103.19 Bernoulli's inequality for negative exponents

Introduction

The standard statement of Bernoulli's inequality is:

If $x > -1$ is a real number and n a positive integer, then $(1+x)^n \geqslant 1 + nx$.

The standard proof is via Mathematical Induction. Here we give a classroom proof for the more general case, namely when n is any integer.

The main result

Theorem: For every real number $x > -1$ and every integer k

$$(1 + x)^k \geqslant 1 + kx. \tag{1}$$

The inequality becomes strict if $x \neq 0$ and $k \neq 0,\ 1$.

Proof: The statement (1) may be reformulated as: for every $x > -1$, the function

$$H : \mathbb{Z} \to \mathbb{R},\ k \to (1 + x)^{-k}(1 + kx),$$

has a global maximum at $k = 0$.

We may again reformulate this into two statements:

the function H is increasing when $k \in \mathbb{Z}, k \leqslant 0$; and

the function H is decreasing when $k \in \mathbb{Z}, k \geqslant 0$.

Now, if $k \leqslant 0$, we may write the statement $H(k-1) \leqslant H(k)$ in the form

$$\frac{1 + (k-1)x}{(1+x)^{k-1}} \leqslant \frac{1+kx}{(1+x)^k}$$

$$\Leftrightarrow \left[1 + (k-1)x\right](1+x) \leqslant 1 + kx$$

$$\Leftrightarrow (k-1)x^2 \leqslant 0.$$

Since this certainly holds for $k \leqslant 0$, our first statement above also holds.

The second statement may be proved similarly, and so we omit its proof.

An application

As an application of Bernoulli's inequality for negative integer exponents, we can show that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ that defines the Euler constant e is strictly increasing, in the following way. First notice that

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$

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$$\Leftrightarrow \left(\frac{n+1}{n}\right)^n < \left(\frac{n+2}{n+1}\right)^{n+1}$$

$$\Leftrightarrow \frac{n+1}{n+2} < \left(\frac{n^2+2n}{n^2+2n+1}\right)^n$$

(after some straightforward rearrangement)

$$= \left(\frac{n^2+2n+1}{n^2+2n}\right)^{-n}$$

$$= \left(1+\frac{1}{n^2+2n}\right)^{-n}.$$

If we now apply inequality (1) we get

$$\left(1+\frac{1}{n^2+2n}\right)^{-n} \geqslant 1-\frac{n}{n^2+2n}$$

$$= \frac{n^2+n}{n^2+2n}$$

$$= \frac{n+1}{n+2};$$

and this inequality proves what we want.

Using calculus to strengthen the main result

By using calculus we can prove a stronger version of the theorem, as follows:

For all real numbers x and α such that $x > -1$, $x \neq 0$ and $\alpha \notin [0, 1]$, we have:

$$(1+x)^\alpha > 1+\alpha x. \tag{2}$$

This is easily seen because the functions $f(x) = (1+x)^\alpha$ and $g(x) = 1+\alpha x$, where $x > -1$, have graphs tangent to each other at the point $(0, 1)$, and the graph of f otherwise stays above the graph of g since for $x > -1$, f is strictly convex and g is linear.

Notice that, if $0 < \alpha < 1$, f is concave; hence the inequality (2) is reversed.

Finally, for $\alpha = 0$ or 1, the inequality (2) becomes an equality.

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