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Entropy, minimal surfaces and negatively curved manifolds

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Abstract. Taubes [Minimal surfaces in germs of hyperbolic 3-manifolds. *Proceedings of the Casson Fest, Geom. Topol. Monogr.* **7** (2004), 69–100 (electronic)] introduced the space of minimal hyperbolic germs with elements consisting of the first and second fundamental form of an equivariant immersed minimal disk in hyperbolic 3-space. Herein, we initiate a further study of this space by studying the behavior of a dynamically defined function which records the entropy of the geodesic flow on the associated Riemannian surface. We provide a useful estimate on this function which, in particular, yields a new proof of Bowen's theorem on the rigidity of the Hausdorff dimension of the limit set of quasi-Fuchsian groups. These follow from new lower bounds on the Hausdorff dimension of the limit set which allow us to give a quantitative version of Bowen's rigidity theorem. To demonstrate the strength of the techniques, these results are generalized to convex-cocompact surface groups acting on n-dimensional CAT (-1) Riemannian manifolds.

1. *Introduction*

Given a convex-cocompact hyperbolic 3-manifold M and a π_1-injective mapping $f : \Sigma \to M$ of a closed surface Σ into M, the general existence and regularity theory developed by Meeks *et al* [**MSY82**], Sacks and Uhlenbeck [**SU82**], Freedman *et al* [**FHS83**], and Osserman and Gulliver [**Gul77**] furnishes the existence of an immersed minimal surface $\Sigma \to M$ in the homotopy class of f which minimizes area among all maps in the homotopy class. Motivated by this proliferation of closed minimal surfaces in hyperbolic 3-manifolds, Taubes [**Tau04**] constructed the space of minimal hyperbolic germs $\mathcal{H}$ which is a deformation space whose typical element consists of a Riemannian metric and symmetric 2-tensor (g, B), which together are the induced metric and second fundamental form of a minimal immersion of Σ into a potentially incomplete hyperbolic 3-manifold. The present paper is the beginning of a deeper investigation of the space $\mathcal{H}$ and its relationship to the PSL$(2, \mathbb{C})$-character variety $\chi(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{C}))$.

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We begin §3 with a study of a dynamically defined function on the space of minimal hyperbolic germs $\mathcal{H}$. A pair $(g, B) \in \mathcal{H}$ satisfies a trio of equations, the most important of which for us is the *Gauss* equation:

$$K_g = -1 - \tfrac{1}{2}\|B\|_g^2.$$

Here, K_g is the sectional curvature of the metric g. In particular, all elements of the space $\mathcal{H}$ are Riemannian surfaces whose sectional curvature is bounded above by -1. We define a function,

$$E : \mathcal{H} \to \mathbb{R},$$

which records the topological entropy of the geodesic flow on the unit tangent bundle of the Riemannian surface (Σ, g). Our analysis shows that E is greater than or equal to one, and the subspace along which it is equal to one is precisely the Fuchsian space $\mathcal{F}$ of constant negative curvature metrics on Σ. The (non-Riemannian) Hessian of E along $\mathcal{F}$ yields a well-defined non-negative symmetric bilinear form on the tangent space to $\mathcal{F}$ that is shown to be bounded below by the Weil–Petersson metric. Hence, the Hessian of E yields a metric on $\mathcal{F}$ that is invariant under the mapping class group of Σ. We also identify an open subset of minimal hyperbolic germs along which E has no critical points; the almost-Fuchsian germs $\mathcal{AF} \subset \mathcal{H}$ defined by the condition that $\|B\|_g^2 < 2$. These results should be compared to the analogous study of the behavior of the Hausdorff dimension of the limit set defined on the space of quasi-Fuchsian representations, see [**BT08**, **Bri10**, **McM08**]. Indeed, general theory identifies the function E as recording the Hausdorff dimension of the Gromov boundary $\partial_\infty(\widetilde{\Sigma})$ computed in the Gromov metric.

Next, in §§4 and 5 we turn to a study of the interaction between the intrinsic dynamics induced on Σ by a pair $(g, B) \in \mathcal{H}$ with the dynamics of a hyperbolic 3-manifold M in which (Σ, g) appears as an isometrically immersed minimal surface with second fundamental form B. The central result is a new lower bound on the Hausdorff dimension of the limit set of a quasi-Fuchsian group.

THEOREM. *Let Γ be a quasi-Fuchsian group and $\Sigma \to \Gamma\backslash\mathbb{H}^3$ a π_1-injective minimal surface with induced metric g and second fundamental form B. Then,*

$$\frac{1}{\mathrm{Vol}(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g \leq \mathrm{H} \cdot \dim(\Lambda_\Gamma),$$

where dV_g is the volume form of g and $\mathrm{H} \cdot \dim(\Lambda_\Gamma)$ is the Hausdorff dimension of the limit set of Γ. Furthermore, equality holds if and only if Γ is Fuchsian.

Setting $\mathrm{H} \cdot \dim(\Lambda_\Gamma) = 1$ in the above theorem gives a new proof of Bowen's theorem on the Hausdorff dimension of quasi-circles [**Bow79**].

COROLLARY. *A quasi-Fuchsian group Γ is Fuchsian if and only if $\mathrm{H} \cdot \dim(\Lambda_\Gamma) = 1$.*

Furthermore, the techniques we use allow a quantification of Bowen's theorem which states that if the Hausdorff dimension of the limit set of a quasi-Fuchsian group is very near one, under the additional hypothesis that the quasi-Fuchsian group admits a stable, incompressible minimal surface with a lower bound on the injectivity radius, then the group in question is very close to being Fuchsian. In particular, we obtain estimates on

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geometric quantities of the quotient 3-manifold such as the convex core diameter and volume, the Teichmüller distance between the conformal boundaries, and the principal curvatures of the stable minimal surface. All of these quantities go to zero in a controlled way as the Hausdorff dimension of the limit set goes to one. In particular, we obtain the conclusion that in this setting the stable, incompressible minimal surface is unique.

In §6 we further explore the relationship between $\mathcal{H}$ and the space of quasi-Fuchsian representations. In particular, it has been known since Uhlenbeck [**Uhl83**] that the subset $\mathcal{AF} \subset \mathcal{H}$ of almost-Fuchsian germs defined by the condition $\|B\|_g^2 < 2$ corresponds to an open subset of quasi-Fuchsian representations; on this subset we show that the action $(g, B) \to (g, -B)$ corresponds to switching the conformal boundary components in the Bers' simultaneous uniformation parameterization of those quasi-Fuchsian manifolds. We mention that the failure of this action to globally extend in this fashion comes from critical points of the *monodromy* map,

$$\Phi : \mathcal{H} \to \chi(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{C})).$$

These critical points were characterized by Taubes [**Tau04**] as corresponding to minimal surfaces admitting non-trivial deformations that preserve area to second order, known as Jacobi fields. Hence, the failure of geometric phenomena to transfer between $\mathcal{H}$ and $\chi(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{C}))$ is governed by the same mechanism that leads to bifurcations in the minimal surface problem: this fascinating behavior is not well understood and only a few cases have been analyzed; we mention [**BT84**] and [**JLPP95**]. We also observe, as a simple consequence of Sard's theorem, that the set of representations $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{C})$ admitting equivariant minimal immersions of a disk which have a non-trivial equivariant Jacobi field are measure zero in the character variety $\chi(\pi_1(\Sigma), \mathrm{Psl}(2, \mathbb{C}))$. This raises basic questions concerning the structure of the critical values of Φ. How singular are they as subsets of the character variety? What are the components of the complement of all critical values?

Finally, in §7 we extend the results in §5 from quasi-Fuchsian groups acting on $\mathbb{H}^3$ to convex-cocompact surface groups acting isometrically on any n-dimensional CAT(-1) Riemannian manifold. The generalization of Bowen's theorem in this setting is due to Bonk and Kleiner [**BK04**], whose theorem is considerably more general than ours because it applies to general quasi-convex, cocompact group actions on any CAT(-1) metric space. That being said, our approach also yields a new lower bound on the Hausdorff dimension of the limit set in the n-dimensional CAT(-1) Riemannian setting; our methods are less general but provide more information. Perhaps the work of Mese on minimal surfaces in Alexandrov spaces [**Mes01**] could be utilized to extend the techniques here to surface group actions on more general metric spaces, although we have not considered that here. In the course of this section, we show that given a convex-cocompact representation $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$ into the isometry group of a CAT(-1) Riemannian manifold X, any equivariant branched minimal immersion is without ramified branch points; in particular, no such branched immersion has false branched points where the immersion is a branched covering onto its image. This is a straightforward application of a previous result of Gulliver and Tomi [**GT89**].

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1.1. *Plan of paper.* Section 2 contains preliminary information regarding minimal surfaces and the geometry and dynamics of groups acting on CAT(-1) spaces. In §3 we study the dynamically defined entropy function on the space $\mathcal{H}$ of minimal hyperbolic germs and prove the aforementioned properties. Next, §4 begins the investigation of the relationship between minimal surfaces and the geometry of hyperbolic 3-manifolds; this culminates in §5 with a new proof of Bowen's theorem on Hausdorff dimension of quasi-circles. In §6 we prove that the action induced by switching conformal boundary components in the simultaneous uniformization parameterization of quasi-Fuchsian space can be seen, at least for some special quasi-Fuchsian manifolds, in terms of an associated $\mathbb{Z}/(2\mathbb{Z})$-action on minimal hyperbolic germs. Finally, in §7 we provide the generalization of the results in §5 to the n-dimensional CAT(-1) Riemannian setting. In the appendix, various technical results needed throughout the paper are collected.

2. *Preliminaries*

Throughout, Σ denotes a smooth, closed, connected, oriented surface of genus greater than one. The universal cover of Σ is denoted $\widetilde{\Sigma}$. Whenever Σ is endowed with a Riemannian metric, we equip $\widetilde{\Sigma}$ with the pullback metric so that the covering projection is a local isometry.

Hyperbolic 3-space $\mathbb{H}^3$ is the unique three-dimensional, 1-connected, complete Riemannian manifold of sectional curvature -1. Given an immersion

$$f : \widetilde{\Sigma} \to \mathbb{H}^3,$$

let N be a locally defined unit normal vector field. Given coordinate vector fields ∂_1 and ∂_2, the *second fundamental form* of the immersion f is the symmetric 2-tensor on $\widetilde{\Sigma}$ defined by

$$B_{ij} = B(\partial_i, \partial_j) = \langle \nabla_{df(\partial_i)} N, df(\partial_j) \rangle,$$

where ∇ denotes the Levi-Civita covariant derivative associated to the Riemannian metric $\langle -, - \rangle$ on $\mathbb{H}^3$. As B is a symmetric, covariant 2-tensor, at each point it has a pair of real eigenvalues λ_1 and λ_2: these are the *principal curvatures* of the immersion. The *mean curvature* H is the trace of B: in terms of the principal curvatures,

$$H = \lambda_1 + \lambda_2.$$

If g denotes the pullback of the metric on $\mathbb{H}^3$ via the immersion f, then the equations of Gauss and Codazzi relate g and B via

$$K_g = -1 + \lambda_1 \lambda_2, \tag{2.1}$$

$$(\nabla_{\partial_i} B)_{jk} - (\nabla_{\partial_j} B)_{ik} = 0. \tag{2.2}$$

Above, K_g denotes the sectional curvature of the metric g and ∇ its Levi-Civita covariant derivative.

The immersion $f : \widetilde{\Sigma} \to \mathbb{H}^3$ is a *minimal surface* if the mean curvature vanishes identically, $H = 0$. This condition is equivalent to the immersion f being critical for area with respect to all compactly supported variations. The following existence theorem combines the work of Meeks *et al* [**MSY82**] and Gulliver [**Gul77**] (see also [**SU82**, **FHS83**]).

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THEOREM 2.1. *Let (M, g) be a compact three-dimensional Riemannian manifold with $\pi_2(M) = 0$. If M has boundary, then assume that ∂M is mean convex. Then, given $f : \Sigma \to M$ such that $f_* : \pi_1(\Sigma) \to \pi_1(M)$ is injective, there exists an area minimizing immersion $g : \Sigma \to M$ in the homotopy class of f.*

Remark. The condition that ∂M is mean convex means that any deformation of ∂M in the outward normal direction is area non-decreasing. The hypothesis also allows that the boundary be non-smooth, provided it satisfies a natural convexity condition (see [**MY82**] for details). In particular, boundaries of open convex sets satisfy this condition.

Now we can introduce the space of minimal hyperbolic germs constructed by Taubes.

Let $(g, B) \in \Gamma(S^2_{>0}T^*\Sigma) \oplus \Gamma(S^2T^*\Sigma)$ be a pair consisting of a Riemannian metric and symmetric 2-tensor on Σ. Such a pair is called a *minimal hyperbolic germ* if B is traceless with respect to g and the Gauss–Codazzi equations (2.1) and (2.2) are satisfied. Letting $\mathrm{Diff}_0(\Sigma)$ be the space of orientation preserving diffeomophisms of Σ isotopic to the identity, the space $\mathcal{H}$ of minimal hyperbolic germs is the quotient,

$$\mathcal{H} = \{\text{minimal hyperbolic germs}\}/\ \mathrm{Diff}_0(\Sigma),$$

with $\mathrm{Diff}_0(\Sigma)$ acting by pullback on the pair of tensors (g, B). By abuse of notation, we shall write a pair $(g, B) \in \mathcal{H}$ to mean that the orbit of the pair belongs to $\mathcal{H}$. The following fundamental theorem of surface theory [**Uhl83**] shows that every element $(g, B) \in \mathcal{H}$ can be integrated to an immersed minimal disk in $\mathbb{H}^3$ with first and second fundamental form (g, B).

THEOREM 2.2. *Let $(g, B) \in \mathcal{H}$. Then there exists an immersion $f : \widetilde{\Sigma} \to \mathbb{H}^3$ whose induced metric and second fundamental form coincide with the lifts of g and B to $\widetilde{\Sigma}$. Furthermore, if $O \in \mathbb{H}^3$ is chosen along with a preferred oriented orthonormal frame $\{F_1, F_2, N\} \subset T_O\mathbb{H}^3$, then the map f is uniquely determined by fixing $p \in \widetilde{\Sigma}$ and an oriented orthonormal frame $\{E_1, E_2\} \subset T_p\widetilde{\Sigma}$ and requiring that:*

- $f(p) = O$,
- $df(E_i) = F_i$.

The above theorem justifies calling $\mathcal{H}$ the space of minimal hyperbolic germs.

The following fundamental theorem is due to Taubes.

THEOREM 2.3. [**Tau04**] *The space of minimal hyperbolic germs $\mathcal{H}$ is a smooth, oriented manifold of dimension $12g - 12$ where g is the genus of Σ.*

The Teichmüller space $\mathcal{T}$ is the space of isotopy classes of complex structures agreeing with the orientation of Σ, which by the Köebe uniformization theorem can also be described as the space of isotopy classes of Riemannian metrics of constant curvature -1. For the sake of context, the space of isotopy classes of metrics of constant curvature -1 will be called the Fuchsian space, denoted by $\mathcal{F}$. As such, the Fuchsian space includes into $\mathcal{H}$ via the map

$$\begin{aligned} \mathcal{F} &\longrightarrow \mathcal{H} \\ g &\mapsto (g, 0). \end{aligned}$$

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Given a complex structure $\sigma \in \mathcal{T}$, Kodaira–Spencer deformation theory identifies the fiber of the holomorphic cotangent bundle over σ as the space of holomorphic quadratic differential $\alpha = \alpha(z)\, dz^2$ on the Riemann surface (Σ, σ).

The space $\mathcal{H}$ admits an important map to $T^*\mathcal{T}$: given $(g, B) \in \mathcal{H}$, let $[g] \in \mathcal{T}$ denote the conformal structure induced by the Riemannian metric g. If (x_1, x_2) are local, isothermal coordinates for the metric g, Hopf observed in [**Hop54**] that the Codazzi equations along with the fact that B is trace free imply that the expression

$$\alpha(g, B) = (B_{11} - iB_{12})(x_1, x_2)\, dz^2$$

defines a holomorphic quadratic differential on $(\Sigma, [g])$ (where $z = x_1 + ix_2$) that satisfies $\mathrm{Re}(\alpha) = B$. This assignment defines a smooth map,

$$\begin{aligned} \Psi : \mathcal{H} &\longrightarrow T^*\mathcal{T} \\ (g, B) &\mapsto ([g], \alpha). \end{aligned}$$

We will repeatedly use the following important fact: Suppose α and α' are two holomorphic quadratic differentials on a fixed Riemann surface (Σ, σ) that are in the image of the map Ψ. If $\mathrm{Re}(\alpha) = B$ and $\mathrm{Re}(\alpha') = B'$ and $(g, B), (g', B') \in \mathcal{H}$ are the corresponding elements of $\mathcal{H}$, then the metrics g and g' are conformal to one another and belong to the conformal class defined by the complex structure σ. In particular, paths of the form $(g_t, tB) \subset \mathcal{H}$ satisfy the property that the conformal class of g_t is constant. Among minimal germs $(g, B) \in \mathcal{H}$ such that the conformal class of g is fixed, the above fact reduces the the Gauss equation to a scalar equation, which greatly simplifies the analysis.

The natural action of the circle on $T^*\mathcal{T}$ induces an action of $U(1)$ on $\mathcal{H}$ making the above mapping equivariant; under this action the metric g is left completely unchanged. Hence, the $U(1)$-orbit $(g, \theta \cdot B)$ of minimal hyperbolic germs is a 1-parameter family of isometric minimal surfaces. Furthermore, the action is free on the complement of the locus of pairs of the form $(g, 0) \in \mathcal{H}$.

2.1. *Almost-Fuchsian germs.* A minimal germ $(g, B) \in \mathcal{H}$ is called almost-Fuchsian if $\|B\|_g^2 < 2$. We denote the set of almost-Fuchsian germs by $\mathcal{AF}$. These minimal germs directly correspond to hyperbolic 3-manifolds via the following theorem due to Uhlenbeck.

THEOREM 2.4. [**Uhl83**] *Let $(g, B) \in \mathcal{AF}$. Then the metric*

$$G(g, B) = dt^2 + g(\cosh(t)\mathbb{I}(-) + \sinh(t)\mathbb{S}(-), \cosh(t)\mathbb{I}(-) + \sinh(t)\mathbb{S}(-))$$

is a complete hyperbolic metric on $\Sigma \times \mathbb{R}$ where $\mathbb{S}$ is the $(1, 1)$ tensor associated to B by raising an index and $\mathbb{I}$ is the identity. Furthermore:

- *G(g,B) is quasi-isometric to a Fuchsian metric. Hence, it is quasi-Fuchsian.*
- *The slice $\Sigma \times \{0\}$ is an embedded, least area minimal surface in $\Sigma \times \mathbb{R}$ with induced metric and second fundamental form (g, B). This is the only closed minimal surface of any kind in $(\Sigma \times \mathbb{R}, G)$.*

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2.2. *Limit sets of discrete groups and* CAT(−1) *spaces.* In this section we will introduce the necessary ingredients from the theory developed by Patterson [**Pat76**], Sullivan [**Sul84**], Bourdon [**Bou95**], and Coornaert [**Coo93**].

Let (X, d) be a proper, CAT(−1) metric space (for our needs we may assume this to be a 1-connected, complete Riemannian manifold of sectional curvature ≤ -1). Given $p \in X$, the geometric (or visual) boundary $\partial_{p,\infty}(X)$ of X is the space of equivalence classes of geodesic rays based at X. Two rays $\gamma, \eta : [0, \infty) \to X$ are equivalent if there exists $K > 0$ such that $d(\gamma(t), \eta(t)) < K$ for all t. The *Gromov product* at p is defined by

$$(x, y)_p = \tfrac{1}{2}(d(x, p) + d(y, p) - d(x, y)).$$

In a CAT(−1) space, this product extends to the geometric boundary via

$$(\eta, \gamma)_p = \lim_{t\to\infty} (\eta(t), \gamma(t))_p$$

for $\eta, \gamma \in \partial_{p,\infty}(X)$. Using the Gromov product, we define the Gromov metric on the geometric boundary by

$$d_p(\eta, \gamma) = \begin{cases} e^{-(\eta,\gamma)_p} & \eta \neq \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

As $p \in X$ varies, the Gromov metrics are all bi-Lipschitz equivalent to one another.

Suppose $\Gamma < \mathrm{Isom}(X)$ is a discrete, convex-cocompact subgroup; this means there is a geodesically convex, Γ-invariant subset of X upon which Γ acts cocompactly. Define the orbit counting function associated to Γ by

$$N_\Gamma(x, R) = |\{\gamma \in \Gamma \mid d(x, \gamma(x)) < R\}|.$$

Then the volume entropy of Γ is

$$\delta(\Gamma) = \lim_{R\to\infty} \frac{\log(N_\Gamma(R, x))}{R}.$$

This number is independent of $x \in X$ and is a measure of the dynamical complexity of the group Γ. The *limit set* Λ_Γ of Γ is the set of accumulation points of Γ-orbits of a fixed point $x \in X$ in $\partial_\infty(X)$. Equivalently, Λ_Γ is the smallest non-empty, closed Γ-invariant subset of $\partial_\infty(X)$. We will use the following theorem (see [**Coo93**]).

THEOREM 2.5. *Let $\Gamma < \mathrm{Isom}(X)$ be a discrete, convex-cocompact subgroup of isometries of a proper,* CAT(−1) *metric space X. Then*

$$\delta(\Gamma) = \mathrm{H} \cdot \dim(\Lambda_\Gamma).$$

Here, the Hausdorff dimension is computed using any of the Gromov metrics on $\partial_\infty(X)$.

3. *Entropy*

In this section, we define the entropy function on the space of minimal hyperbolic germs and discuss some interesting properties.

Given $(g, B) \in \mathcal{H}$, the Gauss equation reads:

$$K_g = -1 - \tfrac{1}{2}\|B\|_g^2.$$

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Thus, every minimal hyperbolic germ refines the structure of a closed Riemannian surface with sectional curvature bounded above by -1. We remark that, by basic comparison geometry, such Riemannian surfaces are proper, locally CAT(-1) metric spaces, and the theory described in §2.2 applies to their universal covers.

Definition 3.1. Given $(g, B) \in \mathcal{H}$, let $B_g(p, R)$ be the metric ball in the universal cover $\widetilde{\Sigma}$ of radius R centered at a basepoint $p \in \widetilde{\Sigma}$. Define the volume entropy as the quantity

$$E(g, B) = \limsup_{R\to\infty} \frac{\log|B_g(p, r)|}{R},$$

where $|B_g(p, R)|$ is the Riemannian volume of the ball centered at p of radius R.

Manning introduced this quantity in [**Man79**] and showed the limit exists and is independent of basepoint. Furthermore, in the case where the manifold in question has negative curvature, Manning showed that this quantity equals the topological entropy of the geodesic flow defined on the unit tangent bundle. Katok *et al* [**KKW91**] show that given a C^∞-perturbation of a metric of negative curvature, the topological entropy of the geodesic flow also varies smoothly. Hence, we have the following proposition.

PROPOSITION 3.2. *The volume entropy*

$$E : \mathcal{H} \to \mathbb{R}$$

is a smooth, non-negative function on the space of minimal hyperbolic germs. This function equals the topological entropy h_{top} of the geodesic flow on the unit tangent bundle.

We will simply refer to this function as the *entropy* of the minimal hyperbolic germ.

We begin with an important lower bound on the entropy.

THEOREM 3.3. *The entropy satisfies*

$$E(g, B) \geq \frac{1}{\mathrm{Vol}(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g,$$

and $E(g, B) = 1$ if and only if $B = 0$. Furthermore, equality is achieved if and only if $E(g, B) = 1$.

Before beginning the proof, we introduce an estimate of Manning which easily yields the theorem. Let (Σ, g) be a Riemannian surface with strictly negative sectional curvature. On the unit tangent bundle of Σ, the (normalized) *Liouville measure* m_L is a probability measure invariant under the geodesic flow. In a local trivialization, m_L is a constant multiple of the product of Riemannian volume on Σ with the standard angle measure on the circle giving it total measure 2π. The measure-theoretic entropy of a metric of constant sectional curvature -1 with respect to Liouville measure equals $\sqrt{-2\pi\chi(\Sigma)}$.

THEOREM 3.4. [**Man81**] *Let (Σ, g) be a Riemannian surface of negative curvature. Then*

$$\frac{1}{\sqrt{\mathrm{Vol}(g)}} \int_\Sigma \sqrt{-K_g}\, dV_g \leq h(m_L),$$

where $h(m_L)$ is the measure-theoretic entropy of the geodesic flow with respect to Liouville measure.

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Remark. Note that if $K = -1$ in the above formula, the inequality becomes equality: $h(m_L) = \sqrt{\mathrm{Vol}(g)} = \sqrt{-2\pi\chi(\Sigma)}$.

We now give the proof of Theorem 3.3.

Proof. Let (g, B) be a minimal hyperbolic germ. By the Gauss equation,

$$K_g = -1 - \tfrac{1}{2}\|B\|_g^2.$$

Applying Theorem 3.4 and inserting the above expression for K_g, we obtain the inequality

$$\frac{1}{\sqrt{\mathrm{Vol}(g)}} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g \leq h(m_L), \tag{3.1}$$

where $h(m_L)$ is the measure-theoretic entropy of the geodesic flow with respect to Liouville measure for the metric g. By the variational principle (see [**KH95**]),

$$h(m_L) \leq h_{top}\left(\frac{1}{\mathrm{Vol}(g)} g\right),$$

where $h_{top}((1/\mathrm{Vol}(g))g)$ is the topological entropy of the geodesic flow for the normalized Riemannian metric $(1/\mathrm{Vol}(g))g$. But, since g has negative curvature,

$$E\left(\frac{1}{\mathrm{Vol}(g)} g,\, B\right) = h_{top}\left(\frac{1}{\mathrm{Vol}(g)} g\right)$$

by [**Man79**]. Furthermore, the entropy scales via

$$E\left(\frac{1}{\mathrm{Vol}(g)} g,\, B\right) = \sqrt{\mathrm{Vol}(g)}\, E(g, B).$$

Returning to line (3.1), the previous lines imply

$$\begin{aligned}
\frac{1}{\sqrt{\mathrm{Vol}(g)}} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g &\leq h(m_L) \\
&\leq h_{top}\left(\frac{1}{\mathrm{Vol}(g)} g\right) \\
&= E\left(\frac{1}{\mathrm{Vol}(g)} g,\, B\right) \\
&= \sqrt{\mathrm{Vol}(g)}\, E(g, B).
\end{aligned}$$

Dividing by $\sqrt{\mathrm{Vol}(g)}$ proves the inequality asserted in Theorem 3.3.

If $E(g, B) = 1$, then

$$\frac{1}{\mathrm{Vol}(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g \leq 1.$$

This implies $\|B\|_g^2 = 0$.

For the other direction, if $\|B\|_g^2 = 0$, then the surface has constant sectional curvature -1 and one may compute directly that $E(g, B) = 1$. In this case, the volume of a ball of radius R in the universal cover is asymptotically e^R.

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For the final statement, we invoke a deep theorem of Katok [**Kat82**]. For a closed surface of genus greater than one, equality holds in

$$h(m_L) \leq h_{top}\left(\frac{1}{\mathrm{Vol}(g)}g\right)$$

if and only if the metric is constant negative curvature. This completes the proof. □

Next, we show that critical points of the restriction of entropy to the space of almost-Fuchsian germs occur precisely at the Fuchsian germs.

THEOREM 3.5. *Consider the restriction of the entropy to the space of almost-Fuchsian hyperbolic germs:*

$$E : \mathcal{AF} \to \mathbb{R}.$$

This function is critical at (g, B) *if and only if* $B = 0$*, hence if and only if the germ is Fuchsian. In particular, the entropy is strictly monotone increasing along rays* $(g_t, tB) \in \mathcal{AF}$ *for* $t > 0$ *provided* $\|tB\|^2_{g_t} < 2$.

Before we prove the theorem, we need to introduce a very useful formula due to Katok *et al* [**KKW91**] for the first variation of the topological entropy of the geodesic flow on a manifold with negative curvature. Throughout the rest of this section, dots over a function dependent on a single real parameter $t \in \mathbb{R}$ represent successive derivatives with respect to t.

THEOREM 3.6. [**KKW91**] *Let* g_t *be a smooth path of negatively curved Riemannian metrics on a closed manifold* M*. If* $h_{top}(g_t)$ *is the topological entropy of the geodesic flow on* $T^1(M)$ *for the metric* g_t*, then*

$$\frac{d}{dt}h_{top}(g_t)\bigg|_{t=0} = -\frac{h_{top}(g_0)}{2}\int_{T^1(M)}\frac{d}{dt}g_t(v, v)\bigg|_{t=0} d\mu_0.$$

Here, μ_0 *is the Bowen–Margulis measure of maximal entropy for the geodesic flow arising from the metric* g_0.

Remark. Since we will use none of its properties, we will not define the Bowen–Margulis measure. Details about its properties and construction can be found in [**Mar04**], although a considerably easier construction mirroring [**Pat76**] can be used for negatively curved Riemannian manifolds.

Proof of Theorem 3.5. We already know from Theorem 3.3 that the entropy function is critical at Fuchsian hyperbolic germs. Observe that the pair $(g_t, tB) \in \mathcal{AF}$ satisfies

$$\Psi : \mathcal{H} \to T^*\mathcal{T}$$
$$(g_t, tB) \mapsto ([g_t], t\alpha).$$

Thus, the conformal class of the metric g_t is constant. Therefore, if h is the hyperbolic metric conformal to g_t, there exists a path of smooth functions $u_t : \Sigma \to \mathbb{R}$ such that $g_t = e^{2u_t}h$. We have two expressions for the sectional curvature of $g_t = e^{2u_t}h$:

$$-1 - \frac{t^2}{2}e^{-4u_t}\|B\|^2_h = K_{g_t} = e^{-2u_t}(-\Delta_h u_t - 1), \tag{3.2}$$

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where Δ_h is the Laplace–Beltrami operator associated to the metric h. For ease of calculation, we rewrite (3.2) as

$$e^{2u_t} + \frac{t^2}{2} e^{-2u_t} \|B\|_h^2 = \Delta_h u_t + 1.$$

Taking the time derivative with respect to t reveals

$$2\dot{u}_t e^{2u_t} + te^{-2u_t} \|B\|_h^2 - t^2 \dot{u}_t e^{-2u_t} \|B\|_h^2 = \Delta_h \dot{u}_t.$$

Rewriting the left-hand side and evaluating at $t = t_0$,

$$\dot{u}_t e^{2u_{t_0}} (2 - t_0^2 e^{-4u_{t_0}} \|B\|_h^2) + t_0 e^{-2u_{t_0}} \|B\|_h^2 = \Delta_h \dot{u}_t. \tag{3.3}$$

At a maximum, $\Delta_h \dot{u}_t \leq 0$, which implies that the left-hand side of (3.3) is non-positive. Applying the hypothesis

$$e^{-4u_{t_0}} t_0^2 \|B\|_h^2 = \|t_0 B\|_{g_{t_0}}^2 < 2$$

implies that $\dot{u}_t \leq 0$. Futhermore, if $\dot{u}_t = 0$ everywhere then equation (3.3) implies that $B = 0$. Thus, the tensor

$$\frac{d}{dt} g_t \Big|_{t=t_0} = 2\dot{u}_t g_{t_0}$$

is negative semi-definite and vanishes identically if and only if $t = 0$. Applying Theorem 3.6,

$$\frac{d}{dt} E(g_t, tB) \Big|_{t=t_0} = \frac{d}{dt} h_{top}(g_t) \Big|_{t=t_0} = -\frac{h_{top}(g_{t_0})}{2} \int_{T^1(M)} 2\dot{u}_t d\mu_{t_0} \geq 0,$$

with equality if and only if $t_0 = 0$. Hence, the entropy is strictly monotone increasing along the ray (g_t, tB) for $t > 0$, which completes the proof. □

Now we show that the entropy function yields a metric on Teichmüller space $\mathcal{T}$ whose norm is bounded below by the Weil–Petersson norm. Recall that given a point $\sigma \in \mathcal{T}$, the cotangent space to $\mathcal{T}$ at σ is identified, via Kodaira–Spencer deformation theory (see [**Kod05**]), with the space of holomorphic quadratic differentials on the Riemann surface (Σ, σ). The uniformization theorem furnishes a unique hyperbolic metric h_σ in the conformal class of metrics defined by σ. Given two holomorphic quadratic differentials α and β, the Weil–Petersson Hermitian pairing is defined by

$$\langle \alpha, \beta \rangle_{WP} = \int_\Sigma \frac{\alpha \overline{\beta}}{h_\sigma}.$$

This defines a Kähler metric on the Teichmüller space whose geometry has been intensely studied (for a nice survey see [**Wol10**]). A number of geometrically defined potential functions for the Weil–Petersson metric have been found; it seems probable, although we have not found a proof, that the entropy function defined here is yet another potential. Before we prove this theorem, we need to describe a key formula due to Pollicott [**Pol94**] from which the theorem will follow easily.

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THEOREM 3.7. [**Pol94**] *Let g_t be a smooth path of Riemannian metrics of negative curvature on a closed manifold M. Then,*

$$\frac{d^2}{dt^2}h_{top}(g_t)\Big|_{t=0} \geq h_{top}(g_0)\left(\mathrm{Var}\left(\frac{\dot{g}(v,v)}{2}\right)+2\left(\int_{T^1(M)}\frac{\dot{g}(v,v)}{2}d\mu_0\right)^2\right)$$
$$+h_{top}(g_0)\left(-\int_{T^1(M)}\frac{\ddot{g}(v,v)}{2}d\mu_0+\frac{1}{4}\int_{T^1(M)}(\dot{g}(v,v))^2 d\mu_0\right).$$

Here, μ_0 is the Bowen–Margulis measure of maximal entropy for the geodesic flow associated to the metric g_0. Further, dots refer to t derivatives evaluated at $t=0$.

In the above formula, we have not defined the term $(\mathrm{Var}(\dot{g}(v,v)/2))$. The reader should see [**Pol94**] for details and definitions; for us, the only thing we will need is that $\mathrm{Var}(0)=0$.

THEOREM 3.8. *The Hessian of the entropy function defines a metric on the Fuchsian space $\mathcal{F}\subset\mathcal{H}$. Furthermore, the norm of this metric is bounded below by 2π times the norm defined by the Weil–Petersson metric.*

Proof. By Theorem 3.5, the entropy function attains a minimum along the Fuchsian locus $\mathcal{F}$, and thus its Hessian is a well-defined non-negative quadratic form on the tangent space. Given a holomorphic quadratic differential α on a Riemann surface (Σ,σ), for small enough $t>0$ we have the almost-Fuchsian germ $(e^{2u_t}h, t\alpha)\in\mathcal{AF}$, where h is the hyperbolic metric uniformizing (Σ,σ). Recalling equation (3.3) and using $\|B\|_h^2=2\|\alpha\|_h^2$ yields

$$\dot{u_t}e^{2u_t}(2-2t^2e^{-4u_t}\|\alpha\|_h^2)+2te^{-2u_t}\|\alpha\|_h^2=\Delta_h\dot{u_t}. \tag{3.4}$$

The maximum principle implies $\dot{u}_t=0$ at $t=0$. Hence, all terms in Theorem 3.7 vanish except for the third containing a second derivative. Differentiating (3.4) again with respect to t and evaluating at $t=0$ we obtain

$$\Delta_h\ddot{u}_0=2\ddot{u}_0+2\|\alpha\|_h^2, \tag{3.5}$$

where above we have used the fact that $u_0=0$ and $\dot{u}_t=0$ at $t=0$. Integrating (3.5) with respect to the Riemannian volume form of h shows

$$\int_\Sigma \ddot{u}_0\,dV_h=-\int_\Sigma\|\alpha\|_h^2\,dV_h.$$

Now, letting $g_t=e^{2u_t}h$, the fact that $\dot{u}_0=0$ implies that

$$\ddot{g}_0=2\ddot{u}_0h.$$

Moreover, the Bowen–Margulis measure μ_0 for the hyperbolic metric h is equal to the Liouville measure on the unit tangent bundle $T^1\Sigma$. Thus,

$$\begin{aligned}\int_{T^1\Sigma}\frac{\ddot{g}_0(v,v)}{2}\,d\mu_0&=\int_{T^1\Sigma}\ddot{u}_0h(v,v)\,d\mu_0\\&=\int_{T^1\Sigma}\ddot{u_0}\,d\mu_0\\&=-2\pi\int_\Sigma\|\alpha\|_h^2\,dV_h\\&=-2\pi\|\alpha\|_{WP}^2.\end{aligned}$$

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Thus, Theorem 3.7 reveals

$$\frac{d^2}{dt^2} E(g_t, tB)\Big|_{t=0} = \frac{d^2}{dt^2} h_{top}(g_t)\Big|_{t=0} \geq 2\pi \|\alpha\|^2_{WP},$$

which completes the proof. □

4. *From germs to hyperbolic 3-manifolds*

4.1. *The* $\mathrm{Isom}^+(\mathbb{H}^3)$ *character variety.*

The $\mathrm{Isom}^+(\mathbb{H}^3)$-character variety

$$\chi(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$$

is the quotient of the space of representations $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$ by an equivalence relation. Two representations ρ and ρ' are equivalent if ρ' is contained in the closure of the ρ-orbit under the action of $\mathrm{Isom}^+(\mathbb{H}^3)$ by conjugation. For the details concerning the following discussion, see [**Gol04**]. Via the identification $\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$, the set of homomorphisms

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$$

has the structure of an affine algebraic set. The set of irreducible representations comprises those $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$ which do not fix a point in $\partial_\infty(\mathbb{H}^3)$. The set of irreducible representations

$$\mathrm{Hom}^{irr}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$$

is a complex manifold of complex dimension $-3\chi(\Sigma) + 3$ upon which the (algebraic) action of $\mathrm{Isom}^+(\mathbb{H}^3)$ by conjugation is free and proper. In particular, the orbit of an irreducible representation is closed. The quotient

$$\mathrm{Hom}^{irr}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))/\,\mathrm{Isom}^+(\mathbb{H}^3) \subset \chi(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3)),$$

is a complex manifold of complex dimension $-3\chi(\Sigma)$.

4.2. *Integrating minimal germs.*

As recorded in Theorem 2.2, every $(g, B) \in \mathcal{H}$ can be integrated to an immersed minimal surface in $\mathbb{H}^3$ with induced metric and second fundamental form (g, B). Furthermore, this immersion is unique up to an isometry of $\mathbb{H}^3$. Since the data arises from tensors on a closed surface Σ, this minimal immersion is equivariant for a representation $\rho : \pi_1(\Sigma) \to \mathrm{Isom}^+(\mathbb{H}^3)$.

Given $(g, B) \in \mathcal{H}$ we first describe how to obtain the representation $\rho : \pi_1(\Sigma) \to \mathrm{Isom}^+(\mathbb{H}^3)$. Let

$$f(g, B) : \widetilde{\Sigma} \to \mathbb{H}^3$$

be an immersion described above and select $\tilde{p} \in \widetilde{\Sigma}$ such that

$$f(\tilde{p}) = O \in \mathbb{H}^3,$$
$$df(\tilde{p})(E_i) = F_i \in T_O\mathbb{H}^3,$$

where $\{E_i\} \subset T_{\tilde{p}}\widetilde{\Sigma}$ constitute an oriented orthonormal frame at $\tilde{p} \in \widetilde{\Sigma}$ and $\{F_1, F_2, N\} \subset T_O\mathbb{H}^3$ is an oriented orthonormal frame at $O \in \mathbb{H}^3$. Now let $\gamma \in \pi_1(\Sigma)$. Then $f \circ \gamma$

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defines a new immersion also with induced metric and second fundamental form (g, B). Thus, by Theorem 2.2 there exists a unique $\rho(\gamma) \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that

$$f(\gamma(\widetilde{p})) = \rho(\gamma) f(\widetilde{p}),$$
$$df \circ d\gamma(X) = d(\rho(\gamma)) \circ df(X),$$

for all $X \in T_{\widetilde{p}}\widetilde{\Sigma}$. This assignment defines a smooth map,

$$\Phi : \mathcal{H} \longrightarrow \chi(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3)). \tag{4.1}$$

We call the map Φ the *monodromy* map. Note that Φ is well defined since changing a pair (g, B) by a diffeomorphism isotopic to the identity produces a conjugate representation. An important feature of this map is the following theorem of Taubes.

THEOREM 4.1. [**Tau04**] *The image of* Φ *consists solely of irreducible representations.*

As a result of the discussion in the previous section, the above theorem implies that the map Φ takes values in a smooth, complex manifold.

5. *Limit sets of quasi-Fuchsian groups*

We begin this section with an overview of discrete subgroups acting isometrically on hyperbolic space. A Kleinian group is a discrete (torsion-free) subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{PSL}(2, \mathbb{C})$ of orientation-preserving isometries of hyperbolic 3-space. Given a Kleinian group Γ, the action on $\mathbb{H}^3$ extends to an action on the conformal boundary $\partial_\infty(\mathbb{H}^3) \simeq \mathbb{CP}^1$ by Möbius transformations. This action divides $\partial_\infty(\mathbb{H}^3)$ into two disjoint subsets: $\Lambda(\Gamma)$ and $\Omega(\Gamma)$. The *limit set* $\Lambda(\Gamma)$ is defined to be the smallest non-empty, Γ-invariant closed subset of $\partial_\infty(\mathbb{H}^3)$. The *domain of discontinuity* $\partial_\infty(\mathbb{H}^3)\backslash\Lambda(\Gamma) = \Omega(\Gamma)$ is the largest open set on which Γ acts properly discontinuously. The quotient $M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold with *holonomy* group Γ.

A discrete, faithful representation $\rho : \pi_1(\Sigma) \to \mathrm{Isom}^+(\mathbb{H}^3)$ is *quasi-Fuchsian* if and only if Λ_Γ is a Jordan curve and $\Omega(\Gamma)$ consists of precisely two invariant, 1-connected components. The representation ρ is *Fuchsian* if Λ_Γ is a round circle. The quotient $\Omega(\Gamma)/\Gamma = X^+ \cup \overline{X^-}$ is a disjoint union of two marked Riemann surfaces $(X^+, \overline{X^-})$, each diffeomorphic to Σ, where the bar over X^- denotes the surface with the opposite orientation. The marking, which is a choice of homotopy equivalence $f^\pm : \Sigma \to X^\pm$, is determined by the requirement that $f^\pm_* = \rho$. Conversely, we have Bers' simultaneous uniformization theorem.

THEOREM 5.1. [**Ber60**] *Given an ordered pair of marked Riemann surfaces* $(X^+, \overline{X^-})$ *each diffeomorphic to* Σ*, there exists an isomorphism* $\rho : \pi_1(\Sigma) \to \Gamma$ *onto a quasi-Fuchsian group* Γ*, unique up to conjugation in* $\mathrm{PSL}(2, \mathbb{C})$*, such that* $\Omega(\Gamma)/\Gamma = X^+ \cup \overline{X^-}$.

Every quasi-Fuchsian representation is irreducible. Thus, quasi-Fuchsian space $\mathcal{QF}$, which consists of conjugacy classes of all quasi-Fuchsian representations, lies in the submanifold of irreducible representations as an open subset,

$$\mathcal{QF} \subset \mathrm{Hom}^{irr}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))/\mathrm{Isom}^+(\mathbb{H}^3) \subset \chi(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3)),$$

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and thus inherits a complex structure. With respect to this complex structure, the bijection provided by Theorem 5.1 becomes a biholomorphism,

$$\mathcal{QF} \simeq \mathcal{T} \times \overline{\mathcal{T}}.$$

The complex structure on $\mathcal{T}$ is the one arising from Kodaira–Spencer deformation theory (see [**Kod05**]).

If $\rho \in \mathcal{QF}$ is a quasi-Fuchsian representation, the fiber $\Phi^{-1}(\rho)$ of the map Φ from (4.1) consists of $\pi_1(\Sigma)$-injective minimal immersions $\Sigma \to \rho(\pi_1(\Sigma))\backslash\mathbb{H}^3$. The general existence theorem (Theorem 2.1) guarantees that this set is always non-empty.

We first show that the dynamics of a quasi-Fuchsian representation ρ are at least as complicated as the induced dynamics on an invariant minimal surface in $\Phi^{-1}(\rho)$.

THEOREM 5.2. *Let $\rho \in \mathcal{QF}$ be a quasi-Fuchsian representation and $(g, B) \in \Phi^{-1}(\rho)$. Then,*

$$\frac{1}{\mathrm{Vol}(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g \leq \mathrm{H} \cdot \dim(\Lambda_\Gamma),$$

with equality if and only if B is identically zero, which holds if and only if ρ is Fuchsian.

Proof. Given $(g, B) \in \Phi^{-1}(\rho)$, let $\widetilde{\Sigma} \subset \mathbb{H}^3$ be the $\rho(\pi_1(\Sigma)) = \Gamma$-invariant minimal disk with induced metric and second fundamental form (g, B) and fix $x \in \widetilde{\Sigma}$. Given $R > 0$, define

$$\widetilde{N}_\Gamma(R) := \{\gamma \in \Gamma \mid d_g(x, \gamma(x)) \leq R\}$$

and

$$N_\Gamma(R) := \{\gamma \in \Gamma \mid d_{\mathbb{H}^3}(x, \rho(\gamma)(x)) \leq R\},$$

which denote the number of Γ-orbits within distance R of x with the distance computed in the induced metric g and the hyperbolic metric, respectively. Every point at distance R from x in the metric g is at distance less than or equal to R in the hyperbolic metric. Thus,

$$\widetilde{N}_\Gamma(R) \leq N_\Gamma(R).$$

Taking logarithms of each side, dividing by R, and then letting $R \to \infty$, the left-hand side converges to the entropy $E(g, B)$. Since quasi-Fuchsian groups are convex-cocompact, Theorem 2.5 implies that the right-hand side converges to the Hausdorff dimension of the limit set of Γ (in fact, this was proved much earlier by Sullivan [**Sul84**]). This proves:

$$E(g, B) \leq \mathrm{H} \cdot \dim(\Lambda_\Gamma).$$

Applying Theorem 3.3,

$$\frac{1}{\mathrm{Vol}(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g \leq E(g, B) \leq \mathrm{H} \cdot \dim(\Lambda_\Gamma).$$

Furthermore, another appeal to Theorem 3.3 shows that the first inequality is an equality if and only if $B = 0$, but this occurs if and only if the minimal surface $\widetilde{\Sigma}$ is totally geodesic, which exists if and only if Γ is Fuchsian. This completes the proof. □

As a corollary we obtain a new proof of Bowen's theorem on the Hausdorff dimension of quasi-circles, proved in [**Bow79**].

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COROLLARY 5.3. *A quasi-Fuchsian representation $\rho \in \mathcal{QF}$ is Fuchsian if and only if* $\mathrm{H} \cdot \dim(\Lambda_\Gamma) = 1$.

Proof. If ρ is Fuchsian the result is immediate. Meanwhile, if $\mathrm{H} \cdot \dim(\Lambda_\Gamma) = 1$, then Theorem 5.2 forces $B = 0$ for any $(g, B) \in \Phi^{-1}(\rho)$. Since there must exist some $(g, B) \in \Phi^{-1}(\rho)$, this implies that ρ leaves invariant a totally geodesic surface; thus, ρ is Fuchsian. □

The previous corollary can be quantified using the estimate in Theorem 5.2. The following result is a quantitative version of Bowen's rigidity for Hausdorff dimension of the limit set. Using results in [**HW13**, **GHW10**], we can give bounds on many geometric quantities pertinent in the study of quasi-Fuchsian groups. First, we need a definition.

Definition 5.4. Fix $\varepsilon_0 > 0$. Let $\mathcal{QF}_{\varepsilon_0} \subset \mathcal{QF}$ be the set of all quasi-Fuchsian groups such that there exists a π_1-injective, stable, immersed minimal surface with induced metric g having injectivity radius at least ε_0.

We remark that every quasi-Fuchsian group Γ such that the injectivity radius of the quotient $\mathbb{H}^3/\Gamma$ is greater than ε_0 belongs to the subset $\mathcal{QF}_{\varepsilon_0}$.

The following is the aforementioned quantitative rigidity theorem.

THEOREM 5.5. *Fix $\varepsilon_0 > 0$ and suppose $\Gamma \in \mathcal{QF}_{\varepsilon_0}$. Let*

$$f : \Sigma \to \mathbb{H}^3/\Gamma$$

be any π_1-injective, stable, immersed minimal surface with induced metric and second fundamental form (g, B) such that the injectivity radius of g is at least ε_0.

Then, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon_0, \varepsilon) > 0$ such that if

$$\mathrm{H} \cdot \dim(\Lambda_\Gamma) < 1 + \delta,$$

the following properties are satisfied.

(1) *The second fundamental form satisfies $\|B\|_g^2 \leq 2\varepsilon$. In particular, when $\varepsilon < 1$ the group Γ is almost-Fuchsian.*

(2) *If $\varepsilon < 1$, then $f : \Sigma \to \mathbb{H}^3/\Gamma$ is the unique closed minimal surface in $\mathbb{H}^3/\Gamma$. Furthermore, f is an embedding.*

(3) *If $\varepsilon < 1$, the diameter and volume of the convex core $\mathcal{C}(\Gamma)$ of $\mathbb{H}^3/\Gamma$ satisfy*

$$\mathrm{Diam}(\mathcal{C}(\Gamma)) \leq \frac{1}{2} \log\left(\frac{1+\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}\right)$$

and

$$\mathrm{Vol}(\mathcal{C}(\Gamma)) \leq 2\pi\chi(\Sigma)\left(\frac{\sqrt{\varepsilon}}{1-\varepsilon} + \frac{1}{2}\log\left(\frac{1+\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}\right)\right).$$

(4) *If $\varepsilon < 1$, the Teichmüller distance between the conformal boundary components of $\mathbb{H}^3/\Gamma$ is at most*

$$\log\left(\frac{1+\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}\right).$$

(5) *If $\varepsilon < 1$ and h is the hyperbolic metric in the conformal class of g, then $\mathbb{H}^3/\Gamma$ is $(1+\varepsilon)$ bi-Lipschitz to the Fuchsian manifold which uniformizes h.*

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Proof. Item (2) is an immediate corollary of the fact that Γ is almost-Fuchsian. The estimates in (3) and (4) follow immediately from the estimate on $\|B\|_g$ using the estimates obtained by Huang and Wang and Guo *et al* in [**HW13**, **GHW10**]. Finally, item (5) follows from a theorem in [**Uhl83**]. Hence, the only thing to prove is item (1).

An immediate application of a theorem of Schoen [**Sch83**] implies that there exists a uniform constant $C_1 > 0$, independent of (g, B), such that the following C^0-estimate holds:

$$\|B\|^2_{C^0} \leq C_1. \tag{5.1}$$

By Theorem 5.2,

$$\frac{1}{\mathrm{Vol}(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|^2_g}\, dV_g \leq \mathrm{H} \cdot \dim(\Lambda_\Gamma).$$

Now let $\varepsilon > 0$; we shall find $\delta > 0$ to prove the desired estimate in item (1).

Consider the subset of Σ,

$$F_\varepsilon = \{p \in \Sigma \mid \varepsilon < \|B\|^2_g\}.$$

Let $E_\varepsilon = \Sigma \backslash F_\varepsilon$. Using (5.1), the Gauss equation,

$$K_g = -1 - \tfrac{1}{2}\|B\|^2_g,$$

implies that there exists a uniform constant C_2 such that $|K_g| \leq C_2$. In the appendix, we show that (5.1) implies that if there exists $p \in F_{2\varepsilon}$, then there exists an $R = R(\varepsilon, \varepsilon_0) > 0$, which depends only on ε and ε_0, such that $B_g(p, R) \subset F_\varepsilon$. Next, bounded sectional curvature in addition to the assumption that the injectivity radius of (Σ, g) is at least ε_0 implies local non-collapsing of volume. Consequently, there exists a uniform constant $\kappa = \kappa(\varepsilon, \varepsilon_0) > 0$ such that whenever $F_{2\varepsilon}$ is non-empty,

$$\mathrm{Vol}(F_\varepsilon) > \mathrm{Vol}(B_g(p, R)) > \kappa.$$

Hence,

$$\begin{aligned}
\mathrm{H} \cdot \dim(\Lambda_\Gamma) &\geq \frac{1}{\mathrm{Vol}(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|^2_g}\, dV_g \\
&= \frac{1}{\mathrm{Vol}(g)} \left(\int_{F_\varepsilon} \sqrt{1 + \frac{1}{2}\|B\|^2_g}\, dV_g + \int_{E_\varepsilon} \sqrt{1 + \frac{1}{2}\|B\|^2_g}\, dV_g \right) \\
&> \frac{1}{\mathrm{Vol}(g)} \left(\mathrm{Vol}(F_\varepsilon) \sqrt{1 + \frac{1}{2}\varepsilon} + \mathrm{Vol}(E_\varepsilon) \right) \\
&> \frac{1}{\mathrm{Vol}(g)} \left(\mathrm{Vol}(F_\varepsilon) \left(1 + \frac{\sqrt{2} - 1}{2}\varepsilon \right) + \mathrm{Vol}(E_\varepsilon) \right) \\
&> \frac{1}{\mathrm{Vol}(g)} \left(\mathrm{Vol}(g) + \kappa\varepsilon \frac{\sqrt{2} - 1}{2} \right).
\end{aligned}$$

Hence, we obtain the estimate

$$\mathrm{H} \cdot \dim(\Lambda_\Gamma) > 1 + \frac{\kappa\varepsilon}{\mathrm{Vol}(g)} \cdot \frac{\sqrt{2} - 1}{2}. \tag{5.2}$$

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Observe that the local non-collapsing of volume implies a uniform lower bound for $\mathrm{Vol}(g)$. Let $\delta > 0$ be defined as

$$\delta := \frac{\kappa\varepsilon}{\mathrm{Vol}(g)} \cdot \frac{\sqrt{2}-1}{2},$$

and assume the limit set satisfies

$$\mathrm{H} \cdot \dim(\Lambda_\Gamma) < 1 + \delta.$$

Then, the estimate (5.2) implies

$$1 + \delta < \mathrm{H} \cdot \dim(\Lambda_\Gamma) < 1 + \delta,$$

which is a contradiction. Hence, we conclude that $F_{2\varepsilon}$ is empty and $\|B\|_g^2 \leq 2\varepsilon$. This completes the proof. □

Remark. The assumption that the injectivity radius of (Σ, g) is bounded below seems essential for the argument since it seems very possible to pack a lot of the mass of $\|B\|_g$ into a region of extremely small volume but controlled diameter. Nonetheless, the veracity of the purely quasi-Fuchsian group statement that small Hausdorff dimension implies almost-Fuchsian is unclear from our approach.

Remark. This result can also be interpreted as a uniqueness result for the solutions to the boundary value problem for a minimal disk in $\mathbb{H}^3$ asymptotic to the very fractal boundary curve that is the limit set of Γ. Results of this type in minimal surface theory often rest on the hypothesis that the boundary curve is rather smooth but has controlled total geodesic curvature. Here, the Hausdorff dimension being near one seems to be playing the role of a geodesic curvature bound.

A final corollary is the following L^1-bound on the norm of the second fundamental form of a π_1-injective minimal surface in a quasi-Fuchsian manifold.

COROLLARY 5.6. *If $\rho \in \mathcal{QF}$ is a quasi-Fuchsian representation and $(g, B) \in \Phi^{-1}(\rho)$, then*

$$\frac{1}{\mathrm{Vol}(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g < 2.$$

Proof. This follows from Theorem 5.2 since $\mathrm{H} \cdot \dim(\Lambda_\Gamma) < 2$. □

This is a necessary condition for a minimal hyperbolic germ to have quasi-Fuchsian monodromy; it is unknown to what extent this condition is sufficient. It is interesting to note that this condition is trivially satisfied by almost-Fuchsian germs. In this regard, this inequality suggests that for minimal surfaces in any quasi-Fuchsian manifold, there is a remnant of the almost-Fuchsian condition.

6. *Comparing actions on $\mathcal{QF}$ and $\mathcal{H}$*

The space $\mathcal{QF}$ of quasi-Fuchsian representations possesses an anti-holomorphic involution ι which acts by

$$\iota(X, \overline{Y}) = (Y, \overline{X}),$$

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where $(X, \overline{Y}) \in \mathcal{T} \times \overline{\mathcal{T}} \simeq \mathcal{QF}$. Meanwhile, the space of minimal germs $\mathcal{H}$ also carries an involution given by the restriction of the U(1)-action to multiplication by -1 sending a germ (g, B) to $(g, -B)$. The following theorem shows that these actions are actually intertwined, at least on the almost-Fuchsian germs.

THEOREM 6.1. *For all* $(g, B) \in \mathcal{AF}$,

$$\Phi((g, -B)) = \iota \circ \Phi((g, B)),$$

where Φ *is the map defined in (4.1)*

Proof. Theorem 2.4 explicitly expresses the almost-Fuchsian metric on $\Sigma \times \mathbb{R}$ corresponding to the germ $(g, B) \in \mathcal{AF}$. Fock [**Foc07**] discovered a complex analytic way to express this metric. Namely, write $g = e^{2u}|dz|^2$ in conformal coordinates and let $\alpha = \Psi((g, B))$ be the quadratic differential whose real part is B. Then,

$$G((g, B)) = dt^2 + e^{2u}|\cosh(t)\, dz + \sinh(t)e^{-2u}\overline{\alpha}\, d\overline{z}|^2$$

expresses the almost-Fuchsian metric (expanding out this expression, it is equal to the one appearing in Theorem 2.4). The Beltrami differential

$$\mu = e^{-2u}\overline{\alpha}$$

has the property that the metrics

$$|dz \pm \mu\, d\overline{z}|^2$$

furnish conformal metrics on the two components of the domain of discontinuity for the almost-Fuchsian group corresponding to (g, B). Now, sending (g, B) to $(g, -B)$ sends α to $-\alpha$ which changes μ to $-\mu$. Hence, the mapping ι interchanges the conformal structures on the domain of discontinuity. This completes the proof. □

If the mapping Φ were an immersion everywhere, then an analytic continuation argument would show that these actions intertwine on the whole quasi-Fuchsian space. However, as we now explain, this is likely not the case and is related to the bifurcation problem in minimal surface theory.

A minimal hyperbolic germ is non-degenerate if the second variation of area has zero nullity. More precisely, given the minimal immersion $f : \widetilde{\Sigma} \to \mathbb{H}^3$ corresponding to a minimal hyperbolic germ $(g, B) \in \mathcal{H}$, let ν be a unit normal vector field to the image of f and let $u \in C^\infty(\Sigma)$. Then, lifting u periodically to the universal cover $\widetilde{\Sigma}$, take a normal variation f_t of f such that

$$\left.\frac{d}{dt} f_t\right|_{t=0} = u\nu.$$

Then, the well-known formula for the second variation of area (see [**CM11**]) gives

$$\left.\frac{d^2}{dt^2}\mathrm{Area}(f_t^* G_{\mathbb{H}^3})\right|_{t=0} = \int_\Sigma -u\Delta_g u - (\|B\|_g^2 - 2)u^2\, dV_g.$$

Here, $G_{\mathbb{H}^3}$ is the Riemannain metric on $\mathbb{H}^3$. A minimal hyperbolic germ is non-degenerate if and only if the *Jacobi operator*,

$$L_{(g,B)} = -\Delta_g - (\|B\|_g^2 - 2),$$

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has no non-zero eigenfunctions with eigenvalue 0, that is, there are no non-zero solutions to $L_{(g,B)}u = 0$. Solutions to $L_{(g,B)}u = 0$ are called *Jacobi fields.*

The following theorem of Taubes shows the special significance of degenerate minimal germs.

THEOREM 6.2. [**Tau04**] *The vector space of Jacobi fields for the operator $L_{(g,B)}$ is in bijection with the kernel of the differential of the map*

$$\Phi : \mathcal{H} \to \chi(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{C})),$$

at the germ (g, B).

An argument using the continuity method in [**Uhl83**] shows that there do exist minimal germs for which the associated Jacobi operator admits non-trivial solutions. Using the above theorem, an application of Sard's theorem shows that in terms of representations, this phenomenon is non-generic.

THEOREM 6.3. *The set of conjugacy classes of representations $\rho : \pi_1(\Sigma) \to \mathrm{Isom}^+(\mathbb{H}^3)$ which admit a degenerate, equivariant minimal immersion $f : \widetilde{\Sigma} \to \mathbb{H}^3$ has measure zero in the character variety.*

Proof. By Theorem 6.2, a representation is a critical value of the map Φ if and only if it admits a degenerate, equivariant minimal immersion. By Theorem 4.1, Φ takes values in the smooth (even complex) manifold of irreducible representations. Thus, by Sard's theorem, the image of Φ is measure zero in the character variety, which proves the theorem. □

Remark. Though we do not pursue this here, it is remarked by Uhlenbeck [**Uhl83**] that the map Φ is real analytic, although this would require a careful proof that the moduli space $\mathcal{H}$ is real analytic; this should be an application of the real analytic implicit function theorem in the Banach manifolds which Taubes uses to construct $\mathcal{H}$. With these technicalities assumed, the critical sets of Φ should then be analytic sets.

7. *From* $\mathbb{H}^3$ *to* CAT(−1) *Riemannian manifolds*

In this section, we will generalize Theorem 5.2 and the subsequent corollaries to the setting of a convex-cocompact representations $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$, where X is a CAT(−1) Riemannian n-manifold and $n \geq 3$.

Definition 7.1. An n-dimensional Riemannian manifold is CAT(−1) if it is 1-connected and complete with sectional curvature less than or equal to -1.

Let $f : \widetilde{\Sigma} \to X$ be a minimal immersion of the universal cover of Σ into a CAT(−1) Riemannian n-manifold X. We pause to review the basic submanifold theory. Let ∇ denote the Levi-Civita connection of X, and ∇^T the component of ∇ tangential to the image of f. Then, the second fundamental form is the symmetric 2-tensor with values in the normal bundle given by

$$B(X, Y) = \nabla_X Y - \nabla^T_X Y,$$

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where $X, Y \in \Gamma(f^*TX)$ are tangent to the image of f. Thus, B is a vector-valued symmetric 2-tensor, which we write in a local trivialization as $(B_1, \ldots, B_{n-2})$. Then each B_i is an ordinary real-valued 2-tensor. Denoting by g the induced metric of the immersion f, the immersion is *minimal* if the trace with respect to the metric g of each B_i vanishes:

$$\mathrm{tr}_g B_i = H_i = 0.$$

For each i, H_i is the ith mean curvature of the immersion. Denoting the Riemannian metric on X by angled brackets $\langle\ ,\ \rangle$, the Gauss equation reads:

$$K_g = \mathrm{Sec}_X(\partial_1, \partial_2) - \langle B(\partial_1, \partial_2), B(\partial_1, \partial_2)\rangle + \langle B(\partial_1, \partial_1), B(\partial_2, \partial_2)\rangle,$$

where K_g is the sectional curvature of g and $\mathrm{Sec}_X(\partial_1, \partial_2)$ is the sectional curvature of the 2-plane spanned by $\{\partial_1, \partial_2\}$ computed in the Riemannian metric of X. Choosing isothermal coordinates on $\widetilde{\Sigma}$ for the metric g and writing the result with respect to an orthonormal framing of the normal bundle, the minimality of f implies

$$B(\partial_1, \partial_1) = -B(\partial_2, \partial_2).$$

This verifies that the Gauss equation in this setting is

$$K_g = \mathrm{Sec}_X(\partial_1, \partial_2) - \tfrac{1}{2}\|B\|_g^2.$$

Since the sectional curvature of X is bounded above by -1, as in the hyperbolic case the sectional curvature of g is also bounded above by -1.

Definition 7.2. Let X be an n-dimensional CAT(-1) Riemannian manifold. A discrete, faithful representation $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$ with $\rho(\pi_1(\Sigma)) = \Gamma$ is convex-cocompact if there is a geodesically convex, Γ-invariant subset $K \subset X$ upon which the action of Γ is cocompact.

Now, let $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$ be a convex-cocompact representation. Given a marked conformal structure $\sigma \in \mathcal{T}$ in the Teichmüller space of Σ, it is proved by Goldman and Wentworth [**GW07**] that there exists a ρ-equivariant harmonic map,

$$f_\rho : (\widetilde{\Sigma}, \sigma) \to X.$$

Formally, f_ρ is a minimizer for the Dirichlet energy,

$$E(u) := \tfrac{1}{2}\int_F \|du\|^2\, dV_g,$$

where $u : \widetilde{\Sigma} \to X$ is any smooth ρ-equivariant map, $F \subset \widetilde{\Sigma}$ is a fundamental domain for the action of $\pi_1(\Sigma)$, and the integrand is the Hilbert–Schmidt norm of the differential du times the volume element constructed using any Riemannian metric g in the conformal class of σ. This is independent of such a choice since the energy with two-dimensional domain is conformally invariant.

Furthermore, results of Al'ber and Hartman [**Har67**] guarantee that the harmonic map f_ρ is unique unless f_ρ maps onto a single geodesic. In our situation this never occurs; by equivariance, this would imply that the image of ρ consists of translations along a single geodesic. Hence, ρ maps $\pi_1(\Sigma)$ faithfully onto an abelian group, which is impossible.

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Thus, we conclude that for each convex-cocompact $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$, there exists a unique ρ-equivariant harmonic map. The regularity theory for harmonic maps implies (see Tromba [**Tro92**] for a careful proof in the $X = \mathbb{H}^2$ case) that this assignment defines a smooth function on Teichmüller space, called the *energy functional*:

$$\mathcal{E}_\rho : \mathcal{T} \to \mathbb{R}_{\geq 0},$$

which records the energy of the unique ρ-equivariant harmonic map $f_\rho : (\widetilde{\Sigma}, \sigma) \to X$.

Also due to Goldman–Wentworth is the following crucial theorem.

THEOREM 7.3. [**GW07**] *If $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$ is convex-cocompact, the the energy functional $\mathcal{E}_\rho$ is a proper function on Teichmüller space.*

As remarked in [**GW07**], this implies the existence of a critical point (not necessarily unique) $\sigma_\rho \in \mathcal{T}$ for $\mathcal{E}_\rho$. By Sacks and Uhlenbeck [**SU82**], σ_ρ is critical for $\mathcal{E}_\rho$ if and only if

$$f_\rho : (\widetilde{\Sigma}, \sigma_\rho) \to X$$

is a conformal harmonic map which implies that f is a *branched minimal immersion*. Recall that a branched immersion from an oriented surface Σ to any manifold N is a C^1-mapping $f : \Sigma \to N$ that is an immersion on the complement of a finite set of points $\{p_i\}$ and such that the differential vanishes at each p_i. The set $\{p_i\}$ is the set of branch points.

In the present setup, a theorem of Gulliver and Tomi [**GT89**] allows us to rule out particular types of branch points. The question of whether branch points can be entirely avoided seems very difficult; in dimension three we can invoke Theorem 2.1 to exclude branch points for branched minimal immersions that are minima for the area functional.

Before we state the theorem, we need a definition. Let $f : \Sigma \to N$ be a branched immersion and let $p \in \Sigma$ be a branch point. Then on every sufficiently small neighborhood of p, the mapping f is $\ell + 1$ to one for some non-negative integer ℓ. The number ℓ is the *order of ramification* of the branch point p and if there exists a branch point p with order of ramification $\ell > 0$, we say that f is ramified. If none of the branch points of f are ramified, we say that f is an unramified branched immersion.

THEOREM 7.4. [**GT89**] *Let Σ be a smooth, closed, oriented surface and N a Riemannian manifold. Suppose*

$$f : \Sigma \to N$$

is a branched minimal immersion which induces an isomorphism $f_* : \pi_1(\Sigma) \to \pi_1(N)$. Then f is an unramified branched immersion.

Remark. This theorem is not stated exactly this way in [**GT89**]. First, they restrict to surfaces with boundary as they have applications to the Plateau problem in mind. Nonetheless, the proof they present works, and is substantially simplified, in the closed case. For completeness, we present this simplified proof, assuming a certain factorization theorem for branched immersions, in an appendix at the end of this paper. Second, they prove the theorem more generally for any branched immersion which has the *unique continuation property*, although they note that branched minimal immersions are premier examples of this phenomenon. The unique continuation property guarantees that the branched immersion is uniquely determined by its value restricted to any open subset of Σ.

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We can finally prove the following theorem.

THEOREM 7.5. *Let X be an n-dimensional* CAT(-1) *Riemannian manifold. Let $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$ be a convex-cocompact representation. Then there exists a ρ-equivariant unramified branched minimal immersion.*

$$f : \widetilde{\Sigma} \to X.$$

Proof. By Theorem 7.3, the energy functional $\mathcal{E}_\rho : \mathcal{T} \to \mathbb{R}$ is proper, and, by the discussion that follows Theorem 7.3, this implies the existence of a ρ-equivariant branched minimal immersion:

$$f : \widetilde{\Sigma} \to X.$$

As ρ is discrete and faithful, the mapping f descends to the quotient as a branched minimal immersion:

$$f' : \Sigma \to X/\Gamma,$$

where $\Gamma = \rho(\pi_1(\Sigma))$. Since ρ is faithful and $f'_* = \rho$, it follows that f' induces an isomorphism on the level of fundamental group:

$$f'_* : \pi_1(\Sigma) \to \Gamma.$$

Thus, the hypotheses of Theorem 7.4 are satisfied, which implies that f' is an unramified branched immersion, and hence, so is f. This completes the proof. □

With the above discussion in place, the following theorem is a generalization of Theorem 5.2.

THEOREM 7.6. *Let X be an n-dimensional* CAT(-1) *Riemannian manifold. Let $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$ be a convex-cocompact representation with $\rho(\pi_1(\Sigma)) = \Gamma$. Let $f : \Sigma \to X/\Gamma$ be a π_1-injective branched minimal immersion with induced metric g and second fundamental form B which are well defined away from the branch points. Let $\hat{\Sigma} = \Sigma \backslash \{p_1, \ldots, p_k\}$ where $\{p_1, \ldots, p_k\} \subset \Sigma$ is the locus of branch points. Then,*

$$\frac{1}{\mathrm{Vol}(g)} \int_{\hat{\Sigma}} \sqrt{-\mathrm{Sec}_X(\partial_1, \partial_2) + \frac{1}{2}\|B\|_g^2}\, dV_g \leq \mathrm{H} \cdot \dim(\Lambda_\Gamma).$$

Remark. In the above theorem, the Hausdorff dimension of Λ_Γ is computed with respect to any Gromov metric on $\partial_\infty(X)$. All Gromov metrics belong to a single bi-Lipschitz class, so the choice is immaterial.

Proof. First note that by Theorem 7.5, there exists a π_1-injective branched minimal immersion in the quotient X/Γ:

$$f : \Sigma \to X/\Gamma.$$

If the dimension of X is equal to three, then Theorem 2.1 implies that f can be taken to be an immersion and the proof of the theorem follows exactly as in Theorem 5.2. So, assume the dimension of X is greater than three.

The strategy is as follows: Let $\mathfrak{B} := \{p_1, \ldots, p_k\}$ be the branching locus. For any small $\varepsilon > 0$, we will construct an immersion that is equal to f away from ε-neighborhoods

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of the p_i and whose induced metric has negative sectional curvature. We then apply the argument of Theorem 5.2 to these perturbed surfaces; a simple limiting argument will complete the proof. We state the exact requirements for the perturbation in the following claim, relegating the tangential proof to the appendix.

Claim. Let (M, h) be a Riemannian manifold with sectional curvature at most -1 and of dimension at least four, and suppose

$$f : \Sigma \to M,$$

is a branched minimal immersion. Then there exists $\varepsilon_0 > 0$ and smooth maps

$$f_\varepsilon : \Sigma \to M$$

for all $\varepsilon \in [0, \varepsilon_0]$, satisfying the following properties.

(1) The maps f_ε are immersions for $\varepsilon > 0$ and $f_0 = f$. Denote the induced metrics by $f_\varepsilon^* h = g_\varepsilon$. Also, let $f^* h = g$ be the induced metric via f on the complement of the branch points $\mathcal{B}$.

(2) The maps f_ε satisfy $f_\varepsilon = f$ on $\Sigma(\varepsilon)$ where

$$\Sigma(\varepsilon) = \Sigma \setminus \left\{ \bigcup B_g(p_i, 3\varepsilon) \right\},$$

with the union taken over the set of branch points $\mathcal{B}$.

(3) Let K_ε be the Gauss curvature of the metric g_ε and K_g be the Gauss curvature of the metric g on the complement of the branch points $\mathcal{B}$. Then $K_\varepsilon \to K_g$ pointwise on the complement of $\mathcal{B}$ in Σ. Note that this formally follows from the previous property.

(4) There exists $\kappa(\varepsilon_0) > 0$ such that $K_\varepsilon(p) < -\kappa(\varepsilon_0)$ for all $\varepsilon \in (0, \varepsilon_0]$ and for all $p \in \Sigma$.

Assuming the claim, the proof of the theorem is as follows.

The exact same argument as in Theorem 5.2 implies

$$\frac{1}{\mathrm{Vol}(g_\varepsilon)} \int_\Sigma \sqrt{-K_\varepsilon}\, dV_{g_\varepsilon} \leq E(g_\varepsilon) \leq \mathrm{H} \cdot \dim(\Lambda_\Gamma), \tag{7.1}$$

where $E(g_\varepsilon)$ is the volume entropy of the induced metric g_ε from the immersion f_ε.

Now recall the definition

$$\Sigma(\varepsilon) = \Sigma \setminus \left\{ \bigcup B_g(p_i, 3\varepsilon) \right\},$$

where the union is taken over all the branch points. By property (2) of the claim, $g_\varepsilon = g$ on $\Sigma(\varepsilon)$; thus,

$$\frac{1}{\mathrm{Vol}(g_\varepsilon)} \int_{\Sigma(\varepsilon)} \sqrt{-K_g}\, dV_g \leq \frac{1}{\mathrm{Vol}(g_\varepsilon)} \int_\Sigma \sqrt{-K_\varepsilon}\, dV_{g_\varepsilon}.$$

Applying (7.1) yields

$$\frac{1}{\mathrm{Vol}(g_\varepsilon)} \int_\Sigma \chi_\varepsilon \sqrt{-K_g}\, dV_g \leq \mathrm{H} \cdot \dim(\Lambda_\Gamma), \tag{7.2}$$

where χ_ε is the characteristic function of $\Sigma(\varepsilon)$.

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For ε varying in any compact set including zero, the volume satisfies $\mathrm{Vol}(g_\varepsilon) > C$ for some $C > 0$ independent of ε. Hence, the bound

$$\frac{1}{\mathrm{Vol}(g_\varepsilon)}\chi_\varepsilon\sqrt{-K_g} \leq \frac{1}{C}\sqrt{-K_g}$$

is valid on all of Σ. Applying the Cauchy–Schwarz inequality reveals

$$\left(\int_\Sigma \sqrt{-K_g}\, dV_g\right)^2 \leq \mathrm{Vol}(g)\int_\Sigma -K_g\, dV_g.$$

Furthermore, the Gauss equation implies a uniform upper bound on $\mathrm{Vol}(g)$. Additionally, the Gauss–Bonnet theorem (for surfaces with cone singularities) implies

$$\int_\Sigma -K_g\, dV_g < \infty,$$

which ensures that $\sqrt{-K_g}$ is integrable with respect to dV_g. Since

$$\frac{1}{\mathrm{Vol}(g_\varepsilon)}\chi_\varepsilon\sqrt{-K_g} \to \frac{1}{\mathrm{Vol}(g)}\sqrt{-K_g}$$

pointwise on the complement of the branch points, the dominated convergence theorem implies

$$\frac{1}{\mathrm{Vol}(g_\varepsilon)}\int_\Sigma \chi_\varepsilon\sqrt{-K_g}\, dV_g \to \frac{1}{\mathrm{Vol}(g)}\int_\Sigma \sqrt{-K_g}\, dV_g$$

as $\varepsilon \to 0$. Hence, letting $\varepsilon \to 0$ in (7.2) yields

$$\frac{1}{\mathrm{Vol}(g)}\int_{\hat{\Sigma}} \sqrt{-K_g}\, dV_g \leq \mathrm{H}\cdot\dim(\Lambda_\Gamma).$$

This completes the proof. □

Just as in the quasi-Fuchsian case, the above theorem easily implies an *a priori* normalized L^1-bound on the second fundamental form of any equivariant branched minimal immersion.

COROLLARY 7.7. *Let X be an n-dimensional* CAT(-1) *Riemannian manifold. Let $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$ be a convex-cocompact representation with $\rho(\pi_1(\Sigma)) = \Gamma$. Let $f : \Sigma \to X/\Gamma$ be a π_1-injective branched minimal immersion with induced metric g and second fundamental form B. Let $\hat{\Sigma} = \Sigma\backslash\{p_1, \ldots, p_k\}$ where $\{p_1, \ldots, p_k\} \subset \Sigma$ is the locus of branch points. Then,*

$$\frac{1}{\sqrt{2}\,\mathrm{Vol}(g)}\int_{\hat{\Sigma}} \|B\|_g\, dV_g < \frac{1}{\mathrm{Vol}(g)}\int_{\hat{\Sigma}} \sqrt{-\mathrm{Sec}_X(\partial_1, \partial_2) + \frac{1}{2}\|B\|_g^2}\, dV_g \leq \mathrm{H}\cdot\dim(\partial_\infty(X)).$$

Proof. The result immediately follows from the obvious estimate

$$\mathrm{H}\cdot\dim(\Lambda_\Gamma) \leq \mathrm{H}\cdot\dim(\partial_\infty(X)).$$

□

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We close with the promised generalization of Bowen's rigidity theorem on Hausdorff dimension of quasi-circles.

COROLLARY 7.8. *Let* $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(X)$ *be a convex-cocompact representation with* $\rho(\pi_1(\Sigma)) = \Gamma$. *Then* $\mathrm{H} \cdot \dim(\Lambda_\Gamma) = 1$ *if and only if there exists a Fuchsian group* Γ_0, *an isomorphism*

$$\phi : \Gamma_0 \to \pi_1(\Sigma),$$

and a $\rho \circ \phi$*-equivariant, totally geodesic embedding*

$$j : \mathbb{H}^2 \to X.$$

Proof. If there exists a $\rho \circ \phi$-equivariant, totally geodesic embedding

$$j : \mathbb{H}^2 \to X,$$

then j extends to a bi-Lipschitz map

$$\overline{j} : \partial_\infty(\mathbb{H}^2) \to \Lambda_\Gamma$$

where $\partial_\infty(\mathbb{H}^2)$ and Λ_Γ are equipped with their natural (bi-Lipschitz equivalence classes of) Gromov metrics. Since the Hausdorff dimension of $\partial_\infty(\mathbb{H}^2)$ in any of these metrics is one and Hausdorff dimension is a bi-Lipschitz invariant,

$$\mathrm{H} \cdot \dim(\Lambda_\Gamma) = 1.$$

In the other direction, assume $\mathrm{H} \cdot \dim(\Lambda_\Gamma) = 1$. Then Theorem 7.6 implies the estimate

$$\frac{1}{\mathrm{Vol}(g)} \int_{\hat{\Sigma}} \sqrt{-\mathrm{Sec}_X(\partial_1, \partial_2) + \frac{1}{2}\|B\|_g^2}\, dV_g \leq \mathrm{H} \cdot \dim(\Lambda_\Gamma) = 1,$$

where (g, B) are the induced metric and second fundamental form, defined on $\hat{\Sigma}$, of any π_1-injective, branched minimal immersion

$$f : \Sigma \to X/\Gamma.$$

We claim that f is actually an immersion. Suppose to the contrary that f has branch points.

Since X is CAT(-1), $\mathrm{Sec}_X(\partial_1, \partial_2) \leq -1$. The only possibility is $\mathrm{Sec}_X(\partial_1, \partial_2) = -1$ and $\|B\|_g = 0$. Then, the Gauss equation becomes

$$K_g = \mathrm{Sec}_X(\partial_1, \partial_2) - \tfrac{1}{2}\|B\|_g^2 = -1,$$

which is valid away from the branching locus of f. Hence f is a branched, totally geodesic immersion whose induced metric away from the branch points is hyperbolic. This implies that the pullback of the metric tensor on X via f is a hyperbolic metric on Σ with conical singularities. The cone points correspond to the branch points and each cone point has cone angle greater than or equal to 2π.

Let $p \in \Sigma$ be a branch point of the map f. A branched hyperbolic metric on a surface with cone singularities at least 2π induces a locally CAT(-1) distance on Σ, which is uniquely geodesic. Let $N \subset \Sigma$ be a small metric disk about the branch point $p \in \Sigma$. Then, for each $q \in \partial N$, we may consider the unique geodesic

$$\gamma_{pq} : I \to \Sigma.$$

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As q ranges over all points of ∂N, this produces a radial foliation of N by geodesic segments. Since the map f is a totally geodesic immersion on the complement of the branch points, the composition

$$\eta_{pq} := f \circ \gamma_{pq} : I \to X$$

is a geodesic in X emanating radially from $f(p)$.

Now, consider the 2-plane distribution $\mathcal{D}$ of $f^*T(X/\Gamma)$ over $N\backslash\{p\}$ which at $q \in N\backslash\{p\}$ is given by the 2-plane

$$\mathcal{D}_q := df(T_q(\Sigma)).$$

Since f is, in particular, a branched minimal immersion, it follows from [**BW03**, p. 224] that $\mathcal{D}$ extends smoothly over $p \in N$ to a smooth 2-plane distribution $\mathcal{D}$ in $f^*T(X/\Gamma)$ defined over all of N. Let $\mathcal{D}_p \subset T_{f(p)}(X/\Gamma)$ denote the value of the distribution $\mathcal{D}$ at p viewed as a tangent plane at the point $f(p) \in X$. By continuity, the initial tangent vector $v_{pq} \in T_{f(p)}(X/\Gamma)$ of each geodesic segment

$$\eta_{pq} : I \to X$$

actually lies in the 2-plane $\mathcal{D}_p \subset T_{f(p)}(X/\Gamma)$. This implies, after potentially shrinking the size of N, that there exists a disk $B \subset T_{f(p)}(X/\Gamma)$ centered at $0 \in T_{f(p)}(X/\Gamma)$ such that

$$f(N) \subset \exp_{f(p)}(B).$$

Hence, choosing B sufficiently small so that $\exp_{f(p)}|_B$ is a diffeomorphism onto its image, this implies that $f(N)$ is contained in the smoothly embedded disk $\exp_{f(p)}(B) \subset X$. Thus, we conclude that f is not injective in any neighborhood of the branch point $p \in \Sigma$.

However, Theorem 7.5 states that the branch points of f must be unramified, which means that there exists a neighborhood of $p \in \Sigma$ where f is injective. This contradiction establishes that f is without branch points. Thus,

$$f : \Sigma \to X/\Gamma$$

is a totally geodesic immersion such that the induced metric on Σ is hyperbolic.

Next, pick $p \in \widetilde{\Sigma}$ and consider the 2-plane $P \subset T_{f(p)}X$ tangent to $f(\widetilde{\Sigma})$ at $f(p)$, where here we use the same name for the lifted map,

$$f : \widetilde{\Sigma} \to X.$$

Since X has negative sectional curvature and f is totally geodesic, the exponential map from $f(p)$ in the directions spanned by P gives a diffeomorphism between P and $f(\widetilde{\Sigma})$, which proves that f is actually an embedding. Thus, $f : \widetilde{\Sigma} \to X$ is a ρ-equivariant, totally geodesic isometric embedding of a complete, 1-connected Riemannian surface with sectional curvature -1 into X.

Hence, there exists a Fuchsian group Γ_0, an isomorphism $\phi : \Gamma_0 \to \pi_1(\Sigma)$, and a ϕ-equivariant isometry

$$h : \mathbb{H}^2 \to \widetilde{\Sigma}.$$

Precomposing f by h gives an isometric, totally geodesic map

$$j := f \circ h : \mathbb{H}^2 \to X,$$

which is equivariant for the representation $\rho \circ \phi : \Gamma_0 \to \Gamma$. This completes the proof. □

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We close the paper with a series of remarks about the results we have obtained.

Remark. We emphasize that these results give a new proof of rigidity of Hausdorff dimension for convex-cocompact closed surface subgroups of rank-one Lie groups of non-compact type. This includes quasi-Fuchsian groups in real hyperbolic space $\mathbb{H}^n$ and complex quasi-Fuchsian groups in complex hyperbolic space $\mathbb{CH}^n$. In addition, the lower bounds we obtain give a geometric explanation for why the Hausdorff dimension of the limit set grows as the lattice of orbits becomes more geometrically distorted in X.

Remark. The applications of the techniques here have not been exhausted: given a discrete, faithful surface group representation into the isometry group of some manifold X for which the associated energy functional on Teichmüller space is proper, one obtains an equivariant unramified branched minimal surface. For example, if $X = G/K$ is a higher rank symmetric space, the situation is more complicated because of the existence of flats, and we would no longer make a statement about the Hausdorff dimension of the limit set, but rather about the growth of orbits directly. In any case, there is a wealth of examples (Hitchin representations, Anosov representations, and, more generally, well-displacing representations) of this type due to the work of Lauborie [**Lab08**]; we plan to study these problems in a future paper.

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A. *Appendix*

A.1. *Gradient estimate at scale epsilon.*

In the proof of the quantitative Bowen rigidity Theorem 5.5, we promised the following proposition. We refer back to the proof for the notation.

PROPOSITION A.1. *Fix an $\varepsilon > 0$ and assume there exists $p \in F_{2\varepsilon}$. Then there exists $R = R(\varepsilon) > 0$ such that $B_g(p, R) \subset F_\varepsilon$.*

Proof. First, recall that by the theorem of Scheon [**Sch83**], there exists $C_1 > 0$ such that

$$\|B\|_g^2 < C_1. \tag{A.1}$$

Then, the Gauss equation,

$$K_g = -1 - \tfrac{1}{2}\|B\|_g^2,$$

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implies that there exists $C_2 > 0$ such that $-C_2 < K_g \leq -1$. Writing g as a conformal deformation of the hyperbolic metric h in the same conformal class $g = e^{2u}h$, we may express the Gauss equation with respect to h via

$$K_g = -e^{2u}(\Delta_h u + 1). \tag{A.2}$$

Using the uniform bounds on K_g and applying the maximum and minimum principle to (A.2) implies that there exists $C_3 > 0$ and the following uniform bound,

$$-C_3 < u \leq 0.$$

Hence, the injectivity radius of (Σ, h) is at least that of (Σ, g). Recall that there exists a holomorphic quadratic differential α such that $\frac{1}{2}\|B\|_g^2 = \|\alpha\|_g^2$. The uniform bound on the conformal factor u in tandem with (A.1) implies that there exists a uniform bound on $\|\alpha\|_h^2$. Hence, over the hyperbolic surface (Σ, h), the set of holomorphic quadratic differentials whose real part appears as the second fundamental form B of a stable, immersed minimal surface is compact. Since the injectivity radius of (Σ, h) is uniformly bounded below, the Mumford compactness theorem implies that the metrics h live in a compact set in the moduli space of all hyperbolic metrics on Σ. Hence, we may define

$$R := \min_{(g,B)} \{R > 0 \mid \|B\|_g(x) > \varepsilon \ \forall x \in B_g(p, R)\}. \tag{A.3}$$

Here, the minimum is taken over all (g, B) first and second fundamental forms of stable, immersed minimal surfaces in quasi-Fuchsian 3-manifolds such that the injectivity radius of g has a uniform lower bound, and points $p \in \Sigma$ such that $p \in F_{2\varepsilon}$; namely, that $\|B\|_g(p) > 2\varepsilon$. Certainly, at any pair (g, B) that are the first and second fundamental forms of a stable, immersed minimal surface in a quasi-Fuchsian 3-manifold, there exists such an $R > 0$ simply by continuity. By our previous discussion, such (g, B) vary over a compact set; hence, we conclude that the minimum in (A.3) is attained and hence $R > 0$. This completes the proof. □

A.2. *Gulliver–Tomi theorem.* In this section of the appendix we provide a simplified proof of the Gulliver–Tomi theorem. Let Σ be a closed, oriented surface of genus greater than one and M any n-dimensional manifold for $n \geq 2$.

Definition A.2. A C^1-mapping $f : \Sigma \to M$ is a branched immersion if there exists a finite set of points $p_1, \ldots, p_k \in \Sigma$ such that $f|_{\Sigma \setminus \{p_1,\ldots,p_k\}}$ is an immersion. Furthermore, for each i there exist positive integers q_i, open sets $U_i \subset \Sigma$, $V_i \subset M$ containing p_i, $f(p_i)$, respectively, and coordinate charts $\phi^i : U_i \to \mathbb{C}$ and $\eta^i : V_i \to \mathbb{R}^n$ such that in these coordinates:

$$\begin{gathered} f^1(x+iy) = \mathrm{Re}((x+iy)^{q_i}) + o(|x+iy|^{q_i}), \\ f^2(x+iy) = \mathrm{Im}((x+iy)^{q_i}) + o(|x+iy|^{q_i}), \\ f^j = o(|x+iy|^{q_i}), \quad \partial f^j/\partial x = o(|x+iy|^{q_i-1}), \\ \partial f^j/\partial y = o(|x+iy|^{q_i-1}) \quad 3 \leq j \leq n. \end{gathered}$$

The points p_i are the branch points of the immersion f. Each number $q_i - 1$ is the order, or index, of the branch point p_i.

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Examples of branched immersions include non-constant holomorphic maps between Riemann surfaces, or, more generally, the critical points of energy functionals which we consider in this paper, which are minimal surfaces on the complement of the branch points.

Let $f : \Sigma \to M$ be a branched immersion and let $p \in \Sigma$ be a branch point. Then on every neighborhood of p, the mapping f is $\ell + 1$ to one for some non-negative integer ℓ. The number ℓ is the *order of ramification* of the branch point p. If none of the branch points of f are ramified, we call f an unramified branched immersion.

A remarkable theory of branched immersions has been developed by many mathematicians, most notably, Gulliver *et al* [**GOR73**]. One of the key elements is a factorization theorem for branched immersions with the unique continuation property. First, recall that f has the unique continuation property if f is uniquely determined by its value on any open set $U \subset \Sigma$.

Define an equivalence relation on non-branch points of Σ as follows: $x \sim y$ for $x, y \in \Sigma$ if and only if there exist open sets $U, V \subset \Sigma$ containing x and y, respectively, and an orientation-preserving homeomorphism $h : U \to V$ such that $f|_U = f \circ h$. The following theorem is proved by Gulliver *et al* [**GOR73**].

THEOREM A.3. *Let $f : \Sigma \to M$ be a branched immersion with the unique continuation property. The quotient*

$$\Sigma^{\sim} := \Sigma / \sim$$

is a closed, oriented surface, the quotient map $\pi : \Sigma \to \Sigma^{\sim}$ is a branched covering, and there exists a unique $f^{\sim} : \Sigma^{\sim} \to M$ such that $f = f^{\sim} \circ \pi$. Furthermore, the branch points of π coincide with those of f and the order of ramification at each branch point is also equal.

Remark. If $f : \Sigma \to \Sigma'$ is a branched covering, then $\Sigma^{\sim} = \Sigma'$, $f = \pi$, and $f^{\sim} = \mathrm{Id}$.

Now, the theorem we wish to prove follows quite rapidly from the Riemann–Hurwitz formula.

THEOREM A.4. *Let $f : \Sigma \to M$ be a branched immersion with the unique continuation property such that $f_* : \pi_1(\Sigma) \to \pi_1(M)$ is an isomorphism. Then $f : \Sigma \to M$ has no ramified branch points, and thus f is an unramified branched immersion.*

Proof. Let $p_i \subset \Sigma$ be the branch points of f with order of ramification $\ell_i - 1$. Form the ramification divisor

$$D_f = \sum (\ell_i - 1)[p_i].$$

Then $\mathrm{Deg}(D_f) = \sum(\ell_i - 1)$ and f is unramified if and only if $\mathrm{Deg}(D_f) = 0$. If $\pi : \Sigma \to \Sigma^{\sim}$ is the branched covering provided by Theorem A.3, then the Riemann–Hurwitz formula implies that there exists $N > 0$ such that

$$\chi(\Sigma) = N\chi(\Sigma^{\sim}) - \mathrm{Deg}(D_f).$$

Since f is an isomorphism on the fundamental group and $f = f^{\sim} \circ \pi$, it follows that $f^{\sim}_* : \pi_1(\Sigma^{\sim}) \to \pi_1(M)$ is surjective and $\pi_* : \pi_1(\Sigma) \to \pi_1(\Sigma^{\sim})$ is injective. The injectivity of π_* implies that $\chi(\Sigma^{\sim}) < 0$, since a closed, oriented surface group of genus greater than

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one cannot inject into the trivial group or into $\mathbb{Z} \oplus \mathbb{Z}$. Thus, the Riemann–Hurwitz formula implies that

$$\begin{aligned} 2 - 2g &= N(2 - 2\widetilde{g}) - \mathrm{Deg}(D_f) \\ &\leq (2 - 2\widetilde{g}) - \mathrm{Deg}(D_f), \end{aligned}$$

where $\widetilde{g}$ is the genus of $\Sigma^{\sim}$. This inequality implies that

$$0 \leq \mathrm{Deg}(D_f) \leq 2(g - \widetilde{g}). \tag{A.4}$$

Finally, since $\pi_1(\Sigma^{\sim})$ surjects onto $\pi_1(M) \simeq \pi_1(\Sigma)$, the minimal cardinality of a generating set of $\pi_1(\Sigma^{\sim})$, which is $2\widetilde{g}$, is at least as large as the minimal cardinality of a generating set of $\pi_1(\Sigma)$, which is $2g$. Thus,

$$2\widetilde{g} \geq 2g.$$

Combining this with (A.4) implies that $g = \widetilde{g}$. Hence, (A.4) implies that $\mathrm{Deg}(D_f) = 0$, which, as stated previously, implies that f has no ramification points. This proves the theorem. □

A.3. *Perturbing branched immersions.* In this section of the appendix, we give a complete proof of the claim contained in the proof of Theorem 7.6. We restate the result here as a proposition.

PROPOSITION A.5. *Let (M, h) be a Riemannian manifold with sectional curvature at most -1 and of dimension at least four, and suppose*

$$f : \Sigma \to M$$

is a branched minimal immersion. Then there exist $\varepsilon_0 > 0$ and smooth maps

$$f_\varepsilon : \Sigma \to M$$

for all $\varepsilon \in [0, \varepsilon_0]$ satisfying the following properties.

(1) *The maps f_ε are immersions for $\varepsilon > 0$ and $f_0 = f$. Denote the induced metrics by $f_\varepsilon^* h = g_\varepsilon$. Also, let $f^* h = g$ be the induced metric via f on the complement of the branch points $\mathcal{B}$.*

(2) *The maps f_ε satisfy $f_\varepsilon = f$ on $\Sigma(\varepsilon)$ where*

$$\Sigma(\varepsilon) = \Sigma \setminus \left\{ \bigcup B_g(p_i, 3\varepsilon) \right\},$$

with the union taken over the set of branch points $\mathcal{B}$.

(3) *Let K_ε be the sectional curvature of the metric g_ε and K_g be the sectional curvature of the metric g on the complement of the branch points $\mathcal{B}$. Then $K_\varepsilon \to K_g$ pointwise on the complement of $\mathcal{B}$ in Σ. Note that this formally follows from the previous property.*

(4) *There exists $\kappa(\varepsilon_0) > 0$ such that $K_\varepsilon(p) < -\kappa(\varepsilon_0)$ for all $\varepsilon \in (0, \varepsilon_0]$ and for all $p \in \Sigma$.*

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Proof. Let $\varepsilon > 0$ and select $p_i \in \mathfrak{B}$. First we need a bump function:

$$\eta_\varepsilon(|t|) = \begin{cases} 1 & |t| < \varepsilon \\ 0 & |t| > 2\varepsilon \end{cases}$$

where also $0 \leq \eta_\varepsilon \leq 1$. We claim that there exists a function $q(t) : \mathbb{R} \to \mathbb{R}$ such that $q(0) = 0$, $q(t) > 0$ if $t > 0$, and, furthermore, as $\varepsilon \to 0$,

$$\|q(\varepsilon)\eta_\varepsilon(t)\|_{C^3(\mathbb{R})} \to 0. \tag{A.5}$$

This can be achieved by choosing

$$q(t) = e^{-1/t}.$$

Pick a coordinate chart on the ball of radius 3ε (choosing ε small enough so that p_i is the only branch point in the chart) about p_i sending p_i to 0. Also, choose a coordinate chart of small radius about the image $f(p_i)$ sending $f(p_i)$ to 0. In these coordinates, write

$$\begin{aligned} f : B_{\mathbb{R}^2}(0, r) &\to B_{\mathbb{R}^n}(0, r) \\ f(u, v) &\mapsto (f^1(u, v), \ldots, f^n(u, v)), \end{aligned}$$

where $r > 0$ is some small number on which the coordinate chart is defined. By the normal form for a branched immersion near a branched point (see Definition A.2), we may assume that the first derivatives of f^1 and f^2 have an isolated zero at $(0, 0)$. Next, define a perturbation of f via

$$f_\varepsilon(u, v) = f(u, v) + (0, 0, \ldots, q(\varepsilon) \cdot \eta_\varepsilon(|(u, v)|) \cdot u, q(\varepsilon) \cdot \eta_\varepsilon(|(u, v)|) \cdot v).$$

Note that this is where we use that the dimension of X is at least four. Here, we equip the image of a small ball about p_i with the induced Riemannian metric so that the coordinate chart is a local isometry, and $|(u, v)|$ is the distance from 0 to (u, v). Observe that $f_0 = f$. Furthermore, since f has a branch point at p_i,

$$\begin{aligned} \left.\frac{\partial f_\varepsilon}{\partial u}\right|_{(0,0)} &= (0, 0, \ldots, q(\varepsilon), 0), \\ \left.\frac{\partial f_\varepsilon}{\partial v}\right|_{(0,0)} &= (0, 0, \ldots, 0, q(\varepsilon)). \end{aligned}$$

The first thing to observe is that f_ε is now an immersion at $(0, 0)$. Next, again by the normal form for branched immersions, the projection of f onto the first two factors,

$$\pi \circ f(u, v) = (f^1(u, v), f^2(u, v)),$$

is an immersion on $B(0, r')\backslash\{0\}$ for some $0 < r' < r$. Choosing $2\varepsilon = r'/2$, it follows from the definition of our bump function η_ε that $f_\varepsilon = f$ on the complement of $B(0, r'/2)$, but f is already known to be an immersion on $B(0, r')\backslash\{0\}$, and thus f_ε is an immersion everywhere. Repeating this process at each branch point produces an immersion $f_\varepsilon : \Sigma \to X/\Gamma$ such that $f_\varepsilon = f$ outside of the union of the 2ε-neighborhoods of the branch points. This takes care of all the points except the last.

By (A.5), the convergence

$$\|f_\varepsilon - f\|_{C^3(\Sigma)} \to 0$$

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as $\varepsilon \to 0$ is immediate. Let g_ε be the induced metric via the immersion f_ε. On the punctured surface $\hat{\Sigma}$, there is C^2-convergence of the Riemannian metrics

$$(\hat{\Sigma}, g_\varepsilon) \to (\hat{\Sigma}, g),$$

and hence C^2-convergence of the associated volume elements,

$$dV_{g_\varepsilon} \to dV_g.$$

Since g has sectional curvature less than -1, for all $p \in \hat{\Sigma}$ and for all $r > 0$ small enough,

$$\mathrm{Vol}_g(B_g(p, r)) \geq \pi r^2 + \beta r^4,$$

for some constant $\beta > 0$. Since the volume elements converge, it follows that for $\varepsilon > 0$ small enough, and for all $p \in \hat{\Sigma}$, and all $r > 0$ small enough,

$$\mathrm{Vol}_{g_\varepsilon}(B_{g_\varepsilon}(p, r)) \geq \pi r^2 + \beta' r^4,$$

for some smaller constant $\beta' > 0$. The Taylor expansion of the volume of balls (see [**Gra04**]) implies that there exists some $\kappa > 0$ such that the scalar curvature satisfies $R_{g_\varepsilon}(p) < -\kappa$ for all $\varepsilon > 0$ small enough and all $p \in \hat{\Sigma}$. Since we are on a surface, this implies that the sectional curvature of g_ε satisfies

$$K_{g_\varepsilon} < -\kappa',$$

for some $\kappa' > 0$. This completes the proof. □

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