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The Euler characteristic and Euler defect for comodules over Euler coalgebras

by

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Abstract

Let K be a field. We study a class of left C-comodules over a basic left Euler coalgebra C by means of the Euler $\mathbb{Z}$-bilinear form $\widehat{b}_C$ associated to C, the Euler characteristic $\chi_C(M,N)$ of left C-comodules M,N, and the defect $\partial_C(M,N) \in \mathbb{Z}$ associated to any computable Euler pair (M,N) of left C-comodules. We show that $\widehat{b}_C(\mathbf{lgth}\,M, \mathbf{lgth}\,N) = \chi_C(M,N) + \partial_C(M,N)$, for any computable Euler pair (M,N) of comodules over a left Euler coalgebra C. One of the main results of the paper asserts that the defect $\partial_C(M,N)$ is zero and $\widehat{b}_C(\mathbf{lgth}\,M, \mathbf{lgth}\,N) = \chi_C(M,N)$, if the comodules M,N are finite-dimensional.

Key Words: coalgebra, comodule, injective resolution, Euler characteristic, integral bilinear form, Grothendieck group, Betti number

Mathematics Subject Classification 2000: 16G60, 16W30, 16S50

1. Introduction

Let C be a K-coalgebra and let C–comod be the category of finite-dimensional left C-comodules. In this paper we continue our study of left C-comodules over a basic left Euler coalgebra C (in the sense of [12]) by means of the Euler $\mathbb{Z}$-bilinear form $b_C : \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} \longrightarrow \mathbb{Z}$ (2.7), the Euler characteristic $\chi_C(M,N)$ of comodules M,N, and the defect $\partial_C(M,N)$, we have started in [10], [11], and [12]. More precisely, following [12] we introduce the concept of a computable Euler pair of C-comodules (not necessarily finite-dimensional) and, for a left Euler coalgebra C and any computable Euler pair (M,N), we define the defect $\partial_C(M,N) \in \mathbb{Z}$ of (M,N) by means of an inverse ${}_CF^{-} \in \mathbb{M}_{I_C}(\mathbb{Z})$ (2.2) of the composition length matrix ${}_CF \in \mathbb{M}_{I_C}(\mathbb{Z})$ (2.1) of C. We show in Theorem 2.8 that every pair (M,N) of finite-dimensional C-comodules M,N over a left Euler coalgebra C is a computable Euler pair, the defect $\partial_C(M,N)$ of any such a pair (M,N) is zero, and the following

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equality holds

$$b_C(\mathbf{lgth}\,M,\mathbf{lgth}\,N) = \chi_C(M,N) = \sum_{j=0}^{\infty}(-1)^j \dim_K \mathrm{Ext}^j_C(M,N),$$

for all comodules M,N in C–comod. Moreover, we show in Theorem 4.4 that, given a computable Euler pair (M,N) of comodules over a left Euler coalgebra C, the following equality holds

$$\widehat{b}_C(\mathbf{lgth}\,M,\mathbf{lgth}\,N) = \chi_C(M,N) + \partial_C(M,N),$$

where $\widehat{b}_C$ is an extension of the Euler $\mathbb{Z}$-bilinear form b_C (2.7) defined by the formula (3.3). We also show in Theorem 4.4 that the defect $\partial_C(M,N)$ is zero, for a large class of computable Euler pairs (M,N) containing all pairs of finite-dimensional comodules.

Here we follow the notation introduced in [10], [11], [12], and [14]. Throughout this paper, we assume that C is a basic K-coalgebra with a fixed left comodule decompositions

$$\mathrm{soc}_C C = \bigoplus_{j\in I_C} S(j) \qquad \text{and} \qquad {}_C C = \bigoplus_{j\in I_C} E(j) \tag{1.1}$$

where I_C is a set, $S(j)$ with $j \in I_C$ are pairwise non-isomorphic simple comodules in C–comod, and $E(j)$ is the injective envelope of $S(j)$ in C–Comod, for each $j \in I_C$. We denote by C–Comod the category of left C-comodules and by $K_0(C) = K_0(C\text{–comod})$ the Grothendieck group of the category C–comod of finite-dimensional left C-comodules. We recall that there is a group isomorphism

$$\mathbf{lgth} : K_0(C) \xrightarrow{\simeq} \mathbb{Z}^{(I_C)} \tag{1.2}$$

that associates to any X in C–comod the composition length vector $\mathbf{lgth}\,X \in \mathbb{Z}^{(I_C)}$, see [10]. Throughout we identify the groups $K_0(C)$ and $\mathbb{Z}^{(I_C)}$ via the isomorphism **lgth**.

2. A left inverse of the composition length matrix of a left Euler coalgebra

Throughout this paper we fix a field K and we denote by C a K-coalgebra. We freely apply the representation theory terminology applied in [1] and [16], and the coalgebra and comodule theory terminology introduced in [2], [6], [9], and [10]-[14].

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A coalgebra C is defined to be **pointed** if the simple C-comodules are one-dimensional, and C is **basic** if $\dim_K S = \dim_K \mathrm{End}_C S$, for any simple left C-comodule S. We recall from [12, Definition 4.15] that a basic coalgebra C with decompositions (1.1) is a left Euler coalgebra if, for all $i, j \in I_C$, $\dim_K \mathrm{Hom}_C(E(i), E(j))$ is finite, the space $\mathrm{Ext}^m_C(S(i), S(j))$ is zero, for m sufficiently large, and every simple left C-comodule $S(j)$ admits an injective resolution $0 \longrightarrow S(j) \longrightarrow E^{(j)}_0 \longrightarrow E^{(j)}_1 \longrightarrow \ldots \longrightarrow E^{(j)}_n \longrightarrow E^{(j)}_{n+1} \longrightarrow \ldots$ such that $E^{(j)}_m$ is socle-finite, for $m \geq 0$, and for each $i \in I_C$ there exists $m_{ji} \geq 0$ such that $\mathrm{Hom}_C(E^{(j)}_r, E(i)) = 0$, for all $r \geq m_{ji}$.

It is easy to see that if C is a basic left Euler K-coalgebra then, for any pair M, N of comodules in C–comod, $\mathrm{Ext}^m_C(M,N) = 0$, for m sufficiently large, and the Euler characteristic

$$\chi_C(M,N) = \sum_{m=0}^{\infty} (-1)^m \dim_K \mathrm{Ext}^m_C(M,N)$$

of the pair M, N is a well-defined integer, see [12, Theorem 4.18 (c2)].

We recall from [12] that, given a basic left Euler coalgebra C with fixed decompositions (1.1), the **left composition length matrix** of C is defined to be the $I_C \times I_C$ matrix

$$(2.1) \qquad {}_C F = [\ell_{ij}]_{i,j \in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z}),$$

with $\ell_{ji} = \frac{1}{s_i} \cdot \dim_K \mathrm{Hom}_C(E(j), E(i))$, where $s_i = \dim_K S(i) = \dim_K \mathrm{End}_C S(i)$, for all $i, j \in I_C$. In other words, the matrix ${}_C F$ has the form

$${}_C F = \begin{bmatrix} \vdots \\ \mathbf{e}(j) \\ \vdots \end{bmatrix}$$

that is, the j-th row of ${}_C F$ is the composition length vector

$$\mathbf{e}(j) = \mathbf{lgth}\, E(j) \in \mathbb{Z}^{I_C}$$

of the indecomposable injective left C-comodule $E(j)$, with $\mathbf{e}(j)_i = \ell_{ji}$.

Following the proof of Theorem 4.18 in [12], for a basic left Euler coalgebra C, we construct a left inverse

$$(2.2) \qquad {}_C F^- = [\ell^-_{jp}]_{i,p \in I_C} = \begin{bmatrix} \vdots \\ \ell^-_j \\ \vdots \end{bmatrix} \in \mathbb{M}_{I_C}(\mathbb{Z})$$

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of ${}_C F$ as follows. Given $j \in I_C$, the jth row ℓ_j^- of ${}_C F^-$ has the form

$$\ell_j^- = [\ldots, \ell_{jp}^-, \ldots]_{p \in I_C}, \tag{2.3}$$

with

$$\ell_{jp}^- = \sum_{m=0}^{\infty} (-1)^m d_{mp}^{(j)}, \tag{2.4}$$

where $\sum_{m=0}^{\infty} (-1)^m d_{mp}^{(j)}$ is a finite sum of integers $d_{mp}^{(j)} \geq 0$ defined in [12, (4.21)] as follows. Given $j \in I_C$, we fix a minimal injective resolution

$$0 \longrightarrow S(j) \xrightarrow{h_0^{(j)}} E_0^{(j)} \xrightarrow{h_1^{(j)}} E_1^{(j)} \xrightarrow{h_2^{(j)}} \ldots \xrightarrow{h_n^{(j)}} E_n^{(j)} \xrightarrow{h_{n+1}^{(j)}} E_{n+1}^{(j)} \longrightarrow \ldots \tag{2.5}$$

of the simple C-comodule $S(j)$, with $E_0^{(j)} = E(j)$ and a decomposition

$$E_m^{(j)} = \bigoplus_{p \in I_C} E(p)^{d_{mp}^{(j)}} = \bigoplus_{p \in I_m^{(j)}} E(p)^{d_{mp}^{(j)}} \tag{2.6}$$

of $E_m^{(j)}$, for $m \geq 1$, where $I_m^{(j)}$ is a finite subset of I_C, $d_{mp}^{(j)}$ is a positive integer, for each $p \in I_m^{(j)}$, and we set $d_{mp}^{(j)} = 0$, for any $p \in I_C \setminus I_m^{(j)}$. The comodule $E_m^{(j)}$ is socle-finite and has such form, because C is a left Euler coalgebra. This also implies that, given $j, p \in I_C$, the space $\mathrm{Hom}_C(E_m^{(j)}, E(p))$ vanish, for m sufficiently large. Hence, $S(p)$ is not embedded into $E_m^{(j)}$ and $d_{mp}^{(j)} = 0$, for m sufficiently large. Consequently, the sum $\ell_{jp}^- = \sum_{m=0}^{\infty} (-1)^m d_{mp}^{(j)}$ is an integer and the matrix ${}_C F^- = [\ell_{jp}^-]_{i,p \in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ (2.2) is well-defined, see also [12, Theorem 4.18].

Following [12, (4.6)], we define the **Euler $\mathbb{Z}$-bilinear form**

$$b_C : \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} \longrightarrow \mathbb{Z} \tag{2.7}$$

of the coalgebra C by the formula

$$\begin{aligned} b_C(x,y) &= x \cdot ({}_C S \cdot {}_C F^{-tr}) \cdot y^{tr} \\ &= (x \cdot {}_C S \cdot {}_C F^{-tr}) \cdot y^{tr} \\ &= x \cdot ({}_C S \cdot {}_C F^{-tr} \cdot y^{tr}), \end{aligned}$$

for $x, y \in \mathbb{Z}^{(I_C)}$, where ${}_C F^-$ is the matrix (2.2), ${}_C F^{-tr} = ({}_C F^-)^{tr}$, and ${}_C S = \mathrm{diag}[s_j]$ is the $I_C \times I_C$ diagonal matrix with $s_j = \dim_K S(j) = \dim_K \mathrm{End}_C S(j)$ on the (j,j) entry.

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The **Euler quadratic form** $q_C : \mathbb{Z}^{(I_C)} \longrightarrow \mathbb{Z}$ of C is defined by the formula $q_C(x) = b_C(x,x) = x \cdot {}_C S \cdot {}_C F^{-tr} \cdot x^{tr}$, for $x \in \mathbb{Z}^{(I_C)}$.

Now we prove that the matrix ${}_C F^-$ is a left inverse of the composition length matrix (2.1) of C. This is part (a) of the following important result that improves [12, Theorem 4.18].

Theorem 2.8 *Let C be a basic left Euler K-coalgebra. Let ${}_C F = [\ell_{ij}]_{i,j\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ be the composition length matrix* (2.1) *of C and ${}_C F^- = [\ell^-_{jp}]_{i,p\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ the matrix* (2.2).

(a) *The matrix ${}_C F^-$ is a left inverse of the matrix ${}_C F$ in $\mathbb{M}_{I_C}(\mathbb{Z})$.*

(b) *For any pair $j, p \in I_C$, we have*

$$\ell^-_{jp} = \frac{1}{s_p} \cdot \chi_C(S(p), S(j)), \quad \text{where} \quad s_p = \dim_K S(p) = \dim_K \mathrm{End}_C S(p). \tag{2.9}$$

(c) *If $j \in I_C$ is such that* $\mathrm{inj.dim} S(j)$ *is finite, then the jth row ℓ^-_j of the matrix ${}_C F^-$ has at most finitely many non-zero entries.*

(d) *For each pair M, N of comodules in C–*comod*, the following equality holds*

$$b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \sum_{m=0}^{\infty} (-1)^m \dim_K \mathrm{Ext}^m_C(M,N). \tag{2.10}$$

Proof: (a) We follow the proof of [12, Theorem 4.18 (a)]. To show that the equality ${}_C F^- \cdot {}_C F = E$ holds, we prove that, given a pair of indices $i, j \in I_C$, we have

$$\sum_{p\in I_C} \ell^-_{jp} \ell_{pi} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \tag{2.11}$$

It follows from the definition of the matrix ${}_C F$ of C that, for each $j \in I_C$, the j-th row of the matrix ${}_C F$ is the length vector $\mathbf{e}(j) = \mathbf{lgth}\, E(j) \in \mathbb{Z}^{I_C}$ of the indecomposable injective left C-comodule $E(j)$, i.e. $\mathbf{e}(j)_i = \ell_{ji}$.

We recall from [12, Proposition 2.6(b)] that

$$\ell_i(N) = \mathrm{lgth}_{B_i} \mathrm{Hom}_C(N, E(i)),$$

for any C-comodule N.

Fix a pair of indices $i, j \in I_C$ and fix a minimal injective resolution (2.5) of the simple C-comodule $S(j)$. It has the properties stated in [12, Definition 4.15(c)]. Since C is a left Euler coalgebra the resolution (2.5) is socle-finite and, given $i \in I_C$, we find an $m_{ji} \geq 1$ such that

$$\ell_i(E^{(j)}_r) = \mathrm{lgth}_{B_i} \mathrm{Hom}_C(E^{(j)}_r, E(i)) = 0,$$

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for $r \geq m_{ji}+1$, where $B_i = \mathrm{End}_C E(i)$. Since $E(i)$ is injective then, for $n = m_{ji}$, the sequence

$$0 \longrightarrow \mathrm{Hom}_C(E_n^{(j)}, E(i)) \longrightarrow \ldots \longrightarrow \mathrm{Hom}_C(E_0^{(j)}, E(i)) \longrightarrow \mathrm{Hom}_C(S(j), E(i)) \longrightarrow 0$$

induced by the resolution (2.5) of $S(j)$ is exact.

Recall that $\ell_i(E_r^{(j)}) = \mathrm{lgth}_{B_i}\mathrm{Hom}_C(E_r^{(j)}, E(i))$, for any $r \geq 0$. Then, by summing up the B_i-lengths, we get

$$\ell_i(S(j)) = \sum_{m=0}^{m_{ji}} (-1)^m \ell_i(E_m^{(j)}) = \sum_{m=0}^{\infty} (-1)^m \ell_i(E_m^{(j)}), \tag{2.12}$$

because $\ell_i(E_r^{(j)}) = \mathrm{lgth}_{B_i}\mathrm{Hom}_C(E_r^{(j)}, E(i)) = 0$, for all $r \geq m_{ji}+1$. Since

$$E_m^{(j)} = \bigoplus_{p \in I_C} E(p)^{d_{mp}^{(j)}} = \bigoplus_{p \in I_m^{(j)}} E(p)^{d_{mp}^{(j)}},$$

where $I_m^{(j)} \subseteq I_C$ is a finite subset of I_C, then, for any $m \geq 0$, the composition length vector

$$\mathbf{lgth}\, E_m^{(j)} = (\ell_i(E_m^{(j)}))_{i \in I_C} \in \mathbb{Z}^{I_C}$$

of the comodule $E_m^{(j)}$ has the form

$$\mathbf{lgth}\, E_m^{(j)} = \sum_{p \in I_C} d_{mp}^{(j)} \cdot \mathbf{lgth}\, E(p) = \sum_{p \in I_C} d_{mp}^{(j)} \cdot \mathbf{e}(p) = \sum_{p \in I_m^{(j)}} d_{mp}^{(j)} \cdot \mathbf{e}(p)$$

and its ith coordinate has the form

$$(\mathbf{lgth}\, E_m^{(j)})_i = \sum_{p \in I_m^{(j)}} d_{mp}^{(j)} (\mathbf{lgth}\, E(p))_i = \sum_{p \in I_C} d_{mp}^{(j)} \mathbf{e}(p)_i = \sum_{p \in I_C} d_{mp}^{(j)} \ell_{pi} = \sum_{p \in I_m^{(j)}} d_{mp}^{(j)} \ell_{pi}. \tag{2.13}$$

Note that the sums have only finitely many non-zero summands. Since, given $m \geq m_{ji}+1$, we have $(\mathbf{lgth}\, E_m^{(j)})_i = \ell_i(E_m^{(j)}) = 0$, then $\sum_{p \in I_C} d_{mp}^{(j)} \ell_{pi} = 0$ and

$$d_{mp}^{(j)} \ell_{pi} = 0,$$

for $m \geq m_{ji}+1$ and all $p \in I_C$.

It follows that

$$\sum_{m=0}^{m_{ji}} (-1)^m \sum_{p \in I_m^{(j)}} d_{mp}^{(j)} \ell_{pi} = \sum_{m=0}^{m_{ji}} (-1)^m \sum_{p \in I(j,i)} d_{mp}^{(j)} \ell_{pi} = \sum_{p \in I(j,i)} \sum_{m=0}^{m_{ji}} (-1)^m d_{mp}^{(j)} \ell_{pi},$$

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where $I(j,i) = I_0^{(j)} \cup I_1^{(j)} \cup \ldots \cup I_{m_{ji}}^{(j)}$ is a finite subset of I_C, and therefore the sums have only a finite number of non-zero summands. In view of (2.12) and (2.13), we get a sequence of equalities

$$
\begin{aligned}
\ell_i(S(j)) &= \sum_{m=0}^{m_{ji}} (-1)^m \sum_{p \in I_m^{(j)}} d_{mp}^{(j)} \ell_{pi} \\
&= \sum_{p \in I(j,i)} \sum_{m=0}^{m_{ji}} (-1)^m d_{mp}^{(j)} \ell_{pi} \\
&= \sum_{p \in I_C} \sum_{m=0}^{\infty} (-1)^m d_{mp}^{(j)} \ell_{pi} \\
&= \sum_{p \in I_C} (\sum_{m=0}^{\infty} (-1)^m d_{mp}^{(j)}) \ell_{pi} \\
&= \sum_{p \in I_C} \ell_{jp}^{-} \ell_{pi},
\end{aligned}
\tag{2.14}
$$

because $\ell_{jp}^{-} = \sum_{m=0}^{\infty} (-1)^m d_{mp}^{(j)} \in \mathbb{Z}$, for $j, p \in I_C$. This shows that the sum $\sum_{p \in I_C} \ell_{jp}^{-} \ell_{pi}$ is finite and, given $i, j \in I_C$, the integer $\ell_{jp}^{-} \ell_{pi}$ is zero, for all but a finite number of indices $p \in I_C$. Since $\ell_i(S(j)) = 1$, for $j = i$, and $\ell_i(S(j)) = 0$, for $j \neq i$, then (2.14) implies the equality (2.11) and, consequently, we get ${}_C F^{-} \cdot {}_C F = E$, that is, the matrix ${}_C F^{-}$ is a left inverse of ${}_C F$.

(b) To prove the formula (2.9), we note that the integer $d_{mp}^{(j)}$ that appears in the formula (2.4) is just the mth Bass number $\mu_m(S(p), S(j))$ (=Betti number) of the pair $(S(p), S(j))$ of the simple C-comodules $S(p)$ and $S(j)$, see [7], [8] and [11, p. 480]. It follows from [11, (4.23)] that, given $m \geq 0$ and $j, p \in I_C$, we get

$$
\begin{aligned}
d_{mp}^{(j)} = \mu_m(S(p), S(j)) &= \frac{1}{s_p} \cdot \dim_K \mathrm{Hom}_C(S(p), E_m^{(j)}) \\
&= \frac{1}{s_p} \cdot \dim_K \mathrm{Ext}_C^m(S(p), S(j)).
\end{aligned}
$$

Hence, the integer ℓ_{jp}^{-} has the form

$$
\ell_{jp}^{-} = \sum_{m=0}^{\infty} (-1)^m d_{mp}^{(j)} = \sum_{m=0}^{\infty} (-1)^m \frac{1}{s_p} \cdot \dim_K \mathrm{Ext}_C^m(S(p), S(j)) = \frac{1}{s_p} \cdot \chi_C(S(p), S(j)).
$$

This finishes the proof of (b).

(c) Assume that $j \in I_C$ and $\mathrm{inj.dim}\, S(j)$ is finite. Then the injective resolution (2.5) of $S(j)$ in C–Comod is finite and has the form

$$
0 \longrightarrow S(j) \xrightarrow{h_0^{(j)}} E_0^{(j)} \xrightarrow{h_1^{(j)}} E_1^{(j)} \xrightarrow{h_2^{(j)}} \ldots \xrightarrow{h_{n_j}^{(j)}} E_{n_j}^{(j)} \longrightarrow 0,
$$

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where $E_m^{(j)}$ is a finite direct sum of the comodules $E(p)$, for $m = 0,\ldots,n_j$. Hence, the module $E_0^{(j)} \oplus E_1^{(j)} \oplus \ldots \oplus E_{n_j}^{(j)}$ has only finitely many direct summands of the form $E(p)$, up to isomorphism. Therefore $\mathrm{Ext}_C^m(S(p),S(j)) = 0$, if $m \geq 0$ and $E(p)$ is not a summand of the module $E_0^{(j)} \oplus E_1^{(j)} \oplus \ldots \oplus E_{n_j}^{(j)}$. This shows that the pth coordinate $\ell_{jp}^- = \frac{1}{s_p} \cdot \chi_C(S(p),S(j))$ of the vector (2.3) is zero, if $E(p)$ is not a summand of the module $E_0^{(j)} \oplus E_1^{(j)} \oplus \ldots \oplus E_{n_j}^{(j)}$. Consequently, the number of non-zero coordinates ℓ_{jp}^- of the vector (2.3) is finite.

(d) First we show that the equality (2.10) holds, for any pair $M = S(p)$ and $M = S(j)$ of simple C-comodules. In this case, we have $e_p = \mathbf{lgth}\, M$ and $e_j = \mathbf{lgth}\, N$ and, in view of (b), we get

$$\begin{aligned} b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) &= b_C(e_p, e_j) \\ &= e_p \cdot {}_C D \cdot {}_C F^{-tr} \cdot e_j^{tr} \\ &= (e_p \cdot s_p \cdot {}_C F^{-tr}) \cdot e_j^{tr} \\ &= (s_p \cdot \ell_{\bullet p}^{-tr}) \cdot e_j^{tr} \\ &= s_p \cdot \ell_{jp}^- \\ &= s_p \cdot \tfrac{1}{s_p} \cdot \chi_C(S(p),S(j)) \\ &= \chi_C(M,N), \end{aligned}$$

where $\ell_{\bullet p}^-$ is the pth column of the matrix ${}_C F^-$ and $\ell_{\bullet p}^{-tr} = [\ldots,\ell_{ip}^-,\ldots]_{i \in I_C}$ is the pth row of the matrix ${}_C F^{-tr}$. Since both sides of the equality $b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M,N)$ are additive functions of M and N, and the equality holds if M and N are simple comodules then an easy induction on $\mathrm{lgth}\, M + \mathrm{lgth}\, N \geq 2$ shows that the equality holds for all M and N in C–comod. This finishes the proof of the theorem. □

We end this section by showing that the matrix ${}_C F$ has a two-sided inverse, if C is a simply dualising Euler coalgebra in the following sense.

Definition 2.15 A coalgebra C is defined to be a **simply dualising Euler coalgebra** if

(a) C is both left and right Euler coalgebra, and

(b) $\dim_K \mathrm{Ext}_C^m(S,S') = \dim_K \mathrm{Ext}_{C^{op}}^m(DS',DS)$, for all $m \geq 0$ and all simple left C-comodules S and S', where $D : C\text{–comod} \longrightarrow C^{op}\text{–comod}$ is the standard duality, see [10].

The class of simply dualising Euler coalgebras contains the incidence coalgebras $K^{\square}I$ of intervally finite posets (see [12], [13], [15]) and semiperfect (left or right) coalgebras of finite global dimension, see [4].

Theorem 2.16 *Let C be a pointed simply dualising Euler K-coalgebra. Let ${}_C F = [\ell_{ij}]_{i,j \in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ be the composition length matrix* (2.1) *of C. Then the left*

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inverse ${}_CF^- = [\ell^-_{jp}]_{j,p\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ (2.2) *of* ${}_CF$ *is also its right inverse, that is, the equality* ${}_CF \cdot {}_CF^- = E$ *holds in* $\mathbb{M}_{I_C}(\mathbb{Z})$.

Proof: Since C is pointed, it is basic [11, Lemma 2.1] and both left and right Euler coalgebra. Then, by Theorem 2.8 (a), the matrix ${}_CF^- = [\ell^-_{jp}]$ (2.2), with $\ell^-_{jp} = \chi_C(S(p),S(j))$, is a left inverse of ${}_CF = [\ell^-_{jp}]$ and the matrix $\widetilde{F}^-_C = {}_{C^{op}}F^- = [\widetilde{\ell}^-_{jp}]$, with $\widetilde{\ell}^-_{jp} = \chi_{C^{op}}(DS(p),DS(j))$, is a left inverse of the composition length matrix $F_C = {}_{C^{op}}F = [\widetilde{\ell}_{jp}]$ of C^{op}, with $\widetilde{\ell}_{jp} = \dim_K \mathrm{Hom}_{C^{op}}(\widetilde{E}(j),\widetilde{E}(p))$, for all $i,p \in I_C = I_{C^{op}}$, where $\widetilde{E}(j),\widetilde{E}(p)$ are the images of $E(j),E(p)$ under the duality C-inj $\to C^{op}$-inj given in [3]. Hence we get

$$\widetilde{\ell}_{ij} = \dim_K \mathrm{Hom}_{C^{op}}(\widetilde{E}(i),\widetilde{E}(j)) = \dim_K \mathrm{Hom}_C(E(j),E(i)) = \ell_{ji},$$

for every pair $i,j \in I_C$. This yields the equality $F^{tr}_C = {}_CF$. Since $\dim_K \mathrm{Ext}^m_C(S,S') = \dim_K \mathrm{Ext}^m_{C^{op}}(DS',DS)$, for all $m \geq 0$ and all simple left C-comodules S and S', by our assumption, then $\chi_{C^{op}}(DS(p),DS(j)) = \chi_C(S(j),S(p))$, for all $j,p \in I_C = I_{C^{op}}$, and we get $(\widetilde{F}^-_C)^{tr} = {}_CF^-$. Hence, the equality $\widetilde{F}^-_C \cdot F_C = E$ yields $E = F^{tr}_C \cdot (\widetilde{F}^-_C)^{tr} = {}_CF \cdot (\widetilde{F}^-_C)^{tr} = {}_CF \cdot {}_CF^-$, that is, ${}_CF^-$ is a right inverse of ${}_CF$. This finishes the proof. □

3. Euler comodules over a left Euler coalgebra

We recall from [12, Section 2] that a left C-comodule M is defined to be **computable** if, for each $j \in I_C$, the jth coordinate $\ell_j(M) = \frac{1}{s_j} \cdot \dim_K \mathrm{Hom}_C(M,E(j))$ of the composition length vector **lgth**M of M is finite.

For a left C-comodule N, we fix a minimal injective resolution

$$0 \longrightarrow N \xrightarrow{h^N_0} E^N_0 \xrightarrow{h^N_1} E^N_1 \xrightarrow{h^N_2} \ldots \xrightarrow{h^N_n} E^N_n \xrightarrow{h^N_{n+1}} E^N_{n+1} \longrightarrow \ldots \tag{3.1}$$

of N in C–Comod. We call the resolution **socle-finite and computable**, if the injective C-comodules $E^N_0, E^N_1, E^N_2, \ldots$ are socle-finite and computable.

It is easy to see that, for any computable comodule M and a comodule N such that the injective resolution (3.1) is socle-finite and computable the K-dimension of $\mathrm{Ext}^m_C(M,N)$ is finite, for any $m \geq 0$, see [12, Corollary 3.3].

Now we generalise Theorem 2.8 (d) to the class of Euler comodules in the following sense.

Definition 3.2 Let C be a basic left Euler K-coalgebra. Let ${}_CF \in \mathbb{M}_{I_C}(\mathbb{Z})$ be the composition length matrix of C, ${}_CF^-$ the left inverse of ${}_CF$ defined by the formulae (2.2)-(2.4), and let ${}_CF^{-tr} = ({}_CF^-)^{tr}$ be the transpose of the matrix ${}_CF^-$. A comodule N in C–Comod is defined to be an **Euler comodule** if

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(a) N is finitely copresented (see [12], [14]), and

(b) the minimal injective resolution (3.1) of N in C–Comod is socle-finite, computable, and **weakly bounded** in the sense that, for any $i \in I_C$, there exists an $m_i \geq 1$ such that $\mathrm{Hom}_C(E_m^N, E(i)) = 0$, for all $m \geq m_i$.

Note that the class of Euler comodules over a left Euler coalgebra C contains all simple comodules, the comodules of finite dimension, socle-finite injective comodules, the syzygy comodules $\Omega_m N = \Omega^{-m} N = \mathrm{Im} h_m$, with $m \geq 1$, for any comodule N in C–comod, and the (socle) finitely copresented comodules N in C–Comod that admit a socle-finite and computable injective resolution of finite length.

Following [12, Section 4], we set

$$\widehat{b}_C(v,w) := v \cdot {}_C S \cdot ({}_C F^{-tr} \cdot w^{tr}) = v \cdot {}_C S \cdot (w \cdot {}_C F^{-})^{tr}, \tag{3.3}$$

for a pair of vectors $v, w \in \mathbb{Z}^{I_C}$, if the multiplication $v \cdot ({}_C F^{-tr} \cdot w^{tr}) = v \cdot (w \cdot {}_C F^{-})^{tr}$ is a well-defined integer.

Now we extend Theorem 2.8 (d) from comodules in C–comod to Euler comodules as follows.

Theorem 3.4 *Let C be a basic left Euler K-coalgebra. Let ${}_C F = [\ell_{ij}]_{i,j \in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ be the composition length matrix* (2.1) *of C and ${}_C F^{-} = [\ell^{-}_{jp}]_{j,p \in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ the matrix* (2.2). *Assume that N is an Euler comodule, $\mathbf{lgth}\, N = (\ell_j(N))_{j \in I_C} \in \mathbb{Z}^{I_C}$ is the composition length vector of N, and d_j^N is the integer defined by the formula* (3.10).

(a) *For any index $j \in I_C$, the jth coordinate $\ell_j(N)$ of the vector $\mathbf{lgth} N$ has the form $\ell_j(N) = \sum\limits_{p \in I_C} d_p^N \ell_{pj}$, where $d_p^N = \frac{1}{s_p} \cdot \chi_C(S(p), N)$.*

(b) *For any index $j \in I_C$, the following equality holds*

$$\widehat{b}_C(\mathbf{lgth}\, S(j), \mathbf{lgth}\, N) := e_j \cdot {}_C S \cdot [{}_C F^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}] = s_j \cdot d_j^N = \chi_C(S(j), N), \tag{3.5}$$

if the comodule N is finite-dimensional, or $\dim_K N = \infty$ and ${}_C F \cdot {}_C F^{-} = E$ (that is, ${}_C F^{-}$ is the right inverse of ${}_C F$), where $e_j = \mathbf{lgth}\, S(j)$.

(c) *If ${}_C F \cdot {}_C F^{-} = E$ and M is a computable left C-comodule, we have*

$$\widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, E(j)) = s_j \cdot \ell(M)_j = \dim_K \mathrm{Hom}_C(M, E(j)) = \chi_C(M, E(j)),$$

for any index $j \in I_C$.

(d) *Assume that M is a computable comodule and N is an Euler comodule in C–Comod. The following equality holds*

$$\widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M, N) = \sum_{m=0}^{\infty} (-1)^m \dim_K \mathrm{Ext}_C^m(M, N), \tag{3.6}$$

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in any of the following three cases:

(d1) *both comodules M and N are finite-dimensional.*

(d2) ${}_C F \cdot {}_C F^- = E$, $\dim_K M$ *is finite, and* $\dim_K N$ *is infinite.*

(d3) ${}_C F \cdot {}_C F^- = E$, $\dim_K M$ *is infinite, and* $\operatorname{inj.dim} N$ *is finite.*

Proof: (a) It follows from the definition of the composition factors matrix ${}_C F$ of C that, for each $j \in I_C$, the j-th row of the matrix ${}_C F$ is the composition length vector $\mathbf{e}(j) = \mathbf{lgth}\, E(j) \in \mathbb{Z}^{I_C}$ of the indecomposable injective left C-comodule $E(j)$, i.e. $\mathbf{e}(j)_p = \ell_{jp}$, for all $p \in I_C$. We also recall that the ith coordinate $\ell_i(N)$ of the vector $\mathbf{lgth}\, N$ is defined by the formula

$$\ell_i(N) = \operatorname{lgth}_{B_i} \operatorname{Hom}_C(N, E(i)) = \frac{1}{s_i} \cdot \dim_K \operatorname{Hom}_C(N, E(i)),$$

where $B_i = \operatorname{End}_C E(i)$ and $s_i = \dim_K S(i) = \dim_K \operatorname{End}_C S(i)$, see [12, Proposition 2.6(b)].

We assume that N is an Euler comodule and we fix a minimal injective resolution (3.1) of N. Fix an index $j \in I_C$ and fix an integer $m_{j,N} \geq 1$ such that

$$\ell_j(E_r^N) = \operatorname{lgth}_{B_j} \operatorname{Hom}_C(E_r^N, E(j)) = 0,$$

for $r \geq m_{j,N} + 1$. Since $E(j)$ is injective then the sequence

$$0 \longrightarrow \operatorname{Hom}_C(E_n^N, E(j)) \longrightarrow \ldots \longrightarrow \operatorname{Hom}_C(E_0^N, E(j)) \longrightarrow \operatorname{Hom}_C(S(j), E(j)) \longrightarrow 0,$$

with $n = m_{j,N}$, induced by the resolution (3.1) of N is exact.

Recall that $\ell_j(E_r^N) = \operatorname{lgth}_{B_j} \operatorname{Hom}_C(E_r^N, E(j))$, for any $r \geq 0$. Then, by summing up the B_j-lengths, we get

$$\ell_j(N) = \sum_{m=0}^{m_{j,N}} (-1)^m \ell_j(E_m^N) = \sum_{m=0}^{\infty} (-1)^m \ell_j(E_m^N), \tag{3.7}$$

because $\ell_j(E_r^N) = \operatorname{lgth}_{B_j} \operatorname{Hom}_C(E_r^N, E(j)) = 0$, for all $r \geq m_{j,N} + 1$. Since the injective comodule E_m^N is socle-finite then, given $m \geq 0$, there is a decomposition

$$E_m^N = \bigoplus_{p \in I_m^N} E(p)^{d_{mp}^N} = \bigoplus_{p \in I_C} E(p)^{d_{mp}^N}, \tag{3.8}$$

where $d_{mp}^N \geq 0$ are integers and $I_m^N \subseteq I_C$ is a finite subset of I_C such that $d_{mp}^N = 0$, for any $p \in I_C \setminus I_m^N$. Hence, for any $m \geq 0$, the composition length vector

$$\mathbf{lgth}\, E_m^N = (\ell_i(E_m^N))_{i \in I_C} \in \mathbb{Z}^{I_C}$$

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of the comodule E^N_m has the form

$$\mathbf{lgth}\, E^N_m = \sum_{p\in I_C} d^N_{mp}\cdot \mathbf{lgth}\, E(p) = \sum_{p\in I_C} d^N_{mp}\cdot \mathbf{e}(p) = \sum_{p\in I^N_m} d^N_{mp}\cdot \mathbf{e}(p)$$

and its jth coordinate has the form

(3.9)

$$(\mathbf{lgth}\, E^N_m)_j = \sum_{p\in I_C} d^N_{mp}\cdot (\mathbf{lgth}\, E(p))_j = \sum_{p\in I_C} d^N_{mp}\cdot \mathbf{e}(p)_j = \sum_{p\in I_C} d^N_{mp}\ell_{pj} = \sum_{p\in I^N_m} d^N_{mp}\ell_{pj}.$$

Note that the sums have only finitely many non-zero summands. Since, given $m \geq m_{j,N}+1$, we have $\mathrm{Hom}_C(E^N_m, E(j)) = 0$, then $(\mathbf{lgth}\, E^N_m)_j = \ell_j(E^N_m) = 0$. Hence $\sum_{p\in I_C} d^N_{mp}\ell_{pj} = 0$ and

$$d^N_{mp}\ell_{pj} = 0,$$

for $m \geq m_{j,N}+1$ and all $p \in I_C$. It follows that

$$\begin{aligned}\sum_{m=0}^{m_{j,N}} (-1)^m \sum_{p\in I^N_m} d^N_{mp}\ell_{pj} &= \sum_{m=0}^{m_{j,N}} (-1)^m \sum_{p\in I(j,N)} d^N_{mp}\ell_{pj}\\ &= \sum_{p\in I(j,N)} \sum_{m=0}^{m_{j,N}} (-1)^m d^N_{mp}\ell_{pj}\\ &= \sum_{p\in I_C} \sum_{m=0}^{\infty} (-1)^m d^N_{mp}\ell_{pj},\end{aligned}$$

where $I(j,N) = I^N_0 \cup I^N_1 \cup \ldots \cup I^N_{m_{j,N}}$ is a finite subset of I_C, and therefore the infinite sum $\sum_{p\in I_C} \sum_{m=0}^{\infty} (-1)^m d^N_{mp}\ell_{pj}$ has only a finite number of non-zero summands.

Given $p \in I_C$, we set

$$d^N_p = \sum_{m=0}^{\infty} (-1)^m d^N_{mp}.$$

Note that, for $p \in I_C$ and N, the integer $m_{p,N}$ is chosen in such a way that $\mathrm{Hom}_C(E^N_m, E(p)) = 0$, for all $m \geq m_{p,N}+1$. It follows that, given $m \geq m_{p,N}+1$, the injective comodule E^N_m has no direct summand isomorphic to $E(p)$ and, hence, $d^N_{mp} = 0$. Consequently,

$$d^N_p = \sum_{m=0}^{\infty} (-1)^m d^N_{mp} = \sum_{m=0}^{m_{p,N}} (-1)^m d^N_{mp} \tag{3.10}$$

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and d_p^N is an integer. Hence, in view of (3.7) and (3.9), we get the equalities

$$\begin{aligned}
\ell_j(N) &= \sum_{m=0}^{m_{j,N}} (-1)^m \sum_{p\in I_m^N} d_{mp}^N \ell_{pi} \\
&= \sum_{p\in I(j,N)} \sum_{m=0}^{m_{j,N}} (-1)^m d_{mp}^N \ell_{pj} \\
&= \sum_{p\in I_C} \sum_{m=0}^{\infty} (-1)^m d_{mp}^N \ell_{pj} \\
&= \sum_{p\in I_C} (\sum_{m=0}^{\infty} (-1)^m d_{mp}^N) \ell_{pj} \\
&= \sum_{p\in I_C} d_p^N \ell_{pj}.
\end{aligned} \tag{3.11}$$

This shows that the sum $\sum_{p\in I_C} d_p^N \ell_{pi}$ has only a finite number of non-zero summands and, given $i, j \in I_C$, the integer $d_p^N \ell_{pi}$ is zero, for all but a finite number of indices $p \in I_C$.

Now we note that the integer d_{mp}^N that appear in the formula (3.8) is just the mth Bass number $\mu_m(S(p), S(j))$ (=Betti number) of the pair $(S(p), N)$ of the comodules $S(p)$ and N, in the sense of [11, p. 480]. It follows from [11, (4.23)] that, given $m \geq 0$ and $p \in I_C$, we get

$$d_{mp}^N = \mu_m(S(p), N) = \frac{1}{s_p} \cdot \dim_K \mathrm{Hom}_C(S(p), E_m^N) = \frac{1}{s_p} \cdot \dim_K \mathrm{Ext}_C^m(S(p), N).$$

Hence, the integer d_p^N (3.10) has the form

$$d_p^N = \sum_{m=0}^{\infty} (-1)^m d_{mp}^N = \sum_{m=0}^{\infty} (-1)^m \frac{1}{s_p} \cdot \dim_K \mathrm{Ext}_C^m(S(p), N) = \frac{1}{s_p} \cdot \chi_C(S(p), N).$$

This finishes the proof of (a).

(b) Note that $e_i \cdot {}_C F^- = \ell_i^- = [\ldots, \ell_{ip}^-, \ldots]_{p\in I_C}$ is the ith row of the matrix ${}_C F^-$ and $\ell_j^{-tr} = (\ell_j^-)^{tr}$ is the ith column of the matrix ${}_C F^{-tr}$, see (2.2).

First we prove (b) in case the comodule N is finite-dimensional. We apply the induction on the length $n = \mathrm{lgth} N$ of N. If $n = 1$, the comodule N is simple. Then $N \cong S(i)$, $\mathbf{lgth}\, N = e_i$, for some $i \in I_C$, and we have

$$\begin{aligned}
\widehat{b}_C(e_j, \mathbf{lgth}\, N) &= e_j \cdot {}_C S \cdot [{}_C F^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}] \\
&= e_j \cdot {}_C S \cdot [e_i \cdot {}_C F^-]^{tr} \\
&= e_j \cdot {}_C S \cdot \ell_i^{-tr} = s_j \cdot \ell_{ij}^- \\
&= s_j \cdot d_j^N,
\end{aligned}$$

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because $d_j^N = d_j^{S(i)} = \ell_{ij}^-$, by (a) and (2.9). The inductive step easily follows, because all terms in the formula (3.5) are additive functions of N.

Now we prove (b) in case the comodule N is infinite-dimensional and ${}_CF \cdot {}_CF^- = E$. In view of (a), we get

$$\begin{aligned} e_j \cdot {}_CS \cdot [{}_CF^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}] &= \sum_{i \in I_C} s_j \ell_{ij}^- \ell_i(N) = \sum_{i \in I_C} s_j \ell_{ij}^- \sum_{p \in I_C} d_p^N \ell_{pi} \\ &= \sum_{p \in I_C} (\sum_{i \in I_C} \ell_{pi} \ell_{ij}^-) s_j d_p^N = s_j d_j^N, \end{aligned}$$

because the assumption ${}_CF \cdot {}_CF^- = E$ yields $\sum_{i \in I_C} \ell_{pi} \ell_{ij}^- = \begin{cases} 0, & \text{for } p \neq j, \\ 1, & \text{for } p = j. \end{cases}$

(c) Let M be a computable C-comodule and fix $j \in I_C$. Assume that $E = {}_CF \cdot {}_CF^-$, that is, the matrix ${}_CF^-$ is a right inverse of ${}_CF$ in $\mathbb{M}_{I_C}(\mathbb{Z})$. It follows that $\mathbf{e}(j) \cdot {}_CF^- = e_j$, with $\mathbf{e}(j) = \mathbf{lgth}\, E(j)$, and we get

$$\begin{aligned} \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, E(j)) &= (\mathbf{lgth}\, M) \cdot {}_CS \cdot [{}_CF^{-tr} \cdot \mathbf{e}(j)^{tr}] \\ &= (\mathbf{lgth}\, M) \cdot {}_CS \cdot [\mathbf{e}(j) \cdot {}_CF^-]^{tr} \\ &= (\mathbf{lgth}\, M) \cdot (s_j \cdot e_j^{tr}) \\ &= \ell(M)_j \cdot s_j \\ &= \dim_K \mathrm{Hom}_K(N, E(j)) \\ &= \chi_C(N, E(j)), \end{aligned}$$

and we are done.

(d) In case (d1), the equality (3.6) is a consequence of Theorem 2.8 (d). To prove (d) in cases (d2) and (d3), assume that ${}_CF \cdot {}_CF^- = E$, M is a computable comodule, and N is an Euler comodule in C–Comod.

(d2) Assume that $\dim_K M$ is finite and $\dim_K N$ is infinite. We prove the equality (3.6) by induction on the length $\mathrm{lgth}\, M \geq 1$ of M. If $\mathrm{lgth}\, M = 1$ then M is a simple comodule, there is an isomorphism $M \cong S(j)$, for some $j \in I_C$, and $\mathbf{lgth}\, M = \mathbf{lgth}\, S(j) = e_j$ is the jth standard basis vector e_j of the group $\mathbb{Z}^{(I_C)}$. Then the equality (3.6) holds for $\mathrm{lgth}\, M = 1$, by (b). Since both sides of the equality (3.6) are additive functions of the variable M then the inductive step easily follows and we are done.

(d3) Assume that assume that $\dim_K M$ is infinite and the dimension $s_N = \mathrm{inj.dim}\, N \geq 0$ is finite. Then the injective resolution (2.5) of N in C–Comod is finite of the form

$$0 \longrightarrow N \xrightarrow{h_0^N} E_0^N \xrightarrow{h_1^N} E_1^N \xrightarrow{h_2^N} \dots \xrightarrow{h_{s_N}^N} E_{s_N}^N \longrightarrow 0,$$

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and $E_m^N = \oplus_{p\in I_m^{(j)}} E(p)^{d_{mp}^{(j)}}$ is the finite direct sum (2.6), for $m = 0,\ldots,s_N$. It follows that $\mathbf{lgth}\, E_m^{(j)} = \sum_{p\in I_m^{(j)}} d_{mp}^{(j)} \cdot \mathbf{e}(p)$ and, by applying (c), we get

$$\begin{aligned}
\widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, E_m^{(j)}) &= \textstyle\sum_{p\in I_m^{(j)}} d_{mp}^{(j)} \cdot (\mathbf{lgth}\, M) \cdot {}_C S \cdot [{}_C F^{-tr} \cdot \mathbf{e}(p)^{tr}] \\
&= \textstyle\sum_{p\in I_m^{(j)}} d_{mp}^{(j)} \cdot \chi_C(M, E(p)) \\
&= \chi_C(M, \oplus_{p\in I_m^{(j)}} E(p)^{d_{mp}^{(j)}}) \\
&= \chi_C(N, E_m^N),
\end{aligned}$$

for $m = 0,\ldots,s_N$. Since the injective resolution of N is finite, the additivity of the function $\mathbf{lgth}$ yields $\mathbf{lgth}\, N = \sum_{m=0}^{s_N} (-1)^m \cdot \mathbf{lgth}\, E_m^N$ and, by the additivity of $\widehat{b}_C(\mathbf{lgth}\, M, -)$ and $\chi_C(M,-)$, we get

$$\begin{aligned}
\widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) &= \sum_{m=0}^{s_N} (-1)^m \cdot \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, E_m^N) \\
&= \sum_{m=0}^{s_N} (-1)^m \cdot \chi_C(M, E_m^N) \\
&= \chi_C(M, N).
\end{aligned}$$

This finishes the proof of (c) and completes the proof of the theorem. □

The properties of the pairs of comodules M, N that are not of any of the type (d1), (d2), and (d3) studied in Theorem 3.4 are discussed in the following section and in particular in Remark 4.9.

4. Computable Euler pairs of comodules and the Euler defect

We recall from [12, Definition 4.12] that a pair M N of left C-comodule is defined to be an **Euler pair** if $\dim_K \mathrm{Ext}_C^m(M,N)$ is finite, for all $m \geq 0$, and $\mathrm{Ext}_C^m(M,N) = 0$, for m sufficiently large. Hence the Euler characteristic $\chi_C(M,N)$ of the Euler pair M, N is an integer.

It follows from Theorems 2.8 and 3.4 that, over a left Euler coalgebra C, each of the pairs M, N studied in (d1), (d2) and (d3) of Theorems 3.4 (d) is an Euler pair such that $\widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M,N)$. Our aim of this section is to study a class of so called computable Euler pairs M, N over a left Euler coalgebra C such that the above equality holds, up to a well-defined integer $\partial_C(M,N)$, called the **Euler defect** of M, N, see [12, Definition 4.12]. More precisely, we show that $\widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M,N) + \partial_C(M,N)$. In particular, this covers some of the pairs M, N studied in Theorems 3.4 (d3), when we drop the assumption that $\mathrm{inj.dim}\, N$ is finite.

Following [12, Theorem 4.18], we introduce the following concept.

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Definition 4.1 Let C be a K-coalgebra and let M and N be left C-comodules.

(a) The pair M, N is called a **computable Euler pair**, if the following conditions are satisfied:

(a1) M and N are computable.

(a2) The minimal injective resolution (3.1) of N is socle-finite and computable.

(a3) There exists a minimal integer $m_0 \geq 0$ such that $\mathrm{Ext}^m_C(M,N) = 0$ and $\mathrm{Hom}_C(M, E^N_m) = 0$, for any $m \geq 1 + m_0$.

(b) We denote by $\deg(M,N)$ the minimal integer m_0 with this property, and we call it the **Euler defect degree** of the pair M,N.

Note that, (a1) and (a2) yield $\dim_K \mathrm{Ext}^m_C(M,N)$ is finite, for $m \geq 0$, see [12, p. 52]. Then every computable Euler pair M, N is an Euler pair and $\chi_C(M,N)$ is defined. Note also that each of the pairs M, N over a left Euler coalgebra C studied in (d1), (d2) and (d3) of Theorems 3.4 (d) is a computable Euler pair. In Example 4.10 below, we present a pair of computable comodules M, N that is not an Euler pair and such that $\mathrm{Ext}^m_C(M,N) \cong K$ and $\mathrm{Hom}_C(M, E^N_m) \neq 0$, for any $m \geq 1$.

By adapting [12, Definition 4.12] to any computable Euler pair we introduce the concept of an Euler defect as follows.

Definition 4.2 Let C be a left Euler K-coalgebra, ${}_CF = [\ell_{ij}]_{i,j\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ the composition length matrix (2.1) of C, and ${}_CF^- = [\ell^-_{jp}]_{i,p\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ the matrix (2.2). Assume that M, N is a computable Euler pair of comodules in C–Comod such that the multiplication $(\mathbf{lgth}\, M) \cdot [{}_CF^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}]$ is defined. The **Euler defect** of the pair M, N is defined to be the integer

$$\begin{aligned} \partial_C(M,N) &= (-1)^{m_0+1} \cdot \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, \Omega_{m_0+1}N) \\ &= (-1)^{m_0+1} \cdot (\mathbf{lgth}\, M) \cdot {}_CS \cdot [{}_CF^{-tr} \cdot (\mathbf{lgth}\, \Omega_{m_0+1}N)^{tr}], \end{aligned} \tag{4.3}$$

where ${}_CF^{-tr} = ({}_CF^{-1})^{tr}$, $m_0 = \deg(M,N) \geq 0$ is the Euler defect degree of the pair M, N, and $\Omega_{m_0+1}N = \mathrm{Im} d^N_{m_0+1}$ is the m_0+1-cosyzygy comodule of N.

By the formula (4.7) proved below and the assumption that the multiplication $(\mathbf{lgth}\, M) \cdot [{}_CF^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}]$ is defined, the defect $\partial_C(M,N)$ is a well-defined integer.

Now we generalise [12, Theorem 4.18] as follows.

Theorem 4.4 *Let C be a basic left Euler K-coalgebra, ${}_CF = [\ell_{ij}]_{i,j\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ the composition length matrix* (2.1) *of C, and ${}_CF^- = [\ell^-_{jp}]_{i,p\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ the matrix* (2.2)*. Assume that ${}_CF \cdot {}_CF^- = E$ and M, N is a computable Euler pair of comodules in C–*Comod *such that $(\mathbf{lgth}\, M) \cdot [{}_CF^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}]$ is defined. Let $\partial_C(M,N)$ be the Euler defect* (4.3)*.*

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(a) *If* $m_0 = \deg(M,N) \geq 0$ *is the Euler defect degree of the pair* M, N *then*

$$\begin{aligned} &\widehat{b}_C(\mathbf{lgth}\,M, \mathbf{lgth}\,E^N_{m_0+s}) = 0, \text{ and} \\ &\partial_C(M,N) = (-1)^{m_0+s} \cdot \widehat{b}_C(\mathbf{lgth}\,M, \mathbf{lgth}\,\Omega_{m_0+s}N), \end{aligned} \tag{4.5}$$

for $s = 1,2,3,4,\ldots$.

(b) *The following equality holds*

$$\widehat{b}_C(\mathbf{lgth}\,M, \mathbf{lgth}\,N) = \chi_C(M,N) + \partial_C(M,N). \tag{4.6}$$

(c) *If any of the conditions* (d1)-(d3) *in Theorem* 3.4 *is satisfied, for the comodules* M, N *then* $\partial_C(M,N) = 0$ *and the equality* $\widehat{b}_C(\mathbf{lgth}\,M, \mathbf{lgth}\,N) = \chi_C(M,N)$ *holds.*

Proof: Let M, N be a computable Euler pair of comodules in C–Comod such that $(\mathbf{lgth}\,M) \cdot [{}_C F^{-tr} \cdot (\mathbf{lgth}\,N)^{tr}]$ is defined and let $m_0 = \deg(M,N) \geq 0$ be the Euler defect degree of M, N.

(a) For any $s = 1,2,3,4,\ldots$, we set

$$L_{m_0+s} = \mathbf{lgth}\,\Omega_{m_0+s}N \text{ and } w^{(m_0+s)} = \mathbf{lgth}\,L_{m_0+s} = \mathbf{lgth}\,\Omega_{m_0+s}N.$$

If $w^{(m_0+1)} = 0$, there is nothing to prove. Assume that $w^{(m_0+1)} \neq 0$ and note that the comodules $L_{m_0+1} \subseteq E^N_{m_0+1}$ and $L_{m_0+2} = \mathrm{Im}h^{(m_0+2)} \cong E^N_{m_0+1}/L_{m_0+1}$ are computable. It follows that $\mathbf{lgth}\,E^N_{m_0+1} = w^{(m_0+1)} + w^{(m_0+2)}$ and

$$E^N_{m_0+1} = \bigoplus_{p \in I^N_{m_0}+1} E(p)^{d^N_{m_0+1\,p}} = \bigoplus_{p \in I_C} E(p)^{d^N_{m_0+1\,p}}$$

where $d^N_{m_0+1\,p} \geq 0$ are integers and $I^N_{m_0+1} \subseteq I_C$ is a finite subset of I_C such that $d^N_{m_0+1\,p} = 0$, for any $p \in I_C \setminus I^N_{m_0+1}$. Since the pair M, N is computable then

$$0 = \mathrm{Hom}_C(M, E^N_{m_0+1}) = \bigoplus_{p \in I^N_{m_0+1}} \mathrm{Hom}_C(M, E(p))^{d^N_{m_0+1\,p}}.$$

Consequently,

$$0 = \dim_K E^N_{m_0+1} = \sum_{p \in I_C} d^N_{m_0+1\,p} \cdot \dim_K \mathrm{Hom}_C(M, E(p)) = \sum_{p \in I_C} d^N_{m_0+1\,p} \cdot s_p \cdot \ell_p(M).$$

On the other hand, since we assume that ${}_C F \cdot {}_C F^{-} = E$ then, by Theorem 3.4 (c), we get $\widehat{b}_C(\mathbf{lgth}\,M, \mathbf{lgth}\,E(p)) = s_p \cdot \ell_p(M)$ and

$$\begin{aligned} \widehat{b}_C(\mathbf{lgth}\,M, \mathbf{lgth}\,E^N_{m_0+1}) &= \sum_{p \in I_C} \widehat{b}_C(\mathbf{lgth}\,M, d^N_{m_0+1\,p} \cdot \mathbf{lgth}\,E(p))) \\ &= \sum_{p \in I_C} d^N_{m_0+1\,p} \cdot \widehat{b}_C(\mathbf{lgth}\,M, \mathbf{lgth}\,E(p))) \\ &= \sum_{p \in I_C} d^N_{m_0+1\,p} \cdot s_p \cdot \ell_p(M) = 0. \end{aligned}$$

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It follows that

$$0 = \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, E^N_{m_0+1}) = \widehat{b}_C(\mathbf{lgth}\, M, w^{(m_0+1)}) + \widehat{b}_C(\mathbf{lgth}\, M, w^{(m_0+2)}),$$

that is, $\widehat{b}_C(\mathbf{lgth}\, M, w^{(m_0+1)}) = -\widehat{b}_C(\mathbf{lgth}\, M, w^{(m_0+2)})$. Hence we get

$$\begin{aligned} \partial_C(M,N) &= (-1)^{m_0+1} \cdot \widehat{b}_C(\mathbf{lgth}\, M, w^{(m_0+1)}) \\ &= -(-1)^{m_0+1} \cdot \widehat{b}_C(\mathbf{lgth}\, M, w^{(m_0+2)}) \\ &= (-1)^{m_0+2} \cdot \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, \Omega_{m_0+2} N), \end{aligned}$$

and (a) follows for $s = 2$. The proof for $s \geq 3$ is analogous and we leave it to the reader.

(b) The additivity of the function **lgth** yields

$$\mathbf{lgth}\, N = \sum_{r=0}^{m_0} (-1)^r \cdot \mathbf{lgth}\, E^N_r + (-1)^{m_0+1} \cdot \mathbf{lgth}\, L_{m_0+1}. \tag{4.7}$$

It follows that the integer $(\mathbf{lgth}\, M) \cdot [{}_C F^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}]$ is defined if and only if $(\mathbf{lgth}\, M) \cdot [{}_C F^{-tr} \cdot (\mathbf{lgth}\, \Omega_{m_0+1} N)^{tr}]$ is defined, because $(\mathbf{lgth}\, M) \cdot [{}_C F^{-tr} \cdot (\mathbf{lgth}\, E^N_r)^{tr}]$ is defined, for $r = 0, \ldots, m_0$, by Theorem 3.4 (c) and (d).

By our assumption, M, N is a computable Euler pair. Then $\mathrm{Hom}_C(M, E^N_r) = 0$, for all $r \geq m_0 + 1$, and the complex

$$\begin{aligned} 0 \to \mathrm{Hom}_C(M, E_0) \xrightarrow{\widetilde{h}_1} \mathrm{Hom}_C(M, E_1) \xrightarrow{\widetilde{h}_2} \ldots \\ \longrightarrow \mathrm{Hom}_C(M, E_{m_0-1}) \xrightarrow{\widetilde{h}_{m_0}} \mathrm{Hom}_C(M, E_{m_0}) \to 0 \end{aligned}$$

consists of finite-dimensional K-vector spaces. Since $\mathrm{Ext}^m_C(M,N)$ is the mth homology group of this complex, for any $m \geq 0$, then we have a chain of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_C(M,N) & \longrightarrow & \mathrm{Hom}_C(M, E^N_0) & \longrightarrow & \mathrm{Im}\widetilde{h}^N_1 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathrm{Im}\widetilde{h}^N_1 & \longrightarrow & \mathrm{Ker}\widetilde{h}^N_2 & \longrightarrow & \mathrm{Ext}^1_C(M,N) & \longrightarrow & 0 \\
0 & \longrightarrow & \mathrm{Ker}\widetilde{h}^N_2 & \longrightarrow & \mathrm{Hom}_C(M, E^N_1) & \longrightarrow & \mathrm{Im}\widetilde{h}^N_2 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathrm{Im}\widetilde{h}^N_2 & \longrightarrow & \mathrm{Ker}\widetilde{h}^N_3 & \longrightarrow & \mathrm{Ext}^2_C(M,N) & \longrightarrow & 0 \\
0 & \longrightarrow & \mathrm{Ker}\widetilde{h}^N_3 & \longrightarrow & \mathrm{Hom}_C(M, E^N_2) & \longrightarrow & \mathrm{Im}\widetilde{h}^N_3 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathrm{Im}\widetilde{h}^N_3 & \longrightarrow & \mathrm{Ker}\widetilde{h}^N_4 & \longrightarrow & \mathrm{Ext}^3_C(M,N) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \mathrm{Ker}\widetilde{h}^N_{m_0} & \longrightarrow & \mathrm{Hom}_C(M, E^N_{m_0-1}) & \longrightarrow & \mathrm{Im}\widetilde{h}^N_{m_0} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathrm{Im}\widetilde{h}^N_{m_0} & \longrightarrow & \mathrm{Hom}_C(M, E^N_{m_0}) & \longrightarrow & \mathrm{Ext}^{m_0}_C(M,N) & \longrightarrow & 0
\end{array}$$

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of finite-dimensional K-vector spaces. In view of (4.7) and the equality

$$\dim_K \mathrm{Hom}_C(M, E_r^N) = \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, E_r^N)$$

proved in Theorem 3.4 (d), summing up the dimensions yields

$$\begin{aligned}
\chi_C(M,N) &= \textstyle\sum_{r=0}^{m_0}(-1)^r \cdot \dim_K \mathrm{Hom}_C(M, E_r^N) \\
&= \textstyle\sum_{r=0}^{m_0}(-1)^r \cdot \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, E_r^N) \\
&= \widehat{b}_C(\mathbf{lgth}\, M, \textstyle\sum_{r=0}^{m_0}(-1)^r \cdot \mathbf{lgth}\, E_r^N) \\
&= \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N + (-1)^{m_0} \cdot \mathbf{lgth}\, L_{m_0+1}) \\
&= b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) + (-1)^{m_0} \cdot \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, \Omega_{m_0+1} N) \\
&= b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) - \partial_C(M,N).
\end{aligned}$$

Hence, we get $b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M,N) + \partial_C(M,N)$ and (b) follows.

Since the statement (c) is a consequence of (b) and Theorem 3.4 (d) then the proof is complete. □

We say that the Euler defect $\partial_C(M,N)$ of a computable Euler pair M, N is an **additive function** of the variable M, if $\partial_C(M,N) = \partial_C(M',N) + \partial_C(M'',N)$, for any short exact sequence $0 \to M' \to M \to M'' \to 0$ of computable comodules in C–Comod such that M', N and M'', N are computable Euler pairs. We also say that the Euler defect $\partial_C(M,N)$ is an **additive function** of the variable N, if $\partial_C(M,N) = \partial_C(M,N') + \partial_C(M,N'')$, for any short exact sequence $0 \to N' \to N \to N'' \to 0$ of computable comodules in C–Comod such that M, N' and M, N'' are computable Euler pairs.

Corollary 4.8 *Assume that C is a left Euler coalgebra as in Definition* 4.2 *such that ${}_CF \cdot {}_CF^- = E$. Then the Euler defect $\partial_C(M,N)$ of a computable Euler pair M, N is an additive function of each of the variables M and N.*

Proof: Assume that M, N is a computable Euler pair in C–Comod and $0 \to M' \to M \to M'' \to 0$ is an exact sequence of computable comodules such that M', N and M'', N are computable Euler pairs. It is easy to show that $m_0 := \deg(M,N) \leq \deg(M',N)$ and $\deg(M,N) \leq \deg(M'',N)$. It follows from (4.5) that there exists $s \geq 1$ such that

$$\partial_C(M',N) = (-1)^{m_0+s} \cdot \widehat{b}_C(\mathbf{lgth}\, M', \mathbf{lgth}\, \Omega_{m_0+s} N),$$
$$\partial_C(M'',N) = (-1)^{m_0+s} \cdot \widehat{b}_C(\mathbf{lgth}\, M'', \mathbf{lgth}\, \Omega_{m_0+s} N), \text{ and}$$
$$\partial_C(M,N) = (-1)^{m_0+s} \cdot \widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, \Omega_{m_0+s} N).$$

On the other hand, by definition,

$$(\mathbf{lgth}\, M) \cdot [{}_CF^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}],$$
$$(\mathbf{lgth}\, M') \cdot [{}_CF^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}],$$
$$(\mathbf{lgth}\, M'') \cdot [{}_CF^{-tr} \cdot (\mathbf{lgth}\, N)^{tr}]$$

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are well-defined integers and the equality $\mathbf{lgth}\,M = \mathbf{lgth}\,M' + \mathbf{lgth}\,M''$ holds. It follows from the formula (4.7) that

$(\mathbf{lgth}\,M)\cdot[{}_C F^{-tr}\cdot(\mathbf{lgth}\,\Omega_{m_0+s}N)^{tr}]$,
$(\mathbf{lgth}\,M')\cdot[{}_C F^{-tr}\cdot(\mathbf{lgth}\,\Omega_{m_0+s}N)^{tr}]$,
$(\mathbf{lgth}\,M'')\cdot[{}_C F^{-tr}\cdot(\mathbf{lgth}\,\Omega_{m_0+s}N)^{tr}]$

are defined, because $(\mathbf{lgth}\,M)\cdot[{}_C F^{-tr}\cdot(\mathbf{lgth}\,E_r^N)^{tr}]$, $(\mathbf{lgth}\,M')\cdot[{}_C F^{-tr}\cdot(\mathbf{lgth}\,E_r^N)^{tr}]$, and $(\mathbf{lgth}\,M'')\cdot[{}_C F^{-tr}\cdot(\mathbf{lgth}\,E_r^N)^{tr}]$ are defined, for $r = 0,\ldots,m_0$, see Theorem 3.4 (c) and (d). Consequently, we get

$$\begin{aligned}\partial_C(M,N) &= (-1)^{m_0+s}(\mathbf{lgth}\,M)\cdot[{}_C F^{-tr}\cdot{}_C S\cdot(\mathbf{lgth}\,\Omega_{m_0+s}N)^{tr}]\\ &= (-1)^{m_0+s}(\mathbf{lgth}\,M')\cdot[{}_C F^{-tr}\cdot{}_C S\cdot(\mathbf{lgth}\,\Omega_{m_0+s}N)^{tr}]\\ &\quad +(-1)^{m_0+s}(\mathbf{lgth}\,M'')\cdot[{}_C F^{-tr}\cdot{}_C S\cdot(\mathbf{lgth}\,\Omega_{m_0+s}N)^{tr}]\\ &= \partial_C(M',N)+\partial_C(M'',N).\end{aligned}$$

The additivity of $\partial_C(M,N)$ at N follows in a similar way. □

Remark 4.9 (a) We do not know if there exists a computable Euler pair M, N of indecomposable socle-finite comodules over a left Euler basic K-coalgebra C, with ${}_C F\cdot{}_C F^- = E$, such that the Euler defect $\partial_C(M,N)$ is non-zero. It follows from Theorem 3.4 (d) that if M, N is such a pair then $\dim_K M = \infty$ and $\mathrm{inj.dim}\,N = \infty$.

(b) Note that if M is computable and $\dim_K M$ is infinite then $\ell_p(M) \neq 0$, for infinitely many indices $p \in I_C$, and the sum

$$\begin{aligned}\widehat{b}_C(\mathbf{lgth}\,M,\mathbf{lgth}\,S(j)) &= (\mathbf{lgth}\,M)\cdot{}_C S\cdot[{}_C F^{-tr}\cdot e_j^{tr}]\\ &= (\mathbf{lgth}\,M)\cdot{}_C S\cdot(\ell_j^-)^{tr}\\ &= s_j\cdot\sum_{p\in I_C}\ell_p(M)\cdot\ell_{jp}^-\end{aligned}$$

has no sense, if $\ell_p(M)\cdot\ell_{jp}^-$ is non-zero, for infinitely many indices $p \in I_C$. Obviously, the sum has a sense if and only if the set $\{p \in I_C;\ell_p(M)\cdot\ell_{jp}^-\}$ is finite. In the particular case when $M = E(i)$ is an indecomposable injective comodule, with $i \in I_C$, we have

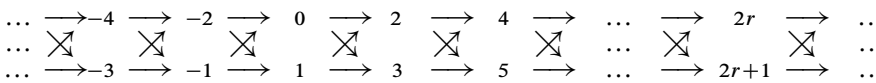
$$\widehat{b}_C(\mathbf{lgth}\,E(i),\mathbf{lgth}\,S(j)) = s_j\cdot\sum_{p\in I_C}\ell_p(E(i))\cdot\ell_{jp}^- = s_j\cdot\sum_{p\in I_C}\ell_{ip}\ell_{jp}^-,$$

and the sum has no sense if and only if $\ell_{ip}\ell_{jp}^-$ is non-zero, for infinitely many indices $p \in I_C$. Note that, if $\dim_K E(i)$ is finite, then Theorem 3.4 (d1) applies and we get the equality $\widehat{b}_C(\mathbf{lgth}\,E(i),\mathbf{lgth}\,S(j)) = \chi_C(E(i),S(j))$.

The following example illustrates the situation of 4.9 (b) and constructs pairs of computable Euler comodules M, N such that the pair M, N is not a computable Euler pair and $\widehat{b}_C(\mathbf{lgth}\,E(j),\mathbf{lgth}\,S(j))$, $\partial_C(M,N)$ are not defined.

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Example 4.10 Let $C = K^{\square}I$ be the incidence coalgebra of the infinite poset I whose Hasse quiver Q_I has the form



It is shown in [12, Example 4.26] that C is a left and right Euler coalgebra that is left and right locally artinian, see [5], and C is neither left nor right semiperfect. Moreover, the matrices ${}_CF = \mathfrak{c}_I^{tr}, {}_CF^- = \mathfrak{c}_I^{-tr} \in \mathbb{M}_I(\mathbb{Z}) = \mathbb{M}_{\mathbb{Z}}(\mathbb{Z})$ are constructed in [12, p. 65]. It is easy to see that ${}_CF^-$ is a unique left inverse and a right inverse of ${}_CF$.

Note that C is basic, $\dim_K C = \aleph_0$, $\operatorname{soc}_C C = \bigoplus_{p \in I_C} S(p)$, where $I_C = \mathbb{Z}$ is the set of integers, and there is a group isomorphism $K_0(C) \cong \mathbb{Z}^{(I_C)} = \mathbb{Z}^{(\mathbb{Z})}$.

It is clear that the left comodules $M = E(0)$ and $N = S(0)$ are computable Euler comodules. It follows from the shape of the matrices ${}_CF^{tr} = \mathfrak{c}_I, {}_CF^{-tr} = \mathfrak{c}_I^{-1} \in \mathbb{M}_I(\mathbb{Z})$ constructed in [12, Example 4.26] that $\mathbf{lgth}\, E(0)$ is the 0th column of $\mathfrak{c}_I$ and ${}_CF^{-tr} \cdot e_0 = (e_0 \cdot {}_CF^-)^{tr}$ is the 0th column of $\mathfrak{c}_I^{-1}$, that is,

$$\mathbf{lgth}\, E(0) = (\ldots,1,1,1,\ 1,1,1,\ 1,\ 1,1,0,0,0,0,\ldots) \text{ and}$$
$$(e_0 \cdot {}_CF^-)^{tr} = (\ldots,1,1,-1,-1,1,1,-1,-1,1,0,0,0,0,\ldots).$$

It follows that $\widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = (\mathbf{lgth}\, E(0)) \cdot (e_0 \cdot {}_CF^-)^{tr}$ is not defined. Moreover, using the minimal injective resolution of the simple C-comodule $S(0)$ constructed in [12, Example 4.26], we show that $\operatorname{Hom}_C(E(0), S(0)) = 0$, $\operatorname{Hom}_C(E(0), E_m^{S(0)}) \neq 0$, and $\operatorname{Ext}_C^m(E(0), S(0)) \cong K$, for any $m \geq 1$. It follows that the Euler characteristic $\chi_C(E(0), S(0))$ is not defined, $M = E(0)$, $N = S(0)$ is not a computable Euler pair in the sense of Definition 4.1, and the Euler defect $\partial_C(M,N)$ of the pair $M = E(0)$, $N = S(0)$ is not defined.

Remark 4.11 The results of the paper are applied in [13] and [15] to comodules over the incidence coalgebras $K^{\square}I$ of intervally finite posets I. They are also used in the study of the Coxeter transformation and almost split sequences for comodules, see [4].

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