorname{ht}\mathfrak{p}$. It now follows from the last theorem that no prime ideal $\mathfrak{p}$ of R containing $M:L$ is the centre of a Krull valuation of T on R. This implies that the set $\Sigma(T, L, M)$ is empty, and so M is a reduction of L.

3. *Theorems on Koszul complexes and Grassmann domains*

The central theorem of this section is the following.

THEOREM 3·1. *Let R be an* LQU*-domain, and let $x_1, \ldots, x_n$ be elements of R which generate a proper ideal I of height h. Let*

$$\mathbf{K}: \qquad 0 \longrightarrow K_n \xrightarrow{d_n} K_{n-1} \xrightarrow{d_{n-1}} K_{n-2} \xrightarrow{d_{n-2}} \ldots \xrightarrow{d_1} K_0 \longrightarrow 0$$

be the Koszul Complex of R with respect to the sequence $x_1, \ldots, x_n$. Then, if $B_r = d_{r+1}(K_{r+1})$ and $Z_r = \operatorname{Ker} d_r$, B_r is a reduction of Z_r for $r > n-h$.

The proof of this theorem depends on the choice of a suitable test domain for B_r, and most of the proof is concerned with proving that the ring T chosen satisfies the conditions for it to be a test domain. We commence by describing the ring T.

We first indicate some changes of notation which are made below for typographical reasons. We will be concerned below with an $r \times n$ matrix $\mathbf{X}_r$ whose elements are independent indeterminates X_{ij} over R. Since the suffixes i, j occurring in this symbol may themselves carry suffixes, we will write $X(i, j)$ in place of X_{ij}. For the same reason, the elements x_i occurring in the statement of the theorem above will be denoted below by $x(i)$.

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We continue by introducing the homogeneous Grassmann ring $G_{n,r}(R)$ over R which is defined to be the graded R-algebra over R generated by the $r \times r$ minors of $\mathbf{X}_r = (X(i,j))$, the minors being given degree 1. That this ring is graded is most easily seen by considering it as a sub-ring of the polynomial ring $R[X(i,j)]$. In this ring the elements of $G_{n,r}(R)$ all have degrees multiples of r, and we divide degrees by r. We now introduce further notation. The minor equal to the determinant of the sub-matrix of $\mathbf{X}_r$ formed by columns $i_1, \ldots, i_r$ of $\mathbf{X}_r$, in that order, will be denoted by $M(i_1, \ldots, i_r)$. Finally, we take $G_{n,0}$ to be the ring $R[t]$, where t is an indeterminate over R.

We now define graded ring homomorphisms

$$D_r : G_{n,r}(R) \to G_{n,r-1}(R)$$

as follows. First suppose that $r > 1$. Then $D_r(M(i_1, \ldots, i_r))$ is defined to be the minor of the matrix obtained by replacing the last row of $\mathbf{X}_r$ by the row $x(1), \ldots, x(n)$. Thus,

$$D_r(M(i_1, \ldots, i_r)) = \sum (-1)^{j-1} x(i_j)\, M(i_1, \ldots, \hat{i}_j, \ldots, i_r),$$

where the circumflex denote that the index below it is to be omitted, and the minor is to be considered as a minor of $\mathbf{X}_{r-1}$. This defines a map of the free R-module of elements of degree 1 of $G_{n,r}$ into the similar module for $G_{n,r-1}$ and this map can be extended to a graded ring-homomorphism of $G_{n,r}$ into $G_{n,r-1}$, since any polynomial relation between the elements $M(i_1, \ldots, i_r)$ is still satisfied by the elements $D_r(M(i_1, \ldots, i_r))$. Finally, if $r = 1$, we define $D_1(x(1,j)) = x(j)\,t$. It is clear that the maps D_r satisfy $D_{r-1} D_r = 0$.

Now consider what happens in degree 1. We can identify the module of elements of degree 1 in $G_{n,r}$ with K_r. Further, with this identification, the map D_r, restricted to elements of degree 1, coincides with d_r. Hence we can identify B_r, Z_r with sub-modules of the modules of elements of degree 1 in $G_{n,r}$. We now define T to be the graded sub-ring of $G_{n,r}$ generated by B_r. Then T is a domain, and further $T_0 = R$, $T_1 = B_r$. Hence we only have to verify that for all discrete valuation rings O_v between R and its field of fractions K, $O_v T$ is normal. This follows from the following lemma using results of Hochster [3] and Igusa [5], which we will indicate below.

LEMMA 3·2. *If O_v, T are as above, then $O_v T$ is isomorphic to $G_{n-1,r}(O_v)$. Also KT is isomorphic to $G_{n-1,r}(K)$. As a consequence, $O_v T$ is integrally closed in its field of fractions.*

By permuting the values of the suffix j if necessary, we can assume that $v(x(1)) \leqslant v(x(j))$ for $j > 1$. Now consider the matrix

$$\begin{pmatrix} X(1,1) & \ldots & X(1,n) \\ X(r,1) & \ldots & X(r,n) \\ x(1) & \ldots & x(n) \end{pmatrix}$$

whose $(r+1)$-rowed minors generate $O_v T$. Now, by hypothesis, $x(j) = a(j)\,x(1)$ with $a(j)$ in O_v for $j = 2, \ldots, n$. Subtract $a(j) \times$ column 1 from column j for $j = 2, \ldots, n$. Then we obtain a matrix whose $(r+1)$-rowed minors still generate $O_v T$ over O_v. But the $(r+1)$-rowed minors of this matrix are simply the r-rowed minors of the matrix

$$\begin{pmatrix} Y(1,2) & \ldots & Y(1,n) \\ Y(r,2) & \ldots & Y(r,n) \end{pmatrix}$$

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multiplied by $x(1)$, where $Y(i,j) = X(i,j) - a(j)\,X(i,1)$. Since we can treat the elements $Y(i,j)$ $(i = 1,\ldots,r;\ j = 2,\ldots,n)$ as independent indeterminates over O_v, it follows that $O_v T$ is isomorphic to $G_{n-1,r}(O_v)$. The proof that KT is isomorphic to $G_{n-1,r}(K)$ is similar.

We note next that by Igusa's Theorem [**5**], $G_{n,r}(K)$ is integrally closed in its field of fractions. Next, for any ring R, $G_{n,r}(R) = R \otimes_Z G_{n,r}(Z)$, and $G_{n,r}(Z)$ is a free Z-module. Hence since $G_{n,r}(K)$ is normal for all choices of the field K, it follows by lemma 3·14 of [**3**] that $G_{n,r}(R)$ is normal if R is normal, in particular if $R = O_v$. (Alternatively, we can observe that, if p is a generator of the maximal ideal of O_v and k_v is the residue field of O_v, then $G_{n,r}(O_v)/pG_{n,r}(O_v)$ is isomorphic to $G_{n,r}(k_v)$ and so is a domain. Hence, by Lemma 1·4, $G_{n,r}(O_v)$ is normal.) Replacing n by $n-1$, we have proved the last statement of the Lemma.

This completes the proof that T is a test domain.

To complete the proof of Theorem 3·1 we first require the following elementary lemma (for a geometrical version, see Hodge and Pedoe[**4**], volume 2, chapter 14, p. 313). For completeness we give a brief indication of the proof.

LEMMA 3·3. *If k is any field, the field of fractions K of $G_{n,r}(k)$ has transcendence degree $r(n-r)+1$ over k.*

Let $\mathbf{X}$ be the matrix $(X(i,j))$. If we replace $X(i,j)$ by 0 if $i \neq j \leqslant r$ and $X(i,i)$ by X if $i \leqslant r$, then we obtain a matrix whose r-rowed minors generate the field $k(X, X(i,j))$ where i runs from 1 to r and j runs from $r+1$ to n. Hence this field has transcendence degree $r(n-r)+1$. Since the ring generated by these minors is a homomorphic image of $G_{n,r}(k)$, it follows that the transcendence degree of the field of fractions of the latter over k is at least $r(n-r)+1$.

On the other hand, if we multiply $\mathbf{X}$ by the inverse of the matrix $\mathbf{Y}$ formed by the first r rows and columns of $\mathbf{X}$, we obtain a matrix of the above type, with 1 replacing $\mathbf{X}$ and certain r-rowed minors of $\mathbf{X}$ multiplied by $(\det \mathbf{Y})^{-1}$ replacing $X(i,j)$. It is easy to see that K is generated over k by the entries of this matrix, together with the inverse of $\det(\mathbf{Y})$. Since the number of entries in this matrix not equal to 0 or 1 is at most $r(n-r)$, it follows that K has transcendence degree at most $r(n-r)+1$ over k.

We can now complete the proof of Theorem 3·1. First we note that $\dim T$ is equal to $r(n-r-1)+1$. Next, suppose that $\mathfrak{p}$ is any prime ideal of R containing I, and that $Q = R_{\mathfrak{p}}$, $\mathfrak{m}$ is the maximal ideal of Q and $k = Q/\mathfrak{m}$. Then $T' = T_{\mathfrak{p}}/\mathfrak{m}T_{\mathfrak{p}}$ is a homomorphic image of $G_{n,r+1}(k)$, and so, if $\mathfrak{P}$ is a minimal prime ideal of T', its field of fractions has transcendence degree at most $(r+1)(n-r-1)+1$ over k. This implies that $s(T, B_r, \mathfrak{p})$ is at most $(r+1)(n-r-1)+1$. Hence

$$s(T, B_r, \mathfrak{p}) - \dim T \leqslant (r+1)(n-r-1) - r(n-r-1) = n-r-1.$$

But Z_r/B_r is annihilated by I, which has height h. Hence, if $r > n-h$,

$$s(T, B_r, \mathfrak{p}) - \dim T < h-1,$$

and by Theorem 2·5, this implies that B_r is a reduction of Z_r.

The above theorem should, perhaps, be compared with the well-known result which states that, if I has grade g, then $H_r(\mathbf{K})$ vanishes if $r > n-g$ and further $H_{n-g}(\mathbf{K}) \neq 0$. It follows that the above theorem is incomplete in the sense that it gives no

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information as to whether B_{n-h} is a reduction of Z_{n-h} or not. In fact both possibilities can occur. First suppose that $n = h$. Then $B_0 = I$ and $Z_0 = R$. Hence in this case, B_0 is not a reduction of Z_0. For the contrary case, take R to be a local domain of dimension 1, and suppose that $x_1, \ldots, x_n$ is a minimal basis of the maximal ideal $\mathfrak{m}$ of R, n being >1. We can identify K_{n-1} with a free R-module of rank n with basis $u_1, \ldots, u_n$ and identify B_{n-1} with the free module generated by the single element $\Sigma x_i u_i$. Then Z_{n-1} consists of all elements $\Sigma y_i u_i$ of K_{n-1} where $x_i y_j - x_j y_i = 0$ for $i, j = 1, \ldots, n$. It is clear that this implies that $y_i = a x_i$ $(i = 1, \ldots, n)$ where a belongs to $\mathfrak{m}^{-1}$. Now consider the product $\mathfrak{m}\mathfrak{m}^{-1}$. This must be $\mathfrak{m}$, since otherwise $\mathfrak{m}$ would be invertible, and therefore generated by one element. Hence a is integrally dependent on R, and so $\mathfrak{m}^{-1}$ is an integral ring extension of R. It follows that $O_v Z_{n-1} = O_v B_{n-1}$ for all valuation rings O_v contained in the field of fractions of R and containing R, and hence B_{n-1} is a reduction of Z_{n-1}.

We now leave Koszul complexes and prove a theorem on Grassmann domains $G_{n,r}(R)$ which can be considered as a generalization of Theorem 3·1 above. In order to prove this theorem, we need a theorem of Hodge concerning such domains which we now describe. (The theorem can be found in [**4**], vol. 2, as theorem 1 on p. 378. A modern treatment can be found in [**2**] as theorem 2·1′ on p. 142.) Suppose that we have an ordered product $M_1 M_2 \ldots M_s$ of $r \times r$ minors of the $r \times n$ matrix $\mathbf{X}_r$ already considered. We can associate with this product a tableau $\mathbf{T}$, that is an $r \times s$ matrix whose entries are integers in the range 1 to n, the jth column consisting of the numbers $j(1), \ldots, j(r)$ where the columns of M_j are columns $j(1), \ldots, j(r)$ of $\mathbf{X}_r$. A tableau is termed standard if its columns, read from top to bottom, are strictly increasing, and its rows, read from left to right, are increasing. Then Hodge's theorem states that the products of minors corresponding to standard tableaux form a free basis of the R-module G_s composed of the elements of $G_{n,r}(R)$ which are homogeneous of degree s (implying that this module is free).

We now indicate briefly the theorem which is our objective, and also introduce some notation. We first suppose that R is fixed and is an LQU-domain. We also suppose that $x(1), \ldots, x(n)$ are elements of R which generate an ideal I of height h. However, we will allow $I = R$, when h is not strictly defined. Here n is fixed and we also fix an integer r. We will write G for the ring $G_{n,r}(R)$, re-introducing the full notation whenever r or n is altered, and $G(S)$ for $G_{n,r}(S)$ if R is replaced by another ring S. Let B_r be the module of r-boundaries considered above, and consider it as an R-module embedded in the module of homogeneous elements of G of degree 1. Then B_r generates an ideal of G, which we will denote by B. Next consider the homomorphism D_r of G into $G_{n,r-1}(R)$. We will denote the kernel of this homomorphism by C. Note that C is a prime ideal of G containing B. Then our objective is to prove that, if $r > n-h$, C is the integral closure of B. A proof of this result, in the special case where $n = h$, $r = 1$, and R was a quasi-unmixed local ring, was the main result of [**11**].

We introduce two further bits of notation. We will write $\mathbf{X}$ in place of $\mathbf{X}_r$, if the value of r is the given one, and we will write $\mathbf{X}/\mathbf{x}$ for the $(r+1) \times n$ matrix formed by adding the row $x(1), \ldots, x(n)$ as an $(r+1)$st row to $\mathbf{X}$.

LEMMA 3·4. *Suppose that one of the elements $x(i)$ is a unit. Then*

$$C = B.$$

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We may assume that $x(1)$ is a unit without loss of generality. Now consider the matrix $\mathbf{X}/\mathbf{x}$. Multiply column 1 by $x(1)^{-1}x(j)$ and subtract from column j for $j = 2, \ldots, n$ and then multiply column 1 by $x(1)^{-1}$. The result is to replace the last row of $\mathbf{X}/\mathbf{x}$ by the row $(1, 0, \ldots, 0)$ and the entries in the first r rows by elements $Y(i,j)$ where $Y(i,1) = x(1)^{-1}X(i,1)$ and, if $j > 1$, $Y(i,j) = X(i,j) - x(1)^{-1}x(j)\,X(i,j)$. Then the map of G into itself induced by the map $X(i,j) \to Y(i,j)$ is an automorphism of G which sends B, C into the corresponding ideals when $x(1), \ldots, x(n)$ are replaced by $1, 0, \ldots, 0$. Hence we will assume these values for $x(j)$ $(j = 1, \ldots, n)$.

In this case, the map D_r will map $M(j(1), \ldots, j(r))$ into zero if $j(1) > 1$ and $M(1, j(2), \ldots, j(r))$ into the element $M(j(2), \ldots, j(r))$ of $G_{n,r-1}(R)$. Further, B is generated by the minors $M(j(1), \ldots, j(r))$ with $j(1) > 1$. Now consider a standard tableau $\mathbf{T}$. If the first row of $\mathbf{T}$ contains an entry > 1, it is clear that the corresponding product of minors is in B. If the first row consists entirely of 1's, then the corresponding product of minors is mapped onto the product of minors of the matrix derived from $\mathbf{X}$ by deleting the first column and the last row, and corresponding to the standard tableau obtained by deleting the first row from $\mathbf{T}$ and replacing $1, \ldots, n$ by $2, \ldots, n$. Now these products form a free basis of the module of elements of degree s in a ring isomorphic to $G_{n-1,r-1}(R)$, and different products corresponding to tableaux with first row consisting of 1's map into different elements of this basis. Hence no linear combination of such products other than 0 maps into zero. Hence $C = B$.

LEMMA 3·5. *If N is large enough, C/B is annihilated by the ideal $(x(1)^N, \ldots, x(n)^N)$.*

By symmetry, it is sufficient to prove that $x(1)^N$ annihilates C/B for large N. Now let S be the multiplicatively closed set consisting of the powers of $x(1)$. If we replace R by R_S, then $C(R_S)/B(R_S) = 0$. This implies that $(C/B)_S = 0$ and hence that C/B is annihilated by a power of $x(1)$.

THEOREM 3·6. *Let R be an* LQU*-domain, G be the Grassmann domain generated over R by the $r \times r$ minors of the $r \times n$ matrix $X = (X(i,j))$ where the elements $X(i,j)$ are indeterminates over R and let I be an ideal of R of height h generated by $x(1), \ldots, x(n)$. Let B, C be as defined above. Then, if $r > n-h$, C is the integral closure of B.*

Since C, being prime, is integrally closed, it is sufficient to prove that, for all valuations v associated with B, $O_v C = O_v B$. Let $\mathfrak{P}$ be the centre of such a valuation. Then, by Macadam's theorem,

$$\operatorname{ht} \mathfrak{P} = s(B, \mathfrak{P}).$$

Now B is generated by the $(r+1) \times (r+1)$ minors of the $(r+1) \times n$ matrix $\mathbf{X}/\mathbf{x}$, and hence it follows that $s(B, \mathfrak{P}) \leqslant (r+1)(n-r-1)+1$. Now consider the ideal Ω generated by the $r \times r$ minors of $\mathbf{X}$, which has height $r(n-r)+1$. If $\mathfrak{p}$ is any prime ideal of R of height h, then it is clear that $\mathfrak{p}G+\Omega$ is a prime ideal of height $h+r(n-r)+1$ and hence, if $h > n-r$, has height $> s(B, \mathfrak{P})$. It follows that $\mathfrak{P}$ cannot contain all the elements $M(j(1), \ldots, j(r))$ and $x(1), \ldots, x(n)$. Suppose first that it does not contain all the minors; to be definite, suppose it does not contain $M(1, 2, \ldots, r) = M$. Let S be the m.c.s. consisting of the powers of M, and consider the ring G_S. Then v is an associated valuation of BG_S. Further, since $h > n-r$, it follows that not more than $r-1$ of the elements $x(j)$ can be zero, and this implies that M is not mapped into zero by D_r. Hence M does not belong to C and the ideal

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C_S is a proper prime ideal of G_S of the same height as C. Now consider G_S. Multiplying X by the inverse of the $r \times r$ matrix formed by the columns of M we see that G_S is generated over $R[M^{-1}]$ by the $r \times r$ minors of a matrix of the form $(\mathbf{I}_r, \mathbf{Y})$ where $\mathbf{Y}$ is an $r \times n-r$ matrix with entries $Y(i,j)$, and hence by the elements $Y(i,j)$ themselves. It follows that the elements $Y(i,j)$ are algebraically independent and hence G_S is a polynomial extension of $R[M^{-1}]$. Now consider B_S. It is generated by the $(r+1) \times (r+1)$ minors of the matrix $(\mathbf{I}_r, \mathbf{Y})/\mathbf{x}$. If we now subtract $x(j) \times$ row j from row $r+1$ for each j from 1 to r, the last row of this matrix is replaced by

$$(0, \ldots, 0, z(r+1), \ldots, z(n))$$

where $z(j) = x(j) - \Sigma\, x(i)\, Y(i,j)$, the sum being from $i = 1$ to r, and it is clear that B_S is generated by $n-r$ elements and so $s(B_S, \mathfrak{P}_S)$ is at most $n-r$ and hence $< h$. Hence $\mathfrak{P}_S$ cannot contain IG_S and therefore $\mathfrak{P}$ cannot contain I. It now follows from Lemma 3·5, since O_v contains $x(j)^{-1}$ for some j, that $CO_v = BO_v$ for all associated valuations of B and hence that C is the integral closure of B.

COROLLARY. *$(B^N)^*$ is C-primary for all N.*

To prove this result, we have to prove that every associated valuation of B has centre C. Since C is, by the Theorem, the radical of B, it follows that the centres of such valuations all contain C. Now C does not meet R. Hence, if S is the set of all non-zero elements of R, C has the same height as the ideal CG_S of G_S. But $G_S = G_{n,r}(K)$ and G_S/C_S is isomorphic to $G_{n-1,r-1}(K)$ by Lemma 3·2. Hence, since we are dealing with finitely generated extensions of K, and G_S, G_S/C_S have, respectively, transcendence degrees equal to $r(n-r)+1$ and $(r-1)(n-r)+1$ over K, it follows that the height of C is $n-r < h$. Now it follows from the last paragraph above that if the centre $\mathfrak{P}$ of an associated valuation v does not contain Ω, then $\mathfrak{P}$ has height at most $n-r$ and so must be C. If it does contain Ω, then it cannot contain all the elements $x(1), \ldots, x(n)$. Suppose it does not contain $x(1)$, then, by localizing at $\mathfrak{P}$, and subtracting suitable multiples of the first column of $\mathbf{X}/\mathbf{x}$ from the others, we see that $B_{\mathfrak{p}}$ is generated by the minors of an $r \times (n-1)$ matrix and consequently $s(B_{\mathfrak{p}}, PG_{\mathfrak{p}})$ is at most $r(n-r-1)+1$ which is less than the height of Ω. Hence we have proved that the only possible centre of v is C.

We conclude this section with some observations on Theorem 3·6 and its Corollary. Once we have proved that the associated valuations of B all have centre C, which meets R in (O), it follows that these valuations are the associated valuations of the ideal $CG(K)$ of the Grassmann domain $G(K) = G_{n,r}(K)$ over K. Write C in place of $CG(K)$. Then the ring $G(K)_C$ is a regular local ring of dimension $n-r$. For C has height $n-r$ and $C = B$ in this case. Hence the maximal ideal of $G(K)_C$ is generated by the elements $z_{r+1}, \ldots, z_n$ defined in the proof of the theorem. It follows that the set of valuations associated with B consists of one valuation, taking zero value on K.

Our second observation concerns the restriction that R be a domain. This can be dropped, if we reinterpret C as the inverse image of the radical of $G_{n,r-1}$ under the map D_r. The proof can be reduced to the domain case by considering the rings $R/\mathfrak{p}$ in place of R, where $\mathfrak{p}$ ranges over the minimal prime ideals of R in much the same way as in [**11**].

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