Ergod. Th. & Dynam. Sys. (1997), **17**, 1437–1448
Printed in the United Kingdom © *1997 Cambridge University Press*

An invariant of basic sets of Smale flows

MICHAEL C. SULLIVAN
Department of Mathematics, Southern Illinois University, Carbondale IL 62901-4408, USA
(e-mail: msulliva@math.siu.edu)

(*Received* 4 *August* 1995 *and accepted in revised form* 25 *August* 1996)

Abstract. We consider one-dimensional flows which arise as hyperbolic invariant sets of a smooth flow on a manifold. Included in our data is the twisting in the local stable and unstable manifolds. A topological invariant sensitive to this twisting is obtained.

1. *Introduction*

One-dimensional flows have been classified by the work of Bowen and Franks [**4**], Parry and Sullivan [**9**], and Franks [**7**]. Here, we consider the problem when some additional information is included. For a one-dimensional flow which is a hyperbolic invariant set of a flow on a manifold, it is usual to study the first return map on a Markov partition of a cross-section. It is possible to assign an orientation to the elements of the partition. Then the first return map will either reverse or preserve orientation on each of the partition elements. We define a *twist matrix* $A(t)$ by incorporating orientation information into the incidence matrix of a Markov partition. Then the computation $\det(I - A(t)) \bmod(t^2 = 1)$ gives our invariant. This result was conjectured in [**11**]. The invariant of [**9**], now known as the Parry–Sullivan invariant, can be recovered by setting $t = 1$. We shall call the invariant of this paper the *full Parry–Sullivan invariant*.

This paper is organized as follows. §2 gives basic definitions and results of the theory of Smale flows and maps. §3 develops the notion of a subshift of finite type with parity. We prove a modified version of the theorem of Williams that classifies non-negative integral matrices according to the topological conjugacy classes of their associated subshifts of finite type. This is used in §4 in the proof our main result, Theorem 4.1. In §5 we give an application for Smale flows in three dimensions. Here basic sets can be modeled with templates, which are branched 2-manifolds with semi-flows whose periodic orbits are ambient isotopic to those of the original flow.

2. *Basic definitions and background*

If the chain-recurrent set $\mathcal{R}$ of a flow has a hyperbolic structure then a result of Smale's shows that $\mathcal{R}$ is the union of a finite collection of disjoint invariant compact sets, called

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

basic sets, each of which contains a dense orbit [**10**]. Bowen showed that one-dimensional basic sets are homeomorphic to suspensions of *subshifts of finite type* [**3**], defined below.

Let S be a finite set of symbols and let $\rightarrowtail$ be a relation on S. Let $\Sigma = \{\boldsymbol{s} \in \prod_{-\infty}^{+\infty} S \mid s_i \rightarrowtail s_{i+1}, \forall i\}$. We often say Σ is the set of allowed sequences. Let $\sigma : \Sigma \to \Sigma$ be the rightward shift map, $\sigma(\boldsymbol{s}) = \boldsymbol{s}'$, where $s_i' = s_{i-1}$. Then (σ, Σ) is a subshift of finite type. When Σ is given the topology induced by the product topology it is a Cantor set.

Given an $n \times n$ matrix A of non-negative integers one can construct a subshift of finite type through the *edge shift* formulation, as described in §3. If the matrix entries are all zeros and ones then one can also define a subshift of finite type through *vertex shifts*. Let

$$\Sigma_A = \left\{\boldsymbol{x} \in \Pi_{-\infty}^{\infty}\{1, \ldots, n\} \mid A_{x_k x_{k+1}} = 1 \text{ for all } k\right\}$$

and define the right shift map $\sigma : \Sigma_A \to \Sigma_A$ by $\sigma(\boldsymbol{x}) = \boldsymbol{y}$, where $y_k = x_{k-1}$. Given a matrix of zeros and ones the edge and vertex subshifts associated to it are topologically conjugate.

To define the *suspension flow* of σ let $X_A = \Sigma_A \times [0, 1]/\sim$, where $\sim$ identifies $(\boldsymbol{x}, 1)$ with $(\sigma(\boldsymbol{x}), 0)$ and the flow ϕ_t on X_A is given by $\phi_t(\boldsymbol{x}, s) = (\boldsymbol{x}, s + t)$ for $s + t \in [0, 1)$ and for other t by using the identification as needed.

Two matrices are said to be *flow equivalent* if their induced subshifts of finite type give rise to topologically equivalent suspension flows. A matrix A is *irreducible* if for each (i, j) there is a power n such that the (i, j) entry of A^n is non-zero. In terms of the corresponding subshift and suspension, irreducibility is equivalent to the existence of a dense orbit and so we are dealing with a single basic set. The suspension flow of a permutation matrix consists of a single closed orbit. Permutation matrices are thus said to form the *trivial flow equivalence class*.

THEOREM 2.1. (Franks) *Suppose that A and B are non-negative irreducible integer matrices neither of which is in the trivial flow equivalence class. The matrices A and B are flow equivalent if and only if*

$$\det(I_n - A) = \det(I_m - B)$$

and

$$\frac{Z^n}{(I_n - A)Z^n} \cong \frac{Z^m}{(I_m - B)Z^m},$$

where n and m are the sizes of A and B respectively and I_n and I_m are identity matrices.

The main tool in the proof of Theorem 2.1 is Theorem 2.2 below, a generalization of which we shall need later. First we define some more terms.

Two non-negative square integer matrices A and B are *strong shift equivalent* if there exist non-negative square integer matrices $A = A_1, A_2, \ldots, A_{k+1} = B$ and non-negative integer matrices $R_1, S_1, R_2, S_2, \ldots, R_k, S_k$ such that $A_i = R_i S_i$ and $A_{i+1} = S_i R_i$ for $i = 1, \ldots, k$.

THEOREM 2.2. (Williams) *Suppose A and B are non-negative square integer matrices and σ_A and σ_B are the corresponding subshifts of finite type. Then σ_A is topologically conjugate to σ_B if and only if A is strong shift equivalent to B.*

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

The gist of this result is as follows. The matrices A and B can be thought of as incidence matrices for maps on a finite open-closed partition of the Cantor sets Σ_A and Σ_B. The equivalence relation is generated by these operations: relabeling the partition elements, refining a partition element into two open-closed sets, and combining two partition elements into one. These can each be realized by the matrix move that generates strong shift equivalence; see [**12**] or Appendix A of [**5**].

3. *Subshifts with parity*

The purpose of this section is to prove the following analog of Theorem 2.2.

THEOREM 3.1. *Parity matrices are parity-wise topologically conjugate if and only if they are strong shift equivalent.*

We now present the relevant definitions. A *subshift of finite type with parity*, or a *shift with parity* for short, is a subshift of finite type together with a *parity map* on the symbol set, $p : S \to \{-1, 1\}$. We define a function $p : \Sigma \to \{-1, 1\}$ by $p(\boldsymbol{s}) = p(s_0)$. Let $\mathcal{C} = \{\boldsymbol{s} \in \Sigma | \boldsymbol{s} \text{ is periodic}\}/\sigma$, be called the set of cycles. Define $p : \mathcal{C} \to \{-1, 1\}$ by $p(\boldsymbol{c}) = p(s_0)p(s_1)\cdots p(s_{n-1})$, where $\boldsymbol{s} = \overline{s_0 s_1 \cdots s_{n-1}}$ is a periodic element of Σ of least period n that represents the class $\boldsymbol{c}$.

Definition 3.2. Let (σ, p) and (σ', p') be shifts with parity. Then they are parity-wise conjugate if there is a homeomorphism $h : \Sigma \to \Sigma'$ such that

$$h \circ \sigma = \sigma' \circ h \quad \text{and} \quad p = p' \circ h.$$

It follows that parity is preserved under the induced homeomorphism on the cycle sets. If only the first equation holds, σ and σ' are topologically conjugate.

It is technically necessary to define what it means to iterate shifts with parity. If $(\sigma : \Sigma \to \Sigma, p : S \to \{-1, 1\})$ is a shift with parity then its kth iterate $(\sigma^{\langle k \rangle} : \Sigma^{\langle k \rangle} \to \Sigma^{\langle k \rangle}, p^{\langle k \rangle} : S^{\langle k \rangle} \to \{-1, 1\})$, is defined as follows. Let $S^{\langle k \rangle} = \{(s_0, \ldots, s_{k-1}) \in S^k | s_i \rightarrowtail s_{i+1}, i = 0, \ldots, k-2\}$. We also let $(s_0, \ldots, s_{k-1}) \overset{\langle k \rangle}{\rightarrowtail} (s'_0, \ldots, s'_{k-1})$ mean $s_{k-1} \rightarrowtail s'_0$. These give a subshift of finite type:

$$\sigma^{\langle k \rangle} : \Sigma^{\langle k \rangle} \to \Sigma^{\langle k \rangle}.$$

Let $p^{\langle k \rangle}((s_0, \ldots, s_{k-1})) = p(s_0)\cdots p(s_{k-1})$. We can extend $p^{\langle k \rangle}$ to cycles in $\Sigma^{\langle k \rangle}$.

A subshift with parity can be specified by giving a labeled directed graph or a matrix whose elements take the form $a + bt$, where a and b are non-negative integers.

Given a graph G with edges labeled ± 1 and vertices numbered 1 through n, we define an $n \times n$ matrix $M(t)$ by $M_{ij} = a_{ij} + b_{ij}t$ with a_{ij} the number of edges from vertex i to j with label $+1$, and b_{ij} the number of such edges labeled -1. Matrices of this form will be called *parity matrices*. Given a parity matrix we can construct a labeled graph.

We go from an $n \times n$ parity matrix to the shift with parity as follows. Let S be a set of $\sum_{ij} a_{ij} + b_{ij}$ symbols; think of them as edges on the graph. We partition S into $2n^2$ disjoint, possibly empty, subsets. Call them A_{ij} and B_{ij} for i and j equal to 1 through n, with $|A_{ij}| = a_{ij}$ and $|B_{ij}| = b_{ij}$. Also, let $S_{ij} = A_{ij} \cup B_{ij}$. Now we define the relation

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

$\rightarrowtail$ on S. Let $s \rightarrowtail s'$, read s can be followed by s', mean $s \in S_{ik}$ and $s' \in S_{kj}$ for some i, j and k. The set of allowed sequences and the shift map are given in the usual way. The parity map is defined by

$$p(s) = \begin{cases} 1 & \text{if } s \in A_{ij} \text{ for some } i \text{ and } j, \\ -1 & \text{if } s \in B_{ij} \text{ for some } i \text{ and } j. \end{cases}$$

Remark. If the entries of a parity matrix are all zeros, ones and t's, and we then set t equal to minus one, the parity matrix becomes the *structure matrix* first defined by Bowen and Franks [**4**] and [**5**, p. 79]. In their context, parity corresponded to whether the shift map was orientation preserving or not. This is also how we shall use parity in the next section.

LEMMA 3.3. *Let (σ, p) and (σ', p') be the (edge) shifts with parity for the $n \times n$ parity matrices M and M^2 respectively. Then $(\sigma^{\langle 2\rangle}, p^{\langle 2\rangle})$ is parity-wise conjugate to (σ', p').*

Proof. Let $a_{ij} + b_{ij}t = M_{ij}$ and $a'_{ij} + b'_{ij}t = (M^2)_{ij}$. Let $(S, \rightarrowtail, \Sigma)$, $(S', \overset{'}{\rightarrowtail}, \Sigma')$ and $(S^{\langle 2\rangle}, \overset{\langle 2\rangle}{\rightarrowtail}, \Sigma^{\langle 2\rangle})$, have their usual meanings.

Partition $S^{\langle 2\rangle}$ and S' as follows. Since σ and σ' are edge shifts we can define maps *from* and *to*, each defined on S and S' and with values in $\{1, \ldots, n\}$ so that for s_1 and s_2 in S

$$to(s_1) = from(s_2) \iff s_1 \rightarrowtail s_2,$$

and for s'_1 and s'_2 in S'

$$to(s'_1) = from(s'_2) \iff s'_1 \overset{'}{\rightarrowtail} s'_2.$$

The partition is given by

$$\begin{aligned}
A^{\langle 2\rangle}_{ij} &= \{(s_0, s_1) \in S^{\langle 2\rangle} \mid from(s_0) = i, to(s_1) = j, \text{ and } p^{\langle 2\rangle}((s_0, s_1)) = 1\}, \\
B^{\langle 2\rangle}_{ij} &= \{(s_0, s_1) \in S^{\langle 2\rangle} \mid from(s_0) = i, to(s_1) = j, \text{ and } p^{\langle 2\rangle}((s_0, s_1)) = -1\}, \\
A'_{ij} &= \{(s_0, s_1) \in S' \mid from(s_0) = i, to(s_1) = j, \text{ and } p'((s_0, s_1)) = 1\}, \\
B'_{ij} &= \{(s_0, s_1) \in S' \mid from(s_0) = i, to(s_1) = j, \text{ and } p'((s_0, s_1)) = -1\}.
\end{aligned}$$

From the definition of matrix multiplication and the fact the $t^2 = 1$ we have,

$$\begin{aligned}
|A^{\langle 2\rangle}_{ij}| &= \sum_{k=1}^{n} a_{ik}a_{kj} + b_{ik}b_{kj} = a'_{ij} = |A'_{ij}| \\
|B^{\langle 2\rangle}_{ij}| &= \sum_{k=1}^{n} a_{ik}b_{kj} + b_{ik}a_{kj} = b'_{ij} = |B'_{ij}|.
\end{aligned}$$

Clearly, $|S^{\langle 2\rangle}| = |S'|$. Hence we can find a bijection $H : S^{\langle 2\rangle} \to S'$ such that $H(A^{\langle 2\rangle}_{ij}) = A'_{ij}$ and $H(B^{\langle 2\rangle}_{ij}) = B'_{ij}$. This bijection induces a homeomorphism $h : \Sigma^{\langle 2\rangle} \to \Sigma'$ given by

$$h(\ldots(s_0, s_1)(s_2, s_3)\ldots) = (\ldots H(s_0, s_1)H(s_2, s_3)\ldots).$$

This gives us a parity-wise conjugacy. □

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

Definition 3.4. Strong shift equivalence is defined as in §2, but now the R_i and S_i matrices can have entries of the form $a + bt$ for non-negative integers a and b.

We break up the proof of Theorem 3.1 into a series of lemmas. The presentation we give follows the pattern of the proof of Theorem 2.2 given in [**5**, Theorem A.1], which the reader will want to have to hand.

LEMMA 3.5. *If A and B are parity matrices with $A = RS$ and $B = SR$, where R and S are as in Definition 3.4, then $(\sigma_A, \Sigma_A, p_A)$ is parity-wise topologically conjugate to $(\sigma_B, \Sigma_B, p_B)$.*

Proof. See the second paragraph of the proof of Theorem A.1 in [**5**]. To check that the conjugacy given there is parity-wise use our Lemma 3.3. □

By induction, Lemma 3.5 gives the 'if' direction of Theorem 3.1.

LEMMA 3.6. *If A is an $n \times n$ parity matrix then A is strong shift equivalent to a matrix A' whose entries are zeros, ones and t's.*

Proof. The proof is so similar to the proof of Lemma A.2 in [**5**] that we merely note that one replaces R_{ij} on the bottom of page 99 with

$$R_{ij} = \begin{cases} 1 & \text{if } U_j \text{ is on the edge emanating from } V_i \text{ and the edge is labeled } +1, \\ t & \text{if } U_j \text{ is on the edge emanating from } V_i \text{ and the edge is labeled } -1, \\ 0 & \text{otherwise.} \end{cases}$$

The reader should have no trouble reconstructing the details. □

LEMMA 3.7. *Suppose $A(t)$ is an $n \times n$ matrix of zeros, ones and t's. Let $k = \sum_{ij} A_{ij}(1)$, the number of edges. Then $A(t)$ is shift equivalent to a $k \times k$ matrix $B(t)$ where each row of $B(t)$ has only zeros and ones or only zeros and t's.*

Proof. In the graph associated with $A(t)$, number the edges by $e_{(i-1)n+j}$ for the edge starting at vertex i and ending on vertex j. Now renumber the edges $e_1, \ldots, e_k$, so as to preserve the previous order. Let p give the label ± 1 of the edges.

Let R be the $n \times k$ matrix given by

$$R_{ij} = \begin{cases} 1 & \text{if } e_j \text{ emanates from } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let S be the $k \times n$ matrix given by

$$S_{ij} = \begin{cases} 1 & \text{if } e_i \text{ ends on } v_j \text{ and } p(e_i) = 1, \\ t & \text{if } e_i \text{ ends on } v_j \text{ and } p(e_i) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $A = RS$. The ith row of R is a list of the edges emanating from v_i, at most one of which may end at v_j. The edges which end at v_j are listed according to their parity in the jth column of S. Since $RS = \sum_k R_{ik} S_{kj}$, we have $(RS)_{ij} = 0$ if and only if there is an edge connecting v_i and v_j. It should also be clear that the parity carries through correctly. Thus, $A = RS$.

Since each edge ends at only one vertex, each row of S will be all zeros except for a single one or t. From this we can conclude that $B = SR$ has the desired form. □

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

Example. Let

$$A(t) = \begin{bmatrix} 1 & t \\ t & 0 \end{bmatrix}.$$

Then

$$R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & t \\ t & 0 \end{bmatrix}.$$

Thus, we get

$$B(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & t \\ t & t & 0 \end{bmatrix}.$$

In the next lemma and in §4 will use *Markov partitions* for homoemorphisms of Cantor sets; see [**5**] for the definition. Let $A(t)$ be a matrix of the form of Lemma 3.7. A Markov partition of Σ_A can be constructed as follows. Let

$$u_i = \{\boldsymbol{a} \in \Sigma_A \mid a_0 = i\}.$$

Then $U = \{u_1, \ldots, u_k\}$ is the canonical Markov partition associated with $A(t)$. We give U a parity convention $p : U \to \{\pm 1\}$ induced by noting that p on Σ_A is constant on elements of U.

Conversely, given a Markov partition and a parity convention an associated parity matrix is given by

$$A_{ij}(U) = \begin{cases} 1 & \text{if } u_i \cap \sigma(u_j) \neq \phi \text{ and } p(u_j) = 1, \\ t & \text{if } u_i \cap \sigma(u_j) \neq \phi \text{ and } p(u_j) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

One checks that given U, $A(U)$ generates the same Markov partition it came from.

Definition 3.8. Let U and V be Markov partitions of the same shift space. We say that U *refines* V (denoted $U > V$) if each element of U is contained in an element of V. By $U \cap V$ we shall mean the Markov partition given by $\{R_i \cap R'_j \mid R_i \in U, R'_j \in V\}$. We also let $U(m, n)$ denote the Markov partition given by $\sigma^m(U) \cap \cdots \cap \sigma^n(U)$ for integers m and n, $m < n$.

LEMMA 3.9. *Suppose U and V are Markov partitions for (σ, Σ, p) with associated matrices A and B respectively. If $U > V$ and either $V(0, 1) > U$ or $V(-1, 0) > U$, then there are matrices R and S of zeros, ones and t's, with $A = SR$ and $B = RS$.*

Proof. The proof is similar to Lemma A.3 of [**5**]. One must change the matrix S_{kl} on page 101 to

$$S_{kl} = \begin{cases} 1 & \text{if } \phi \neq v_r \cap \sigma(v_l) \subset u_k \text{ and } p(u_k) = 1, \text{ where } u_k \subset v_r, \\ t & \text{if } \phi \neq v_r \cap \sigma(v_l) \subset u_k \text{ and } p(u_k) = -1, \text{ where } u_k \subset v_r, \\ 0 & \text{otherwise.} \end{cases}$$

□

The remainder of the proof of Theorem 3.1 follows from induction; see [**5**].

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

4. *The Main Theorem*

Consider a one-dimensional basic set along with the local stable manifolds of the orbits. Call such an object a *ribbon set*. For a flow on a 3-manifold a ribbon set would consist of annuli, Möbius bands, and infinite strips. Two ribbon sets are *topologically equivalent* if there is an orbit preserving homeomorphism between them that respects the flow direction.

A twist matrix $A(t)$ of a basic set is a parity matrix for a first return map of a cross-section, where the parity is determined by whether the map reverses or preserves orientation on a given partition element. The Markov partition of the Cantor set must be fine enough for the parity to be uniform on each piece, but this can always be done; see [**5**].

Two twist matrices are in the same twist-wise flow equivalence class if they can be realized as twist matrices of topologically equivalent ribbon sets.

THEOREM 4.1. *If two twist matrices $A(t)$ and $B(t)$ are in the same twist-wise flow equivalence class then* $\det(I - A(t)) = \det(I - B(t)) \bmod(t^2 = 1)$.

We define three moves or relations on twist matrices.

- The shift move: $A(t) \overset{\text{s}}{\sim} B(t)$ if there exist matrices R and S as in §3 such that $A(t) = RS$ and $B(t) = SR$.
- The expansion move: $A(t) \overset{\text{e}}{\sim} B(t)$ if

$$A(t) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} 0 & a_{11} & \cdots & a_{1n} \\ 1 & 0 & \cdots & 0 \\ 0 & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{nn} \end{bmatrix},$$

  or vice versa. This notion was first introduced in [**9**].
- The twist move: $A(t) \overset{\text{t}}{\sim} B(t)$ if

$$B(t) = \begin{bmatrix} a_{11} & ta_{12} & \cdots & ta_{1n} \\ ta_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ ta_{n1} & a_{2n} & \cdots & a_{nn} \end{bmatrix},$$

  where $A(t)$ is the same as above. All we have done is multiplied the first row and column of $A(t)$ by t and used $t^2 = 1$.

The expansion and twist moves can be visualized on graphs as follows. In Figure 1(a) a new vertex and edge labeled $+1$ have been inserted. This is expansion. In Figure 1(b) the parity of the edges adjoining the vertex have all been flipped, except for the edge coming back to its starting place. This is the twist move. We can apply the expansion and twist moves to any vertex by first relabeling the vertices with shift moves.

We claim that the three relations, shift, expansion and twist, generate the twist-wise flow equivalence relation and that $\det(I - A(t)) \bmod(t^2 = 1)$ is invariant under them.

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

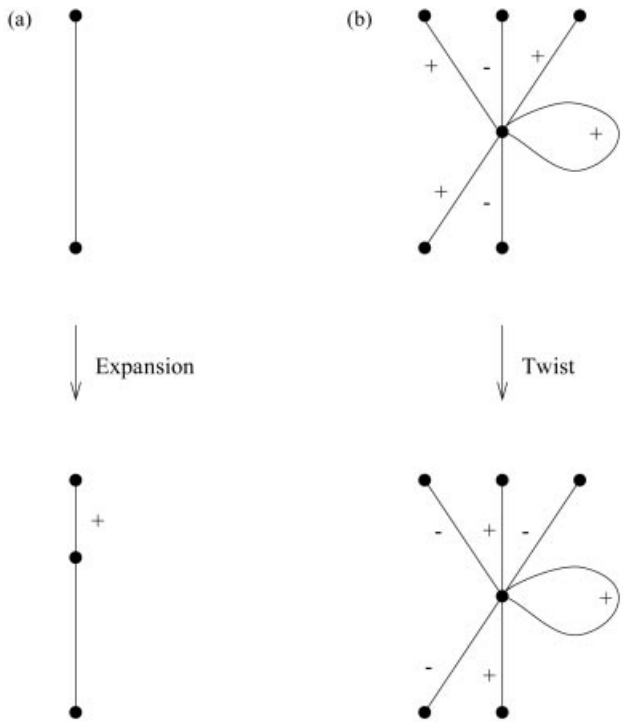


FIGURE 1. Flow direction is down and counter clockwise.

LEMMA 4.2. *The shift, expansion and twist moves leave* $\det(I - A(t))$ *invariant* $\mathrm{mod}(t^2 = 1)$.

Proof. If $A(t) \overset{s}{\sim} B(t)$ then they have the same set of non-zero eigenvalues. Hence, $\det(I - A(t)) = \det(I - B(t))$. Let $A(t) \overset{e}{\sim} B(t)$. Computing $\det(I - B(t))$ along the first column gives $\det(I - B(t)) = (I - A(t))_{11} + \det(I - A(t)) - (I - A(t))_{11}$, where $(I - A(t))_{11}$ is the $(1, 1)$ cofactor of $I - A(t)$. If we apply the twist move to $I - A(t)$ directly the result is still $I - B(t)$. Thus, $\det(I - B(t)) = t^2 \det(I - A(t)) = \det(I - A(t))$, modulo $t^2 = 1$. □

LEMMA 4.3. *Two twist matrices are in the same twist-wise flow equivalence class if and only if they can be connected by a finite sequence of shift, expansion and twist moves.*

Proof. Theorem 3.1 shows that applying the shift move does not alter the twist-wise flow equivalence class. In the expansion case it is clear from Figure 1(a) that the underlying one-dimensional flow (the inverse limit of the semi-flow on the graph) is unchanged. For the twist move think of the one-dimensional flow as a basic set embedded in a flow on a manifold. The local stable manifolds form a ribbon set. The twist move can be realized as an isotopy of the ribbon set, as Figure 2 tries to show, so twist-wise topological flow equivalence is preserved.

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

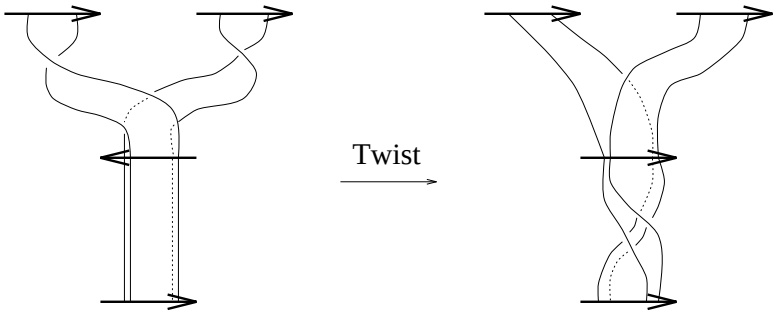


FIGURE 2. Flow direction is down.

For the converse we consider the self-homeomorphism of a Cantor set cross-section given by the first return map of a basic set. Let $C = C_1 \cup C_2$ be an open-closed decomposition of a Cantor set C and let $f : C \to C$ be a homeomorphism. The *expansion* along C_1 is defined as follows. Let $i : C_1 \to C_1'$ be an identification between C_1 and a copy C_1' of C_1. Let $C' = C_1 \cup C_1' \cup C_2$ be a disjoint union. Define $f' : C' \to C'$ by i on C_1, $f \circ i^{-1}$ on C_1' and f on C_2; see Figure 3. For a twist matrix $A(t)$ of zeros, ones and t's, using the vertex shift, expansion along the block $\{\cdots.1\cdots\}$ gives the subshift corresponding to the expansion of A. Using the edge shift for $A(t)$ with parity, expansion corresponds to expansion along $\bigcup\{\cdots.k\cdots\}$ over k, where edge k emanates from vertex 1. The new first edge has parity $+1$.

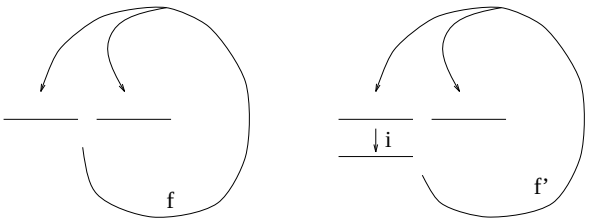


FIGURE 3. Expansion.

Suppose that the orientation of f on C_1 is constant. Reversing the orientation assigned to C_1 will be called *twisting* C_1. If we twist the block $\{\cdots.1\cdots\}$ of a subshift with parity given by $A(t)$ the resulting subshift with parity corresponds to the subshift with parity associated to the matrix resulting from applying the twist move to $A(t)$.

Suppose then that $A(t)$ and $B(t)$ are twist-wise equivalent. We can then regard them as arising from partitions of different cross-sections of the same underlying flow, ϕ_t.

Let C_A and C_B be Cantor set cross-sections corresponding to A and B respectively. Since C_A and C_B are zero-dimensional we can take them to be disjoint.

Let C_{A_0} be the subset of C_A that meets C_B under ϕ_t before coming back to C_A.

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

Let C_{A_1} be the subset of C_A that passes through C_A exactly once before meeting C_B. Continuing this until we exhaust C_A allows us to write $C_A = C_{A_1} \cup \cdots \cup C_{A_n}$, a finite disjoint union, where C_{A_i} is the subset of C_A that passes through C_A exactly i times before meeting C_B.

Likewise partition $C_B = C_{B_1} \cup \cdots \cup C_{B_m}$, but where C_{B_i} is the subset of C_B exactly i times before meeting C_A under the reverse flow ϕ_{-t}.

We can now make Markov partitions for C_A and C_B that are refinements of the partitions just described, such that the parity matrices A' and B' of these Markov partitions induced by the flow have only zeros, ones and t's as entries. It is clear that A' and B' are parity-wise topologically conjugate to A and B respectively.

Now we use twist moves to 'comb out' any twisting between the layers of C_A and C_B. If the map induced by the flow reverses orientation between a partition element $C_{A_{1i}}$ in C_{A_1} and one in C_{A_0} then apply the twist move to the offending partition element of C_{A_1}. If need be we first refine $C_{A_{1i}}$ so that only the twisted edge emanates from it. This pushes the twist to the left in Figure 4. With repeated applications all twisting between the layers of C_A can be combed to the far left. Likewise we comb out all twisting between layers of C_B to the far right.

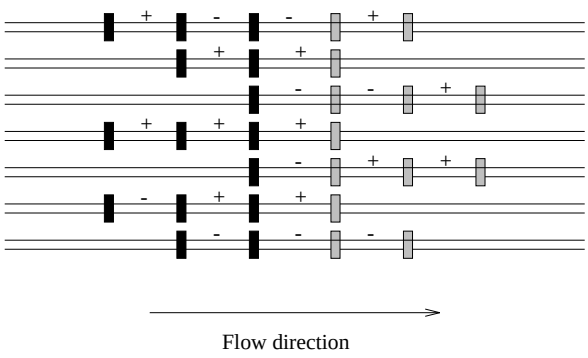


FIGURE 4. Black boxes represent C_A, gray boxes C_B.

We momentarily regard $C_A \cup C_B$ as a single cross-section. If there is any twisting between C_{A_0} and C_{B_0}, push it away, say to the far right.

Now apply (reverse) expansion moves to collapse C_A to C_{A_0} and C_B to C_{B_0}. But C_{B_0} is just a forward translate of C_{A_0}. Thus the A' and B' matrices, and hence A and B, are in the same twist-wise flow equivalence class. □

Lemmas 4.2 and 4.3 combine to prove Theorem 4.1.

5. *Templates*

A template is a smooth branched 2-manifold which supports a semi-flow. For a Smale flow on a 3-manifold, non-trivial basic sets can be modeled by templates in the following sense. The inverse limit of the semi-flow of the template recovers the basic set. Further,

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

any finite link of closed orbits in the basic set is ambient isotopic to a corresponding link on the template. Also, the twisting in the local stable manifolds is preserved in the twisting of the bands of the template. The foundations of template theory are in [**1, 2**]; also see [**8**].

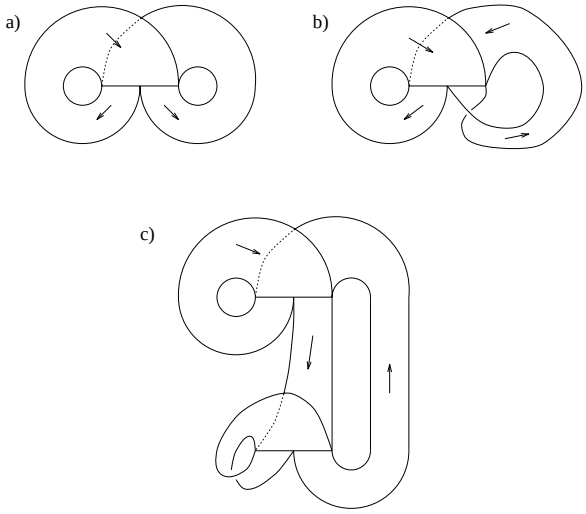


FIGURE 5. Templates.

Figure 5(a) is the Lorenz template and Figure 5(b) is called the Horseshoe template. The classification theorem of [**7**] depends only on an incidence matrix which is $\left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right]$ for both these templates. However, the twist matrix is $\left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right]$ for the Lorenz template and $\left[\begin{smallmatrix}1 & 1\\ t & t\end{smallmatrix}\right]$ for the Horseshoe template. Thus, $\det(I - A(t))$ is -1 for the Lorenz template and $-t$ for the Horseshoe, and they are distinguished. More generally, consider a template with a single branch line with n bands emenating from it. Each band loops back and stretches over the entire branch line. So, the dynamics of a return map are a full shift on n symbols. Suppose k of the bands have an even number of half twists while $l = n - k$ have an odd number of half twists. Then the twist matrix is

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \\ t & \cdots & t \\ \vdots & & \vdots \\ t & \cdots & t \end{bmatrix}.$$

Thus, the full Parry–Sullivan invariant is $1-k-lt$. However, the template in Figure 5(c), though unorientable, has the same invariant as the Lorenz template, -1.

Note added. The set $\{1, t\}$ under multiplication modulo $t^2 = 1$ is, of course, a group isomorphic to $\mathbb{Z}_2$. The entries of the parity matrix are just elements of the group ring

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press

$\mathbb{Z}\mathbb{Z}_2$. The only properties of $\mathbb{Z}_2$ used in §3 were group properties. Hence the definitions are meaningful and the analogous results hold true for any group G. If G is Abelian then an analog of the Parry–Sullivan invariant should hold as well. Here, however, one has to replace the twist move with a g-move for each non-identity element $g \in G$ consisting of multiplying the first row of the matrix by g and the first column by g^{-1}.

REFERENCES

[1] J. Birman and R. Williams. Knotted periodic orbits in dynamical systems I: Lorenz knots. *Topology* **22** (1983), 47–82.

[2] J. Birman and R. Williams. Knotted periodic orbits in dynamical systems II: knot holders for fibered knots. *Contemporary Math.* **20** (1983).

[3] R. Bowen. One-dimensional hyperbolic sets for flows. *J. Differential Equations* **12** (1972), 173–179.

[4] R. Bowen and J. Franks. Homology for zero dimensional basic sets. *Ann. Math.* **106** (1977), 73–92.

[5] J. Franks. *Homology and Dynamical Systems (C.B.M.S. 49).* American Mathematical Society, 1980.

[6] J. Franks. Symbolic dynamics in flows on three-manifolds. *Tran. Amer. Math. Soc.* **279**(1) (1983), 231–236.

[7] J. Franks. Flow equivalence of subshifts of finite type. *Ergod. Th. & Dynam. Sys.* **4** (1984), 53–66.

[8] R. Ghrist, P. Holmes and M. Sullivan. Knots and links in three-dimensional flows. *Lecture Notes in Mathematics, vol. 1654*. Springer, 1997.

[9] W. Parry and D. Sullivan. A topological invariant of flows on 1-dimensional spaces. *Topology* **14** (1975), 297–299.

[10] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

[11] M. Sullivan. A zeta-function for flows with positive templates. *Top. Appl.* **66** (1995) 199–213.

[12] R. Williams. Classification of subshifts of finite type. *Ann. Math.* **98** (1973), 120–153; Errata **99** (1974), 380–381.

https://doi.org/10.1017/S0143385797097617 Published online by Cambridge University Press