Math. Proc. Camb. Phil. Soc. (1975), **78**, 117
MPCPS 78–10
Printed in Great Britain

The self-intersection formula and the 'formule-clef'

By A. T. LASCU, D. MUMFORD and D. B. SCOTT
Université de Montréal, Harvard University, University of Sussex

(*Received* 25 *September* 1974)

Introduction. We shall consider exclusively algebraic non-singular quasi-projective irreducible varieties over an algebraically closed field. If V is such a variety $\mathscr{A}(V)$ will be the Chow ring of rational equivalence classes of cycles of V and

$$\phi_*: \mathscr{A}(V_1) \to \mathscr{A}(V_2)$$

the group homomorphism defined by any proper morphism $\phi: V_1 \to V_2$. Also

$$\phi^*: \mathscr{A}(V_2) \to \mathscr{A}(V_1)$$

denotes the ring homomorphism defined by ϕ.

Let X, Y be two varieties. Assume that Y is a subvariety of X (by subvariety, we will always mean *closed* subvariety), let $i: Y \to X$ be the inclusion and E the normal bundle of Y in X. Then Grothendieck conjectured the *self-intersection formula*

$$i^*i_*(y) = yc_r(E), \tag{1}$$

for any $y \in \mathscr{A}(Y)$, where $r = \operatorname{codim}_X(Y)$ and $c_r(E)$ is the rth Chern class of E.

Consider the blowing-up diagram

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle \rho_E}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array}$$

where $Y' = P(E)$ is the projective bundle associated to E. The normal bundle of Y' in X' is the tautological bundle $\check{L}_E$ of E and one has the exact sequence

$$0 \to \check{L}_E \to \rho_E^* E \to E^{(1)} \to 0$$

which defines $E^{(1)}$ (4). Then Grothendieck also conjectured:

$$f^*i_*(y) = j_*(\rho^*(y)\, c_{r-1}(E^{(1)})) \tag{2}$$

for any $y \in \mathscr{A}(Y)$, which he called the 'formule-clef' for calculating $\mathscr{A}(X')$.

Our aim is to prove both formulae. Especially (1) looks so innocuous that it is hard to believe it is not false or trivial; for instance they are both well known in singular cohomology when the ground field is $\mathbb{C}$. They were first conjectured by Grothendieck in 1957 ((2), exposé 0), and were proven *modulo torsion* by a very roundabout method in (2), cf. esp. exposé XIV (4·4). They were subsequently used in (6). Their analogues in

https://doi.org/10.1017/S0305004100051550 Published online by Cambridge University Press

étale cohomology are apparently to be published in SGA5. Our proof is completely elementary, but requires a good deal of manipulation.

1. *Preliminaries.* The notation will be that of (7) as far as possible. If E is a vector bundle

$$C(t,E) = \sum_{i=0}^{r} c_i(E)t^i,$$

where $r = \text{rank}(E)$, will be its Chern polynomial and $[\lambda C](t,E)$ the polynomial in which an operator λ has been applied to its coefficients. The point of this notation is that if a suitable element x is substituted for t then in $[\lambda C](x,E)$ the operator λ does not apply to the powers of x. The reversed Chern polynomial is defined as

$$\bar{C}(t,E) = t^r C(t^{-1},E).$$

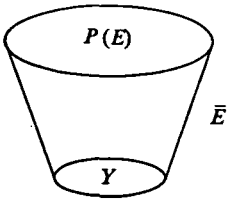
One has

$$[\rho_E^* C](t,E) = C(t,\rho_E^* E) = C(t,E^{(1)}).C(t,\check{L}_E)$$

by applying the additivity formula to the exact sequence of the introduction. Let $\xi_E = c_1(L_E)$ so that $C(t,\check{L}_E) = 1-t\xi_E$. Multiplying by $(1-t\xi_E)^{-1}$ and comparing constant terms, we get:

$$c_{r-1}(E^{(1)}) = \sum_{i=0}^{r-1} \rho_E^* c_i(E).\xi_E^{r-1-i}. \tag{1·1}$$

Let $G = E \oplus \mathbf{1}_Y$ where $\mathbf{1}_Y$ is the trivial line bundle over Y. Then $\bar{E} = P(G)$ is the projective closure of E. The canonical inclusions of E and $\mathbf{1}_Y$ in G give two subvarieties of $\bar{E}$ isomorphic to $P(E)$ and Y respectively, which can be called the 'roof' and the 'floor' of $\bar{E}$. We shall identify the roof with $P(E)$ and the floor with Y.



If $\bar{\imath}: Y \to \bar{E}$ and $\bar{\jmath}: P(E) \to \bar{E}$ are the inclusion maps then

$$\rho_G \bar{\imath} = Id_Y, \quad \rho_G \bar{\jmath} = \rho_E, \tag{1·2}$$

where ρ_G is the projection of $P(G)$. One has

$$\bar{\jmath}_*(1) = \xi_G. \tag{1·3}$$

and

$$\bar{\imath}_*(1) = [\rho_G^* \bar{C}](\xi_G, E) \tag{1·4}$$

according to Scott's formula ((5) and (7)). The first equality implies

$$\bar{\imath}^*(\xi_G) = 0 \tag{1·5}$$

https://doi.org/10.1017/S0305004100051550 Published online by Cambridge University Press

because $P(E) \cap Y = \varnothing$ in $\bar{E}$. One can now prove the self intersection formula for the special case where Y is regarded as a subvariety of the variety $\bar{E}$ as follows.† For any $y \in \mathscr{A}(Y)$, $\bar{\imath}_*(y) = \bar{\imath}_*\bar{\imath}^*\rho_G^*(y)$ by (1·2). By projection formula $\bar{\imath}_*(y) = \rho_G^*(y).\bar{\imath}_*(1)$ hence

$$\bar{\imath}_*(y) = \rho_G^*(y).[\rho_G^*\bar{C}](\xi_G, E), \tag{1·6}$$

using also (1·4). Applying $\bar{\imath}^*$ on both sides and taking into account (1·5),

$$\bar{\imath}^*\bar{\imath}_*(y) = \bar{\imath}^*\rho_G^*(y).\bar{\imath}^*\rho_G^* c_r(E) = \bar{\imath}^*\rho_G^*(yc_r(E)) = yc_r(E),$$

since $\bar{\imath}^*\rho_G^* = Id$. Hence

$$\bar{\imath}^*\bar{\imath}_*(y) = y.c_r(E) \tag{1·7}$$

for any $y \in \mathscr{A}(Y)$.

A useful remark in this situation is that $\bar{\jmath}_*$ is injective. In fact, if α is any element of $\mathscr{A}(P(E))$ then $\alpha = \sum_0^{r-1} \rho_E^* a_i.\xi_E^i$, so $\alpha = \bar{\jmath}^* \sum_0^{r-1} \rho_G^* a_i.\xi_G^i = \bar{\jmath}^*\beta$, say. Then

$$\bar{\jmath}_*\alpha = \bar{\jmath}_*\bar{\jmath}^*\beta = \beta.\bar{\jmath}_*(1) = \beta\xi_G$$

by (1·3). So $\bar{\jmath}_* \sum_0^{r-1} \rho_E^* a_i.\xi_E^i = \sum_0^{r-1} \rho_G^* a_i.\xi_G^{i+1}$. As the minimal equations of ξ_E and ξ_G are respectively of degrees $r-1$ and r it follows that $\bar{\jmath}_*$ is injective.

LEMMA 1·1. ('Excision lemma' cf. (3), 4–30, Lemma 4). *Let U, V be two varieties, V a subvariety of U, $W = U \setminus V$ and Z a cycle of U such that its restriction Z_W to W is rationally equivalent to zero. Then there exists a cycle Γ of V which is rationally equivalent to Z on U.*

COROLLARY 1·2. *Consider the blowing-up diagram*

$$\begin{array}{ccc} V' & \longrightarrow & U' \\ \downarrow & & \downarrow \phi \\ V & \longrightarrow & U \end{array}$$

and let Z be a cycle of U such that no irreducible component of Z is contained in V. Let Z' be the proper transform of Z by ϕ. Assume that the restriction Z_W of Z to $W = U \setminus V$ is rationally equivalent to zero. Then there is a cycle Γ of V' such that Z' is rationally equivalent to Γ.

Proof. $W' = U' \setminus V'$ is isomorphic to W by ϕ and the restriction $Z'_{W'}$ of Z' to W' corresponds to Z_W. Hence $Z'_{W'}$ is rationally equivalent to zero. Apply (1·1) to Z', U' and V'.

2. *The construction.* Let $\tilde{X} = X \times \mathbb{P}^1$ and blow it up along $Y \times 0$ to get a morphism $\tilde{f}: \tilde{X}' \to \tilde{X}$. One can identify X with $X \times 0$ and its proper transform by $\tilde{f}^{-1}$ with X'.

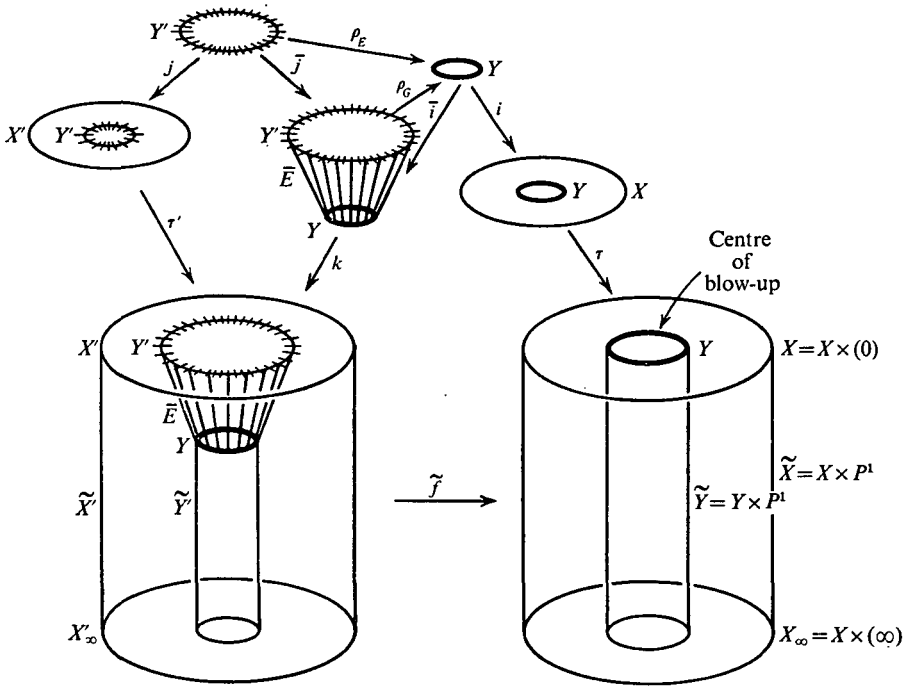
† A no more difficult argument enable us to establish the self intersection formula in the case where X and Y are both projective bundles over the same base space.

https://doi.org/10.1017/S0305004100051550 Published online by Cambridge University Press

Then $\tilde{f}|X' = f$ and the total transform of X is $\tilde{f}^{-1}(X) = X' + \bar{E}$, because the normal bundle of $Y \times 0$ in $\tilde{X}$ is $E \oplus \mathbf{1}_Y$, $Y \times 0$ is a simple subvariety of $X \times 0$ and

$$\operatorname{codim}_{\tilde{X}}(X \times 0) = 1.$$

Also $\bar{E}$ is attached to X' along Y' its roof, $\tilde{Y}'$ is the proper transform of $Y\,(= Y \times \mathbb{P}^1)$ and $\tilde{f}$ induces an isomorphism $\tilde{Y}' \xrightarrow{\sim} \tilde{Y}$.



For convenience we shall list the maps needed in the sequel: The projection

$$\pi' : \tilde{Y}' \to Y.$$

The inclusions†

$$\begin{aligned} i &: Y \to X, \quad \bar{\imath} : Y \to \bar{E}, \quad \tilde{\imath}' : \tilde{Y}' \to \tilde{X}', \\ j &: Y' \to X', \quad \bar{\jmath} : Y' = P(E) \to \bar{E}, \\ k &: \bar{E} \to \tilde{X}, \\ \beta &: X_\infty \to \tilde{X}, \quad \beta' : X'_\infty \to \tilde{X}', \\ \tau &: X \to \tilde{X}, \quad \tau' : X' \to \tilde{X}'. \end{aligned}$$

LEMMA 2·1. $k_*(1) = \beta'_*(1) - \tau'_*(1)$.

Proof. $\tau_*(1) = \beta_*(1)$ and $\tilde{f}^*\tau_*(1) = k_*(1) + \tau'_*(1)$, $\tilde{f}^*\beta_*(1) = \beta'_*(1)$.

LEMMA 2·2. (i) $\bar{\imath}^* = \pi'_* \tilde{\imath}'^* k_*$, (ii) $\tau'^* k_* = j_* \bar{\jmath}^*$.

Proof. (i). Any subvariety Z of $\bar{E}$ properly intersecting Y on $\bar{E}$, also intersects $\tilde{Y}'$ properly on $\tilde{X}'$ because $\operatorname{codim}_{\bar{E}}(Y) = \operatorname{codim}_{\tilde{X}'}(\tilde{Y}')$ and $Z \cap Y = Z \cap \tilde{Y}'$. The proof of (ii) is similar taking into account that $\operatorname{codim}_{\bar{E}}(Y') = \operatorname{codim}_{\tilde{X}'}(X') = 1$.

† β and τ are just abbreviations for 'bottom' and 'top'.

https://doi.org/10.1017/S0305004100051550 Published online by Cambridge University Press

LEMMA 2·3. $\bar{\imath}^* k^* k_* = 0$.

Proof. By Lemma 2·2, (i) for any $u \in \mathscr{A}(\bar{E})$, $\bar{\imath}^* k^* k_* u = \pi'_* \tilde{\imath}'^* k_* k^* k_* u$. Then

$$k_* k^* k_* u = k_*(u)\, k_*(1)$$

by projection formula, $k_*(u).k_*(1) = k_*(u)(\beta'_*(1) - \tau'_*(1))$ by Lemma 2·1 and $k_*(u).\beta'_*(1) = 0$, because $\bar{E} \cap X'_\infty = \varnothing$. It follows that $k_* k^* k_* u = -k_*(u)\tau'_*(1)$. Finally $\pi'_* \tilde{\imath}'^* k_* k^* k_* u = -\pi'_* \tilde{\imath}'^* (k_*(u).\tau'_*(1)) = 0$ because $k_*(u).\tau'_*(1)$ is represented by a cycle on X' which will be disjoint from $\tilde{Y}' = \mathrm{Im}\,(\tilde{\imath}')$.

3. *The self intersection formula.* For any cycle A of X denote $A \times \mathbb{P}^1$ by $\tilde{A}$ and let $\tilde{A}'$ be the proper transform of $\tilde{A}$ by $\tilde{f}$. The following lemma is straightforward to verify:

LEMMA 3·1. (i) *If A is a subvariety of Y, then $\tilde{A}'$ is isomorphic with $\tilde{A}$ and it intersects the floor of $\bar{E}$ along A; moreover $\tilde{A}'.\bar{E}$ is defined on $\tilde{X}'$, $\tilde{A}'.\bar{E} = A$ and $\tilde{A}'.X' = 0$.*

(ii) *If A intersects Y properly on X then the cycle $\tilde{f}^{-1}(\tilde{A})$ is defined and $\tilde{f}^{-1}(\tilde{A}) = \tilde{A}'$; the cycle $\tilde{A}'.\bar{E}$ is defined on $\tilde{X}'$ and $\tilde{A}'.\bar{E} = \rho_G^{-1}(A\,.\,Y)$.*

(iii) *Under the assumption of* (ii) *the cycle $X'.\tilde{A}'$ is defined on $\tilde{X}'$ and $X'.\tilde{A}' = f^{-1}(A)$.*

THEOREM 1. *For any $y \in \mathscr{A}(Y)$*

$$i^* i_*(y) = y c_r(E),$$

where E is the normal bundle of Y in X.

Proof. One can assume that y is the class of a subvariety B of Y. Let B_1 be a cycle of X properly intersecting Y and rationally equivalent to B. According to Lemma 3·1, (i)

$$\tilde{B}'.\bar{E} = B \tag{3·1}$$

and

$$\tilde{B}'.X' = 0. \tag{3·2}$$

By (ii) and (iii) of the same lemma

$$\tilde{f}^{-1}(\tilde{B}_1) = \tilde{B}'_1, \tag{3·3}$$

$$\tilde{B}'_1.\bar{E} = \rho_G^{-1}(B_1\,.\,Y), \tag{3·4}$$

$$\tilde{B}'_1.X' = f^{-1}(B_1). \tag{3·5}$$

Also the class of $B_1\,.\,Y$ in $\mathscr{A}(Y)$ is

$$cl_Y(B_1\,.\,Y) = i^* i_*(y). \tag{3·6}$$

The cycle $\tilde{B} - \tilde{B}_1$ is rationally equivalent to zero on $\tilde{X}$. By Corollary 1·2, with $\phi = \tilde{f}$ and $Z = \tilde{B} - \tilde{B}_1$, there exists a cycle Γ of $\bar{E}$ rationally equivalent to $\tilde{B}' - \tilde{B}'_1$ on $\tilde{X}'$, i.e. in rational equivalence classes

$$cl_{\tilde{X}'}(\tilde{B}') - cl_{\tilde{X}'}(\tilde{B}'_1) = k_*(\gamma), \tag{3·7}$$

where $\gamma \in \mathscr{A}(\bar{E})$ is the class of Γ. According to (3·1) $k^* cl_{\tilde{X}'}(\tilde{B}') = \bar{\imath}_*(y)$. Also from (3·4) and (3·6) one deduces $k^* cl_{\tilde{X}'}(\tilde{B}'_1) = \rho_G^* i^* i_*(y)$. These give, by applying k^* to (3·7),

$$\bar{\imath}_*(y) - \rho_G^* i^* i_*(y) = k^* k_*(\gamma). \tag{3·8}$$

https://doi.org/10.1017/S0305004100051550 Published online by Cambridge University Press

Applying $\bar{i}^*$ on both sides and taking into account that $\bar{i}^* k^* k_* = 0$ by Lemma 2·3, $\bar{i}^* \bar{i}_*(y) - \bar{i}^* \rho_G^* i^* i_*(y) = 0$, hence $\bar{i}^* \bar{i}_*(y) = i^* i_*(y)$ because $\rho_G \bar{i} = Id_Y$. To conclude apply (1·7).

Note. Since $i^* i_*(y) = y c_r(E)$ one can write (3·8) as follows

$$\bar{i}_*(y) = \rho_G^*(y c_r(E)) + k^* k_*(\gamma). \tag{3·9}$$

4. *The 'formule-clef' for* $\mathscr{A}(X')$.

LEMMA 4·1. *With notations of section* 3

$$-\bar{j}^*(\gamma) = \rho_E^*(y) \cdot c_{r-1}(E^{(1)}).$$

Proof. By (3·9) $k^* k_* \gamma = \bar{i}_*(y) - \rho_G^*(y \cdot c_r(E))$. Also

$$k^* k_*(\gamma) = -\gamma \cdot \xi_G = -\gamma \cdot \bar{j}_*(1) = -\bar{j}_* \bar{j}^*(\gamma)$$

using the self-intersection formula, the fact that $\check{L}_G$ is the normal bundle of $\bar{E}$ in $\tilde{X}'$ and the formula $c_1(\check{L}_G) = -\xi_G$ for the first equality; and using (1·3) for the second formula. By (1·6) and (1·3) again:

$$\begin{aligned}
\bar{i}_*(y) - \rho_G^*(y \cdot c_r(E)) &= \rho_G^*(y) \cdot ([\rho_G^* \bar{C}](\xi_G, E) - \rho_G^* c_r(E)) \\
&= \rho_G^*(y) \cdot \xi_G \cdot \left(\sum_{i=0}^{r-1} \xi_G^{r-1-i} \cdot \rho_G^* c_i(E) \right) \\
&= \rho_G^*(y) \cdot \bar{j}_* \bar{j}^* \left(\sum_{i=0}^{r-1} \xi_G^{r-1-i} \cdot \rho_G^* c_i(E) \right).
\end{aligned}$$

But $\bar{j}^* \xi_G = \xi_E$ and $\rho_E = \rho_G j$, so putting everything together and using (1·1), we get:

$$\begin{aligned}
-\bar{j}_* \bar{j}^*(\gamma) &= \bar{i}_*(y) - \rho_G^*(y \cdot c_r(E)) \\
&= \bar{j}_* \left(\rho_E^*(y) \cdot \sum_{i=0}^{r-1} \xi_E^{r-1-i} \cdot \rho_E^* c_i(E) \right) \qquad (4·1) \\
&= \bar{j}_*(\rho_E^*(y) \cdot c_{r-1}(E^{(1)})).
\end{aligned}$$

But $\bar{j}_*$ is injective as remarked in section 1, so we may cancel it in (4·1).

THEOREM 2. *For any* $y \in \mathscr{A}(Y)$

$$j_*(\rho^*(y) \cdot c_{r-1}(E^{(1)})) = f^* i_*(y).$$

Proof. By applying τ'^* to (3·7), $-\tau'^* cl_{\tilde{X}'}(\tilde{B}_1') = \tau'^* k_*(\gamma)$ since $\tilde{B}'$ is a cycle of $\tilde{Y}'$ and $\tilde{Y}' \cap X' = \varnothing$. Also $\tau'^* cl_{\tilde{X}'}(\tilde{B}_1') = f^* i_*(y)$ according to (3·5). Hence

$$-f^* i_*(y) = \tau'^* k_*(\gamma).$$

By (ii) of Lemma 2·2, $\tau'^* k_*(\gamma) = j_* \bar{j}^*(\gamma)$. But in Lemma (4·1) we have shown that $-\bar{j}^*(\gamma) = \rho_E^*(y) c_{r-1}(E^{(1)})$. Putting this together, we get the 'formule-clef'.

https://doi.org/10.1017/S0305004100051550 Published online by Cambridge University Press

REFERENCES

(1) Borel, A. and Serre, J.-P. Le théorème de Riemann-Roch. *Bull. Soc. Math. France* **86** (1958), 97–136.

(2) Berthelot, A., Grothendieck, A. and Illusie, L. Théorie des intersections et théorème de Riemann-Roch. SGA6, Springer Lecture Notes no. 225.

(3) Grothendieck, A. Sur quelques propriétés fondamentales en théorie des intersections. *Anneaux de Chow et applications. Séminaire C. Chevalley*, 2e année (1958).

(4) Grothendieck, A. La théorie des classes de Chern. *Bull. Soc. Math. France* **86** (1958), 137–159.

(5) Ilori, S., Ingleton, A. W. and Lascu, A. T. On a formula of D. B. Scott. *J. London Math. Soc.* (2), **8** (1974), 539–544.

(6) Jouanolou, J. P. Riemann–Roch sans dénominateurs. *Inventiones Math.* **11** (1970), 15–26.

(7) Lascu, A. T. and Scott, D. B. An algebraic correspondence with applications to projective bundles and blowing-up Chern classes. *Annali di Matematica pura ed applicata* (to appear).

https://doi.org/10.1017/S0305004100051550 Published online by Cambridge University Press