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# A local variational principle for conditional entropy

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*Abstract*. For a given factor map $\pi : X \longrightarrow Y$ between two topological dynamical systems and a Borel cover $\mathcal{U}$, two notions of measure-theoretical conditional entropy $h_\mu^+(T, \mathcal{U} \mid Y)$ and $h_\mu^-(T, \mathcal{U} \mid Y)$ for an invariant Borel probability measure $\mu$ are introduced. It is shown that $h_\mu^+(T, \mathcal{U} \mid Y) = h_\mu^-(T, \mathcal{U} \mid Y)$. Moreover, $\max_\mu h_\mu^+(T, \mathcal{U} \mid Y) = h_{\text{top}}(T, \mathcal{U} \mid Y)$ when $\mathcal{U}$ is an open cover. The relative variational principle is a consequence of the results.

## 1. *Introduction*

By a *topological dynamical system* (for short TDS) we mean a pair $(X, T)$ where $X$ is a compact metric space and $T : X \to X$ is a homeomorphism from $X$ to $X$. It is well known that associated to $(X, T)$ there are invariant Borel probability measures. Since the introduction of measure-theoretical entropy for an invariant measure in 1958 [**Kol**] and topological entropy in 1965 [**AKM**], the relationship between these two kinds of entropy has gained a lot of attention.

In 1969 Goodwyn [**Gwyn**] showed that $h_\mu(T) \leq h_{\text{top}}(T)$ and later Goodman [**Gman**] proved $\sup_\mu h_\mu(T) \geq h_{\text{top}}(T)$, where the supremum is taken over all invariant measures, completing the classical variational principle. For a short proof, see [**Mis1**].

The entropy concept can be localized by defining entropy pairs or tuples (even entropy sets [**DYZ**] and entropy points [**YZ**]) both in topological and in measure-theoretical situations; see [**Bla1, Bla2, BL, BHMMR, BGH, HLSY, HY1, HY2, HMY, HMRY**] for related results. To study the relationship between the two kinds of entropy pairs or tuples, one needs a local version of the variational principle. In [**BGH**] the authors showed that for a given TDS $(X, T)$ and an open cover $\mathcal{U}$ there is an invariant measure $\mu$ with $\inf_\alpha h_\mu(T, \alpha) \geq h_{\text{top}}(T, \mathcal{U})$, where the infimum is taken over all partitions which are finer than $\mathcal{U}$. In [**HY2**] the authors showed that, if $\mu$ is an invariant measure and $h_\mu(T, \alpha) > 0$ for each partition $\alpha$ which is finer than $\mathcal{U}$, then $\inf_{\alpha \succeq \mathcal{U}} h_\mu(T, \alpha) > 0$ and $h_{\text{top}}(T, \mathcal{U}) > 0$, providing some kind of converse statement of the above mentioned result of [**BGH**].

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To study the question of whether $\inf_{\alpha\succeq\mathcal{U}} h_\mu(T,\alpha) = h_{\mathrm{top}}(T,\mathcal{U})$ for a given open cover $\mathcal{U}$, Romagnoli [**Rom**] introduced two kinds of measure-theoretical entropy for covers, namely,

$$h_\mu^+(T,\mathcal{U}) = \inf_{\alpha\succeq\mathcal{U}} h_\mu(T,\alpha)$$

and

$$h_\mu^-(T,\mathcal{U}) = \lim_{n\to\infty}\frac{1}{n}\inf_{\alpha\succeq\bigvee_{i=0}^{n-1}T^{-i}\mathcal{U}} H_\mu(\alpha)$$

(where $\alpha$ is a finite Borel partition). He showed that for a TDS $(X,T)$ there is an invariant measure $\mu$ with $h_\mu^-(T,\mathcal{U}) = h_{\mathrm{top}}(T,\mathcal{U})$. As $h_\mu^-(T,\mathcal{U}) \le h_{\mathrm{top}}(T,\mathcal{U})$ for each invariant measure $\mu$ we have

$$\max_\mu h_\mu^-(T,\mathcal{U}) = h_{\mathrm{top}}(T,\mathcal{U}). \tag{1.1}$$

Just recently, Glasner and Weiss [**GW**] showed that, for each invariant measure $\mu$, $h_\mu^+(T,\mathcal{U}) \le h_{\mathrm{top}}(T,\mathcal{U})$ and consequently

$$\max_\mu h_\mu^+(T,\mathcal{U}) = h_{\mathrm{top}}(T,\mathcal{U}). \tag{1.2}$$

In [**HMRY**] the authors showed that if there are a TDS $(X,T)$ and an invariant measure $\mu$ with $h_\mu^+(T,\mathcal{U}) > h_\mu^-(T,\mathcal{U})$ then there is a uniquely ergodic TDS with the same property. Thus, combining (1.1) and (1.2) for the first time in the literature we observe that

$$h_\mu^+(T,\mathcal{U}) = h_\mu^-(T,\mathcal{U}). \tag{1.3}$$

Expressions (1.1) and (1.2) are called *local variational principles for a given open cover*. We remark that the classical variational principle follows from the local ones by some simple arguments.

For a factor map $\pi$ between two TDS $(X,T)$ and $(Y,S)$ one has the notions of relative topological entropy $h_{\mathrm{top}}(T, X \mid Y)$ and relative measure-theoretical entropy $h_\mu(T, X \mid Y)$ for an invariant measure $\mu$. In [**LW, BFF, DS**] the authors obtained the relative variational principle, i.e.

$$\sup_\mu h_\mu(T, X \mid Y) = h_{\mathrm{top}}(T, X \mid Y).$$

Thus it is a natural question if there are local relative variational principles for a given open cover. We will address this question in the current paper.

To study the question we introduce two notions of measure-theoretical conditional entropy for covers, namely $h_\mu^+(T,\mathcal{U} \mid Y)$ and $h_\mu^-(T,\mathcal{U} \mid Y)$. Expressions (1.1), (1.2) and (1.3) are generalized to the relative case. Namely, for a factor map $\pi : X \longrightarrow Y$ and a given open cover $\mathcal{U}$ we show that

$$\max_\mu h_\mu^-(T,\mathcal{U} \mid Y) = h_{\mathrm{top}}(T,\mathcal{U} \mid Y), \quad \max_\mu h_\mu^+(T,\mathcal{U} \mid Y) = h_{\mathrm{top}}(T,\mathcal{U} \mid Y) \tag{1.4}$$

and

$$h_\mu^+(T,\mathcal{U} \mid Y) = h_\mu^-(T,\mathcal{U} \mid Y) =: h_\mu(T,\mathcal{U} \mid Y). \tag{1.5}$$

We remark that the inequalities $h_\mu^-(T,\mathcal{U} \mid Y) \le h_\mu^+(T,\mathcal{U} \mid Y) \le h_{\mathrm{top}}(T,\mathcal{U} \mid Y)$ and the equality (1.5) hold for any finite Borel cover $\mathcal{U}$. Moreover, (1.4) may be false for some finite Borel cover $\mathcal{U}$ (see Proposition A.4 for a more general result).

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Again the relative variational principle follows from the local ones. As it seems that the methods in [**BGH, Rom**] are difficult to generalize to the relative case, we should find some new methods, and thus some of our methods are quite different from the ones used in [**BGH, Rom**].

The applications of the above results to the relative Pinsker factor, the relative entropy tuples both in topological and in measure-theoretical settings as well as the relative disjointness involving topological entropy will be given in a forthcoming paper [**HYZ**] by the current authors.

## 2. *Preliminaries*

Let $(X, T)$ be a TDS and $\mathcal{B}(X)$ be the collection of all Borel subsets. For the measurability we always refer to $\mathcal{B}(X)$. In this article, a *cover* of $X$ is a finite family of Borel subsets of $X$, whose union is $X$. A *partition* of $X$ is a cover of $X$ whose elements are pairwise disjoint. Denote the set of partitions by $\mathcal{P}_X$, the set of covers by $\mathcal{C}_X$ and the set of open covers by $\mathcal{C}_X^o$. Given two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$ (write $\mathcal{U} \succeq \mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Given integers $M, N \in \mathbb{N}$ with $M \leq N$ and $\mathcal{U} \in \mathcal{C}_X$ or $\mathcal{P}_X$, we use the notation $\mathcal{U}_M^N = \bigvee_{n=M}^N T^{-n}\mathcal{U}$.

Let $(X, T)$ and $(Y, S)$ be two TDS. A continuous map $\pi : X \to Y$ is called a *homomorphism* or a *factor map* between $(X, T)$ and $(Y, S)$ if it is onto and $\pi T = S\pi$. In this case we say $(X, T)$ is an *extension* of $(Y, S)$ or $(Y, S)$ is a *factor* of $(X, T)$. If $\pi$ is also injective then it is called an *isomorphism*.

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS. Given $\mathcal{U} \in \mathcal{C}_X$ and $K \subset X$, put $N(\mathcal{U} \mid K) = \min\left\{\text{the cardinality of } \mathcal{F} \mid \mathcal{F} \subset \mathcal{U}, \bigcup_{F\in\mathcal{F}} F \supset K\right\}$. When $K = X$, we write $N(\mathcal{U} \mid K)$ simply by $N(\mathcal{U})$. For $y \in Y$, we write $N(\mathcal{U} \mid y) = N(\mathcal{U} \mid \pi^{-1}(y))$ and $H(\mathcal{U} \mid y) = \log N(\mathcal{U} \mid y)$. Clearly, if there is another cover $\mathcal{V} \succeq \mathcal{U}$ then $H(\mathcal{V} \mid y) \geq H(\mathcal{U} \mid y)$. In fact, for two covers $\mathcal{U}, \mathcal{V}$ we have $H(\mathcal{U} \vee \mathcal{V} \mid y) \leq H(\mathcal{U} \mid y) + H(\mathcal{V} \mid y)$.

We now proceed by defining the topological conditional entropy of a cover $\mathcal{U} \in \mathcal{C}_X$ of $X$ with respect to $(Y, S)$; see [**Mis2**] for the definition of relative entropy of two open covers and [**Bow**, Theorem 17] for the relation of entropies between a TDS and its extensions. Let

$$N(\mathcal{U} \mid Y) = \sup_{y\in Y} N(\mathcal{U} \mid y) \text{ and } H(\mathcal{U} \mid Y) = \log N(\mathcal{U} \mid Y). \tag{2.1}$$

It is easy to see that $a_n = H(\mathcal{U}_0^{n-1} \mid Y)$ is a non-negative sub-additive sequence, i.e. $a_{n+m} \leq a_n + a_m$, for all $n, m \in \mathbb{N}$. It is well known that

$$\lim_{n\to+\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

The *conditional entropy of* $\mathcal{U}$ *with respect to* $(Y, S)$ is then defined by

$$h_{\text{top}}(T, \mathcal{U} \mid Y) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{U}_0^{n-1} \mid Y) = \inf_{n\geq 1} \frac{1}{n} H(\mathcal{U}_0^{n-1} \mid Y). \tag{2.2}$$

Moreover, the *topological conditional entropy of* $(X, T)$ *with respect to* $(Y, S)$ is defined by

$$h_{\text{top}}(T, X \mid Y) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_{\text{top}}(T, \mathcal{U} \mid Y). \tag{2.3}$$

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If $(Y, S)$ is a trivial system, we recover the standard topological entropy. In this case we write $h_{\text{top}}(T, \mathcal{U} \mid Y)$ and $h_{\text{top}}(T, X \mid Y)$ by $h_{\text{top}}(T, \mathcal{U})$ and $h_{\text{top}}(T)$ respectively.

Let $(X, T)$ be a TDS. Denote by $\mathcal{M}(X)$ the set of all Borel probability measures on $X$, $\mathcal{M}(X, T) \subset \mathcal{M}(X)$ the set of $T$-invariant measures, and $\mathcal{M}^e(X, T) \subset \mathcal{M}(X, T)$ the set of ergodic measures. Then both $\mathcal{M}(X)$ and $\mathcal{M}(X, T)$ are convex, compact metric spaces endowed with the weak$^*$-topology.

Given partitions $\alpha, \beta \in \mathcal{P}_X$, $\mu \in \mathcal{M}(X)$ and a $\sigma$-algebra $\mathcal{A} \subset \mathcal{B}(X)$, define

$$H_\mu(\alpha) = \sum_{A \in \alpha} -\mu(A) \log \mu(A), \quad H_\mu(\alpha \mid \mathcal{A}) = \sum_{A \in \alpha} \int_X -\mathbb{E}(1_A \mid \mathcal{A}) \log \mathbb{E}(1_A \mid \mathcal{A}) \, d\mu,$$

$$H_\mu(\alpha \mid \beta) = H_\mu(\alpha \vee \beta) - H_\mu(\beta), \quad H_\mu(\alpha \mid \beta \vee \mathcal{A}) = H_\mu(\alpha \vee \beta \mid \mathcal{A}) - H_\mu(\beta \mid \mathcal{A}),$$

where $\mathbb{E}(1_A \mid \mathcal{A})$ is the expectation of $1_A$ with respect to $\mathcal{A}$. One standard fact states that $H_\mu(\alpha \mid \beta)$ (respectively $H_\mu(\alpha \mid \mathcal{A})$) increases with respect to $\alpha$ and decreases with respect to $\beta$ (respectively $\mathcal{A}$).

When $\mu \in \mathcal{M}(X, T)$ and $\mathcal{A}$ is a $T$-invariant measurable $\sigma$-algebra of $X$, it is not hard to see that $H_\mu(\alpha_0^{n-1} \mid \mathcal{A})$ is a non-negative sub-additive sequence for a given $\alpha \in \mathcal{P}_X$. We define the *conditional entropy of $\alpha$ with respect to $\mathcal{A}$* by

$$h_\mu(T, \alpha \mid \mathcal{A}) = \lim_{n \to +\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1} \mid \mathcal{A}) = \inf_{n \geq 1} \frac{1}{n} H_\mu(\alpha_0^{n-1} \mid \mathcal{A}). \tag{2.4}$$

Moreover, we define the *measurable conditional entropy of $(X, T, \mu)$ with respect to $\mathcal{A}$* by

$$h_\mu(T, X \mid \mathcal{A}) = \sup_{\alpha \in \mathcal{P}_X} h_\mu(T, \alpha \mid \mathcal{A}). \tag{2.5}$$

Particularly, if $\pi : (X, T) \to (Y, S)$ is a factor map between TDS and $\alpha \in \mathcal{P}_X$, we define the *conditional entropy of $\alpha$ with respect to $(Y, S)$* by

$$h_\mu(T, \alpha \mid Y) = h_\mu(T, \alpha \mid \pi^{-1}(\mathcal{B}(Y))) = \lim_{n \to +\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1} \mid \pi^{-1}(\mathcal{B}(Y))). \tag{2.6}$$

Moreover, we define the *measurable conditional entropy of $(X, T, \mu)$ with respect to $(Y, S)$* by

$$h_\mu(T, X \mid Y) = \sup_{\alpha \in \mathcal{P}_X} h_\mu(T, \alpha \mid Y). \tag{2.7}$$

As in the topological case, when $(Y, S)$ is a trivial system, we recover the metric entropy of $T$ with respect to $\mu$. In this case we omit the restriction on $Y$. For the classical theory of measure-theoretical entropy see [**Par, Wal**], and, for the classical theory of topological entropy see [**DGS**] and [**Wal**]. The variational principle for conditional entropy is stated as follows.

THEOREM 2.1. (Variational principle of conditional entropy [**LW, DS**]) *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS. Then for every $\mu \in \mathcal{M}(X, T)$, $h_\mu(T, X \mid Y) \leq h_{\text{top}}(T, X \mid Y)$ and $\sup_{\mu \in \mathcal{M}(X,T)} h_\mu(T, X \mid Y) = h_{\text{top}}(T, X \mid Y)$.*

*Remark.* The classical variational principle follows from Theorem 2.1 by taking $(Y, S)$ to be the trivial system.

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Following the idea of Romagnoli [**Rom**], we define a new notion of $\mu$-measure conditional theoretical entropy for covers, which extends definition (2.4) to covers. It allows us to give a local version (for a fixed open cover $\mathcal{U}$) of Theorem 2.1. Let $\pi : (X, T) \to (Y, S)$ be a factor map and $\mu \in \mathcal{M}(X)$. For $\mathcal{U} \in \mathcal{C}_X$ define

$$H_\mu(\mathcal{U} \mid Y) = \inf_{\alpha \in \mathcal{P}_X, \alpha \succeq \mathcal{U}} H_\mu(\alpha \mid \pi^{-1}\mathcal{B}(Y)). \tag{2.8}$$

In particular, $H_\mu(\alpha \mid Y) \doteq H_\mu(\alpha \mid \pi^{-1}\mathcal{B}(Y))$ for $\alpha \in \mathcal{P}_X$. It is not hard to prove that many of the properties of the conditional function $H_\mu(\alpha \mid Y)$ are extended to $H_\mu(\mathcal{U} \mid Y)$ from partitions to covers; for details see [**Rom**].

We remark that if $\mathcal{U} = \{U_1, \ldots, U_n\}$ with $U_i \in \mathcal{B}(X)$ and $\mu\big(\bigcup_{i=1}^n U_i\big) = 1$, then we can define $H_\mu(\mathcal{U} \mid Y)$ in the same way.

LEMMA 2.2. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS and $\mu \in \mathcal{M}(X)$. If $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, then the following hold:*

(1) $0 \le H_\mu(\mathcal{U} \mid Y) \le \log N(\mathcal{U})$;
(2) *If $\mathcal{U} \succeq \mathcal{V}$, then $H_\mu(\mathcal{U} \mid Y) \ge H_\mu(\mathcal{V} \mid Y)$;*
(3) $H_\mu(\mathcal{U} \vee \mathcal{V} \mid Y) \le H_\mu(\mathcal{U} \mid Y) + H_\mu(\mathcal{V} \mid Y)$;
(4) $H_\mu(T^{-1}\mathcal{U} \mid Y) = H_{T\mu}(\mathcal{U} \mid Y)$.

When $\mu \in \mathcal{M}(X, T)$, one gets easily that $H_\mu(\mathcal{U}_0^{n-1} \mid Y)$ is a sub-additive function of $n \in \mathbb{N}$ from parts (3) and (4) of Lemma 2.2, and we may define the $\mu^-$-conditional entropy of $\mathcal{U}$ with respect to $(Y, S)$ as

$$h_\mu^-(T, \mathcal{U} \mid Y) = \lim_{n \to +\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1} \mid Y) = \inf_{n \ge 1} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1} \mid Y). \tag{2.9}$$

If $(Y, S)$ is a trivial system, we write $H_\mu(\mathcal{U} \mid Y)$ and $h_\mu^-(T, \mathcal{U} \mid Y)$ by $H_\mu(\mathcal{U})$ and $h_\mu^-(T, \mathcal{U})$ respectively. When $\mathcal{U}$ is a partition of $X$, $h_\mu^-(T, \mathcal{U} \mid Y) = h_\mu(T, \mathcal{U} \mid Y)$, which implies that $h_\mu(T, X \mid Y) = \sup_{\mathcal{U} \in \mathcal{C}_X} h_\mu^-(T, \mathcal{U} \mid Y)$. In fact, we may get stronger results.

LEMMA 2.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS and $\mu \in \mathcal{M}(X, T)$. Then*

(1) $h_\mu(T, X \mid Y) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\mu^-(T, \mathcal{U} \mid Y)$;
(2) $h_\mu^-(T, \mathcal{U} \mid Y) \le h_{\text{top}}(T, \mathcal{U} \mid Y)$ *for each $\mathcal{U} \in \mathcal{C}_X$.*

*Proof.* (1) For $\mathcal{U} \in \mathcal{C}_X^o$, let $\alpha$ be the Borel partition generated by $\mathcal{U}$. Then $\alpha \succeq \mathcal{U}$, and hence $h_\mu(T, X \mid Y) \ge h_\mu(T, \alpha \mid Y) \ge h_\mu^-(T, \mathcal{U} \mid Y)$. This implies $h_\mu(T, X \mid Y) \ge \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\mu^-(T, \mathcal{U} \mid Y)$.

Conversely, for a Borel partition $\alpha = \{A_1, A_2, \ldots, A_k\}$ of $X$ and $\epsilon > 0$, we have the following claim.

CLAIM. *There exists an open cover $\mathcal{U}$ of $X$ with $k$ elements such that, for any finite measurable partition $\beta$ finer than $\mathcal{U}$ as a cover, $H_\mu(\alpha \mid \beta) \le \epsilon$.*

*Proof of the Claim.* It is well known that there exists $\delta_1 > 0$ such that if $\beta = \{C_1, C_2, \ldots, C_k\}$ is a measurable partition of $X$ with $k$ elements and $\sum_{i=1}^k \mu(A_i \Delta C_i) < \delta_1$ then $H_\mu(\alpha \mid \beta) \le \epsilon$ (see for example, [**Wal**, Lemma 4.15]). Since $\mu$ is regular, we can take closed subsets $B_i \subset A_i$ with $\mu(A_i \setminus B_i) < \delta_1/(2k^2)$, $i = 1, 2, \ldots, k$.

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Let $B_0 = X \setminus \bigcup_{i=1}^{k} B_i$. Then $\mu(B_0) < \delta_1/(2k)$. Let $U_i = B_0 \cup B_i, i = 1, 2, \ldots, k$. Then $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$ is an open cover of $X$.

Now for any finite measurable partition $\beta$ finer than $\mathcal{U}$ as a cover, we can find a measurable partition $\beta' = \{C_1, C_2, \ldots, C_k\}$ satisfying that $C_i \subset U_i$, $i = 1, 2, \ldots, k$ and $\beta \succeq \beta'$. Hence $H_\mu(\alpha \mid \beta) \leq H_\mu(\alpha \mid \beta')$. Note that $U_i \supset C_i \supset X \setminus \bigcup_{j \neq i} U_j = B_i$. One has

$$\mu(C_i \Delta A_i) \leq \mu(A_i \setminus B_i) + \mu(B_0) < \frac{\delta_1}{2k} + \frac{\delta_1}{2k^2} \leq \frac{\delta_1}{k}.$$

Hence $\sum_{i=1}^{k} \mu(C_i \Delta B_i) < \delta_1$ and $H_\mu(\alpha \mid \beta') \leq \epsilon$. Thus $H_\mu(\alpha \mid \beta) \leq \epsilon$. This ends the proof of the claim. □

Now for $n \in \mathbb{N}$ and a finite measurable partition $\beta_n \succeq \mathcal{U}_0^{n-1}$, since $T^i \beta_n \succeq \mathcal{U}$ for $0 \leq i \leq n-1$, one has

$$\begin{aligned} H_\mu(\alpha_0^{n-1} \mid \pi^{-1}\mathcal{B}(Y)) &\leq H_\mu(\beta_n \mid \pi^{-1}\mathcal{B}(Y)) + H_\mu(\alpha_0^{n-1} \mid \beta_n \vee \pi^{-1}\mathcal{B}(Y)) \\ &\leq H_\mu(\beta_n \mid \pi^{-1}\mathcal{B}(Y)) + \sum_{i=0}^{n-1} H_\mu(\alpha \mid T^i \beta_n) \\ &\leq H_\mu(\beta_n \mid \pi^{-1}\mathcal{B}(Y)) + n\epsilon \quad \text{(by claim)}. \end{aligned}$$

This shows that $H_\mu(\alpha_0^{n-1} \mid \pi^{-1}\mathcal{B}(Y)) \leq H_\mu(\mathcal{U}_0^{n-1} \mid \pi^{-1}\mathcal{B}(Y)) + n\epsilon$. Hence

$$\begin{aligned} h_\mu(T, \alpha \mid Y) &= \lim_{n \to +\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1} \mid \pi^{-1}\mathcal{B}(Y)) \\ &\leq \limsup_{n \to +\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1} \mid \pi^{-1}\mathcal{B}(Y)) + \epsilon \\ &= h_\mu^-(T, \mathcal{U} \mid Y) + \epsilon \leq \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\mu^-(T, \mathcal{U} \mid Y) + \epsilon. \end{aligned}$$

Since $\alpha$ and $\epsilon$ are arbitrary, one has $h_\mu(T, X \mid Y) \leq \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\mu^-(T, \mathcal{U} \mid Y)$. This ends the proof of part (1).

(2) It is enough to show that for each $\mathcal{V} = \{V_1, V_2, \ldots, V_k\} \in \mathcal{C}_X$ there exists $\eta \in \mathcal{P}_X$ finer than $\mathcal{V}$ such that $H_\mu(\eta \mid Y) \leq \log N(\mathcal{V} \mid Y)$ which implies $H_\mu(\mathcal{V} \mid Y) \leq \log N(\mathcal{V} \mid Y)$.

Let $\mu = \int_Y \mu_y \, d\nu(y)$ be the disintegration of $\mu$ over $\nu$ where $\nu = \pi\mu$. For each $y \in Y$, there exists $I_y \subset \{1, \ldots, k\}$ with cardinality $N(\mathcal{V} \mid y)$ such that $\bigcup_{i \in I_y} V_i \supset \pi^{-1}(y)$. Hence we can find $y_1, \ldots, y_s \in Y$ such that for each $y \in Y$, $I_y = I_{y_i}$ for some $i \in \{1, 2, \ldots, s\}$. For $i = 1, \ldots, s$, define $D_i = \{y \in Y : \mu_y(\bigcup_{j \in I_{y_i}} V_j) = 1\}$. Then $D_i$ is measurable for $i = 1, \ldots, s$ and $\nu(\bigcup_{i=1}^{s} D_i) = 1$.

Fix $i \in \{1, \ldots, s\}$. Assume $I_{y_i} = \{k_1 < \cdots < k_{t_i}\}$ where $t_i = N(\mathcal{V} \mid y_i)$. Take $\eta_{y_i} = \{W_1(y_i), \ldots, W_{t_i}(y_i)\}$ where $W_1(y_i) = V_{k_1}$, $W_2(y_i) = V_{k_2} \setminus V_{k_1}$, $\ldots$, $W_{t_i}(y_i) = V_{k_{t_i}} \setminus \bigcup_{j=1}^{t_i - 1} V_{k_j}$. For any $y \in D_i$, since $\mu_y(\bigcup_{j=1}^{t_i} W_j(y_i)) = \mu_y(\bigcup_{j \in I_{y_i}} V_j) = 1$, $\eta_{y_i}$ can be considered as a finite partition of $\pi^{-1}(y)$ (mod $\mu_y$) and

$$H_{\mu_y}(\eta_{y_i}) \leq \log t_i \leq \log N(\mathcal{V} \mid Y). \tag{2.10}$$

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Let $C_i = D_i \setminus \bigcup_{j=1}^{i-1} D_j, i = 1, 2, \ldots, s$ and $A = X \setminus \big(\bigcup_{i=1}^{s} \big(\pi^{-1}(C_i) \cap \bigcup_{j=1}^{t_i} W_j(y_i)\big)\big)$. Set $A_l = A \cap \big(V_l \setminus \bigcup_{j=1}^{l-1} V_j\big), l = 1, 2, \ldots, k$. Then put

$$\eta = \{\pi^{-1}(C_1) \cap W_1(y_1), \ldots, \pi^{-1}(C_1) \cap W_{t_1}(y_1), \ldots \\ \pi^{-1}(C_s) \cap W_1(y_s), \ldots, \pi^{-1}(C_s) \cap W_{t_s}(y_s), A_1, A_2, A_3, \ldots, A_k\}.$$

Clearly, $\eta$ is a finite measurable partition of $X$ finer than $\mathcal{V}$. Now for $y \in C_i, i = 1, 2, \ldots, s$, we have $H_{\mu_y}(\eta) = H_{\mu_y}(\eta_{y_i}) \le \log N(\mathcal{V} \mid Y)$ by inequality (2.10). Hence

$$H_\mu(\eta \mid Y) = \int_Y H_{\mu_y}(\eta)\, d\nu(y) = \int_{\bigcup_{i=1}^{s} C_i} H_{\mu_y}(\eta)\, d\nu(y) \le \log N(\mathcal{V} \mid Y).$$

This ends the proof of Lemma 2.3. □

*Remark.* When $(Y, S)$ is a trivial system, the inequality $h_\mu^-(T, \mathcal{U}) \le h_{\text{top}}(T, \mathcal{U})$ can be obtained easily by the fact that $H_\mu(\alpha) \le \log |\alpha|$ for $\alpha \in \mathcal{P}_X$, where $|\alpha|$ is the cardinality of $\alpha$.

For any $\beta \in \mathcal{P}_X$ and $\theta \in \mathcal{M}(X)$ we define $|\beta|_\theta = |\{B \in \beta : \theta(B) > 0\}|$. Then in the proof of Lemma 2.3 part (2), in fact we have obtained the following result.

COROLLARY 2.4. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS and $\mu \in \mathcal{M}(X, T)$. If $\mu = \int_Y \mu_y\, d\nu(y)$ is the disintegration of $\mu$ over $\nu = \pi\mu$, then for any $\mathcal{V} \in \mathcal{C}_X$, there exists $\eta \in \mathcal{P}_X$ such that $\eta \succeq \mathcal{V}$ and $|\eta|_{\mu_y} \le N(\mathcal{V} \mid Y)$ for $\nu$-a.e. $y \in Y$.*

By parts (1) and (2) of Lemma 2.3, we get easily the inequality stated in Theorem 2.1. In fact we could show the following.

THEOREM 2.5. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS. For every $\mathcal{U} \in \mathcal{C}_X^o$ there exists $\mu \in \mathcal{M}(X, T)$ such that $h_{\text{top}}(T, \mathcal{U} \mid Y) = h_\mu^-(T, \mathcal{U} \mid Y)$.*

Theorem 2.5 shows that $h_{\text{top}}(T, \mathcal{U} \mid Y) \le \sup_\nu h_\nu(T, X \mid Y)$. By taking the supremum over all open covers in Theorem 2.5 and using Lemma 2.3, Theorem 2.1 is followed. Moreover, Theorem 2.5 shows that if there exists $\mathcal{U} \in \mathcal{C}_X^o$ such that $h_{\text{top}}(T, \mathcal{U} \mid Y) = h_{\text{top}}(T, X \mid Y)$, then there exists $\mu \in \mathcal{M}(X, T)$ such that $h_\mu(T, X \mid Y) = h_{\text{top}}(T, X \mid Y)$.

We will delay the proof of Theorem 2.5 until §4. In the next section we will introduce another new notion of $\mu$-measure conditional theoretical entropy for covers, then prove a variational inequality of conditional entropy which parallels a result in [**BGH**]. Note that the method we use is completely different from the one in [**BGH**], which is influenced by [**Mis1**] and [**DS**]. Then we use the inequality to prove Theorem 2.5 in §4. Again the method is different from the one used in [**Rom**].

## 3. *A variational inequality of conditional entropy for $h_\mu^+$*

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS. Given $\mu \in \mathcal{M}(X, T)$ and $\mathcal{U} \in \mathcal{C}_X$ define

$$h_\mu^+(T, \mathcal{U} \mid Y) = \inf_{\alpha \in \mathcal{P}_X, \alpha \succeq \mathcal{U}} h_\mu(T, \alpha \mid Y). \tag{3.1}$$

If $(Y, S)$ is a trivial system, we write $h_\mu^+(T, \mathcal{U} \mid Y)$ by $h_\mu^+(T, \mathcal{U})$. It is clear that $h_\mu^+(T, \mathcal{U} \mid Y) \ge h_\mu^-(T, \mathcal{U} \mid Y)$. Moreover, by Lemma 2.3 part (1), it is not hard to see

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that $h_\mu(T, X \mid Y) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_\mu^+(T, \mathcal{U} \mid Y)$. In this section, we aim to show that, for a given $\mathcal{U} \in \mathcal{C}_X^o$, there exists $\mu \in \mathcal{M}(X, T)$ such that $h_\mu^+(T, \mathcal{U} \mid Y) \geq h_{\text{top}}(T, \mathcal{U} \mid Y)$.

To start, we need several lemmas. The first lemma is an obvious fact.

LEMMA 3.1. *Let $\pi : (X, T) \to (Y, S)$ and $\varphi : (Z, R) \to (X, T)$ be two factor maps between TDS. If $\mu \in \mathcal{M}(Z, R)$, $\nu = \varphi\mu$, $\alpha \in \mathcal{P}_X$ and $\mathcal{U} \in \mathcal{C}_X^o$ then:*
(1) $h_{\text{top}}(R, \varphi^{-1}(\mathcal{U}) \mid Y) = h_{\text{top}}(T, \mathcal{U} \mid Y)$;
(2) $h_\mu(R, \varphi^{-1}(\alpha) \mid Y) = h_\nu(T, \alpha \mid Y)$.

The following lemma is well known.

LEMMA 3.2. *Let $(X, T)$ be a TDS, $\mu \in \mathcal{M}(X)$ and $\alpha \in \mathcal{P}_X$. If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ is an increasing sequence of measurable sub-$\sigma$-algebras with $\mathcal{F}_n \nearrow \mathcal{F}$ (respectively, $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$ is a decreasing sequence of measurable sub-$\sigma$-algebras with $\mathcal{F}_n \searrow \mathcal{F}$), then $H_\mu(\alpha \mid \mathcal{F}_n) \searrow H_\mu(\alpha \mid \mathcal{F})$ (respectively, $H_\mu(\alpha \mid \mathcal{F}_n) \nearrow H_\mu(\alpha \mid \mathcal{F})$).*

We also need the following lemma.

LEMMA 3.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS and $\alpha \in \mathcal{P}_X$. Then the following hold:*
(1) *The function $H_\bullet(\alpha \mid Y)$ is concave on $\mathcal{M}(X)$;*
(2) *The function $h_\bullet(T, \alpha \mid Y)$ and $h_\bullet(T, X \mid Y)$ are affine on $\mathcal{M}(X, T)$.*

*Proof.* (1) Take an increasing sequence of finite Borel partitions $\{\beta_j\}$ of $Y$ with $\lim_{j\to+\infty} \operatorname{diam}(\beta_j) = 0$. Let $\mu = a\nu + (1-a)\eta$, where $\nu, \eta \in \mathcal{M}(X)$ and $0 < a < 1$. Using the concavity of $\phi(t) = -t\log t$, for each $A, B \in \mathcal{B}(X)$ we have

$$\phi\left(\frac{\mu(A\cap B)}{\mu(B)}\right) \geq a\frac{\nu(B)}{\mu(B)}\phi\left(\frac{\nu(A\cap B)}{\nu(B)}\right) + (1-a)\frac{\eta(B)}{\mu(B)}\phi\left(\frac{\eta(A\cap B)}{\eta(B)}\right),$$

i.e.

$$-\mu(A\cap B)\log\frac{\mu(A\cap B)}{\mu(B)} \geq -a\nu(A\cap B)\log\frac{\nu(A\cap B)}{\nu(B)} - (1-a)\eta(A\cap B)\log\frac{\eta(A\cap B)}{\eta(B)}. \tag{3.2}$$

Now, for any $\beta \in \mathcal{P}_X$ using (3.2) we get

$$\begin{aligned} H_\mu(\alpha \mid \beta) &= \sum_{A\in\alpha, B\in\beta} -\mu(A\cap B)\log\frac{\mu(A\cap B)}{\mu(B)} \\ &\geq \sum_{A\in\alpha, B\in\beta} -a\nu(A\cap B)\log\frac{\nu(A\cap B)}{\nu(B)} - (1-a)\eta(A\cap B)\log\frac{\eta(A\cap B)}{\eta(B)} \\ &= aH_\nu(\alpha \mid \beta) + (1-a)H_\eta(\alpha \mid \beta). \end{aligned} \tag{3.3}$$

In particular, replacing $\beta$ by $\pi^{-1}(\beta_j)$ in inequality (3.3), letting $j \to +\infty$ and using Lemma 3.2 one has

$$H_\mu(\alpha \mid Y) \geq aH_\nu(\alpha \mid Y) + (1-a)H_\eta(\alpha \mid Y). \tag{3.4}$$

This ends the proof of part (1).

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(2) Let $\mu = a\nu + (1-a)\eta$, where $\nu, \eta \in \mathcal{M}(X,T)$ and $0 < a < 1$. Using (3.3) we have for each $\gamma, \beta \in \mathcal{P}_X$,

$$\begin{aligned} 0 &\le H_\mu(\gamma \mid \beta) - aH_\nu(\gamma \mid \beta) - (1-a)H_\eta(\gamma \mid \beta) \\ &= (H_\mu(\gamma \vee \beta) - aH_\nu(\gamma \vee \beta) - (1-a)H_\eta(\gamma \vee \beta)) \\ &\quad - (H_\mu(\beta) - aH_\nu(\beta) - (1-a)H_\eta(\beta)) \\ &\le H_\mu(\gamma \vee \beta) - aH_\nu(\gamma \vee \beta) - (1-a)H_\eta(\gamma \vee \beta) \quad \text{(by (3.3))} \\ &\le \phi(a) + \phi(1-a) \quad \text{(well known)}. \end{aligned} \tag{3.5}$$

In particular, replacing $\beta$ by $\pi^{-1}(\beta_j)$ in inequality (3.5), letting $j \to +\infty$ and using Lemma 3.2 one has

$$0 \le H_\mu(\gamma \mid Y) - aH_\nu(\gamma \mid Y) - (1-a)H_\eta(\gamma \mid Y) \le \phi(a) + \phi(1-a).$$

Hence $0 \le H_\mu(\alpha_0^{n-1} \mid Y) - aH_\nu(\alpha_0^{n-1} \mid Y) - (1-a)H_\eta(\alpha_0^{n-1} \mid Y) \le \phi(a) + \phi(1-a)$. Thus

$$h_\mu(T, \alpha \mid Y) = ah_\nu(T, \alpha \mid Y) + (1-a)h_\eta(T, \alpha \mid Y), \tag{3.6}$$

so that $\mu \mapsto h_\mu(T, \alpha \mid Y)$ is affine.

Finally, take an increasing sequence of finite Borel partitions $\{\alpha_i\}$ of $X$ with $\operatorname{diam}(\alpha_j) \to 0$. Replacing $\alpha$ by $\alpha_i$ in the equality (3.6), letting $i \to +\infty$ one has $h_\mu(T, X \mid Y) = ah_\nu(T, X \mid Y) + (1-a)h_\eta(T, X \mid Y)$, i.e. $\mu \mapsto h_\mu(T, X \mid Y)$ is an affine map. □

A real-valued function $f$ defined on a compact metric space $Z$ is called *upper semi-continuous* (for short u.s.c.) if one of the following equivalent conditions holds:
(A1) $\limsup_{z' \to z} f(z') \le f(z)$ for each $z \in Z$;
(A2) for each $r \in \mathbb{R}$ the set $\{z \in Z : f(z) \ge r\}$ is closed.
By (A2), the infimum of any family of u.s.c. functions is again a u.s.c. one; both the sum and supremum of finitely many u.s.c. functions are u.s.c. ones.

LEMMA 3.4. *Let $\pi : (X,T) \to (Y,S)$ be a factor map between TDS and $\alpha \in \mathcal{P}_X$ whose elements are clopen (closed and open) sets of $X$. Then:*
(1) *$H_\bullet(\alpha \mid Y)$ is a u.s.c. function on $\mathcal{M}(X)$;*
(2) *$h_\bullet(T, \alpha \mid Y)$ is a u.s.c. function on $\mathcal{M}(X,T)$.*

*Proof.* (1) To show that $H_\bullet(\alpha \mid Y)$ is a u.s.c. function on $\mathcal{M}(X)$, it remains to show that, for each $\mu \in \mathcal{M}(X)$, $\limsup_{\mu' \to \mu} H_{\mu'}(\alpha \mid Y) \le H_\mu(\alpha \mid Y)$.

Fix $\mu \in \mathcal{M}(X)$ and set $\nu = \pi\mu$. It is well known that for each $n \in \mathbb{N}$ we can choose $\gamma_n \in \mathcal{P}_Y$ satisfying
(i) $\operatorname{diam}(\gamma_n) < 1/n$ and
(ii) $\nu(\partial A) = 0$, for all $A \in \gamma_n$.

Let $\beta_n = \bigvee_{i=1}^n \gamma_n$, $n \in \mathbb{N}$. Then $\{\beta_n\}$ is a sequence of partitions of $Y$ with $\operatorname{diam}(\beta_n) \to 0$ and $\nu(\partial A) = 0$, for all $A \in \beta_n$. Note that $\partial\pi^{-1}(A) \subset \pi^{-1}(\partial A)$ for each $A \in \beta_n$, and $\mu(\partial\pi^{-1}(A)) \le \mu(\pi^{-1}(\partial A)) = \nu(\partial A) = 0$ for each $A \in \beta_n$. Combining the above fact and the assumption of $\alpha$, one has that, for each $n \in \mathbb{N}$,

$$\mu \text{ is a continuous point of function } H_\bullet(\alpha \mid \pi^{-1}(\beta_n)) \text{ on } \mathcal{M}(X). \tag{3.7}$$

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Let $\epsilon > 0$. Since $H_\mu(\alpha \mid Y) = \lim_{n\to+\infty} H_\mu(\alpha \mid \pi^{-1}(\beta_n))$, there exists $M \in \mathbb{N}$ such that $H_\mu(\alpha \mid \pi^{-1}(\beta_M)) \le H_\mu(\alpha \mid Y) + \epsilon$. Hence by (3.7) we get

$$\limsup_{\mu'\to\mu} H_{\mu'}(\alpha \mid Y) \le \limsup_{\mu'\to\mu} H_{\mu'}(\alpha \mid \pi^{-1}(\beta_M)) = H_\mu(\alpha \mid \pi^{-1}(\beta_M)) \le H_\mu(\alpha \mid Y) + \epsilon.$$

Letting $\epsilon \to 0$ we finish the proof of (1).

(2) By (1), $H_\bullet(\alpha_0^{n-1} \mid Y)$ is also a u.s.c. function on $\mathcal{M}(X,T)$. Note that for $\mu \in \mathcal{M}(X,T)$, $h_\mu(T,\alpha \mid Y) = \inf_{n\ge1}(1/n)H_\mu(\alpha_0^{n-1} \mid Y)$, i.e. the function $h_\bullet(T,\alpha \mid Y)$ is the infimum of the family of u.s.c. functions $(1/n)H_\bullet(\alpha_0^{n-1} \mid Y)$ on $\mathcal{M}(X,T)$. By (A2), $h_\bullet(T,\alpha \mid Y)$ is a u.s.c. function on $\mathcal{M}(X,T)$. □

The following lemma plays a key role in our proof of the variational inequality of conditional entropy for $h_\mu^+$. Using it and following Misiurewicz's idea, we can avoid a similar combinatorial lemma in [**BGH**] and can prove the variational inequality of conditional entropy stated in the beginning of this section.

LEMMA 3.5. *Let $\pi : (X,T) \to (Y,S)$ be a factor map between TDS, $y \in Y$ and $\mathcal{U} \in \mathcal{C}_X^o$. Assume $K \in \mathbb{N}$ and $\{\alpha_l\}_{l=1}^K$ is a finite sequence of partitions of $X$ finer than $\mathcal{U}$. Then for each $N \in \mathbb{N}$, there exists a finite subset $B_N \subset \pi^{-1}(y)$ with cardinality $[N(\mathcal{U}_0^{N-1} \mid y)/K]$ such that each atom of $(\alpha_l)_0^{N-1}$ contains at most one point of $B_N$ for $1 \le l \le K$, where $[N(\mathcal{U}_0^{N-1} \mid y)/K]$ is the integer part of $N(\mathcal{U}_0^{N-1} \mid y)/K$.*

*Proof.* Let $N \in \mathbb{N}$. For any $x \in \pi^{-1}(y)$ and $1 \le l \le K$, let $A_l(x)$ be the atom of $(\alpha_l)_0^{N-1}$ containing $x$. Then for any $x_1$ and $x_2$ in $\pi^{-1}(y)$ and $1 \le l \le K$, $x_1$ and $x_2$ are contained in the same atom of $(\alpha_l)_0^{N-1}$ if and only if $A_l(x_1) = A_l(x_2)$.

Let $B'_N$ be a maximal subset of $\pi^{-1}(y)$ such that any atom of $(\alpha_l)_0^{N-1}$ contains at most one point of $B'_N$ for $1 \le l \le K$. We claim that the cardinality of $B'_N$ is at least $[N(\mathcal{U}_0^{N-1} \mid y)/K]$. Assume the contrary, i.e. $B'_N = \{x_1,\dots,x_d\}$ with $d < [N(\mathcal{U}_0^{N-1} \mid y)/K]$.

Set $B_y = \left(\bigcup_{i=1}^d \bigcup_{l=1}^K A_l(x_i)\right) \cap \pi^{-1}(y)$. For any $1 \le i \le d$ and $1 \le l \le K$, $A_l(x_i)$ is an atom of $(\alpha_l)_0^{N-1}$, and thus is contained in an element of $\mathcal{U}_0^{N-1}$. Particularly, $B_y$ is covered by at most $dK < K[N(\mathcal{U}_0^{N-1} \mid y)/K] \le N(\mathcal{U}_0^{N-1} \mid y)$ elements of $\mathcal{U}_0^{N-1}$. Since any subcover of $\mathcal{U}_0^{N-1}$ which covers $\pi^{-1}(y)$ has at least $N(\mathcal{U}_0^{N-1} \mid y)$ elements, we have $\pi^{-1}(y) \setminus B_y \ne \emptyset$.

Choosing an arbitrary point $x \in \pi^{-1}(y) \setminus B_y$, we have $x \notin \bigcup_{i=1}^d \bigcup_{l=1}^K A_l(x_i)$. Note that for any $1 \le i \le d$ and $1 \le l \le K$, $A_l(x) \ne A_l(x_i)$. We conclude that $B'_N \cup \{x\}$ is also a subset of $\pi^{-1}(y)$ such that any atom of $(\alpha_l)_0^{N-1}$ contains at most one point of $B'_N \cup \{x\}$ for $1 \le l \le K$, a contradiction, as $B'_N$ is maximal. Choosing $B_N \subset B'_N$ with cardinality $[N(\mathcal{U}_0^{N-1} \mid y)/K]$ we are done. □

Now, let us prove the variational inequality of conditional entropy for $h_\mu^+$ stated before.

THEOREM 3.6. *Let $\pi : (X,T) \to (Y,S)$ be a factor map between TDS and $\mathcal{U} \in \mathcal{C}_X^o$. Then there exists $\mu \in \mathcal{M}(X,T)$ such that $h_\mu^+(T,\mathcal{U} \mid Y) \ge h_{\mathrm{top}}(T,\mathcal{U} \mid Y)$.*

*Proof.* Let $\mathcal{U} = \{U_1, U_2, \dots, U_d\}$. Define

$$\mathcal{U}^* = \{\alpha \in \mathcal{P}_X : \alpha = \{A_1,\dots,A_d\}, \forall m \in \{1,\dots,d\}, A_m \subset U_m\}. \tag{3.8}$$

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*Case 1.* Assume first that $X$ is zero-dimensional. The family of partitions finer than $\mathcal{U}$, consisting of clopen sets in $\mathcal{U}^*$, is countable; let $\{\alpha_l : l \geq 1\}$ be an enumeration of this family.

Fix $n \in \mathbb{N}$. By definition, there exists $y_n \in Y$ such that $N(\mathcal{U}_0^{n^2+n-1} \mid Y) = N(\mathcal{U}_0^{n^2+n-1} \mid y_n)$. By Lemma 3.5, there exists $C_n = B_{n^2}(T^{-n}\mathcal{U}, \{T^{-n}\alpha_l\}_1^n, y_n) \subset \pi^{-1}(y_n)$ with cardinality $[N(\mathcal{U}_n^{n^2+n-1} \mid y_n)/n]$ such that any atom of $(\alpha_l)_n^{n^2+n-1}$ contains at most one point of $C_n$ for $1 \leq l \leq n$. Let $\nu_n$ be the equidistributed probability measure on $C_n$. Then for each $0 \leq i, l \leq n$, every element of $T^{-i}(\alpha_l)_0^{n^2+n-1}$ (finer than $(\alpha_l)_n^{n^2+n-1}$) contains at most one atom of the discrete measure $\nu_n$. Since

$$N(\mathcal{U}_0^{n^2+n-1} \mid y_n) \leq N(\mathcal{U}_0^{n-1} \mid y_n) N(\mathcal{U}_n^{n^2+n-1} \mid y_n) \leq d^n N(\mathcal{U}_n^{n^2+n-1} \mid y_n),$$

one has

$$N(\mathcal{U}_n^{n^2+n-1} \mid y_n) \geq d^{-n} N(\mathcal{U}_0^{n^2+n-1} \mid y_n) = d^{-n} N(\mathcal{U}_0^{n^2+n-1} \mid Y).$$

Thus, for any $1 \leq i, l \leq n$ we have

$$\begin{aligned} H_{T^i\nu_n}((\alpha_l)_0^{n^2+n-1}) &= H_{\nu_n}(T^{-i}(\alpha_l)_0^{n^2+n-1}) \\ &= \log\left[\frac{N(\mathcal{U}_n^{n^2+n-1} \mid y_n)}{n}\right] \geq \log\left[\frac{N(\mathcal{U}_0^{n^2+n-1} \mid Y)}{nd^n}\right]. \end{aligned}$$

Since $\nu_n$ is supported by $\pi^{-1}(y_n)$, $T^i\nu_n$ is supported by $\pi^{-1}(S^i y_n)$, for $1 \leq i, l \leq n$ we have

$$H_{T^i\nu_n}((\alpha_l)_0^{n^2+n-1} \mid Y) = H_{T^i\nu_n}((\alpha_l)_0^{n^2+n-1}) \geq \log\left[\frac{N(\mathcal{U}_0^{n^2+n-1} \mid Y)}{nd^n}\right].$$

Fix $m \in \mathbb{N}$ with $m \leq n$, and let $n^2 + n = km + b$, where $0 \leq b \leq m-1$. Then for $1 \leq i, l \leq n$ we have

$$\begin{aligned} &H_{T^i\nu_n}((\alpha_l)_0^{n^2+n-1} \mid Y) \\ &\quad = H_{T^i\nu_n}\left((\alpha_l)_{km}^{n^2+n-1} \vee \bigvee_{j=0}^{k-1} T^{-mj}(\alpha_l)_0^{m-1} \,\middle|\, Y\right) \\ &\quad \leq H_{T^i\nu_n}((\alpha_l)_{km}^{n^2+n-1} \mid Y) + \sum_{j=0}^{k-1} H_{T^i\nu_n}(T^{-mj}(\alpha_l)_0^{m-1} \mid Y) \\ &\quad \leq \sum_{j=0}^{k-1} H_{T^{i+mj}\nu_n}((\alpha_l)_0^{m-1} \mid Y) + m \log d \quad \text{(by Lemma 2.2 (1) and (4))}. \end{aligned}$$

Summing over $0 \leq i \leq m-1$ for each $1 \leq l \leq n$, we get

$$\begin{aligned} m \log\left[\frac{N(\mathcal{U}_0^{n^2+n-1} \mid Y)}{nd^n}\right] &\leq \sum_{i=0}^{m-1} H_{T^i\nu_n}((\alpha_l)_0^{n^2+n-1} \mid Y) \\ &\leq \sum_{i=0}^{m-1}\sum_{j=0}^{k-1} H_{T^{i+mj}\nu_n}((\alpha_l)_0^{m-1} \mid Y) + m^2 \log d \\ &\leq \sum_{i=0}^{n^2+n-1} H_{T^i\nu_n}((\alpha_l)_0^{m-1} \mid Y) + m^2 \log d. \end{aligned}$$

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Put

$$\mu_n = \frac{1}{n^2+n}\sum_{i=0}^{n^2+n-1} T^i \nu_n.$$

As $H_\bullet((\alpha_l)_0^{m-1} \mid Y)$ is concave on $\mathcal{M}(X)$ (Lemma 3.3 part (1)), we get for each $1 \le l \le n$

$$\begin{aligned} H_{\mu_n}((\alpha_l)_0^{m-1} \mid Y) &\ge \frac{1}{n^2+n}\sum_{i=0}^{n^2+n-1} H_{T^i\nu_n}((\alpha_l)_0^{m-1} \mid Y) \\ &\ge \frac{m}{n^2+n}\left(\log\left[\frac{N(\mathcal{U}_0^{n^2+n-1} \mid Y)}{nd^n}\right] - m\log d\right). \end{aligned} \tag{3.9}$$

Suppose that $\mu_{n_k} \to \mu$ with $\mu \in \mathcal{M}(X)$ in the weak*-topology, where $n_k \to \infty$ as $k \to \infty$. Clearly, $\mu \in \mathcal{M}(X, T)$. Fixing $m \in \mathbb{N}$, we have

$$\lim_{k\to\infty} \frac{1}{n_k^2+n_k}\left(\log\left[\frac{N(\mathcal{U}_0^{n_k^2+n_k-1} \mid Y)}{n_k d^{n_k}}\right] - m\log d\right) = h_{\mathrm{top}}(T, \mathcal{U} \mid Y). \tag{3.10}$$

Since any element of $(\alpha_l)_0^{m-1}$ is clopen, by Lemma 3.4 (1), equation (3.10), and replacing $n$ by $n_k$ in the inequality (3.9) and letting $k \to +\infty$, we get

$$\frac{1}{m} H_\mu((\alpha_l)_0^{m-1} \mid Y) \ge h_{\mathrm{top}}(T, \mathcal{U} \mid Y),$$

for any fixed positive integers $l, m \in \mathbb{N}$.

Now fixing $l \in \mathbb{N}$, and letting $m \to +\infty$, one gets $h_\mu(T, \alpha_l \mid Y) \ge h_{\mathrm{top}}(T, \mathcal{U} \mid Y)$. Since $X$ is zero-dimensional, $(\alpha_l)_{l\in\mathbb{N}}$ is dense in the collection of Borel partitions of $X$ satisfying (3.8) with respect to the distance associated with $L^1(\mu)$. Hence

$$h_\mu^+(T, \mathcal{U} \mid Y) = \inf_{\beta\in\mathcal{P}_X, \beta\succeq\mathcal{U}} h_\mu(T, \beta \mid Y) = \inf_{l\in\mathbb{N}} h_\mu(T, \alpha_l \mid Y) \ge h_{\mathrm{top}}(T, \mathcal{U} \mid Y).$$

This completes the proof of Case 1.

*Case 2.* This is the general case.

Let us recall a well-known fact (see e.g. [**BGH**]): there exist a TDS $(Z, R)$ (where $Z$ is zero-dimensional) and a continuous surjective map $\varphi : Z \to X$ with $\varphi R = T\varphi$.

Then by Lemma 3.1 and the above Case 1, there exists $\nu \in \mathcal{M}(Z, R)$ such that $h_\nu^+(R, \varphi^{-1}(\mathcal{U}) \mid Y) \ge h_{\mathrm{top}}(R, \varphi^{-1}(\mathcal{U}) \mid Y) = h_{\mathrm{top}}(T, \mathcal{U} \mid Y)$. That is, $h_\nu(R, \beta \mid Y) \ge h_{\mathrm{top}}(T, \mathcal{U} \mid Y)$ for every Borel partition $\beta$ of $Z$ finer than $\varphi^{-1}(\mathcal{U})$. Let $\mu = \varphi\nu$. Then for any Borel partition $\alpha$ of $X$ finer than $\mathcal{U}$, $\varphi^{-1}(\alpha)$ is a Borel partition of $Z$ finer than $\varphi^{-1}(\mathcal{U})$. By Lemma 3.1 we have $h_\mu(T, \alpha \mid Y) = h_\nu(R, \varphi^{-1}(\alpha) \mid Y) \ge h_{\mathrm{top}}(T, \mathcal{U} \mid Y)$. Hence $h_\mu^+(T, \mathcal{U} \mid Y) \ge h_{\mathrm{top}}(T, \mathcal{U} \mid Y)$. This completes the proof of the theorem. □

## 4. *A local variational principle of conditional entropy for $h_\mu^-$*

In this section, we shall provide a proof of Theorem 2.5. First, we state some relations between two kinds of measure conditional entropy of covers, then use the relations and Theorem 3.6 to prove Theorem 2.5.

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LEMMA 4.1. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS and $\mu \in \mathcal{M}(X, T)$. Then for $\mathcal{U} \in \mathcal{C}_X$, we have:*

(1) $h_\mu^-(T, \mathcal{U} \mid Y) \le h_\mu^+(T, \mathcal{U} \mid Y)$;

(2) $h_\mu^-(T, \mathcal{U} \mid Y) = (1/M) h_\mu^-(T^M, \mathcal{U}_0^{M-1} \mid Y)$ *for each* $M \in \mathbb{N}$;

(3) $h_\mu^-(T, \mathcal{U} \mid Y) = \lim_{n\to\infty} (1/n) h_\mu^+(T^n, \mathcal{U}_0^{n-1} \mid Y)$.

*Proof.* Parts (1) and (2) are easy to prove. In the following we prove part (3). By parts (1) and (2), we get that, for any $M \in \mathbb{N}$,

$$\begin{aligned} h_\mu^-(T, \mathcal{U} \mid Y) &= \frac{1}{M} h_\mu^-(T^M, \mathcal{U}_0^{M-1} \mid Y) \\ &\le \frac{1}{M} h_\mu^+(T^M, \mathcal{U}_0^{M-1} \mid Y) \le \frac{1}{M} H_\mu(\mathcal{U}_0^{M-1} \mid Y). \end{aligned}$$

Now take the limit when $M \to +\infty$ to conclude argument. □

With the above preparation, now we prove Theorem 2.5.

*Proof of Theorem 2.5.* We follow the arguments in the proof of [**HMRY**, Theorem 4]. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_d\}$. Put

$$\mathcal{U}^* = \{\alpha \in \mathcal{P}_X : \alpha = \{A_1, \ldots, A_d\}, \forall m \in \{1, \ldots, d\}, A_m \subset U_m\}.$$

*Case 1.* Assume first that $X$ is zero-dimensional. The family of partitions finer than $\mathcal{U}$, consisting of clopen sets in $\mathcal{U}^*$, is countable; let $\{\alpha_l : l \ge 1\}$ be an enumeration of this family. Clearly, for any $k \in \mathbb{N}$ and $\mu \in \mathcal{M}(X, T)$

$$h_\mu^+(T^k, \mathcal{U}_0^{k-1} \mid Y) = \inf_{s_k \in \mathbb{N}^k} h_\mu\left(T^k, \bigvee_{i=0}^{k-1} T^{-i} \alpha_{s_k(i)} \,\middle|\, Y\right). \tag{4.1}$$

For any $k \in \mathbb{N}$ and $s_k \in \mathbb{N}^k$, set

$$\begin{aligned} M(k, s_k) \doteq \Bigg\{ \mu \in \mathcal{M}(X, T) : \frac{1}{k} h_\mu\left(T^k, \bigvee_{i=0}^{k-1} T^{-i} \alpha_{s_k(i)} \,\middle|\, Y\right) \\ \ge \frac{1}{k} h_{\text{top}}(T^k, \mathcal{U}_0^{k-1} \mid Y) = h_{\text{top}}(T, \mathcal{U} \mid Y) \Bigg\}. \end{aligned}$$

By Theorem 3.6 we know that there exists $\mu_k \in \mathcal{M}(X, T^k)$ such that

$$h_{\mu_k}^+(T^k, \mathcal{U}_0^{k-1} \mid Y) \ge h_{\text{top}}(T^k, \mathcal{U}_0^{k-1} \mid Y).$$

Since $\bigvee_{i=0}^{k-1} T^{-i} \alpha_{s_k(i)}$ is finer than $\mathcal{U}_0^{k-1}$ for each $s_k \in \mathbb{N}^k$, one has

$$h_{\mu_k}\left(T^k, \bigvee_{i=0}^{k-1} T^{-i} \alpha_{s_k(i)} \,\middle|\, Y\right) \ge h_{\text{top}}(T^k, \mathcal{U}_0^{k-1} \mid Y). \tag{4.2}$$

Let $T^i \mu_k = \mu_k \circ T^{-i}$, $i \in \mathbb{N}$ and

$$\nu_k = \frac{\mu_k + T\mu_k + \cdots + T^{k-1}\mu_k}{k}.$$

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Then $T^i\mu_k \in \mathcal{M}(X, T^k)$ for each $i \in \{0, 1, \ldots, k-1\}$ and $\nu_k \in \mathcal{M}(X, T)$. For $s_k \in \mathbb{N}^k$ and $j = 1, 2, \ldots, k-1$, denote

$$P^j s_k = \underbrace{s_k(k-j)s_k(k-j+1)\ldots s_k(k-1)}_{j}\underbrace{s_k(0)s_k(1)\ldots s_k(k-j-1)}_{k-j} \in \mathbb{N}^k.$$

Set $P^0 s_k = s_k$. It is easy to see that for $j = 0, 1, 2, \ldots, k-1$

$$h_{T^j\mu_k}\Big(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)} \,\Big|\, Y\Big) = h_{\mu_k}\Big(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{P^j s_k(i)} \,\Big|\, Y\Big) \quad \text{(by the definitions)}$$
$$\geq h_{\text{top}}(T^k, \mathcal{U}_0^{k-1} \mid Y) \quad \text{(by (4.2))}.$$

Moreover by Lemma 3.3 part (2) for each $s_k \in \mathbb{N}^k$

$$h_{\nu_k}\Big(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)} \,\Big|\, Y\Big) = \frac{1}{k}\sum_{i=0}^{k-1} h_{T^i\mu_k}\Big(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)} \,\Big|\, Y\Big)$$
$$\geq h_{\text{top}}(T^k, \mathcal{U}_0^{k-1} \mid Y).$$

This implies $\nu_k \in \bigcap_{s_k\in\mathbb{N}^k} M(k, s_k)$. Write $M(k) = \bigcap_{s_k\in\mathbb{N}^k} M(k, s_k)$. Then $M(k)$ is a non-empty subset of $\mathcal{M}(X, T)$.

Since for every $s_k \in \mathbb{N}^k$, $\bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}$ is a clopen cover, hence the map $\mu \mapsto h_\mu\big(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)} \mid Y\big)$ is a u.s.c. function from $\mathcal{M}(X, T^k)$ to $\mathbb{R}$ by Lemma 3.4 part (2). Since $\mathcal{M}(X, T) \subset \mathcal{M}(X, T^k)$, $h_\bullet(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)} \mid Y)$ is also u.s.c. on $\mathcal{M}(X, T)$. Therefore, $M(k, s_k)$ is closed in $\mathcal{M}(X, T)$ for each $s_k \in \mathbb{N}^k$. Thus $M(k)$ is a non-empty closed set of $\mathcal{M}(X, T)$.

Now we show that if $k_1, k_2 \in \mathbb{N}$ with $k_1 | k_2$ then $M(k_2) \subset M(k_1)$. Let $\mu \in M(k_2)$ and $k = k_2/k_1$. For any $s_{k_1} \in \mathbb{N}^{k_1}$, let $s_{k_2} = s_{k_1}\ldots s_{k_1} \in \mathbb{N}^{k_2}$. Then

$$\frac{1}{k_1} h_\mu\Big(T^{k_1}, \bigvee_{i=0}^{k_1-1} T^{-i}\alpha_{s_{k_1}(i)} \,\Big|\, Y\Big) = \frac{1}{k_1}\Big[\frac{1}{k} h_\mu\Big(T^{kk_1}, \bigvee_{j=0}^{k-1} T^{-jk_1}\bigvee_{i=0}^{k_1-1} T^{-i}\alpha_{s_{k_1}(i)} \,\Big|\, Y\Big)\Big]$$
$$= \frac{1}{k_2} h_\mu\Big(T^{k_2}, \bigvee_{i=0}^{k_2-1} T^{-i}\alpha_{s_{k_2}(i)} \,\Big|\, Y\Big) \geq h_{\text{top}}(T, \mathcal{U} \mid Y).$$

Hence $\mu \in M(k_1, s_{k_1})$ for each $s_{k_1} \in \mathbb{N}^{k_1}$. That is, $\mu \in M(k_1)$. This shows that $M(k_2) \subset M(k_1)$.

Since $M(k_1) \cap M(k_2) \supset M(k_1k_2) \neq \emptyset$ for any $k_1, k_2 \in \mathbb{N}$, one has $\bigcap_{k\in\mathbb{N}} M(k) \neq \emptyset$. Take $\mu \in \bigcap_{k\in\mathbb{N}} M(k)$. For any $k \in \mathbb{N}$, by equation (4.1) one has

$$\frac{1}{k} h_\mu^+(T^k, \mathcal{U}_0^{k-1} \mid Y) = \inf_{s_k\in\mathbb{N}^k} \frac{1}{k} h_\mu\Big(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)} \,\Big|\, Y\Big) \geq h_{\text{top}}(T, \mathcal{U} \mid Y).$$

Moveover, by Lemma 4.1 part (3), one gets that

$$h_\mu^-(T, \mathcal{U} \mid Y) = \lim_{k\to+\infty} \frac{1}{k} h_\mu^+(T^k, \mathcal{U}_0^{k-1} \mid Y) \geq h_{\text{top}}(T, \mathcal{U} \mid Y).$$

Following Lemma 2.3 part (2), we have $h_\mu^-(T, \mathcal{U} \mid Y) = h_{\text{top}}(T, \mathcal{U} \mid Y)$. This completes the proof of Case 1.

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*Case 2.* This is the general case.

First, there exists a zero-dimensional TDS $(Z, R)$ and a factor map $\varphi : (Z, R) \to (X, T)$. Let $\mathcal{V} = \varphi^{-1}(\mathcal{U})$. One has $h_{\text{top}}(R, \mathcal{V} \mid Y) = h_{\text{top}}(T, \mathcal{U} \mid Y)$ by Lemma 3.1 part (1). Using Case 1, we know that there is $\nu \in \mathcal{M}(Z, R)$ for which $h_\nu^-(R, \mathcal{V} \mid Y) = h_{\text{top}}(R, \mathcal{V} \mid Y)$. Let $\mu = \varphi\nu$. Note that if $N \in \mathbb{N}$ and $\beta \in \mathcal{P}_X$ is finer than $\mathcal{U}_0^{N-1}$, then $\varphi^{-1}(\beta)$ is a finite partition finer than $\mathcal{V}_0^{N-1}$. Thus $H_\mu(\beta \mid Y) = H_\nu(\varphi^{-1}\beta \mid Y) \geq H_\nu(\mathcal{V}_0^{N-1} \mid Y)$. Hence $H_\mu(\mathcal{U}_0^{N-1} \mid Y) \geq H_\nu(\mathcal{V}_0^{N-1} \mid Y)$ for each $N \in \mathbb{N}$. Thus we get

$$\begin{aligned} h_\mu^-(T, \mathcal{U} \mid Y) &= \lim_{N\to\infty} \frac{1}{N} H_\mu(\mathcal{U}_0^{N-1} \mid Y) \geq \lim_{N\to\infty} \frac{1}{N} H_\nu(\mathcal{V}_0^{N-1} \mid Y) \\ &= h_\nu^-(R, \mathcal{V} \mid Y) = h_{\text{top}}(R, \mathcal{V} \mid Y) = h_{\text{top}}(T, \mathcal{U} \mid Y) \geq h_\mu^-(T, \mathcal{U} \mid Y). \end{aligned}$$

This ends the proof of Theorem 2.5. □

## 5. *Ergodic decompositions*

In this section, we will give the ergodic decompositions of two kinds of measure conditional entropy of covers. Consequently, we can strengthen Theorem 2.5 so that the obtained measure is ergodic.

Let $(X, T)$ be a TDS and $\mu \in \mathcal{M}(X, T)$. Given $\alpha \in \mathcal{P}_X$, put $\alpha^- = \bigvee_{n=1}^{\infty} T^{-i}\alpha$ and $\alpha^T = \bigvee_{n=-\infty}^{+\infty} T^{-i}\alpha$. The following lemma is a classical result in ergodic theory (see for example [**Par**]).

LEMMA 5.1. (Pinsker formula) *Let $\alpha, \beta \in \mathcal{P}_X$. Then*

$$h_\mu(T, \alpha \vee \beta) = h_\mu(T, \beta) + H_\mu(\alpha \mid \beta^T \vee \alpha^-).$$

LEMMA 5.2. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS, $\mu \in \mathcal{M}(X, T)$ and $\alpha \in \mathcal{P}_X$. If $\mu = \int_\Omega \mu_\omega \, dm(\omega)$ is the ergodic decomposition of $\mu$, then*

$$h_\mu(T, \alpha \mid Y) = \int_\Omega h_{\mu_\omega}(T, \alpha \mid Y) \, dm(\omega) \quad \text{and} \quad h_\mu(T \mid Y) = \int_\Omega h_{\mu_\omega}(T \mid Y) \, dm(\omega).$$

*Proof.* First, we take an increasing sequence of finite Borel partitions $\{\beta_j\}$ of $Y$ with $\lim_{j\to+\infty} \text{diam}(\beta_j) = 0$. Then using the Pinsker formula and the dominated convergence theorem, we have

$$\begin{aligned} h_\mu(T, \alpha \mid Y) &= H_\mu(\alpha \mid \alpha^- \vee \pi^{-1}\mathcal{B}(Y)) \\ &= \lim_{j\to\infty} H_\mu(\alpha \mid \alpha^- \vee (\pi^{-1}(\beta_j))^T) \\ &= \lim_{j\to\infty} [h_\mu(T, \alpha \vee \pi^{-1}(\beta_j)) - h_\mu(T, \pi^{-1}(\beta_j))] \\ &= \lim_{j\to\infty} \int_\Omega [h_{\mu_\omega}(T, \alpha \vee \pi^{-1}(\beta_j)) - h_{\mu_\omega}(T, \pi^{-1}(\beta_j))] \, dm(\omega) \\ &= \lim_{j\to\infty} \int_\Omega H_{\mu_\omega}(\alpha \mid \alpha^- \vee (\pi^{-1}\beta_j)^T) \, dm(\omega) \\ &= \int_\Omega \lim_{j\to\infty} H_{\mu_\omega}(\alpha \mid \alpha^- \vee (\pi^{-1}(\beta_j))^T) \, dm(\omega) \\ &= \int_\Omega h_{\mu_\omega}(T, \alpha \mid Y) \, dm(\omega). \end{aligned}$$

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Now we take an increasing sequence of finite Borel partitions $\gamma_j$ of $X$ such that $\lim_{j\to+\infty}\operatorname{diam}(\gamma_j)=0$, then

$$\begin{aligned} h_\mu(T\mid Y) &= \lim_{j\to\infty} h_\mu(T,\gamma_j\mid Y)\\ &= \lim_{j\to\infty}\int_\Omega h_{\mu_\omega}(T,\gamma_j\mid Y)\,dm(\omega)\\ &= \int_\Omega \lim_{j\to\infty} h_{\mu_\omega}(T,\gamma_j\mid Y)\,dm(\omega)\\ &= \int_\Omega h_{\mu_\omega}(T\mid Y)\,dm(\omega). \end{aligned}$$

□

The proof of the following theorem follows the idea of the proof of Lemma 4.8 in [**HY2**].

THEOREM 5.3. *Let $\pi:(X,T)\to(Y,S)$ be a factor map between TDS, $\mu\in\mathcal{M}(X,T)$ and $\mathcal{U}\in\mathcal{C}_X$. If $\mu=\int_\Omega \mu_\omega\,dm(\omega)$ is the ergodic decomposition of $\mu$, then*

$$h^+_\mu(T,\mathcal{U}\mid Y)=\int_\Omega h^+_{\mu_\omega}(T,\mathcal{U}\mid Y)\,dm(\omega)$$

*and*

$$h^-_\mu(T,\mathcal{U}\mid Y)=\int_\Omega h^-_{\mu_\omega}(T,\mathcal{U}\mid Y)\,dm(\omega).$$

*Proof.* Let $\mathcal{U}=\{U_1,U_2,\dots,U_M\}\in\mathcal{C}_X$ and define

$$\mathcal{U}^*=\{\alpha\in\mathcal{P}_X:\alpha=\{A_1,\dots,A_M\},A_m\subset U_m,\forall m\in\{1,\dots,M\}\}.$$

Since $X$ is a compact metric space there exists a sequence of partitions $(\alpha_k:k\in\mathbb{N})$ in $\mathcal{U}^*$ that is $L^1(X,\mathcal{B}(X),\nu)$-dense in $\mathcal{U}^*$ for every $\nu\in\mathcal{M}(X,T)$. In particular, it holds that

$$\text{for every } \nu\in\mathcal{M}(X,T),\ h^+_\nu(T,\mathcal{U}\mid Y)=\inf_{k\in\mathbb{N}} h_\nu(T,\alpha_k\mid Y). \tag{5.1}$$

Write $\alpha_k=\{A_1^k,\dots,A_M^k\}$, $k\in\mathbb{N}$. By Fatou's lemma

$$\begin{aligned} h^+_\mu(T,\mathcal{U}\mid Y) &= \inf_{k\in\mathbb{N}} h_\mu(T,\alpha_k\mid Y)\\ &= \inf_{k\in\mathbb{N}}\int_\Omega h_{\mu_\omega}(T,\alpha_k\mid Y)\,dm(\omega)\quad\text{(Lemma 5.2)}\\ &\ge \int_\Omega \inf_{k\in\mathbb{N}} h_{\mu_\omega}(T,\alpha_k\mid Y)\,dm(\omega)=\int_\Omega h^+_{\mu_\omega}(T,\mathcal{U}\mid Y)\,dm(\omega). \end{aligned}$$

This proves one inequality. To prove the other one, for every $\varepsilon>0$ and $n\in\mathbb{N}$ define $B_n^\varepsilon=\{\omega\in\Omega: h_{\mu_\omega}(T,\alpha_n\mid Y)<h^+_{\mu_\omega}(T,\mathcal{U}\mid Y)+\varepsilon\}$. By (5.1) we know that $m\big(\Omega\Delta\bigcup_{n\in\mathbb{N}}B_n^\varepsilon\big)=0$, so there exists a partition $(\Omega_n:n\in\mathbb{N})$ of $\Omega$ with $m(\Omega_n)>0$, and a sequence of partitions $(\alpha_{k_n}:n\in\mathbb{N})$ such that for $m$-a.e. $\omega\in\Omega_n$ and $n\in\mathbb{N}$ one has $h_{\mu_\omega}(T,\alpha_{k_n}\mid Y)<h^+_{\mu_\omega}(T,\mathcal{U}\mid Y)+\varepsilon$. Therefore, for every $n\in\mathbb{N}$ we can define $\mu_n\in\mathcal{M}(X,T)$ as

$$\mu_n=\frac{1}{m(\Omega_n)}\int_{\Omega_n}\mu_\omega\,dm(\omega).$$

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We deduce

$$\begin{aligned} h_{\mu_n}(T, \alpha_{k_n} \mid Y) &= \frac{1}{m(\Omega_n)} \int_{\Omega_n} h_{\mu_\omega}(T, \alpha_{k_n} \mid Y)\, dm(\omega) \quad \text{(by Lemma 5.2)} \\ &\le \frac{1}{m(\Omega_n)} \int_{\Omega_n} h^+_{\mu_\omega}(T, \mathcal{U} \mid Y)\, dm(\omega) + \varepsilon. \end{aligned}$$

Since $\{\mu_n\}_{n\in\mathbb{N}}$ is mutually singular, there exists a family of Borel sets $\{X_n\}_{n\in\mathbb{N}}$ such that $\mu_n(X_n) = 1$ and $\mu_n(X_k) = 0$ for $k \neq n$. For $1 \le i \le M$ define $A_i = \bigcup_{n\ge 1}(X_n \cap A_i^{k_n})$, then $\alpha = \{A_1, \ldots, A_M\} \in \mathcal{U}^*$ and

$$\begin{aligned} h^+_\mu(T, \mathcal{U} \mid Y) &\le h_\mu(T, \alpha \mid Y) \\ &= \sum_{n\in\mathbb{N}} m(\Omega_n) h_{\mu_n}(T, \alpha \mid Y) \quad \text{(by Lemma 3.3(2))} \\ &= \sum_{n\in\mathbb{N}} m(\Omega_n) h_{\mu_n}(T, \alpha_{k_n} \mid Y) \\ &\le \int_\Omega h^+_{\mu_\omega}(T, \mathcal{U} \mid Y)\, dm(\omega) + \varepsilon. \end{aligned}$$

We conclude that $h^+_\mu(T, \mathcal{U} \mid Y) \le \int_\Omega h^+_{\mu_\omega}(T, \mathcal{U} \mid Y)\, dm(\omega)$.

Finally, using statement (2) of Lemma 4.1 we deduce

$$\begin{aligned} h^-_\mu(T, \mathcal{U} \mid Y) &= \lim_{n\to\infty} \frac{1}{n} h^+_\mu(T^n, \mathcal{U}_0^{n-1} \mid Y) \\ &= \lim_{n\to\infty} \frac{1}{n} \int_\Omega h^+_{\mu_\omega}(T^n, \mathcal{U}_0^{n-1} \mid Y)\, dm(\omega) \\ &= \int_\Omega \lim_{n\to\infty} \frac{1}{n} h^+_{\mu_\omega}(T^n, \mathcal{U}_0^{n-1} \mid Y)\, dm(\omega) \\ &= \int_\Omega h^-_{\mu_\omega}(T, \mathcal{U} \mid Y)\, dm(\omega). \end{aligned}$$

□

As a corollary of Theorem 2.5 and Theorem 5.3, we have the following result.

THEOREM 5.4. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS. For every $\mathcal{U} \in \mathcal{C}_X^o$ there exists $\mu \in \mathcal{M}^e(X, T)$ such that $h_{\text{top}}(T, \mathcal{U} \mid Y) = h^-_\mu(T, \mathcal{U} \mid Y)$. Moreover,*

$$h_{\text{top}}(T, \mathcal{U} \mid Y) = \max_{\mu\in\mathcal{M}(X,T)} h^-_\mu(T, \mathcal{U} \mid Y) = \max_{\mu\in\mathcal{M}^e(X,T)} h^-_\mu(T, \mathcal{U} \mid Y).$$

*Remark.* Theorem 5.4 shows that if there exists $\mathcal{U} \in \mathcal{C}_X^o$ such that $h_{\text{top}}(T, \mathcal{U} \mid Y) = h_{\text{top}}(T, X \mid Y)$, then there exists $\mu \in \mathcal{M}^e(X, T)$ such that $h_\mu(T, X \mid Y) = h_{\text{top}}(T, X \mid Y)$.

## 6. *A local variational principle of conditional entropy for $h^+_\mu$*

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS, $\mu \in \mathcal{M}(X, T)$ and $\mathcal{U} \in \mathcal{C}_X$. In this section, we shall prove that $h^+_\mu(T, \mathcal{U} \mid Y) \le h_{\text{top}}(T, \mathcal{U} \mid Y)$ and $h^+_\mu(T, \mathcal{U} \mid Y) = h^-_\mu(T, \mathcal{U} \mid Y)$ for each $\mu \in \mathcal{M}(X, T)$. Note that the first inequality was proved in [**GW**] when $(Y, S)$ is trivial.

The following is the well-known Rohlin lemma (see [**Gla**] for more details and see [**GW**] for a universal version of it).

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LEMMA 6.1. (Rohlin's lemma) *Let $(X, T)$ be a TDS and $\mu \in \mathcal{M}^e(X, T)$. Suppose $\mu$ is non-atomic (i.e. $\mu(\{x\}) = 0$ for each $x \in X$). Then given any $N \in \mathbb{N}$ and $\epsilon > 0$, there exists a Borel subset $D$ of $X$ such that $D, TD, \ldots, T^{N-1}D$ are pairwise disjoint and $\mu\big(\bigcup_{i=0}^{N-1} T^i D\big) > 1 - \epsilon$.*

With the help of Lemma 6.1 and Theorem 3.6, we have our next theorem.

THEOREM 6.2. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS, $\mu \in \mathcal{M}(X, T)$ and $\mathcal{U} \in \mathcal{C}_X$. Then $h_\mu^+(T, \mathcal{U} \mid Y) \le h_{\text{top}}(T, \mathcal{U} \mid Y)$, and thus when $\mathcal{U} \in \mathcal{C}_X^o$*

$$\max_{\mu \in \mathcal{M}(X,T)} h_\mu^+(T, \mathcal{U} \mid Y) = h_{\text{top}}(T, \mathcal{U} \mid Y).$$

*Proof.* We follow the ideas in [**GW**]. By Theorem 5.3, it remains to show that $h_\mu^+(T, \mathcal{U} \mid Y) \le h_{\text{top}}(T, \mathcal{U} \mid Y)$ for all non-atomic measures $\mu \in \mathcal{M}^e(X, T)$. Now let $\mu \in \mathcal{M}^e(X, T)$ and $\mu$ be non-atomic. Then $\nu = \pi\mu \in \mathcal{M}^e(Y, S)$. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$.

Choose for $\epsilon > 0$ an integer $N$ large enough so that

$$N(\mathcal{U}_0^{N-1} \mid Y) \le 2^{N(h_{\text{top}}(T,\mathcal{U}|Y)+\epsilon)} \quad \text{and} \quad -\left(1 - \frac{1}{N}\right)\log\left(1 - \frac{1}{N}\right) - \frac{1}{N}\log\frac{1}{N} < \epsilon. \tag{6.1}$$

Take small enough $1 > \delta > 0$ such that

$$2\sqrt{\delta}\log k < \epsilon. \tag{6.2}$$

Applying Lemma 6.1 to $\mu$, we can find a Borel subset $D$ of $X$ such that $D, TD, \ldots, T^{N-1}D$ are pairwise disjoint and $\mu\big(\bigcup_{i=0}^{N-1} T^i D\big) > 1 - \delta$. Let $\mu = \int_Y \mu_y \, d\nu(y)$ be the disintegration of $\mu$ with respect to $\nu$. By Corollary 2.4, we can take $\beta \succeq \mathcal{U}_0^{N-1}$ a partition of $X$ such that $|\beta|_{\mu_y} \le N(\mathcal{U}_0^{N-1} \mid Y)$ for $\nu$-a.e. $y \in Y$.

Consider the partition $\beta_D = \{B \cap D : B \in \beta\}$ of $D$. Then each element $P \in \beta_D$ has the form $P = P_{i_0,i_1,\ldots,i_{N-1}} \subseteq \big(\bigcap_{j=0}^{N-1} T^{-j} U_{i_j}\big) \cap D$. Next use the partition $\beta_D$ to define a partition $\alpha = \{A_i : i = 1, \ldots, k\}$ of $X$ by assigning to the set $A_i$ all sets of the form $T^j P_{i_0,i_1,\ldots,i_{N-1}}$ where $i_j = i$ ($j$ can be any number in $[0, N-1]$). On the remainder of the space $\alpha$ can be taken to be any partition refining $\mathcal{U}$. For $\gamma' \in \mathcal{P}_X$ and $R \subset X$, we define $\gamma' \cap R = \{A \cap R : A \in \gamma' \text{ and } A \cap R \neq \emptyset\}$. From the construction of $\alpha$, it is easy to see that $\alpha_0^{N-1} \cap D = \beta_D$. Moreover,

$$|\alpha_0^{N-1} \cap D|_{\mu_y} = |\beta_D|_{\mu_y} \le |\beta|_{\mu_y} \le N(\mathcal{U}_0^{N-1} \mid Y) \text{ for } \nu\text{-a.e. } y \in Y. \tag{6.3}$$

Now, we verify that $h_\mu^+(T, \mathcal{U} \mid Y) \le h_{\text{top}}(T, \mathcal{U} \mid Y) + 3\epsilon$. Let $E = \bigcup_{i=0}^{N-1} T^i D$, then $\mu(E) > 1 - \delta$. Fix an $n \gg N$, let $f_n(x) = (1/n)\sum_{i=0}^{n-1} 1_E(T^i x)$ and let $G_n = \{x \in X : f_n(x) > 1 - \sqrt{\delta}\}$. Since

$$\begin{aligned}\mu(G_n) + (1 - \sqrt{\delta})(1 - \mu(G_n)) &\ge \int_{G_n} f_n(x)\, d\mu(x) + \int_{X \setminus G_n} f_n(x)\, d\mu(x) \\ &= \int_X f_n(x)\, d\mu(x) \\ &= \mu(E) > 1 - \delta,\end{aligned}$$

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one has

$$\mu(G_n) > 1 - \sqrt{\delta}. \tag{6.4}$$

For each $x \in G_n$, let $S_n(x) = \{i \in \{0, 1, \ldots, n-1\} : T^i x \in D\}$ and $U_n(x) = \{i \in \{0, 1, \ldots, n-1\} : T^i x \in E\}$. Note that for any $x \in X$ and $i \in \mathbb{Z}$, if $T^i x \in E$ then there exists $j \in \{0, 1, \ldots, N-1\}$ such that $T^{i-j} x \in D$. Using this fact, it is not hard to see that for each $x \in G_n$, one has

$$U_n(x) \subseteq \bigcup_{j=0}^{N-1} (S_n(x) + j) \cup \{0, 1, \ldots, N-1\}.$$

Since for each $x \in G_n$, $|U_n(x)| = \sum_{i=0}^{n-1} 1_E(T^i x) > n(1 - \sqrt{\delta})$, we have $|\{0, 1, \ldots, n-1\} \setminus U_n(x)| \leq n\sqrt{\delta}$. Therefore, for each $x \in G_n$, one gets

$$\begin{aligned} &\left| \{0, 1, \ldots, n-1\} \setminus \bigcup_{j=0}^{N-1} (S_n(x) + j) \right| \\ &\quad \leq |\{0, 1, \ldots, N-1\} \cup (\{0, 1, \ldots, n-1\} \setminus U_n(x))| \\ &\quad \leq n\sqrt{\delta} + N. \end{aligned} \tag{6.5}$$

Now let $\mathcal{F}_n = \{S_n(x) : x \in G_n\}$. Since for each $F \in \mathcal{F}_n$, $F \cap (F + i) = \emptyset, i = 1, \ldots, N-1$, one has $|F| \leq (n/N) + 1$. Putting $a_n = [n/N] + 1$, then we have

$$|\mathcal{F}_n| \leq \sum_{j=1}^{a_n} \frac{n!}{j!\,(n-j)!} \leq a_n \frac{n!}{a_n!\,(n-a_n)!} \leq n \frac{n!}{a_n!\,(n-a_n)!}.$$

By Stirling's formulation and the latter inequality in (6.1), one has

$$\lim_{n \to +\infty} \frac{1}{n} \log\left( n \frac{n!}{a_n!\,(n-a_n)!} \right) = -\left(1 - \frac{1}{N}\right) \log\left(1 - \frac{1}{N}\right) - \frac{1}{N} \log \frac{1}{N} < \epsilon.$$

Hence we have

$$\limsup_{n \to \infty} \frac{1}{n} \log(|\mathcal{F}_n| + 1) \leq \lim_{n \to +\infty} \frac{1}{n} \log\left( n \frac{n!}{a_n!\,(n-a_n)!} + 1 \right) \leq \epsilon. \tag{6.6}$$

For each $F \in \mathcal{F}_n$, let $B_F = \{x \in G_n : S_n(x) = F\}$. Clearly, $\{B_F\}_{F \in \mathcal{F}_n}$ forms a partition of $G_n$.

For $F \in \mathcal{F}_n$, let $F = \{s_1 < s_2 < \cdots < s_l\}$ and $H_F = \{0, \ldots, n-1\} \setminus \bigcup_{i=0}^{N-1} (F + i)$. Then $l \leq (n/N) + 1$, $|H_F| \leq n\sqrt{\delta} + N$ (see (6.5)) and for $\nu$-a.e. $y \in Y$

$$\begin{aligned} &|\alpha_0^{n-1} \cap B_F|_{\mu_y} \\ &\quad \leq \left| \bigvee_{j=1}^{l} T^{-s_j} (\alpha_0^{N-1} \cap D) \vee \bigvee_{r \in H_F} T^{-r} \alpha \right|_{\mu_y} \quad \left( \text{as } B_F \subset G_n \cap \bigcap_{j=1}^{l} T^{-s_j} D \right) \\ &\quad \leq \prod_{j=1}^{l} |T^{-s_j} (\alpha_0^{N-1} \cap D)|_{\mu_y} \prod_{r \in H_F} |T^{-r} \alpha|_{\mu_y} \\ &\quad = \prod_{j=1}^{l} |\alpha_0^{N-1} \cap D|_{\mu_{T^{s_j} y}} \prod_{r \in H_F} |\alpha|_{\mu_{T^r y}} \\ &\quad \leq N(\mathcal{U}_0^{N-1} \mid Y)^l \, k^{|H_F|} \quad \text{(by (6.3))} \\ &\quad \leq k^{n\sqrt{\delta} + N} \, (N(\mathcal{U}_0^{N-1} \mid Y))^{n/N+1}. \end{aligned} \tag{6.7}$$

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Put $b_n = k^{n\sqrt{\delta}+N}\,(N(\mathcal{U}_0^{N-1} \mid Y))^{n/N+1}$ and $\phi(t) = -t\log t$, then we have

$$\begin{aligned} H_\mu(\alpha_0^{n-1} \cap B_F \mid Y) &= \int_Y H_{\mu_y}(\alpha_0^{n-1} \cap B_F)\, d\nu(y) \\ &\le \int_Y \mu_y(B_F)(\log|\alpha_0^{n-1} \cap B_F|_{\mu_y} - \log\mu_y(B_F))\, d\nu(y) \\ &\le \int_Y \mu_y(B_F)(\log b_n - \log\mu_y(B_F))\, d\nu(y) \quad \text{(by (6.7))} \\ &= \mu(B_F)\log b_n + \int_Y \phi(\mu_y(B_F))\, d\nu(y). \end{aligned} \tag{6.8}$$

Similarly, one has

$$H_\mu(\alpha_0^{n-1} \cap (X \setminus G_n) \mid Y) \le \mu(X \setminus G_n)\log|\alpha_0^{n-1}| + \int_Y \phi(\mu_y(X \setminus G_n))\, d\nu(y). \tag{6.9}$$

Let $\gamma = \{B_F\}_{F\in\mathcal{F}_n} \cup \{X \setminus G_n\}$. Combining (6.8) and (6.9), one gets

$$\begin{aligned} H_\mu(\alpha_0^{n-1} \mid Y) &\le H_\mu(\alpha_0^{n-1} \vee \gamma \mid Y) \\ &= \sum_{F\in\mathcal{F}_n} H_\mu(\alpha_0^{n-1} \cap B_F \mid Y) + H_\mu(\alpha_0^{n-1} \cap (X \setminus G_n) \mid Y) \\ &\le \mu(G_n)\log b_n + \mu(X \setminus G_n)\log|\alpha_0^{n-1}| \\ &\quad + \int_Y H_{\mu_y}(\gamma)\, d\nu(y) \quad \text{(by (6.8) and (6.9))} \\ &\le \log b_n + \sqrt{\delta}n\log k + \int_Y \log|\gamma|\, d\nu(y) \\ &= \log b_n + \sqrt{\delta}n\log k + \log(|\mathcal{F}_n| + 1). \end{aligned} \tag{6.10}$$

Finally, one has

$$\begin{aligned} h_\mu^+(T, \mathcal{U} \mid Y) &\le h_\mu(T, \alpha \mid Y) \\ &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1} \mid Y) \\ &\le \limsup_{n\to\infty} \frac{1}{n}(\log b_n + \log(|\mathcal{F}_n| + 1)) + \sqrt{\delta}\log k \quad \text{(by (6.10))} \\ &= \limsup_{n\to\infty} \frac{1}{n}\Bigg(\log(|\mathcal{F}_n| + 1) + (n\sqrt{\delta} + N)\log k \\ &\quad + \left(\frac{n}{N} + 1\right)\log N(\mathcal{U}_0^{N-1} \mid Y)\Bigg) + \sqrt{\delta}\log k \\ &= \limsup_{n\to\infty} \frac{1}{n}\log|\mathcal{F}_n| + \frac{1}{N}\log N(\mathcal{U}_0^{N-1} \mid Y) + 2\sqrt{\delta}\log k \\ &\le \frac{1}{N}\log N(\mathcal{U}_0^{N-1} \mid Y) + 2\epsilon \quad \text{(by (6.2) and (6.6))} \\ &\le h_{\text{top}}(T, \mathcal{U} \mid Y) + 3\epsilon \quad \text{(by (6.1)).} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, one has $h_\mu^+(T, \mathcal{U} \mid Y) \le h_{\text{top}}(T, \mathcal{U} \mid Y)$. This completes the proof of Theorem 6.2. □

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In the following, we prove that $h_\mu^+(T,\mathcal{U}\mid Y)=h_\mu^-(T,\mathcal{U}\mid Y)$ for any given factor map $\pi:(X,T)\to(Y,S)$, $\mu\in\mathcal{M}(X,T)$ and $\mathcal{U}\in\mathcal{C}_X$. Given two measurable covers $\mathcal{U}=\{U_1,U_2,\dots,U_M\}$ and $\mathcal{V}=\{V_1,V_2,\dots,V_M\}$, we set $\mu(\mathcal{U}\Delta\mathcal{V})=\sum_{m=1}^M\mu(U_m\Delta V_m)$. The proof of the following lemma is completely similar to that of in [**HMRY**, Lemma 9].

LEMMA 6.3. *Let $\pi:(X,T)\to(Y,S)$ be a factor map between TDS. For every $M\geq 1$, for every $\varepsilon>0$, there exists $\delta=\delta(M,\varepsilon)>0$ such that for all $M$-set measurable covers of $X$, $\mathcal{U}=\{U_1,U_2,\dots,U_M\}$ and $\mathcal{V}=\{V_1,V_2,\dots,V_M\}$ satisfying $\mu(\mathcal{U}\Delta\mathcal{V})<\delta$, one has $|h_\mu^+(\mathcal{U},T\mid Y)-h_\mu^+(\mathcal{V},T\mid Y)|\leq\varepsilon$.*

When a TDS is uniquely ergodic we have the following result.

THEOREM 6.4. *Let $\pi:(X,T)\to(Y,S)$ be a factor map between TDS. If $(X,T)$ is uniquely ergodic, then:*
(1) *for each $\mathcal{U}\in\mathcal{C}_X^o$, $h_\mu^-(T,\mathcal{U}\mid Y)=h_\mu^+(T,\mathcal{U}\mid Y)=h_{\mathrm{top}}(T,\mathcal{U}\mid Y)$;*
(2) *for each $\mathcal{V}\in\mathcal{C}_X$, $h_\mu^-(T,\mathcal{V}\mid Y)=h_\mu^+(T,\mathcal{V}\mid Y)$.*

*Proof.* (1) This is a direct consequence of Theorems 2.5 and 6.2.

(2) Fix $\mathcal{V}\in\mathcal{C}_X$ and $\epsilon>0$. Put $R=|\mathcal{V}|$. By Lemma 6.3, there exists $\delta_1>0$ such that for every $\mathcal{U}_1\in\mathcal{C}_X$ with $|\mathcal{U}_1|=R$, when $\mu(\mathcal{U}_1\Delta\mathcal{V})<\delta_1$ one has

$$h_\mu^+(T,\mathcal{V}\mid Y)\leq h_\mu^+(T,\mathcal{U}_1\mid Y)+\epsilon. \tag{6.11}$$

Using Lemma 4.1 part (3), there exists $M\in\mathbb{N}$ such that

$$\frac{1}{M}h_\mu^+\Big(T^M,\bigvee_{m=0}^{M-1}T^{-m}\mathcal{V}\;\Big|\;Y\Big)\leq h_\mu^-(T,\mathcal{V}\mid Y)+\frac{\varepsilon}{2}. \tag{6.12}$$

By Lemma 6.3, there exists $\delta_2>0$ such that for every $\mathcal{U}_2\in\mathcal{C}_X$ with $|\mathcal{U}_2|=R^M$ such that $\mu(\mathcal{U}_2\Delta\bigvee_{m=0}^{M-1}T^{-m}\mathcal{V})<\delta_2$ one has

$$\frac{1}{M}h_\mu^+(T^M,\mathcal{U}_2\mid Y)\leq\frac{1}{M}h_\mu^+\Big(T^m,\bigvee_{m=0}^{M-1}T^{-m}\mathcal{V}\;\Big|\;Y\Big)+\frac{\epsilon}{2}. \tag{6.13}$$

Now we consider a cover $\mathcal{U}$ of $X$ made of open sets with $|\mathcal{U}|=R$ such that $\mu(\mathcal{U}\Delta\mathcal{V})<\min\{\delta_1,\delta_2/R^M\}$. From equation (6.11), we get

$$h_\mu^+(T,\mathcal{V}\mid Y)\leq h_\mu^+(T,\mathcal{U}\mid Y)+\epsilon. \tag{6.14}$$

For measurable sets $A_1$, $A_2$, $B_1$ and $B_2$ one has $\mu((A_1\cap A_2)\Delta(B_1\cap B_2))\leq\mu(A_1\Delta B_1)+\mu(A_2\Delta B_2)$. Thus we have

$$\mu\Big(\bigvee_{m=0}^{M-1}T^{-m}\mathcal{V}\Delta\bigvee_{m=0}^{M-1}T^{-m}\mathcal{U}\Big)\leq R^M\mu(\mathcal{V}\Delta\mathcal{U})<\delta_2.$$

Now from equations (6.12) and (6.13)

$$\frac{1}{M}h_\mu^+\Big(T^M,\bigvee_{m=0}^{M-1}T^{-m}\mathcal{U}\;\Big|\;Y\Big)\leq\frac{1}{M}h_\mu^+\Big(T^M,\bigvee_{m=0}^{M-1}T^{-m}\mathcal{V}\;\Big|\;Y\Big)+\frac{\epsilon}{2}\leq h_\mu^-(T,\mathcal{V}\mid Y)+\epsilon. \tag{6.15}$$

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Therefore, from equations (6.14) and (6.15) and Lemma 4.1

$$\begin{aligned} h_\mu^+(T,\mathcal{V}\mid Y) &\le h_\mu^+(T,\mathcal{U}\mid Y)+\epsilon \\ &= h_\mu^-(T,\mathcal{U}\mid Y)+\epsilon \quad \text{(by (1))} \\ &= \frac{1}{M}h_\mu^-(T^M,\mathcal{U}_0^{M-1}\mid Y)+\epsilon \\ &\le \frac{1}{M}h_\mu^+(T^M,\mathcal{U}_0^{M-1}\mid Y)+\epsilon \\ &\le h_\mu^-(T,\mathcal{V}\mid Y)+2\epsilon \quad \text{(by (6.15))}. \end{aligned}$$

Since $\epsilon$ is arbitrary, one has $h_\mu^+(T,\mathcal{V}\mid Y)=h_\mu^-(T,\mathcal{V}\mid Y)$, completing the proof. □

Now we are ready to show the following theorem.

THEOREM 6.5. *Let $\pi:(X,T)\to(Y,S)$ be a factor map between TDS, $\mu\in\mathcal{M}(X,T)$ and $\mathcal{U}\in\mathcal{C}_X$. Then*

$$h_\mu^+(T,\mathcal{U}\mid Y)=h_\mu^-(T,\mathcal{U}\mid Y).$$

*Proof.* By Theorem 5.3 it is enough to consider an ergodic measure $\mu$. Let $\nu=\pi\circ\mu$. By the relative Jewett–Krieger theorem [**Wei**], there exists a unique ergodic model $\widehat{\pi}:(\widehat{X},\widehat{T},\widehat{\mu})\to(\widehat{Y},\widehat{S},\widehat{\nu})$ of $\pi:(X,T,\mu)\to(Y,S,\nu)$, i.e. there exist measure-theoretical isomorphisms $\phi:(X,T,\mu)\to(\widehat{X},\widehat{T},\widehat{\mu})$ and $\psi:(Y,S,\nu)\to(\widehat{Y},\widehat{S},\widehat{\nu})$ such that the diagram

$$\begin{array}{ccc} (X,T,\mu) & \xrightarrow{\phi} & (\widehat{X},\widehat{T},\widehat{\mu}) \\ \pi\downarrow & & \downarrow\widehat{\pi} \\ (Y,S,\nu) & \xrightarrow{\psi} & (\widehat{Y},\widehat{S},\widehat{\nu}) \end{array}$$

is commutative. Then

$$h_\mu^\pm(T,\mathcal{U}\mid Y)=h_{\widehat{\mu}}^\pm(\widehat{T},\phi(\mathcal{U})\mid\widehat{Y}). \tag{6.16}$$

By Theorem 6.4

$$h_{\widehat{\mu}}^-(\widehat{T},\phi(\mathcal{U})\mid\widehat{Y})=h_{\widehat{\mu}}^+(\widehat{T},\phi(\mathcal{U})\mid\widehat{Y}). \tag{6.17}$$

Therefore, from equations (6.16) and (6.17) we obtain $h_\mu^-(T,\mathcal{U}\mid Y)=h_\mu^+(T,\mathcal{U}\mid Y)$. □

*Remark.* By Theorem 6.5, we can define $h_\mu(T,\mathcal{U}\mid Y)=h_\mu^\pm(T,\mathcal{U}\mid Y)$. Clearly,

$$h_\mu(T,\mathcal{U}\mid Y)=\frac{1}{M}h_\mu\left(T^M,\bigvee_{i=0}^{M-1}T^{-i}\mathcal{U}\;\middle|\;Y\right)$$

for each $M\in\mathbb{N}$.

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A. *Appendix*

In this section we shall prove a stronger version of Theorem 6.2.

Let $X$ be a compact metric space. A subset $A \subset X$ is called *universal measurable* if, for any $\mu \in \mathcal{M}(X)$, $A$ is $\mu$-measurable, i.e. there are a Borel set $A'$ and a $\mu$-zero measure set $B$ with $A = A' \Delta B$.

A *universal measurable cover* of $X$ is a finite family of universal measurable subsets of $X$, whose union is $X$. A *universal measurable partition* of $X$ is a universal measurable cover of $X$ whose elements are pairwise disjoint. Denote the set of universal measurable partitions by $\mathcal{P}_X^u$, and the set of universal measurable covers by $\mathcal{C}_X^u$.

Let $\pi : (X, T) \to (Y, S)$ be a factor map and $\mu \in \mathcal{M}(X, T)$. For $\mathcal{U} \in \mathcal{C}_X$, it is easy to see that

$$h_\mu(T, \mathcal{U} \mid Y) = \inf_{\alpha \in \mathcal{P}_X^u, \alpha \succeq \mathcal{U}} h_\mu(T, \alpha \mid Y). \tag{A.1}$$

LEMMA A.1. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS. Then for any $\mathcal{V} \in \mathcal{C}_X$, there exists $\eta \in \mathcal{P}_X^u$ such that $\eta \succeq \mathcal{V}$ and $N(\eta \mid Y) \le N(\mathcal{V} \mid Y)$. Particularly, when $\mathcal{V} \in \mathcal{C}_X^o$, we can require that $\eta \in \mathcal{P}_X$.*

*Proof.* Let $\mathcal{V} = \{V_1, V_2, \ldots, V_k\}$ and $l = N(\mathcal{V} \mid Y)$. Let $\mathcal{F} = \{\{i_1, i_2, \ldots, i_l\} : 1 \le i_1 < i_2 < \cdots < i_l \le k\}$. For $I = \{i_1, i_2, \ldots, i_l\} \in \mathcal{F}$, denote $Y_I = \left\{y \in Y : \pi^{-1}(y) \subset \bigcup_{i \in I} V_i\right\}$. As $Y_I = Y \setminus \pi\left(X \setminus \bigcup_{i \in I} V_i\right)$, we have $Y_I$ is universal measurable, since the image of a Borel set of $X$ under $\pi$ is universal measurable (when $\mathcal{V} \in \mathcal{C}_X^o$, $Y_I$ is a Borel set of $Y$). Then we define $\beta_I = \left\{V_{i_1}, V_{i_2} \setminus V_{i_1}, \ldots, V_{i_l} \setminus \left(\bigcup_{j=1}^{l-1} V_{i_j}\right)\right\}$, and one has $|\beta_I| \le l$.

Let $\mathcal{F} = \{I_1, I_2, \ldots, I_r\}$, where $r = C_k^l$. Set $Y(1) = Y_{I_1}$, $Y(2) = Y_{I_2} \setminus Y_{I_1}, \ldots,$ $Y(r) = Y_{I_r} \setminus \left(\bigcup_{j=1}^{r-1} Y_{I_j}\right)$. Clearly, $Y(j)$, $j = 1, \ldots, r$, is a universal measurable set of $Y$. Put $\eta_j = \beta_{I_j} \cap \pi^{-1}(Y(j))$, and note that each element of $\eta_j$ is universal measurable, since the pre-image of a universal measurable set of $Y$ is universal measurable.

Since $\bigcup_{j=1}^r Y(j) = \bigcup_{j=1}^r Y_{I_j} = Y$, we have $\eta = \bigcup_{j=1}^r \eta_j \in \mathcal{P}_X^u$ and $\eta \succeq \mathcal{V}$ (when $\mathcal{V} \in \mathcal{C}_X^o$, $\eta \in \mathcal{P}_X$). By the construction of $\eta$, for each $y \in Y$, there is $j \in \{1, 2, \ldots, r\}$ such that $y \in Y(j)$. Therefore

$$|\eta \cap \pi^{-1}(y)| = |\eta_j \cap \pi^{-1}(y)| \le |\eta_j| \le |\beta_{I_j}| \le N(\mathcal{V} \mid Y).$$

This ends the proof. □

The following is a universal version of the Rohlin lemma (see [**GW**] for a proof).

LEMMA A.2. (Rohlin's lemma) *Let $(X, T)$ be a TDS. Then for any given $N \in \mathbb{N}$ and $\epsilon > 0$, there exists a Borel subset $D$ of $X$ such that $D, TD, \ldots, T^{N-1}D$ are pairwise disjoint and $\mu\left(\bigcup_{i=0}^{N-1} T^i D\right) > 1 - \epsilon$ for any non-atomic $\mu \in \mathcal{M}^e(X, T)$.*

Now we are ready to strengthen Theorem 6.2.

THEOREM A.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDS and $\mathcal{U} \in \mathcal{C}_X$. Then $\inf_{\alpha \in \mathcal{P}_X^u, \alpha \succeq \mathcal{U}} \sup_{\mu \in \mathcal{M}(X,T)} h_\mu(T, \alpha \mid Y) \le h_{\text{top}}(T, \mathcal{U} \mid Y)$, and when $\mathcal{U} \in \mathcal{C}_X^o$*

$$\inf_{\alpha \in \mathcal{P}_X, \alpha \succeq \mathcal{U}} \sup_{\mu \in \mathcal{M}(X,T)} h_\mu(T, \alpha \mid Y) = h_{\text{top}}(T, \mathcal{U} \mid Y).$$

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*Proof.* Choose for $\epsilon > 0$ an integer $N$ large enough so that

$$N(\mathcal{U}_0^{N-1} \mid Y) \le 2^{N(h_{\text{top}}(T,\mathcal{U}|Y)+\epsilon)} \text{ and } -\left(1-\frac{1}{N}\right)\log\left(1-\frac{1}{N}\right) - \frac{1}{N}\log\frac{1}{N} < \epsilon.$$

Take small enough $1 > \delta > 0$ such that

$$2\sqrt{\delta}\log k < \epsilon. \tag{A.2}$$

Applying Lemma A.2, we can find a Borel subset $D$ of $X$ such that $D, TD, \dots, T^{N-1}D$ are pairwise disjoint and $\mu\big(\bigcup_{i=0}^{N-1} T^i D\big) > 1-\delta$ for any non-atomic $\mu \in \mathcal{M}^e(X,T)$. By Lemma A.2, we can take $\beta \succeq \mathcal{U}_0^{N-1}$ a universal measurable partition of $X$ such that $|\beta \cap \pi^{-1}(y)| \le N(\mathcal{U}_0^{N-1} \mid Y)$ for $y \in Y$ (when $\mathcal{U} \in \mathcal{C}_X^o$, we can require $\beta \in \mathcal{P}_X$).

Consider the partition $\beta_D = \{B \cap D : B \in \beta\}$ of $D$. Then each element $P \in \beta_D$ has the form $P = P_{i_0,i_1,\dots,i_{N-1}} \subseteq (\bigcap_{j=0}^{N-1} T^{-j}U_{i_j}) \cap D$. Next use the partition $\beta_D$ to define a universal measurable partition $\alpha = \{A_i : i = 1,\dots,k\}$ of $X$ by assigning to the set $A_i$ all sets of the form $T^j P_{i_0,i_1,\dots,i_{N-1}}$ where $i_j = i$ ($j$ can be any number in $[0, N-1]$). On the remainder of the space $\alpha$ can be taken to be any partition refining $\mathcal{U}$. We need to mention that when $\mathcal{U} \in \mathcal{C}_X^o$, we can require $\alpha \in \mathcal{P}_X$. For $\gamma' \in \mathcal{P}_X$ and $R \subset X$, we define $\gamma' \cap R = \{A \cap R : A \in \gamma' \text{ and } A \cap R \neq \emptyset\}$. From the construction of $\alpha$, it is easy to see that $\alpha_0^{N-1} \cap D = \beta_D$. Moreover,

$$|\alpha_0^{N-1} \cap D \cap \pi^{-1}(y)| = |\beta_D \cap \pi^{-1}(y)| \le |\beta \cap \pi^{-1}(y)| \le N(\mathcal{U}_0^{N-1} \mid Y) \quad \text{for } y \in Y. \tag{A.3}$$

Now, following the proof in Theorem 6.2, we have that $h_\mu(T,\alpha \mid Y) \le h_{\text{top}}(T,\mathcal{U} \mid Y) + 3\epsilon$ for all non-atomic measures $\mu \in \mathcal{M}^e(X,T)$. By Theorem 5.3, we have $h_\mu(T,\alpha \mid Y) \le h_{\text{top}}(T,\mathcal{U} \mid Y) + 3\epsilon$ for all $\mu \in \mathcal{M}(X,T)$. Since $\epsilon$ is arbitrary, this finishes the proof of Theorem A.3. □

Note that a TDS $(X,T)$ is *transitive* if there is $x \in X$ such that $\{x, Tx, T^2x, \dots\}$ is dense in $X$. To end the section, we now prove the following very general result concerning the entropy of closed covers. In particular, for any non-periodic transitive system $(X,T)$ with zero entropy, (1.4) is false even for some finite closed cover $\mathcal{U}$ of $X$.

PROPOSITION A.4. *Let $(X,T)$ be a non-periodic transitive system. Then for any $1 > \epsilon > 0$ and $N \in \mathbb{N}$ with $N \ge 2$, there exists a closed cover $\mathcal{U} = \{A_1, A_2, \dots, A_N\}$ of $X$ such that $h_{\text{top}}(T,\mathcal{U}) \ge (1-\epsilon)\log N$.*

*Proof.* We just prove the case of $N = 2$; it works similarly for the general case. Let $\mu \in \mathcal{M}(X,T)$. Then for any transitive point $x$ of $(X,T)$, $\mu(\{x\}) = 0$, since $(X,T)$ is non-periodic. Hence there exist two non-empty open subsets $U_1^1, U_2^1$ such that $U_1^1 \cap U_2^1 = \emptyset$ and $\mu(\text{cl}(U_1^1 \cup U_2^1)) < \epsilon/2$. Put $V_1 = X \setminus \text{cl}(U_1^1 \cup U_2^1)$.

For $N_1 \in \mathbb{N}$ large enough, since

$$\int_X \frac{1}{N_1}\left(\sum_{i=0}^{N_1-1} 1_{V_1}(T^i x)\right) d\mu(x) = \mu(V_1) > 1 - \frac{\epsilon}{2},$$

there exists $x_1 \in X$ with

$$\frac{1}{N_1}\left(\sum_{i=0}^{N_1-1} 1_{V_1}(T^i x_1)\right) > 1 - \frac{\epsilon}{2}.$$

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Let $\mathcal{A}_1 = \{n \in \mathbb{Z}_+ : n \leq N_1 - 1 \text{ and } T^n x_1 \in V_1\}$. Then one has

$$|\mathcal{A}_1| \geq \left(1 - \frac{\epsilon}{2}\right) N_1 \quad \text{and} \quad x_1 \in \bigcap_{n \in \mathcal{A}_1} T^{-n} V_1 \neq \emptyset,$$

and so we can choose for each $s \in \{1,2\}^{\mathcal{A}_1}$ a transitive point $x_s \in \bigcap_{n \in \mathcal{A}_1} T^{-n} V_1$ such that $T^i x_s \neq T^j x_t$, when $s, t \in \{1,2\}^{\mathcal{A}_1}$ and $i, j \in \{0, 1, \ldots, N_1 - 1\}$ with $(i, s) \neq (j, t)$.

Now for each $n \in \mathcal{A}_1$ and $s \in \{1,2\}^{\mathcal{A}_1}$, take an open neighborhood $U_{n,s}$ of the transitive point $T^n x_s$ such that $\mathrm{cl}(U_{n,s}) \subset V_1$ and $U_{n,s} \cap U_{m,t} = \emptyset$ when $s, t \in \{1,2\}^{\mathcal{A}_1}$ and $n, m \in \mathcal{A}_1$ with $(n, s) \neq (m, t)$. Since $\mu(\{T^n x_s\}) = 0$, we can require that

$$\mu\left(\bigcup_{n \in \mathcal{A}_1, s \in \{1,2\}^{\mathcal{A}_1}} \mathrm{cl}(U_{n,s})\right) < \frac{\epsilon}{4}.$$

Let

$$V_2 = V_1 \Big\backslash \left(\bigcup_{n \in \mathcal{A}_1, s \in \{1,2\}^{\mathcal{A}_1}} \mathrm{cl}(U_{n,s})\right).$$

Then $V_2$ is open and $\mu(V_2) > 1 - 3\epsilon/4$. Take

$$U_j^2 = U_j^1 \cup \bigcup_{n \in \mathcal{A}_1, s \in \{1,2\}^{\mathcal{A}_1} : s(n) = j} U_{n,s}, \quad j = 1, 2.$$

Then $U_1^2, U_2^2$ are open with $U_1^2 \cap U_2^2 = \emptyset$, $V_2 \cap (U_1^2 \cup U_2^2) = \emptyset$ and for each $s \in \{1,2\}^{\mathcal{A}_1}$,

$$x_s \in \bigcap_{n \in \mathcal{A}_1} T^{-n} U_{s(n)}^2 \neq \emptyset.$$

For $V_2$, repeating the same process, we obtain some $N_2 > N_1$ large enough and $\mathcal{A}_2 \subset \{0, 1, \ldots, N_2 - 1\}$ with $|\mathcal{A}_2| \geq (1 - 3\epsilon/4)N_2$, and two open subsets $U_1^3 \supset U_1^2$, $U_2^3 \supset U_2^2$ and an open subset $V_3$ such that $U_1^3 \cap U_2^3 = \emptyset$, $\mu(V_3) > 1 - 7\epsilon/8$, $V_3 \cap (U_1^3 \cup U_2^3) = \emptyset$ and for any $s \in \{1,2\}^{\mathcal{A}_2}$,

$$\bigcap_{n \in \mathcal{A}_2} T^{-n} U_{s(n)}^3 \neq \emptyset.$$

Doing induction for each $k \geq 3$, we can obtain some $N_k > N_{k-1}$ large enough and $\mathcal{A}_k \subset \{0, 1, \ldots, N_k - 1\}$ with $|\mathcal{A}_k| \geq (1 - (2^k - 1)\epsilon/2^k)N_k$, and two open subsets $U_1^{k+1} \supset U_1^k, U_2^{k+1} \supset U_2^k$ and an open subset $V_{k+1}$ such that $U_1^{k+1} \cap U_2^{k+1} = \emptyset$, $\mu(V_{k+1}) > 1 - (2^{k+1} - 1)\epsilon/2^{k+1}$, $V_{k+1} \cap (U_1^{k+1} \cup U_2^{k+1}) = \emptyset$ and for any $s \in \{1,2\}^{\mathcal{A}_k}$,

$$\bigcap_{n \in \mathcal{A}_k} T^{-n} U_{s(n)}^{k+1} \neq \emptyset.$$

Put $U_j = \bigcup_{i=1}^{\infty} U_j^i$, $j = 1, 2$, then $U_1, U_2$ are open and $U_1 \cap U_2 = \emptyset$. Moreover, for each $k \in \mathbb{N}$ and $s \in \{1,2\}^{\mathcal{A}_k}$,

$$\bigcap_{n \in \mathcal{A}_k} T^{-n} U_{s(n)} \supseteq \bigcap_{n \in \mathcal{A}_k} T^{-n} U_{s(n)}^{k+1} \neq \emptyset.$$

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Now define the closed cover of $X$ by $\mathcal{U} = \{X \setminus U_1, X \setminus U_2\}$. One has

$$\begin{aligned} h_{\mathrm{top}}(T, \mathcal{U}) &= \lim_{k\to\infty} \frac{1}{N_k} \log N(\mathcal{U}_0^{N_k-1}) \\ &\geq \liminf_{k\to\infty} \frac{1}{N_k} \log N\left(\bigvee_{n\in\mathcal{A}_k} T^{-n}\mathcal{U}\right) \\ &= \liminf_{k\to\infty} \frac{1}{N_k} \log(2^{|\mathcal{A}_k|}) \\ &\geq (1-\epsilon)\log 2. \end{aligned}$$

This finishes the proof of Proposition A.4. □

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