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Conditional symmetries and exact solutions of a nonlinear three-component reaction-diffusion model

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Q-conditional (non-classical) symmetries of the known three-component reaction-diffusion (RD) system [K. Aoki et al. *Theor. Popul. Biol.* **50**, 1–17 (1996)] modelling interaction between farmers and hunter-gatherers are constructed for the first time. A wide variety of Q-conditional symmetries are found, and it is shown that these symmetries are not equivalent to the Lie symmetries. Some operators of Q-conditional (non-classical) symmetry are applied for finding exact solutions of the RD system in question. Properties of the exact solutions (in particular, their asymptotic behaviour) are identified and possible biological interpretation is discussed.

Key words: Reaction-diffusion system, hunter-gatherer–farmer system, Lie symmetry, Q-conditional symmetry, exact solution

2020 Mathematics Subject Classification: 35B06 (Primary); 35K57, 35Cxx

1 Introduction

In [2], a three-component model for describing the spread of an initially localised population of farmers into a region occupied by hunter-gatherers was introduced. Under some assumptions clearly indicated in [2], the spread and interaction between farmers and hunter-gatherers can be modelled as a reaction-diffusion (RD) process in the form of the three-component system of non-linear partial differential equations (PDEs). Recently, the model was used for mathematical description of some other phenomena. For example, a model describing language competition was derived in [36]. The model is based on the three-component system of non-linear PDEs, which has the same structure as the system introduced in [2]; however, some coefficients have opposite signs. Notably, the model proposed in [36] is a modification of another model for language competition developed earlier in [24] (see also the recent papers [15, 23]).

Here we study the original model from the paper [2] used for modelling competition between farmers and hunter-gatherers. The work is a natural continuation of our recent paper [14], in which Lie symmetries and travelling fronts of this model have been studied. After the relevant re-scaling (see [14] for details), the model takes the form of the non-linear RD system

$$\begin{aligned} u_t &= d_1 u_{xx} + u(1-u-a_1 v),\\ v_t &= d_2 v_{xx} + a_2 v(1-u-a_1 v) + uw + a_1 vw,\\ w_t &= d_3 w_{xx} + a_3 w(1-w) - a_4 uw - a_5 vw, \end{aligned} \tag{1.1}$$

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where $u(t,x)$, $v(t,x)$ and $w(t,x)$ are non-dimensional densities of the three populations of initial farmers, converted farmers and hunter-gatherers, respectively (hereafter the lower subscripts t and x mean differentiation w.r.t. these variables). Here (1.1) is called the hunter-gatherer–farmer (HGF) system and one is the main object of investigation in this paper. A detailed explanation of the meaning of the terms and coefficients in (1.1) is presented in [2, 14]. Here we only remind that the diffusivities d_1, d_2 and d_3 are positive constants, while other parameters are non-negative constant, except a_4 that must be a positive constant (otherwise, the carrying capacity of farmers is zero [14]). Obviously, the HGF system (1.1) is not a particular case of the well-known diffusive Lotka–Volterra (DLV) system

$$\begin{aligned} u_t &= d_1 u_{xx} + u(a_1 + b_1 u + c_1 v + h_1 w),\\ v_t &= d_2 v_{xx} + v(a_2 + b_2 u + c_2 v + h_2 w),\\ w_t &= d_3 w_{xx} + w(a_3 + b_3 u + c_3 v + h_3 w), \end{aligned} \tag{1.2}$$

because of the term uw in the second equation of (1.1). Notably, in the special case

$$d_1 = d_2,\ a_1 \neq 0,\ a_2 = 1$$

system (1.1) is reduced to the DLV system by the transformation $u + a_1 v \to v$.

In contrast to our previous study [14], which is devoted to Lie symmetries, here we search for Q-conditional symmetry (non-classical symmetry) of the HGF system (1.1) in the form

$$\begin{aligned} Q = {} & \xi^0(t,x,u,v,w)\partial_t + \xi^1(t,x,u,v,w)\partial_x \\ & + \eta^1(t,x,u,v,w)\partial_u + \eta^2(t,x,u,v,w)\partial_v + \eta^3(t,x,u,v,w)\partial_w,\ \left(\xi^0\right)^2 + \left(\xi^1\right)^2 \neq 0, \end{aligned} \tag{1.3}$$

where the coefficients ξ^0, ξ^1 and η^k $(k = 1, 2, 3)$ should be found using the well-known criterion (hereafter, $\partial_z \equiv \frac{\partial}{\partial z}$, $z = t, x, u, v, w$). The notion of non-classical symmetry was introduced in [8] and plays an important role in investigation of non-linear PDEs (see review [34] and monographs [7, 13, 18] for more details). In particular, having such symmetries in an explicit form (i.e., all the coefficients in operator (1.3) are known functions), one may construct new exact solutions, which are not obtainable by the classical Lie algorithm.

The algorithm for finding Q-conditional symmetry (following [21], we use this terminology instead of non-classical symmetry) of a given PDEs is based on the classical Lie method [30,32]. However, in contrast to the case of Lie symmetry, the corresponding system of determining equations (DEs) is *nonlinear* and its general solution can be found only in exceptional cases. So, obtaining an exhaustive description of Q-conditional symmetry of the given equation is a non-trivial and difficult task. As a result, *scalar PDEs only* were under study for a long time because systems of DEs for systems of PDEs are much more complicated. In the case of scalar PDEs, the first highly non-trivial results were derived in [4, 19, 21, 26, 29, 31, 35, 40] (see also an extensive review on this matter in Chapter 1 of [13] and the most recent papers [5, 25]). To the best of our knowledge, there are only a few papers devoted to the search of Q-conditional symmetries for systems of PDEs published before 2010 [1, 6, 16, 17, 27]. A majority of such papers were published during the current decade [3, 9–12, 37]. It should be stressed that only the papers [1, Section 4.1] and [11] are devoted to construct Q-conditional symmetry operators for a *three-component PDE system* while two-component systems only are under study in other papers. Notably, the conditional symmetries of the three-component Prandtl system derived in [1]

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coincide with the relevant Lie symmetries. In paper [11], several Q-conditional symmetries of the three-component DLV system are found, and it is shown that they are not obtainable by the classical Lie method. Here we make essential progress comparing with the papers cited above because *all possible Q-conditional symmetries* (excepting the so-called no-go case $\xi^0 = 0$) of the HGF system (1.1) are constructed in an explicit form.

It should be also mentioned that there is a further generalisation of the notion of Q-conditional symmetry – generalised conditional symmetry – introduced in [20] (the terminology 'conditional Lie-Bäcklund symmetry' suggested in [39] has the same definition but one is rather misleading). Recently, some two-component systems of evolution equations (including RD equations) have been studied using the generalised conditional symmetry method [22, 38].

The paper is organised as follows. In Section 2, the main theorem about Q-conditional symmetries of the HGF system (1.1) is proved. In Section 3, the most interesting (from applicability point of view) case of system (1.1) is examined. In particular, non-Lie ansätze are derived and applied for reducing the system in question to systems of ODEs. The reduced systems are analysed in order to construct exact solutions, and some highly nontrivial exact solutions are derived and their properties are studied. Finally, we briefly discuss the result obtained and present some conclusions in the last section.

2 Q-conditional symmetries of the HGF system (1.1)

First of all, we remind the reader that similarly to Lie symmetries, Q-conditional symmetries are constructed in the form (1.3) using the invariance criterion, which was used for the first time in the pioneering work [8]. Taking into account the property of the Q-conditional symmetries (operator can be multiplied by an arbitrary smooth function), operator (1.3) has different forms in the cases $\xi^0 \neq 0$ and $\xi^0 = 0$, namely,

$$Q = \partial_t + \xi(t,x,u,v,w)\partial_x + \eta^1(t,x,u,v,w)\partial_u + \eta^2(t,x,u,v,w)\partial_v + \eta^3(t,x,u,v,w)\partial_w, \qquad (2.1)$$

and

$$Q = \partial_x + \eta^1(t,x,u,v,w)\partial_u + \eta^2(t,x,u,v,w)\partial_v + \eta^3(t,x,u,v,w)\partial_w. \qquad (2.2)$$

It turns out that the systems of DEs for finding Q-conditional symmetries (2.1) and (2.2) are essentially different (see, e.g., [13, Section 1.4]). Hereafter, we concentrate ourselves on the case $\xi^0 \neq 0$.

According to the invariance criterion extended on the PDE systems (see, e.g., [13, Section 2.2]), operator (2.1) is the Q-conditional symmetry (non-classical symmetry) for the HGF system (1.1) if the following invariance conditions are satisfied:

$$\begin{aligned}
&\underset{2}{Q}(S_1)\Big|_{\mathcal{M}} \equiv \underset{2}{Q}\left(d_1u_{xx} - u_t + u(1-u-a_1v)\right)\Big|_{\mathcal{M}} = 0,\\
&\underset{2}{Q}(S_2)\Big|_{\mathcal{M}} \equiv \underset{2}{Q}\left(d_2v_{xx} - v_t + a_2v(1-u-a_1v) + uw + a_1vw\right)\Big|_{\mathcal{M}} = 0,\\
&\underset{2}{Q}(S_3)\Big|_{\mathcal{M}} \equiv \underset{2}{Q}\left(d_3w_{xx} - w_t + a_3w(1-w) - a_4uw - a_5vw\right)\Big|_{\mathcal{M}} = 0,
\end{aligned} \qquad (2.3)$$

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where operator $\underset{2}{Q}$ is the second prolongation of the operator Q, the manifold

$$\mathcal{M} = \{S_1 = 0,\ S_2 = 0,\ S_3 = 0,\ Q(u) = 0,\ Q(v) = 0,\ Q(w) = 0\}.$$

Here by the definition $Q(u) = u_t + \xi u_x - \eta^1$, $Q(v) = v_t + \xi v_x - \eta^2$, $Q(w) = w_t + \xi w_x - \eta^3$ and the second prolongation of the operator Q has the form

$$\begin{aligned}\underset{2}{Q} = Q &+ \rho^1_t \tfrac{\partial}{\partial u_t} + \rho^2_t \tfrac{\partial}{\partial v_t} + \rho^3_t \tfrac{\partial}{\partial w_t} + \rho^1_x \tfrac{\partial}{\partial u_x} + \rho^2_x \tfrac{\partial}{\partial v_x} + \rho^3_x \tfrac{\partial}{\partial w_x} + \sigma^1_{tx} \tfrac{\partial}{\partial u_{tx}} + \sigma^2_{tx} \tfrac{\partial}{\partial v_{tx}} \\ &+ \sigma^3_{tx} \tfrac{\partial}{\partial w_{tx}} + \sigma^1_{tt} \tfrac{\partial}{\partial u_{tt}} + \sigma^2_{tt} \tfrac{\partial}{\partial v_{tt}} + \sigma^3_{tt} \tfrac{\partial}{\partial w_{tt}} + \sigma^1_{xx} \tfrac{\partial}{\partial u_{xx}} + \sigma^2_{xx} \tfrac{\partial}{\partial v_{xx}} + \sigma^3_{xx} \tfrac{\partial}{\partial w_{xx}},\end{aligned}$$

where the coefficients ρ^k and σ^k $(k = 1, 2, 3)$ with the relevant indices are calculated by the well-known formulae (see, e.g., [13, 30, 32]).

Now we apply the rather standard procedure for obtaining system of DEs, using the invariance conditions (2.3). From the formal point of view, the procedure is the same as for Lie symmetry search; however, six (not three!) different derivatives, say u_{xx}, v_{xx}, w_{xx}, u_t, v_t and w_t, can be excluded using the manifold $\mathcal{M}$. After straightforward calculations, one arrives at the *non-linear system* of DEs

$$\begin{aligned}
&(1)\ \xi_u = \xi_v = \xi_w = 0,\\
&(2)\ \eta^k_{uu} = \eta^k_{uv} = \eta^k_{uw} = \eta^k_{vv} = \eta^k_{vw} = \eta^k_{ww} = 0,\ k = 1, 2, 3,\\
&(3)\ (d_1 - d_2)\,\xi\,\eta^1_v - 2d_1 d_2 \eta^1_{xv} = 0, \quad (d_1 - d_3)\,\xi\,\eta^1_w - 2d_1 d_3 \eta^1_{xw} = 0,\\
&(4)\ (d_1 - d_2)\,\xi\,\eta^2_u + 2d_1 d_2 \eta^2_{xu} = 0, \quad (d_2 - d_3)\,\xi\,\eta^2_w - 2d_2 d_3 \eta^2_{xw} = 0,\\
&(5)\ (d_1 - d_3)\,\xi\,\eta^3_u + 2d_1 d_3 \eta^3_{xu} = 0, \quad (d_2 - d_3)\,\xi\,\eta^3_v + 2d_2 d_3 \eta^3_{xv} = 0,\\
&(6)\ \xi_t - d_1 \xi_{xx} + 2d_1 \eta^1_{xu} + 2\xi\xi_x = 0,\\
&(7)\ \xi_t - d_2 \xi_{xx} + 2d_2 \eta^2_{xv} + 2\xi\xi_x = 0,\\
&(8)\ \xi_t - d_3 \xi_{xx} + 2d_3 \eta^3_{xw} + 2\xi\xi_x = 0,\\
&(9)\ \eta^1 C^1_u + \eta^2 C^1_v + \eta^3 C^1_w + \left(2\xi_x - \eta^1_u\right) C^1 - \tfrac{d_1}{d_2}\,\eta^1_v C^2 - \tfrac{d_1}{d_3}\,\eta^1_w C^3\\
&\qquad + d_1 \eta^1_{xx} - \eta^1_t - 2\xi_x \eta^1 + \left(\tfrac{d_1}{d_2} - 1\right)\eta^2 \eta^1_v + \left(\tfrac{d_1}{d_3} - 1\right)\eta^3 \eta^1_w = 0,\\
&(10)\ \eta^1 C^2_u + \eta^2 C^2_v + \eta^3 C^2_w + \left(2\xi_x - \eta^2_v\right) C^2 - \tfrac{d_2}{d_1}\,\eta^2_u C^1 - \tfrac{d_2}{d_3}\,\eta^2_w C^3\\
&\qquad + d_2 \eta^2_{xx} - \eta^2_t - 2\xi_x \eta^2 + \left(\tfrac{d_2}{d_1} - 1\right)\eta^1 \eta^2_u + \left(\tfrac{d_2}{d_3} - 1\right)\eta^3 \eta^2_w = 0,\\
&(11)\ \eta^1 C^3_u + \eta^2 C^3_v + \eta^3 C^3_w + \left(2\xi_x - \eta^3_w\right) C^3 - \tfrac{d_3}{d_1}\,\eta^3_u C^1 - \tfrac{d_3}{d_2}\,\eta^3_v C^2\\
&\qquad + d_3 \eta^1_{xx} - \eta^3_t - 2\xi_x \eta^3 + \left(\tfrac{d_3}{d_1} - 1\right)\eta^1 \eta^3_u + \left(\tfrac{d_3}{d_2} - 1\right)\eta^2 \eta^3_v = 0,
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
&C^1 = u(1 - u - a_1 v), \quad C^2 = a_2 v(1 - u - a_1 v) + uw + a_1 vw,\\
&C^3 = a_3 w(1 - w) - a_4 uw - a_5 vw.
\end{aligned} \tag{2.5}$$

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Remark 2.1 *The non-linear system (2.4) is the system of DEs for the general RD system of the form*

$$u_t = d_1 u_{xx} + C^1(u, v, w),$$
$$v_t = d_2 v_{xx} + C^2(u, v, w),$$
$$w_t = d_3 w_{xx} + C^3(u, v, w),$$

where C^k $(k = 1, 2, 3)$ *are arbitrary smooth functions and* $d_k > 0$. *Here we examine this system in the case when the functions* C^k $(k = 1, 2, 3)$ *possess the form (2.5).*

Solving the linear subsystem (1) and (2) of system (2.4), one specifies the form of operator (2.1) as follows

$$\begin{aligned} Q = {} & \partial_t + \xi(t,x)\partial_x + \big(r^1(t,x)u + q^1(t,x)v + h^1(t,x)w + p^1(t,x)\big)\partial_u \\ & + \big(r^2(t,x)v + q^2(t,x)u + h^2(t,x)w + p^2(t,x)\big)\partial_v \\ & + \big(r^3(t,x)w + q^3(t,x)u + h^3(t,x)v + p^3(t,x)\big)\partial_w, \end{aligned} \tag{2.6}$$

where ξ, r^k, q^k, h^k and p^k are unknown smooth functions at the moment.

Now we formulate the main result of this section.

Theorem 2.1 *System (1.1) is invariant under Q-conditional symmetry operator(s) of the form (2.6) if and only if one and the corresponding operator(s) have the forms listed in Table 1.*

The proof of the theorem is presented in the Appendix.

Remark 2.2 *In Table 1, the function* $P(t, x)$ *is an arbitrary solution of the linear diffusion equation* $P_t = d_2 P_{xx}$, $E(\delta_1, \delta_2) = \exp\left[\mu \frac{\delta_2 - \delta_1}{2\delta_1\delta_2}\left(x + \mu \frac{\delta_2 - \delta_1}{2\delta_1} t\right)\right]$, *while* α_i $(i = 1, 2, 3, 4)$ *and* μ *are arbitrary constants.*

Remark 2.3 *Some Q-conditional symmetry operators presented in Table 1 are equivalent to the relevant Lie symmetry operators obtained in [14] provided arbitrary parameters satisfy additional restrictions. For example, operator from Case 6 with* $\alpha_1 = -a_2 d_3$ *is the linear combination of Lie symmetry operators* ∂_t *and* $v\partial_v + w\partial_w$ *(see Case 1 of Table 1 [14]).*

Remark 2.4 *The HGF systems and relevant Q-conditional symmetry operators from Cases 4 and 5 of Table 1 can be reduced by the transformation*

$$u^* = a_1 w,\ v^* = -(u + a_1 v),\ w^* = e^{-t} u,\ x^* = \frac{x}{\sqrt{d}} \tag{2.7}$$

to the subcases of Cases 7 and 5 from Table 2 [11], respectively.

3 Exact solutions of the HGF system

If one compares the HGF systems with the reaction terms arising in Table 1 with its general form (1.1), then it is clear that Cases 2–5 are the most interesting from the applicability point of

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Table 1. *Q-conditional symmetry operators of system (1.1)*

	Reaction terms	Restrictions	Operator(s)
1	$u(1-u)$ $\frac{d_2}{d_1}\,v(1-u)+uw$ $a_3w(1-w)-a_4uw$	$a_3\neq 0$	$\partial_t+\mu\partial_x+(q^2u+p^2)\partial_v,$ $q^2=\left(\alpha_1+\alpha_2e^{\frac{d_2}{d_1}t}\right)E(d_1,d_2),$ $p^2=-\alpha_2e^{\frac{d_2}{d_1}t}E(d_1,d_2)$
2	$u(1-u-a_1v)$ $\frac{d_2-d_3}{d_1-d_3}\,v(1-u-a_1v)+$ $uw+a_1vw$ $-\frac{d_3}{d_1}uw-\frac{a_1d_3}{d_1}vw$	$a_1\neq 0$ $d_1\neq d_2$ $d_1\neq d_3$	$\partial_t-\frac{d_1}{d_1-d_2}\,w(a_1\partial_u-\partial_v)-\frac{d_3}{d_1-d_3}\,w\partial_w;$ $\partial_t-\frac{d_3}{a_1(d_1-d_3)}\,u\,(a_1\partial_u-\partial_v)$ $-\frac{(d_1-d_2)d_3^2}{a_1d_1(d_1-d_3)^2}\,u\partial_w$
3	$u(1-u-a_1v)$ $\frac{d_2}{d_1}\,v(1-u-a_1v)$ $+uw+a_1vw$ $-a_4uw-a_1a_4vw$	$a_1\neq 0$ $d_1\neq d_2$ $a_4d_1\neq d_3$	$(d_1-d_2)\partial_t$ $+\left(\frac{d_2}{a_1}(u-1)+d_2v+\frac{d_1d_3}{a_4d_1-d_3}\,w\right)(a_1\partial_u-\partial_v)$
4	$u(1-u-a_1v)$ $v(1-u-a_1v)$ $+uw+a_1vw$ $-a_4uw-a_1a_4vw$	$a_1\neq 0$ $d_1=d_2=d$ $a_4d\neq d_3$	$\partial_t+((\alpha_1+\alpha_2(a_4d-d_3)\,e^t)u$ $+\alpha_2a_1(a_4d-d_3)\,e^tv+\alpha_2a_1d_3\,e^tw$ $+\alpha_2(d_3-a_4d)\,e^t)(a_1\partial_u-\partial_v)$
5	$u(1-u-a_1v)$ $v(1-u-a_1v)$ $+uw+a_1vw$ $-\frac{d_3}{d}\,uw-a_1\frac{d_3}{d}\,vw$	$a_1\neq 0$ $d_1=d_2=d$	$\partial_t+\mu\partial_x$ $+(\alpha_1u+\alpha_2e^tE(d_3,d)w)\,(a_1\partial_u-\partial_v)$
6	$u(1-u)$ $a_2v(1-u)+uw$ $-a_4uw$	$a_4d_2\neq a_2d_3$ $a_2d_1\neq d_2$ $d_2\neq d_3$	$(d_2-d_3)\partial_t+\left(\alpha_1v+\frac{d_3(a_2d_3+\alpha_1)}{a_4d_2-a_2d_3}\,w+\right.$ $\left.P(t,x)\right)\partial_v-a_2d_3\,w\partial_w,\ a_2P(t,x)=0$
7	$u(1-u)$ $\frac{a_4d_2}{d_3}\,v(1-u)+uw$ $-a_4uw$		$\partial_t+\mu\partial_x+(\alpha_1v+h^2w)\partial_v+\alpha_1w\partial_w,$ $h^2=\alpha_2\exp\left(\frac{a_4d_2+\alpha_1(d_2-d_3)}{d_3}\,t\right)E(d_3,d_2)$
8	$u(1-u)$ $\frac{d_2}{d_1}\,v(1-u)+uw$ $-a_4uw$	$a_4d_1\neq d_3$	$\partial_t+\mu\partial_x+(\alpha_1v+q^2u+p^2)\partial_v+\alpha_1w\partial_w,$ $q^2=\left(\alpha_2+\alpha_3e^{\frac{d_2}{d_1}t}\right)E(d_1,d_2),$ $p^2=-\alpha_3e^{\frac{d_2}{d_1}t}E(d_1,d_2);$ $(d_2-d_3)\partial_t+\left(\alpha_1v+\frac{d_3(d_2d_3+d_1\alpha_1)}{d_2(a_4d_1-d_3)}\,w\right.$ $\left.+\alpha_2u+\alpha_3e^{\frac{d_2}{d_1}t}(u-1)\right)\partial_v-\frac{d_2d_3}{d_1}\,w\partial_w$
9	$u(1-u)$ $\frac{d_2}{d_1}\,v(1-u)+uw$ $-\frac{d_3}{d_1}uw$		$\partial_t+\mu\partial_x+\left(\alpha_1v+q^2u+h^2w+p^2\right)\partial_v+\alpha_1w\partial_w,$ $p^2=-\alpha_3e^{\frac{d_2}{d_1}t}E(d_1,d_2),$ $q^2=\left(\alpha_2+\alpha_3e^{\frac{d_2}{d_1}t}\right)E(d_1,d_2),$ $h^2=\alpha_4\exp\left(\frac{d_2d_3+\alpha_1d_1(d_2-d_3)}{d_1d_3}\,t\right)E(d_3,d_2)$

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Table 1. *Continued.*

	Reaction terms	Restrictions	Operator(s)
10	$u(1-u)$	$d_2 \neq d_3$	$\partial_t + \Big(\alpha_1 v + q^2 u\Big)\partial_v + \Big(\alpha_1 w + q^3 u + \alpha_2\Big)\partial_w,$
	$\frac{d_2-d_3}{d_1}\, v(1-u) + uw$		$q^3 = \alpha_3 e^{-\frac{d_3}{d_1} t} - \alpha_2,\ q^2 = -\frac{d_1}{d_3} q^3;$
	$-\frac{d_3}{d_1} uw$		$\partial_t + \Big(\alpha_1 v + \frac{\alpha_1 d_1 + d_3}{d_3}\, w\Big)\, \partial_v - \frac{d_3}{d_1}\, w\partial_w$
11	$u(1-u)$	$a_2 d_1 \neq d_2$	$\partial_t + \Big(\alpha_1 v - \frac{d_1\alpha_2}{a_2 d_1 - d_2}\, u + P(t,x)\Big)\, \partial_v + \Big(\alpha_2\,(1-u)$
	$a_2 v(1-u) + uw$	$a_2 d_1 \neq d_2 - d_3$	$+ \alpha_1 w\Big)\partial_w;\quad (d_2 - d_3)\partial_t - a_2 d_3 w\partial_w$
	$-\frac{d_3}{d_1} uw$	$d_2 \neq d_3$	$+ \Big(\alpha_1 v + \frac{d_1(a_2 d_3 + \alpha_1)}{d_2 - a_2 d_1}\, w + P(t,x)\Big)\, \partial_v;$
		$a_2 P(t,x) = 0$	$\partial_t + \Big(\frac{a_2 d_3}{d_3 - d_2}\, w + \alpha_1 e^{-\frac{d_3}{d_1} t} u\Big)\partial_w + \Big(-\frac{d_3}{d_1}\, v$
			$+ \frac{\alpha_1 d_1}{a_2 d_1 - d_2} e^{-\frac{d_3}{d_1} t} u + \frac{d_3(a_2 d_1 - d_2 + d_3)}{(a_2 d_1 - d_2)(d_3 - d_2)}\, w + P(t,x)\Big)\partial_v$
12	$u(1-u)$	$a_2 \neq 0$	$\partial_t + \mu\partial_x + \Big(\alpha_1 v + q^2 u\Big)\partial_v + \Big(q^3 u + \alpha_1 w + p^3\Big)\partial_w$
	$a_2 v(1-u) + uw$	$d_2 = d_3 = d$	$q^2 = \alpha_2 E(d_1, d),\ q^3 = \frac{a_2 d_1 - d}{d_1}\, q^2,\ p^3 = \frac{d - a_2 d_1}{d_1}\, q^2$
	$-\frac{d}{d_1} uw$		
13	$u(1-u)$	$d_2 = d_3 = d$	$\partial_t + \mu\partial_x + \Big(\alpha_1 v + q^2 u + h^2 w + P(t,x)\Big)\partial_v +$
	uw		$\Big(q^3 u + \alpha_2 w + p^3\Big)\partial_w,\ h^2 = \frac{d_1(\alpha_1 - \alpha_2)}{d},\ q^3 = -\frac{d}{d_1}\, q^2$
	$-\frac{d}{d_1} uw$		$q^2 = \Big(\alpha_3 e^{-\frac{d}{d_1} t} + \alpha_4\Big)\, E(d_1, d),\ p^3 = \frac{\alpha_4 d}{d_1} E(d_1, d)$

view. In fact, all the other cases of Table 1 lead to the systems involving the autonomous Fisher equation in the HGF system (1.1). The autonomous Fisher equation means that initial farmers u does not interact with converted farmers v. So, it is unlikely that such systems can describe adequately the spread and interaction between farmers and hunter-gatherers.

Here we study in detail the system with the reaction terms from Case 2 of Table 1 because this system admits two Q-conditional symmetries (i.e., possesses a wider symmetry) in contrast to those from Cases 3 to 5. Thus, we examine the system

$$\begin{aligned} u_t &= d_1 u_{xx} + u(1 - u - a_1 v), \\ v_t &= d_2 v_{xx} + \tfrac{d_2 - d_3}{d_1 - d_3}\, v(1 - u - a_1 v) + uw + a_1 vw, \\ w_t &= d_3 w_{xx} - \tfrac{d_3}{d_1} uw - \tfrac{a_1 d_3}{d_1} vw, \end{aligned} \tag{3.1}$$

where $a_1 \neq 0,\ d_1 \neq d_2$ and $d_1 \neq d_3$.

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One can set $a_1 = 1$ without losing a generality because of the transformation $a_1 v \to v$, $a_1 w \to w$, hence system (3.1) and its operators take the forms

$$\begin{aligned} u_t &= d_1 u_{xx} + u(1-u-v),\\ v_t &= d_2 v_{xx} + \tfrac{d_2-d_3}{d_1-d_3}\, v(1-u-v) + uw + vw,\\ w_t &= d_3 w_{xx} - \tfrac{d_3}{d_1} uw - \tfrac{d_3}{d_1} vw, \end{aligned} \tag{3.2}$$

$$\begin{aligned} Q_1 &= \partial_t - \tfrac{d_1}{d_1-d_2}\, w(\partial_u - \partial_v) - \tfrac{d_3}{d_1-d_3}\, w\partial_w,\\ Q_2 &= \partial_t - \tfrac{d_3}{d_1-d_3}\, u\,(\partial_u - \partial_v) - \tfrac{(d_1-d_2)d_3^2}{d_1(d_1-d_3)^2}\, u\partial_w. \end{aligned} \tag{3.3}$$

It can be easily checked using Theorem 1 [14] that system (3.2) admits the trivial Lie symmetry allowing to search only for plane wave solutions, in particular travelling fronts (see [14, Section 4]). Because operators (3.3) present non-Lie symmetries, we construct here exact solutions with more complicated structure.

It is well known that using any Q-conditional symmetry, one can reduce the given two-dimensional system of PDEs to a system of ODEs via the same procedure as for classical Lie symmetries. Thus, to construct an ansatz corresponding to the operator Q, the system of the linear first-order PDEs

$$Q(u) = 0,\ Q(v) = 0,\ Q(w) = 0 \tag{3.4}$$

should be solved.

In the case of the operator Q_1, system (3.4) takes the form

$$u_t = -\tfrac{d_1}{d_1-d_2}\, w, \quad v_t = \tfrac{d_1}{d_1-d_2}\, w, \quad w_t = -\tfrac{d_3}{d_1-d_3}\, w. \tag{3.5}$$

Solving system (3.5), one obtains the ansatz

$$\begin{aligned} u &= \varphi_1(x) + \tfrac{d_1(d_1-d_3)}{d_3(d_1-d_2)}\, \varphi_3(x) \exp\left(\tfrac{d_3}{d_3-d_1}\, t\right),\\ v &= \varphi_2(x) - \tfrac{d_1(d_1-d_3)}{d_3(d_1-d_2)}\, \varphi_3(x) \exp\left(\tfrac{d_3}{d_3-d_1}\, t\right),\\ w &= \varphi_3(x) \exp\left(\tfrac{d_3}{d_3-d_1}\, t\right), \end{aligned} \tag{3.6}$$

where φ_1, φ_2 and φ_3 are unknown functions.

In order to construct the reduced system, we substitute ansatz (3.5) into (3.2). Making the relevant calculations one arrives at the ODE system

$$\begin{aligned} &d_1\varphi_1'' + \varphi_1\left(1-\varphi_1-\varphi_2\right) = 0,\\ &d_2\varphi_2'' + \tfrac{d_2-d_3}{d_1-d_3}\, \varphi_2\left(1-\varphi_1-\varphi_2\right) = 0,\\ &d_1\varphi_3'' - \varphi_3\left(\varphi_1+\varphi_2+\tfrac{d_1}{d_3-d_1}\right) = 0. \end{aligned} \tag{3.7}$$

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Using the same procedure for the operator Q_2, we obtain the ansatz

$$\begin{aligned} u &= \varphi_1(x)\exp\left(\tfrac{d_3}{d_3-d_1}\,t\right),\\ v &= \varphi_2(x)-\varphi_1(x)\exp\left(\tfrac{d_3}{d_3-d_1}\,t\right),\\ w &= \varphi_3(x)+\tfrac{d_3(d_1-d_2)}{d_1(d_1-d_3)}\,\varphi_1(x)\exp\left(\tfrac{d_3}{d_3-d_1}\,t\right), \end{aligned} \tag{3.8}$$

and the ODE system

$$\begin{aligned} &d_1\varphi_1''-\varphi_1\left(\tfrac{d_1}{d_3-d_1}+\varphi_2\right)=0,\\ &d_2\varphi_2''+\tfrac{d_2-d_3}{d_1-d_3}\,\varphi_2\left(1-\varphi_2+\tfrac{d_1-d_3}{d_2-d_3}\,\varphi_3\right)=0,\\ &d_1\varphi_3''-\varphi_2\varphi_3=0. \end{aligned} \tag{3.9}$$

The ODE systems (3.7) and (3.9) are non-linear systems, which are non-integrable. To the best of our knowledge, even particular solutions of these ODE systems are unknown. Thus, we consider some special cases allowing to construct their exact solutions.

Let us consider system (3.7) with the additional restriction $\varphi_2 = 1-\varphi_1$, which essentially simplifies the system. Thus,we immediately obtain

$$\varphi_1 = \alpha x+\beta,\ \varphi_2 = -\alpha x+1-\beta,$$

where α and β are arbitrary constants. Because $\varphi_1+\varphi_2=1$, the third equation of (3.7) has the general solution

$$\begin{aligned} &\varphi_3 = c_1\sin\left(\sqrt{D}\,x\right)+c_2\cos\left(\sqrt{D}\,x\right),\ D=\frac{d_3}{d_1(d_1-d_3)}>0;\\ &\varphi_3 = c_1e^{\sqrt{|D|}\,x}+c_2e^{-\sqrt{|D|}\,x},\ D<0 \end{aligned}$$

(hereafter c_1 and c_2 are arbitrary constants). Substituting the functions φ_1, φ_2 and φ_3 into ansatz (3.6), we observe that

$$(u,v,w)\to(\varphi_1,\varphi_2,0)\ \texttt{if}\ t\to\infty, \tag{3.10}$$

provided $d_1>d_3$ (otherwise, all the components tend to infinity with time).

Because the asymptotic behaviour (3.10) is plausible from the applicability point of view, we concentrate ourselves on this case. Assuming for a simplicity that $c_1=k(d_2-d_1)D>0$, $c_2=0$, $k>0$ and substituting the functions φ_1, φ_2 and φ_3 into ansatz (3.6), we obtain exact solution of the HGF system (3.1)

$$\begin{aligned} u &= -k\sin\left(\sqrt{D}\,x\right)\exp\left(-d_1D\,t\right)+\alpha x+\beta,\\ v &= k\sin\left(\sqrt{D}\,x\right)\exp\left(-d_1D\,t\right)-\alpha x+1-\beta,\\ w &= k(d_2-d_1)D\sin\left(\sqrt{D}\,x\right)\exp\left(-d_1D\,t\right). \end{aligned} \tag{3.11}$$

All the components of solution (3.11) are bounded and nonnegative in the domain

$$\Omega=\left\{(t,x)\in(0,+\infty)\times\left(0,\frac{\pi}{\sqrt{D}}\right)\right\}$$

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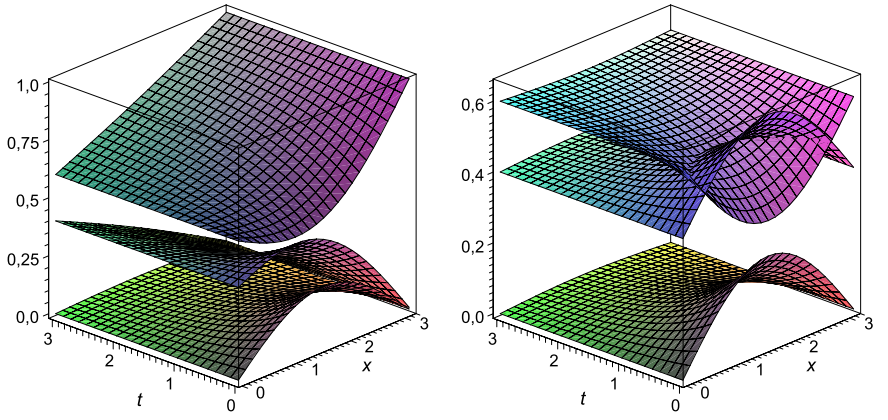


FIGURE 1. Solution (3.11) of the HGF system (3.2) with $\alpha = 1/8$ (a) and $\alpha = 0$ (b). Other parameters are $k = 1/4,\ \beta = 3/5,\ d_1 = 1,\ d_2 = 2,\ d_3 = 1/2$. The upper, middle and lower surfaces represent the functions u, v and w, respectively.

if the additional restrictions

$$\max\left\{0, k - \frac{\alpha\pi}{2\sqrt{D}}\right\} \leqslant \beta \leqslant 1 - \frac{\alpha\pi}{\sqrt{D}},\ \texttt{if}\ \alpha \geqslant 0,$$

$$\max\left\{-\frac{\alpha\pi}{\sqrt{D}}, k - \frac{\alpha\pi}{2\sqrt{D}}\right\} \leqslant \beta \leqslant 1,\ \texttt{if}\ \alpha < 0,$$

take place. Notably, the asymptotic behaviour (3.10) takes the form

$$(u, v, w) \to (\beta, 1 - \beta, 0)\ \texttt{if}\ t \to \infty \tag{3.12}$$

in the border case $\alpha = 0$. Here the point $(\beta, 1 - \beta, 0)$ is nothing else but a steady-state point of system (3.2).

Solution (3.11) has a clear biological interpretation and describes such interaction between farmers and hunter-gatherers that hunter-gatherers disappear while the initial and converted farmers coexist. Moreover, the population of initial farmers is increasing with time, while the number of converted farmers is decreasing. Examples of the solution are presented in Figure 1.

We also point out that the components of solution (3.11) obey the property $u + v = 1$. Interestingly, the numerical solutions presented in Figure 2 [2] (see curves for the components F and C excepting a vicinity of the point $x = l$) possess the same property. We note that the curves representing F and C vanish at the point $x = l$ because the zero Dirichlet conditions are used in [2], hence the property $F + C = 1$ is not valid in the vicinity of the point $x = l$ (in contrast to our solution (3.11)).

Now we turn to the reduced system (3.9) and assume that the functions φ_2 and φ_3 are linearly dependent. One notes that second and third equations of system (3.9) coincide if the restriction

$$\varphi_3 = \frac{d_3(d_2 - d_1)}{d_1(d_1 - d_3)}(\varphi_2 - 1) \tag{3.13}$$

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takes place. Thus, we obtain the single ODE

$$d_1\varphi_2'' = \varphi_2(\varphi_2 - 1). \tag{3.14}$$

The general solution of the non-linear equation (3.14) can be found in the parametric form [33]:

$$\varphi_2 = \pm\frac{3}{2}y,\ x = \sqrt{d_1}\int \frac{dy}{\sqrt{c_1 \pm y^3 - y^2}} + c_2, \tag{3.15}$$

where the integral in the right-hand side can be expressed in terms of elliptic functions. Here the constant c_2 can be removed by applying the space translation $x - c_2 \to x$. Moreover, we can obtain the exact solution of equation (3.14) in an explicit form for some correctly specified values of c_1. For instance, solution (3.15) with $c_1 = 0$ and $c_1 = \frac{4}{27}$ has the forms

$$\varphi_2 = \frac{3}{2}\left(1 + \tan^2\frac{x}{2\sqrt{d_1}}\right) \tag{3.16}$$

and

$$\varphi_2 = \frac{1}{2}\left(-1 + 3\tanh^2\frac{x}{2\sqrt{d_1}}\right), \tag{3.17}$$

respectively. Thus, the solution of the HGF system (3.2) corresponding to the function φ_2 from (3.16) has the form

$$\begin{aligned} u &= \varphi_1(x)\exp\left(\tfrac{d_3}{d_3-d_1}\,t\right),\\ v &= \tfrac{3}{2}\left(1 + \tan^2\tfrac{x}{2\sqrt{d_1}}\right) - \varphi_1(x)\exp\left(\tfrac{d_3}{d_3-d_1}\,t\right),\\ w &= \tfrac{d_3(d_2-d_1)}{2d_1(d_1-d_3)}\left(1 + 3\tan^2\tfrac{x}{2\sqrt{d_1}}\right) + \tfrac{d_3(d_1-d_2)}{d_1(d_1-d_3)}\,\varphi_1(x)\exp\left(\tfrac{d_3}{d_3-d_1}\,t\right), \end{aligned} \tag{3.18}$$

where the function $\varphi_1(x)$ is an arbitrary solution of the linear ODE

$$d_1\varphi_1'' - \varphi_1\left(\frac{3d_3 - d_1}{2(d_3 - d_1)} + \frac{3}{2}\tan^2\frac{x}{2\sqrt{d_1}}\right) = 0. \tag{3.19}$$

It turns out that equation (3.19) can be reduced to the hypergeometric equation:

$$z(z-1)\psi''(z) - \left(z + \frac{1}{2}\right)\psi'(z) - \frac{1}{2}\left(1 - \frac{3d_3 - d_1}{d_3 - d_1}\right)\psi(z) = 0 \tag{3.20}$$

by the substitution [33]

$$z = \sin^2\frac{x}{2\sqrt{d_1}},\ \psi(z) = \cos^2\frac{x}{2\sqrt{d_1}}\,\varphi_1(x). \tag{3.21}$$

Thus, the general solution of the equation (3.20) can be presented in the form of the hypergeometric functions.

Equation (3.20) is integrable in terms of elementary functions provided diffusivities d_1 and d_3 have some correctly specified values. For example, this equation with $d_3 = \frac{5}{9}d_1$ has the solution

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$\psi(z) = (1-z)^{\frac{5}{2}}$. Thus, using formulae (3.8), (3.13), (3.17), (3.18), (3.21) and transformation $x \to \sqrt{d_1}x$, we obtain the exact solution

$$\begin{aligned} u &= c_1 \cos^3 \tfrac{x}{2} \, \exp\left(-\tfrac{5}{4}\, t\right), \\ v &= \tfrac{3}{2}\left(1+\tan^2 \tfrac{x}{2}\right) - c_1 \cos^3 \tfrac{x}{2} \, \exp\left(-\tfrac{5}{4}\, t\right), \\ w &= \tfrac{5(d-1)}{8}\left(1+3\tan^2 \tfrac{x}{2}\right) + c_1 \tfrac{5(1-d)}{4} \cos^3 \tfrac{x}{2} \, \exp\left(-\tfrac{5}{4}\, t\right) \end{aligned} \tag{3.22}$$

of the HGF system

$$\begin{aligned} u_t &= u_{xx} + u(1-u-v), \\ v_t &= d\, v_{xx} + \tfrac{9d-5}{4}\, v(1-u-v) + uw + vw, \\ w_t &= \tfrac{5}{9}\, w_{xx} - \tfrac{5}{9} uw - \tfrac{5}{9} vw, \end{aligned} \tag{3.23}$$

where $d = \frac{d_2}{d_1}$.

Now we observe that the exact solution (3.22) possesses the asymptotic behaviour

$$(u, v, w) \to \left(0,\ \frac{3}{2}\left(1+\tan^2 \frac{x}{2}\right),\ \frac{5(d-1)}{8}\left(1+3\tan^2 \frac{x}{2}\right)\right) \ \texttt{if}\ t \to \infty.$$

In contrast to the exact solution (3.11), a possible interpretation of (3.22) says that the initial farmers disappear while the hunter-gatherers and converted farmers coexist. Moreover, the limiting distribution of the hunter-gatherers and converted farmers is nonconstant. The corresponding domain, in which all the components of solution (3.22) are bounded and nonnegative, can be easily specified if one sets $d > 1$ and $0 < c_1 \leqslant \frac{1}{2}$. Of course, this scenario looks unrealistic; however we have shown mathematically that the HGF model with some coefficients (e.g., as specified in (3.23)) admit a complete disappearance of the initial farmers.

Applying a similar procedure for the function φ_2 from (3.17), the exact solutions

$$\begin{aligned} u &= \cosh^3 \tfrac{x}{2} \, \exp\left(\tfrac{9}{4}\, t\right), \\ v &= \tfrac{1}{2}\left(-1+3\tanh^2 \tfrac{x}{2}\right) - \cosh^3 \tfrac{x}{2} \, \exp\left(\tfrac{9}{4}\, t\right), \\ w &= \tfrac{27(d-1)}{8}\left(1-\tanh^2 \tfrac{x}{2}\right) + \tfrac{9(d-1)}{4} \cosh^3 \tfrac{x}{2} \, \exp\left(\tfrac{9}{4}\, t\right) \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} u &= e^{4t} \sinh \tfrac{x}{2} \cosh^3 \tfrac{x}{2}, \\ v &= \tfrac{1}{2}\left(-1+3\tanh^2 \tfrac{x}{2}\right) - e^{4t} \sinh \tfrac{x}{2} \cosh^3 \tfrac{x}{2}, \\ w &= 6(d-1)\left(1-\tanh^2 \tfrac{x}{2}\right) + 4(d-1)\, e^{4t} \sinh \tfrac{x}{2} \cosh^3 \tfrac{x}{2} \end{aligned} \tag{3.25}$$

of the HGF system (3.2) have been constructed. The exact solution (3.24) is valid if $d_1 = 1$, $d_2 = d$ and $d_3 = \frac{9}{5}$, while (3.25) is valid if $d_1 = 1$, $d_2 = d$ and $d_3 = \frac{4}{3}$. We note that solutions (3.24) and (3.25) are growing unboundedly with time; therefore, their biological interpretation is questionable.

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4 Conclusions

In this paper, Q-conditional (non-classical) symmetry of the non-linear three-component system (1.1) used for describing the spread of farmers into a region occupied by hunter-gatherers was under study. The main result is formulated in Theorem 2.1, and it says that there are exactly 13 inequivalent systems of the form (1.1) admitting Q-conditional symmetry operators of the form (2.1). The result is rather unusual because the corresponding non-linear system of DEs (see (2.4)–(2.5)) was fully integrated without any additional restrictions. For example, the non-linear system of DEs corresponding to the three-component DLV system was not solved in [11] but only under the restriction that the symmetries in question are *Q-conditional symmetries of the first type* (see the definition in [9]). A natural question arises: Why conditional symmetry of a more complicated system can be easier identified than the DLV system (1.2)? We have the following hypothesis: systems involving PDEs with the same structure (the DLV system is a typical example) possesses a wider conditional symmetry comparing with those involving equations with different structures. Roughly speaking, a symmetric structure of PDE systems leads to a wider conditional symmetry.

It is interesting that two systems among the HGF systems admitting Q-conditional symmetry are reducible to the three-component DLV systems. It occurs in Cases 4 and 5 of Table 1, in which the local substitution (2.7) reduces the corresponding systems to those arising in Cases 7 and 5 of Table 2 [11]. However, all other systems of the form (1.1) with the reaction terms listed in Table 1 are not reducible to any DLV system. It means that the relevant Q-conditional symmetries are indeed new and cannot be derived from those presented in [11].

Each Q-conditional symmetry listed in Table 1 can be applied for reduction of the relevant HGF system to a system of ODEs and search for exact solutions. Here the symmetries listed in Case 2 of Table 1 were examined in order to find exact solutions of system (3.2) because the latter is the most interesting among others from both mathematical and applicability point of view. As a result, several solutions in explicit form were derived (see formulae (3.11), (3.22), (3.24) and (3.25)). The most interesting among them is the exact solution (3.11), which describes plausible scenarios of interaction between the three populations and possesses (with correctly specified parameters) similar properties to numerical solutions presented in the pioneering work [2]. In particular, the solution predicts the scenario when hunter-gatherers disappear while the initial and converted farmers coexist and their densities tend with time (see formula (3.12)) to the steady-state point of system (3.2).

Finally, we point out that the following problem is still open: to find Q-conditional symmetries of the HGF system in the so-called no-go case, i.e., to construct operators of the form (2.2). Our experience in the case of two-component RD systems [12] says that some progress can be done in this direction if one applies the definition of Q-conditional symmetry of the first type [9]. Another possibility is to use the method of their equations introduced in [28].

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Conflict of Interest

None.

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Appendix. The proof of Theorem 2.1

In order to prove Theorem 2.1, one needs to solve the system of equations (3)–(11) from (2.4) under restrictions (2.5) on the functions C^k and taking into account that unknown functions have the structure (see (2.6))

$$\xi = \xi(t,x),\ \ \eta^1 = r^1(t,x)u + q^1(t,x)v + h^1(t,x)w + p^1(t,x),$$
$$\eta^2 = r^2(t,x)v + q^2(t,x)u + h^2(t,x)w + p^2(t,x),$$
$$\eta^3 = r^3(t,x)w + q^3(t,x)u + h^3(t,x)v + p^3(t,x).$$

It turns out that essentially different solutions of equations (3)–(11) from (2.4) are obtained depending on parameters d_k and a_i arising in the HGF system (1.1). All such solutions are identified in what follows.

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First of all, one notes that equations (3)–(8) of system (2.4) do not depend on u, v and w, and take the form

$$(d_1-d_2)\,\xi q^1-2d_1d_2q^1_x=0,\ (d_1-d_3)\,\xi h^1-2d_1d_3h^1_x=0, \tag{A1}$$

$$(d_1-d_2)\,\xi q^2+2d_1d_2q^2_x=0,\ (d_2-d_3)\,\xi h^2-2d_2d_3h^2_x=0, \tag{A2}$$

$$(d_1-d_3)\,\xi q^3+2d_1d_3q^3_x=0,\ (d_2-d_3)\,\xi h^3+2d_2d_3h^3_x=0, \tag{A3}$$

$$d_1(\xi_{xx}-2r^1_x)=\xi_t+2\xi\xi_x,\qquad d_2(\xi_{xx}-2r^2_x)=\xi_t+2\xi\xi_x,$$
$$d_3(\xi_{xx}-2r^3_x)=\xi_t+2\xi\xi_x. \tag{A4}$$

On the other hand, equations (9)–(11) of system (2.4) can be splitted with respect to the variables $u, v, w, uv, uw, vw, u^2, v^2$ and w^2. As a result, one obtains the system

$$a_3h^1=0,\ d_3h^1+(a_3d_2+a_1d_3)\,h^2=0,\ (a_5d_2-a_1a_2d_3)\,h^3=0, \tag{A5}$$

$$a_1\,(a_2d_1-d_2)\,q^1=0,\ d_2\,(a_4d_1-2d_3)\,h^1-a_1d_2d_3h^2-d_1d_3q^1=0, \tag{A6}$$

$$d_2\,(a_5d_1-a_1d_3)\,h^1-a_1d_1d_3q^1=0, \tag{A7}$$

$$d_1\,(a_4d_2-a_2d_3)\,h^3+d_2\,(a_5d_1-a_1d_3)\,q^3=0,\ (a_4d_1-d_3)\,q^3=0, \tag{A8}$$

$$a_1q^2+r^1+2\xi_x=0,\ (a_2d_1-d_2)\,q^2-d_1q^3=0, \tag{A9}$$

$$d_3h^3+a_5d_2q^2+2a_3d_2q^3+a_4d_2\left(r^1+2\xi_x\right)=0, \tag{A10}$$

$$d_1h^3-a_1\,(2a_2d_1-d_2)\,q^2+a_1d_1q^3-a_2d_1\left(r^1+2\xi_x\right)=0, \tag{A11}$$

$$(a_2d_1-2d_2)\,q^1-a_1d_2\left(r^2+2\xi_x\right)=0,\ a_1h^3-a_2q^1-a_1a_2\left(r^2+2\xi_x\right)=0, \tag{A12}$$

$$\left(2a_3+a_1\frac{d_3}{d_2}\right)h^3+a_4q^1=-a_5\left(r^2+2\xi_x\right),\ a_4h^1+a_5h^2=-a_3\left(r^3+2\xi_x\right), \tag{A13}$$

$$a_2d_3h^1-(a_5d_2-2a_1a_2d_3)\,h^2-d_3q^1=a_1d_3\left(r^3+2\xi_x\right), \tag{A14}$$

$$(a_4d_2-a_2d_3)\,h^2+a_1d_3q^2+d_3\left(r^1-r^2\right)+d_3\left(r^3+2\xi_x\right)=0, \tag{A15}$$

$$d_1r^1_{xx}-r^1_t-2r^1\xi_x+2\xi_x+\left(\frac{d_1}{d_2}-1\right)q^1q^2+\left(\frac{d_1}{d_3}-1\right)h^1q^3=2p^1+a_1p^2, \tag{A16}$$

$$d_2r^2_{xx}-r^2_t-2r^2\xi_x+2a_2\xi_x+\left(\frac{d_2}{d_1}-1\right)q^1q^2+\left(\frac{d_2}{d_3}-1\right)h^2h^3$$
$$-a_2p^1-2a_1a_2p^2+a_1p^3=0, \tag{A17}$$

$$d_3r^3_{xx}-r^3_t-2r^3\xi_x+2a_3\xi_x+\left(\frac{d_3}{d_1}-1\right)h^1q^3+\left(\frac{d_3}{d_2}-1\right)h^2h^3$$
$$-a_4p^1-a_5p^2-2a_3p^3=0, \tag{A18}$$

$$d_1q^1_{xx}-q^1_t-2q^1\xi_x+\left(1-r^2\right)q^1+\left(r^2-a_2\right)\frac{d_1q^1}{d_2}+\left(\frac{d_1}{d_3}-1\right)h^1h^3=a_1p^1, \tag{A19}$$

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$$d_2 q^2_{xx} - q^2_t - 2q^2\xi_x + \left(a_2 - r^1\right) q^2 + \left(r^1 - 1\right) \frac{d_2 q^2}{d_1} + \frac{d_2 - d_3}{d_3} h^2 q^3 = a_2 p^2 - p^3, \quad \text{(A20)}$$

$$d_3 q^3_{xx} - q^3_t - 2q^3\xi_x + \left(a_3 - r^1\right) q^3 + \left(r^1 - 1\right) \frac{d_3 q^3}{d_1} + \left(\frac{d_3}{d_2} - 1\right) h^3 q^2 = a_4 p^3, \quad \text{(A21)}$$

$$d_1 h^1_{xx} - h^1_t - 2h^1\xi_x + \left(1 - r^3\right) h^1 + \left(r^3 - a_3\right) \frac{d_1 h^1}{d_3} + \left(\frac{d_1}{d_2} - 1\right) h^2 q^1 = 0, \quad \text{(A22)}$$

$$d_2 h^2_{xx} - h^2_t - 2h^2\xi_x + \left(a_2 - r^3\right) h^2 + \left(r^3 - a_3\right) \frac{d_2 h^2}{d_3} + \left(\frac{d_2}{d_1} - 1\right) h^1 q^2 + p^1 + a_1 p^2 = 0, \quad \text{(A23)}$$

$$d_3 h^3_{xx} - h^3_t - 2h^3\xi_x + \left(a_3 - r^2\right) h^3 + \left(r^2 - a_2\right) \frac{d_3 h^3}{d_2} + \left(\frac{d_3}{d_1} - 1\right) q^1 q^3 = a_5 p^3, \quad \text{(A24)}$$

$$d_1 p^1_{xx} - p^1_t - 2p^1\xi_x + p^1 + \left(\frac{d_1}{d_2} - 1\right) p^2 q^1 + \left(\frac{d_1}{d_3} - 1\right) h^1 p^3 = 0, \quad \text{(A25)}$$

$$d_2 p^2_{xx} - p^2_t - 2p^2\xi_x + a_2 p^2 + \left(\frac{d_2}{d_1} - 1\right) p^1 q^2 + \left(\frac{d_2}{d_3} - 1\right) h^2 p^3 = 0, \quad \text{(A26)}$$

$$d_3 p^3_{xx} - p^3_t - 2p^3\xi_x + a_3 p^3 + \left(\frac{d_3}{d_1} - 1\right) p^1 q^3 + \left(\frac{d_3}{d_2} - 1\right) h^3 p^2 = 0. \quad \text{(A27)}$$

Although the above system is very cumbersome, one is highly overdetermined because the equation number is much larger than the number of unknown functions ξ, r^k, q^k, h^k and p^k. Moreover, equations (A5)–(A8) are algebraic (not PDEs !). It allows us to identify all inequivalent solutions of system (A5)–(A27).

First of all, we note that the non-linear system (A1)–(A27) in the case

$$q^k = h^k = 0 \ (k = 1, 2, 3)$$

is reducible to the system of DEs for searching Lie symmetry operators. All possible Lie symmetries were found in [14] (see Table 1 therein).

Now we observe that a linear combination of equations (A9) and (A11) leads to $d_3 h^3 = 0$, i.e., $h^3 = 0$. Moreover, *two essentially different cases,* $a_3 \neq 0$ *and* $a_3 = 0$, follow from the first equation of (A5).

Let us examine in details case $a_3 \neq 0$. Having $h^3 = 0$ and assuming $a_3 \neq 0$, equations (A5)–(A8), (A9) and (A10) immediately lead to

$$q^1 = q^3 = h^1 = h^2 = 0. \quad \text{(A28)}$$

Thus, one needs to set $q^2 \neq 0$ in order to find a non-Lie symmetry; therefore, equations (A9) and (A10) produce restrictions

$$a_2 = \frac{d_2}{d_1}, \ a_5 = a_1 a_4.$$

Moreover, using (A28), we obtain $p^1 = p^3 = 0$ from equations (A19), (A21) and (A23).

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Hence, the system of DEs (A1)–(A27) for finding Q-conditional symmetries of the HGF system (1.1) with $a_3 \neq 0$ takes the form

$$r^2 = r^3 = -2\xi_x, \tag{A29}$$

$$\xi_t - 5d_2\,\xi_{xx} + 2\xi\xi_x = 0,\ \xi_t - 5d_3\,\xi_{xx} + 2\xi\xi_x = 0, \tag{A30}$$

$$d_1(\xi_{xx} - 2r^1_x) = \xi_t + 2\xi\xi_x, \tag{A31}$$

$$a_1q^2 + r^1 + 2\xi_x = 0, \tag{A32}$$

$$d_1r^1_{xx} - r^1_t - 2r^1\xi_x + 2\xi_x = 0, \tag{A33}$$

$$d_2r^2_{xx} - r^2_t - 2r^2\xi_x + 2\frac{d_2}{d_1}\,\xi_x = 0, \tag{A34}$$

$$d_3r^3_{xx} - r^3_t - 2r^3\xi_x + 2a_3\xi_x = 0, \tag{A35}$$

$$(d_1 - d_2)\,\xi q^2 + 2d_1d_2q^2_x = 0, \tag{A36}$$

$$d_2q^2_{xx} - q^2_t - 2q^2\xi_x + \left(\frac{d_2}{d_1} - 1\right)q^2r^1 = 0, \tag{A37}$$

$$d_2p^2_{xx} - p^2_t - 2p^2\xi_x + \frac{d_2}{d_1}\,p^2 = 0,\ \ a_1p^2 = 0. \tag{A38}$$

The corresponding Q-conditional symmetry has the form

$$Q = \partial_t + \xi\,\partial_x + r^1u\partial_u + (r^2v + q^2u + p^2)\partial_v + r^3w\partial_w.$$

Let us integrate system (A29)–(A38). Substituting (A29) into (A34) and using the first equation of (A30), one obtains the overdetermined system

$$\xi_t + 2\xi\xi_x - 5d_2\,\xi_{xx} = 0,\ \xi_{tx} + 2\xi^2_x - d_2\,\xi_{xxx} + 2\frac{d_2}{d_1}\,\xi_x = 0.$$

The general solution of this system is well known (see (2.28) in [4]):

$$\xi = \mu, \tag{A39}$$

where μ is an arbitrary constant. Having (A39), we immediately obtain from equations (A29), (A31)–(A33)

$$r^2 = r^3 = 0,\ r^1_t = r^1_x = 0,\ a_1q^2 + r^1 = 0.$$

If $a_1 \neq 0$, then $d_1 = d_2$ (see (A37)), and the general solution of system (A29)–(A38) leads to the operator

$$Q = \partial_t + \mu\,\partial_x + r^1\left(u\partial_u - \frac{1}{a_1}u\partial_v\right).$$

This is nothing else but Lie symmetry operator (see Case 4 of Table 1 [14]).

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If $a_1 = 0$, then the general solution of system (A29)–(A38) has the form

$$\xi = \mu,\ r^1 = r^2 = r^3 = 0,\ q^1 = q^3 = h^k = 0\ (k = 1, 2, 3),\ p^1 = p^3 = 0,$$

$$q^2 = \left(\alpha_1 + \alpha_2 e^{\frac{d_2}{d_1} t}\right) \exp\left(\mu^2 \frac{(d_1 - d_2)^2}{4 d_1^2 d_2}\, t + \mu \frac{d_2 - d_1}{2 d_1 d_2}\, x\right),$$

$$p^2 = -\alpha_2 \exp\left(\left(\frac{d_2}{d_1} + \mu^2 \frac{(d_1 - d_2)^2}{4 d_1^2 d_2}\right) t + \mu \frac{d_2 - d_1}{2 d_1 d_2}\, x\right),$$

where α_1 and α_2 are arbitrary constants. Thus, Case 1 of Table 1 is obtained, and the case $a_3 \neq 0$ is completely examined.

The second generic case $a_3 = 0$ can be examined in a quite similar way. The second equations of (A5) and (A13)

$$h^1 + a_1 h^2 = 0,\ a_4 h^1 + a_5 h^2 = 0$$

lead to the restriction $a_5 = a_1 a_4$ (otherwise, $h^1 = h^2 = 0 \Rightarrow q^1 = q^2 = q^3 = 0$, so that only Lie symmetries can be derived).

Moreover, one notes from the first equation of (A6) that two possibilities $a_1 \neq 0$ and $a_1 = 0$ should be analysed.

Let us assume that $a_1 \neq 0$. Equations (A6), (A8) and (A9) lead to the restriction $(a_2 d_1 - d_2)(a_4 d_1 - d_3) = 0$ (otherwise, we obtain $h^1 = h^2 = q^1 = q^2 = q^3 = 0$). Thus, we need to consider two subcases:

$$\text{(i) } a_2 d_1 - d_2 \neq 0 \Rightarrow a_4 d_1 = d_3; \quad \text{(ii) } a_2 d_1 = d_2.$$

Subcase (i). From the first equation of (A6), we find $q^1 = 0$, and, as result, $p^1 = p^3 = 0$ (see (A19) and (A24)). Integrating the first equation of (A2), we find

$$q^2 = \varphi(t) \exp\left(\int \frac{d_2 - d_1}{2 d_1 d_2}\, \xi(t, x) dx\right),$$

were $\varphi(t)$ is arbitrary smooth function and $q^3 = (a_2 - \frac{d_2}{d_1})\, q^2$ from the second equation of (A9). Substituting the function q^3 into the first equation of (A3), we have

$$(d_2 - d_3)\, \varphi\, \xi = 0.$$

If $\xi \neq 0$, then only Lie symmetries can be obtained. If $\xi = 0$, then $r^2 = 0$ (see the first equation (A12)) and

$$q^2_x = q^3_x = h^1_x = h^2_x = 0 \text{ (see (A1)–(A3))}.$$

Under the above equalities system (A5)–(A27) essentially simplifies and takes the form

$$q^2_t + \left(1 - \frac{d_1}{d_3}\right)\left(a_2 - \frac{d_2}{d_1}\right) q^2 h^2 - p^2 = 0, \tag{A40}$$

$$\left(a_2 - \frac{d_2}{d_1}\right) h^2_t + a_1 \left(a_2 - \frac{d_2}{d_1}\right)\left(\frac{d_3}{d_1} - 1\right) h^2 q^2 + a_1 \frac{d_3}{d_1}\, p^2 = 0, \tag{A41}$$

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$$q_t^2+\left(\frac{d_2}{d_1}-a_2\right)q^2+a_1\left(\frac{d_2}{d_1}-1\right)\left(q^2\right)^2+\left(a_2-\frac{d_2}{d_1}\right)\left(1-\frac{d_2}{d_3}\right)h^2q^2=0, \tag{A42}$$

$$\left(a_2-\frac{d_2}{d_1}\right)q_t^2+\frac{d_3}{d_1}\left(a_2-\frac{d_2}{d_1}\right)q^2+a_1\left(a_2-\frac{d_2}{d_1}\right)\left(\frac{d_3}{d_1}-1\right)\left(q^2\right)^2=0, \tag{A43}$$

$$h_t^2-h^2-\left(a_2-\frac{d_2}{d_1}\right)\left(\frac{d_1}{d_3}-1\right)\left(h^2\right)^2=0 \tag{A44}$$

$$h_t^2-a_2h^2-\left(a_2-\frac{d_2}{d_1}\right)\left(\frac{d_2}{d_3}-1\right)\left(h^2\right)^2+a_1\left(\frac{d_2}{d_1}-1\right)h^2q^2-a_1p^2=0, \tag{A45}$$

and

$$q^1=r^2=p^1=p^3=0,\ p_t^2=0,\ a_2p^2=0,$$
$$r^1=-a_1q^2,\ q^3=\left(a_2-\frac{d_2}{d_1}\right)q^2,\ h^1=-a_1h^2,\ r^3=\left(a_2-\frac{d_2}{d_1}\right)h^2.$$

Thus, we obtain the overdetermined non-linear system of PDEs with two unknown functions q^2 and h^2 under the restriction $\left(q^2\right)^2+\left(h^2\right)^2\neq 0$ (otherwise, only Lie symmetries can be derived). Note that this system is incompatible in the case $q^2h^2\neq 0$. Integrating system (A40)–(A45) for $q^2=0,\ h^2\neq 0$ and $h^2=0,\ q^2\neq 0$, we obtain first and second operators of Cases 2 of Table 1, respectively.

Subcase (ii) was examined in a similar way. As a result, Cases 3–5 of Table 1 have been derived.

Thus, subcase $a_1\neq 0$ is completely examined and Cases 2–5 of Table 1 were obtained.

Finally, the subcase $a_1=0$ was examined. Because three parameters $a_1,\ a_3$ and a_5 vanish, the system of DEs (A5)–(A27) simplifies essentially. It turns out that the detailed analysis of the system of DEs (A1)–(A27) in this case leads only to the possibility $\xi=\mu$ with μ being an arbitrary constant.

Thus, the system of DEs (A1)–(A27) for finding Q-conditional symmetries of the HGF system (1.1) takes the form

$$r^1=q^1=h^1=p^1=0,\ r^2=const,\ r^3=const, \tag{A46}$$

$$(a_4d_1-d_3)\,q^3=0,\ (a_2d_1-d_2)\,q^2-d_1q^3=0, \tag{A47}$$

$$(a_4d_2-a_2d_3)\,h^2+d_3\left(r^3-r^2\right)=0, \tag{A48}$$

and

$$q_x^2=\frac{\mu(d_2-d_1)}{2d_1d_2}\,q^2,\ h_x^2=\frac{\mu(d_2-d_3)}{2d_2d_3}\,h^2,\ q_x^3=\frac{\mu(d_3-d_1)}{2d_1d_3}\,q^3, \tag{A49}$$

$$d_2q_{xx}^2-q_t^2+\left(a_2-\frac{d_2}{d_1}\right)q^2+\frac{d_2-d_3}{d_3}\,h^2q^3-a_2p^2+p^3=0, \tag{A50}$$

$$d_2h_{xx}^2-h_t^2+\left(a_2-r^3+\frac{d_2r^3}{d_3}\right)h^2=0, \tag{A51}$$

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$$d_3 q^3_{xx} - q^3_t - \frac{d_3}{d_1}\, q^3 - a_4 p^3 = 0, \tag{A52}$$

$$d_2 p^2_{xx} - p^2_t + a_2 p^2 + \left(\frac{d_2}{d_3} - 1\right) h^2 p^3 = 0,\ d_3 p^3_{xx} - p^3_t = 0. \tag{A53}$$

A further integration of equations (A49)–(A53) depends essentially on the parameters a_2 and a_4 in the algebraic equations (A47) and (A48).

In the case $a_2 d_1 \neq d_2$ and $a_4 d_1 \neq d_3$, we immediately obtain $q^2 = q^3 = 0$ from equations (A47) and $a_2 p^2 = 0,\ p^3 = 0$ from equations (A50) and (A52), hence $h^2 \neq 0$ (otherwise Lie symmetries are obtained). At the last step, equation (A48) leads to two subcases, namely $a_4 d_2 - a_2 d_3 = 0$ and $a_4 d_2 - a_2 d_3 \neq 0$. As a result, Cases 6 and 7 of Table 1 have been derived.

In the case $a_2 d_1 = d_2$, we arrive at the requirement $q^3 = p^3 = 0$. Taking into account algebraic equation (A48), one can obtain the restriction: (i) $h^2 = const$ if $a_4 d_1 \neq d_3$; (ii) $r^2 = r^3 = \alpha_1$ if $a_4 d_1 = d_3$.

In subcase (i), the second equation of (A49) and equation (A51) lead to the algebraic conditions

$$\mu\,(d_2 - d_3)\, h^2 = 0, \quad \left(\frac{d_2}{d_1} + \frac{d_2 - d_3}{d_3}\, r^3\right) h^2 = 0,$$

hence $\mu h^2 = 0$. If μ is an arbitrary constant, then $h^2 = 0$, and the first operator listed in Case 8 of Table 1 can be easily obtained. Indeed, integrating the first equation of (A49) we arrive at

$$q^2 = \exp\left(\frac{\mu(d_2 - d_1)}{2 d_1 d_2}\, x\right) \varphi_1(t),$$

where $\varphi_1(t)$ is a smooth function that can be found from the first equation of (A53) using the function

$$p^2 = \frac{d_1}{d_2}\left(-\varphi_1'(t) + \frac{\mu^2 (d_1 - d_2)^2}{4 d_1^2 d_2}\, \varphi_1(t)\right) \exp\left(\frac{\mu(d_2 - d_1)}{2 d_1 d_2}\, x\right).$$

Similarly, the subcase $\mu = 0$ and $h^2 \neq 0$ leads to the second operator in Case 8 of Table 1.

Subcase (ii). Integrating the second equation of (A49), one obtains

$$h^2 = \exp\left(\frac{\mu(d_2 - d_3)}{2 d_2 d_3}\, x\right) \varphi_2(t),$$

where the function $\varphi_2(t)$ can be specified from equation (A51). So, the functions q^2 and p^2 are found in a similar way as in subcase *(i)*. Thus, Case 9 of Table 1 is identified.

Finally, Cases 10–13 of Table 1 were obtained by examination of the last possibility, namely, $a_4 d_1 = d_3$.

The proof is now complete.

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