A caution (or two) regarding mathematical expectation

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Introduction

Computing mathematical expectation for an experiment involving a finite number of numerical outcomes is straightforward. Let X denote the random variable having n possible values $x_1, x_2, x_3, \ldots, x_n$. Letting p_k denote the probability of x_k, the expected value of X is

$$E(X) = \sum_{k=1}^{n} x_k p_k,$$

which can be interpreted as a weighted average of all x_k, where the weight of each outcome is represented by its probability. But caution is required when interpreting the sum if there are infinitely many outcomes and the series fails to converge absolutely.

Three scenarios follow, the last two of which relate to the caution noted. Each scenario involves tossing a fair coin repeatedly until the coin first lands heads, say on the kth toss. The probability of heads first occurring on the kth toss is $p_k = \frac{1}{2^k}$. For each scenario, the random variable X represents your payoff (a win or loss) in pounds sterling, with each x_k being a specified function of k. Your account is adjusted after each trial, up or down as appropriate.

The Article concludes with a second, often overlooked, caution relating to the applicability of mathematical expectation to real-life scenarios.

Scenario 1

Assume you win £2 if k is odd and lose the same if k is even. Your payoff associated with k tosses is therefore $x_k = (-1)^{k-1}2$ which occurs with probability $p_k = \frac{1}{2^k}$ yielding the overall expectation

$$E(X) = \sum_{k=1}^{\infty} x_k p_k = \sum_{k=1}^{\infty} \left[(-1)^{k-1}2\right]\frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{k-1}}$$

$$= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots = \frac{2}{3}.$$

The geometric series converges absolutely and you expect to win, on average, approximately £0.67 per trial, in which case your account will almost certainly increase, non-monotonically, over time.

Scenario 2

The second scenario is the same as the first, except for the payoff values x_k. In the second scenario you win $\frac{2^k}{k}$ pounds if k is odd and lose the same if k is even. As before, the random variable X denotes your gain (or loss)

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after each sequence of tosses terminates. The expression for $E(X)$ simplifies to the alternating harmonic series

$$E(X) = \sum_{k=1}^{\infty} x_k p_k = \sum_{k=1}^{\infty} \left[(-1)^{k-1} \frac{2^k}{k} \right] \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2 \approx 0.69 > 0$$

seeming to suggest that the game is mathematically favourable for you, yielding, in the long run, an almost certain profit. But this is incorrect! The alternating harmonic series converges conditionally because the sum of the absolute values of the terms, the harmonic series, diverges. And for any conditionally convergent series, such as the alternating harmonic, the terms can be rearranged to converge to any specified real number, or be made to diverge to $+\infty$ or $-\infty$. For mathematical expectation there is no natural (native) ordering of terms. The order here yielding the alternating harmonic series is adopted purely for cosmetic reasons. This being so, there can be no well-defined mathematical expectation for Scenario 2 as described. Computer simulations of Scenario 2 support this.

The definition of $E(X)$ for a countable infinity of outcomes should include, if only implicitly, the proviso that a converging series must do so absolutely for there to be a well-defined finite expectation.

Depending on the application, one might choose, within reason, an alternate definition of $E(X)$ in such a way that the infinite series representation of $E(X)$ converges absolutely, making $E(X)$ well-defined. As an example for Scenario 2, one might define $E(X)$ as the expected value of a conditional expectation, denoted here as $E[E(X \mid C)]$. An example follows.

For $i = 1, 2, 3, \ldots$, define the following conditions, shown with their respective probabilities.

$$\text{Condition } C_1 : k \in \{1, 2\} \text{ where } P(C_1) = \frac{1}{2} + \left(\frac{1}{2}\right)^2 = \frac{3}{4}.$$

$$\text{Condition } C_2 : k \in \{3, 4\} \text{ where } P(C_2) = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 = \frac{3}{16}.$$

$$\text{Condition } C_3 : k \in \{5, 6\} \text{ where } P(C_3) = \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 = \frac{3}{64}.$$

In general,

$$\text{Condition } C_i : k \in \{2i - 1, 2i\} \text{ where } P(C_i) = \frac{3}{4^i}.$$

Since $k = 2i - 1$ is twice as likely as $k = 2i$ we have

$$E(X \mid C_i) = \frac{2}{3} \cdot \frac{2^{2i-1}}{2i - 1} - \frac{1}{3} \cdot \frac{2^{2i}}{2i} = \frac{4^i}{3(2i - 1)(2i)}$$

$$E(X \mid C_i) \cdot P(C_i) = \frac{4^i}{3(2i-1)(2i)} \cdot \frac{3}{4^i} = \frac{1}{(2i-1)(2i)} = \frac{1}{2i-1} - \frac{1}{2i}$$

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$$E(X) = E[E(X \mid C)] = \sum_{i=1}^{\infty} E(X \mid C_i) \cdot P(C_i) = \sum_{i=1}^{\infty} \left(\frac{1}{2i-1} - \frac{1}{2i} \right) = \ln 2 > 0.$$

The series converges absolutely to $\ln 2$. (Note that each term of the series is the sum of two consecutive terms of the alternating harmonic series.) If Scenario 2 is simulated this way, the expectation exists and your account will almost certainly increase in time.

Scenario 3

This scenario, arguably the most perplexing of the three, is a variation of 'Beat the House', a paradox appearing in David Gale's *Tracking the automatic ant* [1, p. 8], where Gale describes it as an *antinomy*—an irreconcilable contradiction. It is similar to 'Strange Expectations', a puzzle posed by Ian Hacking [2].

I show you a positive integer which I've chosen by some method unknown to you. You then toss a coin as in the previous scenario to determine your k number, the number of tosses required for the coin to first land heads. If k is exactly one less than the number I've displayed, you must pay me £8^k. If k is exactly one more than what I've displayed, I pay you £8^{k-1}. Otherwise no money is exchanged. From your perspective, is this game mathematically favourable? An analysis of your expectation should provide the answer.

If I display the number 1 then you will not lose money. If you first toss heads on the second toss ($k = 2$) I will pay you $8^{k-1} = 8^{2-1} =$ £8. Under the assumption I'm showing the number 1, you win £8 with probability $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ with no risk of losing. If I choose to display the number 2 then you pay me $8^1 =$ £8 if your k number is 1 and I must pay you $8^2 =$ £64 if your k number is 3. Under the assumption I display the number 2, your expectation is $-8^1\left(\frac{1}{2}\right)^1 + 8^2\left(\frac{1}{2}\right)^3 =$ £4, a favourable expectation from your perspective. In general, if I display the positive integer $\alpha\,(\alpha > 1)$ then, for that displayed number, your expectation is

$$-8^{\alpha-1}\left(\frac{1}{2}\right)^{\alpha-1} + 8^{\alpha}\left(\frac{1}{2}\right)^{\alpha+1} = 4^{\alpha-1} > 0. \tag{1}$$

So, no matter which positive integer I display, your expectation is positive, suggesting that in the long run you will profit from repeated plays of the game.

But now I advise you that the number I display is determined by the exact same method you use to select k. That is, I toss my own fair coin first, letting my k number (α) be the number of tosses necessary for my coin to first land heads. This is the number I display just before you begin tossing your coin. We each choose our own k numbers independently, using exactly the same method. After we toss our coins, I have my k number, α, and you have yours which we can call β. To settle our accounts the k-tosser pays the $(k+1)$-tosser 8^k pounds. The game is completely symmetric! Being so,

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how can the preceding analysis be correct in suggesting you are favoured? If that analysis is correct, then by symmetry, I could make the exact same calculation showing that I am favoured. And symmetry suggests the game should be fair. Something has gone off the rails.

This paradox can be resolved partially by taking a deeper dive into $E(X)$, your expected gain (or loss) per game when the game is repeated indefinitely.

The event of my k number being α and yours β occurs with probability $\frac{1}{2^{\alpha+\beta}}$, in which case your payoff is

$$f(\alpha, \beta) = \begin{cases} -8^{\beta} & \text{if } \alpha = \beta + 1, \\ 8^{\alpha} & \text{if } \beta = \alpha + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Your expectation for Scenario 3 is therefore

$$E(X) = \sum_{\alpha,\beta=1}^{\infty} \frac{f(\alpha, \beta)}{2^{\alpha+\beta}}.$$

For $E(X)$ to be well-defined, the order of terms should be irrelevant, in which case $E(X)$ could be calculated as either

$$\sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{f(\alpha, \beta)}{2^{\alpha+\beta}} \text{ or } \sum_{\beta=1}^{\infty} \sum_{\alpha=1}^{\infty} \frac{f(\alpha, \beta)}{2^{\alpha+\beta}}. \tag{3}$$

The inner sum of the left member of (3) represents $E(X \mid \alpha)$, your conditional expectation, conditioning on my k number α. From (1) it is clear that this conditional expectation increases as α increases. It follows that the double sum left member of (3) representing $E[E(X \mid \alpha)]$ diverges to $+\infty$, indicating the game favours you. But doing an analysis similar to that which leads to (1), one sees that the double sum right member of (3), representing $E[E(X \mid \beta)]$, diverges to $-\infty$ indicating that the game favours me. Since the double sum is order dependent, $E(X)$ is not well-defined.

Yet, the paradox remains. Assume α and β are created independently by coin tossing as has been described. My number α is pasted on my forehead and your number β is pasted on your forehead in such a way that we can see each other's number but not our own. As usual, if our numbers differ by exactly 1, the person with the k number on their forehead pays the person with the $k + 1$ number on their forehead £8^k. By the analysis leading to (1) and the symmetry of the game, we might each view the game as mathematically favourable to ourselves, despite the obvious contradiction. Strange!

Further exploration and a final caution

Scenarios 2 and 3 are elementary illustrations of what may be broadly described as *expectation-based symmetry-breaking paradoxes*. These and

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related paradoxes have been extensively discussed in mathematical and philosophical publications. For examples see [3]. An early, recreational example, 'The Paradox of the Neckties', appears in a 1953 collection of mathematical recreations by Belgian mathematician Maurice Kraitchik [4, pp. 133-134].

> Each of two persons claims to have the finer necktie. They call in a third person who must make a decision. The winner must give his necktie to the loser as consolation. Each of the contestants reasons as follows: 'I know what my tie is worth. I may lose it, but I may also win a better one, so the game is to my advantage.'
>
> How can the game be to the advantage of both?

The problem was later popularised by Martin Gardner [5, p. 106] and is known today as the classic *two-envelopes* or *two-wallets* problem. Variations are discussed in [6, Ch. 5].

Expectation reversal paradoxes are not just theoretical contrivances. *Simpson's Paradox* occurs as a statistical phenomenon when subgroups of a sample space are combined in such a way that the direction and strength between variables appears to change. C. H. Wagner cites examples [7].

> The United States federal tax rate increased from 1974 to 1978. However, the rate decreased for each income bracket.
>
> In 1973 data suggests that men applying to graduate school at the University of California, Berkeley were more likely to be admitted than women. However, when broken down by department, women were admitted at a higher rate than men.
>
> In 1910 the overall tuberculosis mortality rate in New York City was lower than that of Richmond, Virginia. But for both whites and non-whites, Richmond had a lower mortality rate than New York City.

A final caution is appropriate. Applications of mathematical expectation include risk analysis, insurance, investment decisions and more. In *The black swan*, Nassim Nicholas Taleb coins the term 'ludic fallacy' (*ludic* from the Latin *ludus* meaning games), referring to using idealised statistical models in actual settings where they would not apply [8, Ch. 9]. Computation and application of mathematical expectation work well in gambling venues such as casinos where payoff probabilities are controlled. But when applied to the physical world, the *science of uncertainty* is itself highly uncertain due to multiple unknowns and unpredictables. We simply don't know what we don't know and the probabilities involved are highly subjective, varying from one individual to the next. All considered, real-life application of mathematical expectation and other probability-related concepts should be considered prudently.

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