Mathematical Structures in Computer Science (2024), **34**, pp. 375–389
doi:10.1017/S0960129524000033

CAMBRIDGE UNIVERSITY PRESS

PAPER

Parameterized complexity of weighted team definability

Juha Kontinen[1], Yasir Mahmood[2], Arne Meier[3] and Heribert Vollmer[3]

[1]Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland [2]Department of Computer Science, Paderborn University, Paderborn, Germany [3]Institut für Theoretische Informatik, Leibniz Universitfät Hannover, Hanover, Germany

Corresponding author: Arne Meier; Email: meier@thi.uni-hannover.de

(Received 12 April 2023; revised 07 November 2023; accepted 29 January 2024; first published online 20 February 2024)

Abstract

In this article, we study the complexity of weighted team definability for logics with team semantics. This problem is a natural analog of one of the most studied problems in parameterized complexity, the notion of weighted Fagin-definability, which is formulated in terms of satisfaction of first-order formulas with free relation variables. We focus on the parameterized complexity of weighted team definability for a fixed formula φ of central team-based logics. Given a first-order structure $\mathcal{A}$ and the parameter value $k \in \mathbb{N}$ as input, the question is to determine whether $\mathcal{A}, T \models \varphi$ for some team T of size k. We show several results on the complexity of this problem for dependence, independence, and inclusion logic formulas. Moreover, we also relate the complexity of weighted team definability to the complexity classes in the well-known W-hierarchy as well as paraNP.

Keywords: Parameterized complexity; descriptive complexity; weighted definability; team semantics; dependence logic; independence logic; inclusion logic

1. Introduction

In this article, we study the parameterized complexity of weighted team definability for logics in team semantics. Team definability is a natural analog of the notion of Fagin-definability whose weighted version can be used to characterize the W-hierarchy in parameterized complexity (Downey et al. 1998). We give several results on the complexity of this problem for dependence, independence, and inclusion logic formulas.

The birth of the nowadays established logics of dependence and independence can be traced back to the introduction of dependence logic in 2007 (Väänänen 2007). In team semantics, formulas are interpreted by sets of assignments (teams) instead of a single assignment as in Tarski's semantics of first-order logic. Syntactically dependence logic extends first-order logic by new dependence atomic formulas (dependence atoms) $\mathsf{dep}(\mathbf{x};y)$ expressing that the values of variables $\mathbf{x}$ functionally determine the value of the variable y in the team under consideration. Independence and inclusion logics are further extensions of first-order logic by independence atoms $\mathbf{x} \perp_{\mathbf{z}} \mathbf{y}$ and inclusion atoms $\mathbf{x} \subseteq \mathbf{y}$ which essentially correspond to embedded multivalued dependences and inclusion dependences from database theory (Galliani 2012; Grädel and Väänänen 2013).

For the applications, it is important to understand the complexity theoretic aspects of team-based logics. During the past ten years, the expressivity and complexity theoretic aspects of logics

© The Author(s), 2024. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

in first-order (also propositional Yang and Väänänen 2017, modal Hella et al. 2019, 2020, temporal Gutsfeld et al. 2022 and probabilistic Durand et al. 2018) team semantics have been studies extensively (see, e.g., Hannula et al. 2018, Lück 2019, Hannula et al. 2020, Durand et al. 2022). The baseline for these studies is the well-known results stating that the sentences of dependence logic and independence logic are equivalent to existential second-order logic (Grädel and Väänänen 2013), while inclusion logic corresponds to positive greatest fixed point logic and thereby captures **P** over finite (ordered) structures (Galliani and Hella 2013). In team semantics results for sentences of the logic do not immediately extend to open formulas. In particular, the open formulas of dependence logic correspond in expressive power to sentences of $\mathcal{ESO}$ with an extra relation encoding the team that occurs only negatively in the sentence (Kontinen and Väänänen 2009). For independence logic, the requirement of negativity can be lifted (Galliani 2012). For inclusion logic an analogous result shows that any first-order sentence $\varphi(R)$ whose truth is preserved under R-unions can be expressed by an inclusion logic formula $\varphi^*(\mathbf{x})$. In other words, for all $\mathcal{A}$ and teams $T \neq \emptyset$:

$$\mathcal{A}, T \models \varphi^*(\mathbf{x}) \Leftrightarrow \mathcal{A} \models \varphi(\mathrm{rel}(T)/R),$$

where $\mathrm{rel}(T)$ is a relation encoding the team T (Galliani and Hella 2013). These results can be used to relate weighted team definability to weighted Fagin-definablity. However, it is instructive to note that, due to higher expressive power of the logics considered in this article, the syntactic complexity of a formula does not in general correlate with the complexity of the model-checking of the formula. In particular, any formula of dependence and independence logic is logically equivalent to a formula with $\forall\exists$-quantifier prefix (Väänänen 2007, Theorem 6.15) (Kontinen and Väänänen 2009, Theorem 4.9).

A formalism to enhance the understanding of the inherent intractability of computational problems is brought by the framework of parameterized complexity (Downey and Fellows 2013). Here, one aims to find parameters relevant for practice allowing to solve the problem by algorithms running in time $f(k) \cdot n^{O(1)}$, for some computable function f, where k is the parameter value and n is the input length. Problems with such a running time are called *fixed-parameter tractable* (**FPT**) and correspond to efficient computation in the parameterized setting. The problems solvable within the runtimes of the form $f(k) \cdot n^{O(1)}$ with respect to nondeterministic machines belong to the complexity class **paraNP** $\supseteq$ **FPT**. Moreover, restricting the amount of nondeterminism allows to study a subclass $\mathbf{W}[\mathbf{P}] \subseteq$ **paraNP**. The complexity class $\mathbf{W}[\mathbf{P}]$ is defined via nondeterministic machines that have at most $h(k) \cdot \log n$ many nondeterministic steps, where h is a computable function. In between **FPT** and $\mathbf{W}[\mathbf{P}]$, a presumably infinite **W**-hierarchy is contained: $\mathbf{FPT} \subseteq \mathbf{W}[1] \subseteq \mathbf{W}[2] \subseteq \cdots \subseteq \mathbf{W}[\mathbf{P}]$. It is unknown whether any of these inclusions is strict. Showing $\mathbf{W}[1]$-hardness of a problem in parameterized world intuitively corresponds to being intractable in this setting. Moreover, $\mathbf{W}[1]$ contains problems that can be solved by an algorithm whose nondeterministic steps are bounded in terms of the parameter and occur at the end of the computation.

Our contributions. We define and study the parameterized complexity of weighted team definability with respect to formulas of several team-based logics. Moreover, we establish the relationship between our framework and the problem of weighted Fagin definability. In more details, we explore the complexity of weighted team definability in parameterized setting for dependence, independence, and inclusion logic formulas as well as sentences. Thereby, we prove and obtain novel logical characterizations of, and new complete problems for, the aforementioned central parameterized complexity classes, i.e., the **W**-hierarchy, $\mathbf{W}[\mathbf{P}]$, and **paraNP**. Table 1 gives a partial overview of our results concerning weighted team definability.

Related work. The complexity of counting/enumerating satisfying teams for a fixed first-order formula of team-based logic has been studied before (Haak et al. 2019, 2022). Furthermore,

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

Table 1. Partial overview of our results concerning weighted team definability with pointers to the respective theorem or corollary

Logic	$\exists\, \varphi_t$ s.t. p-WT$_{\varphi_t}$ is	Condition	Result
$\mathcal{FO}$	in TC^0	unparameterized, all formulas	Thm. 9
$\mathcal{FO}(\subseteq)$	in **FPT**/**W**[**P**]	all sentences/all formulas	Thm. 10/Thm. 17
	W[1]-hard, $\in$ **W**[2]	any quantifier-free formula, without $\vee$	Cor. 12
	W[t]-complete	for all even $t \in \mathbb{N}$	Cor. 16
$\mathcal{FO}$(dep)	**W**[t]-complete	for all odd $t \in \mathbb{N}$	Cor. 22
	paraNP-complete	sentence/formula	Thm. 18/Thm. 23
$\mathcal{FO}(\bot)$	**W**[t]-complete	for all $t \in \mathbb{N}$	Thm. 24 (1.)
	W[**P**]-complete	formula	Thm. 24 (2.)
	paraNP-complete	sentence/formula	Thm. 24 (3.) / (4.)

there are also recent works on the parameterized complexity model-checking and satisfiability for propositional and first-order team-based logics (Kontinen et al. 2022; Mahmood and Meier 2022; Mahmood and Virtema 2021; Meier and Reinbold 2018). Regarding the descriptive complexity, Downey et al. (1998) explored the logical characterization of the classes in the **W**-hierarchy.

2. Preliminaries

We require a basic knowledge of standard notions from classical complexity theory (Papadimitriou 1994). The classical complexity classes we encounter mostly in this work are **P** and **NP** together with their respective completeness notions, employing polynomial time many-one reductions ($\leq_m^{\mathbf{P}}$). Moreover, we assume the reader is familiar with the basic first-order (predicate) logic (Ebbinghaus and Flum 1995). In the following, we define a few important classes of first-order formulas that are relevant to the results in this work.

FO-Formula Classes. The class of all first-order formulas is denoted by $\mathcal{FO}$. Let τ be a relational vocabulary and $R \in \tau$ be a relation symbol of arity r. An atomic formula is a formula of the form $x = y$ or $R(x_1, \ldots, x_r)$. A literal is an atomic or a negated atomic formula. A quantifier-free formula is a formula that contains no quantifiers and a formula is in negation normal form (NNF) if the negation symbols occurs only front of atoms. A formula φ is in prenex normal form if φ has the form $Q_1 x_1 \ldots Q_n x_n \psi$, where ψ is quantifier free and $Q_1, \ldots, Q_n \in \{\exists, \forall\}$. The classes Σ_0 and Π_0 both consist of quantifier-free formulas. Then, for $t \geq 0$, the class Σ_{t+1} includes all formulas of the form $\exists x_1 \ldots \exists x_\ell \varphi$, where $\varphi \in \Pi_t$. Similarly, Π_{t+1} includes all formulas of the form $\forall x_1 \ldots \forall x_\ell \varphi$, where $\varphi \in \Sigma_t$.

Fagin Definability. The first-order variables range over individual elements of the universe. In second-order logic, one also quantifies relation variables which range over relations on the universe. We now introduce first-order formulas where we also allow relation variables. Let τ be a vocabulary, X_i for $i \leq n$ be free relation variables of arity s_i and $\varphi(X_1, \ldots, X_n)$ be a $\mathcal{FO}$-formula with free relation variables in τ. Notice that $\varphi(X_1, \ldots, X_n)$ does not contain free first-order variables. Moreover, let $\mathcal{A}$ be a τ-structure and $S_i \subseteq A^{s_i}$ be relations over $\mathcal{A}$ for $i \leq n$. Then we say that the tuple $\bar{S} = (S_1, \ldots, S_n)$ is a solution for φ in $\mathcal{A}$ if $\mathcal{A} \models \varphi(\bar{S})$. We call the following decision problem, the problem *Fagin-defined* by φ.

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

Problem:	FD_φ – Fagin definability for fixed $\varphi \in \mathcal{FO}$
Input:	A τ-structure $\mathcal{A}$.
Question:	Is there a solution for φ in $\mathcal{A}$?

Let $\Theta \subseteq \mathcal{FO}$ be a class of formulas, then by FD-Θ we denote the class of all problems FD_φ such that $\varphi \in \Theta$. The following result regarding $\mathcal{FO}$ is known.

Proposition 1. (Flum and Grohe 2006, Cor. 4.35)**.** **NP** = FD-$\mathcal{FO}$ = FD-Π_2.

Next we introduce the following *weighted* version of Fagin definabilty, where we restrict our solution to have a specific size for a single free relation symbol S of arity s. Notice that the inclusion of a single free relation variable is without loss of generality as allowing more than one such variables does not increase the complexity (see Flum and Grohe 2006, p.87).

Problem:	WD_φ – weighted Fagin definability for fixed $\varphi \in \mathcal{FO}$
Input:	A τ-structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Question:	Is there a solution for φ of cardinality k?

As before, for a class $\Theta \subseteq \mathcal{FO}$ of formulas, we denote by WD-Θ the class of all problems WD_φ such that $\varphi \in \Theta$.

Example 2. We assume standard graph theoretic notation and consider loop-free graphs. The problem CLIQUE is defined as follows. Given a graph $\mathcal{G} := (V, E)$ and $k \in \mathbb{N}$. Is there a set $S \subseteq V$ such that $|S| = k$ and $(u, v) \in E$ for every $x, y \in S$? Then CLIQUE is WD_{φ_c}, where

$$\varphi_c(X) := \forall x \forall y \big((X(x) \wedge X(y) \wedge x \neq y) \rightarrow E(xy) \big).$$

Consequently, CLIQUE is in WD-Π_1.

Moreover, Let DOMINATINGSET be the problem to determine if a graph $\mathcal{G}$ contains a set $S \subseteq V$ such that $|S| = k$ and every vertex in $V \setminus S$ is incident to some vertex in S? Then DOMINATINGSET is in WD-Π_2 since the problem is WD_{φ_d}, where

$$\varphi_d(X) := \forall x \exists y \big(X(y) \wedge (E(x, y) \vee x = y) \big).$$

Parameterized Complexity Theory. A *parameterized problem* (PP) $P \subseteq \Sigma^* \times \mathbb{N}$ is a subset of the cross-product of an alphabet and the natural numbers. For an *instance* $(x, k) \in \Sigma^* \times \mathbb{N}$, k is called the (value of the) *parameter*. A *parameterization* is a polynomial-time computable function that maps a value from $x \in \Sigma^*$ to its corresponding $k \in \mathbb{N}$. The problem P is said to be *fixed-parameter tractable* (or in the class **FPT**) if there exists a deterministic algorithm $\mathcal{A}$ and a computable function f such that for all $(x, k) \in \Sigma^* \times \mathbb{N}$, algorithm $\mathcal{A}$ correctly decides the membership of $(x, k) \in P$ and runs in time $f(k) \cdot |x|^{O(1)}$. The problem P belongs to the class **XP** if $\mathcal{A}$ runs in time $|x|^{f(k)}$ on a deterministic machine. Abusing a little bit of notation, we write $\mathcal{C}$-machine for the type of machines that decide languages in the class $\mathcal{C}$, and we will say a function f is *$\mathcal{C}$-computable* if it can be computed by a machine on which the resource bounds of the class $\mathcal{C}$ are imposed. The class **paraNP** includes problems decidable by a nondeterministic algorithm $\mathcal{A}$ which runs in time $f(k) \cdot |x|^{O(1)}$ for some computable function f. One can define a parameterized complexity class **para**$\mathcal{C}$ corresponding to a complexity class $\mathcal{C}$ via a *precomputation on the parameter*.

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

Definition 3. Let $\mathcal{C}$ be any complexity class. Then **para**$\mathcal{C}$ is the class of all PPs $P \subseteq \Sigma^* \times \mathbb{N}$ such that there exists a computable function $\pi : \mathbb{N} \to \Delta^*$ and a language $L \in \mathcal{C}$ with $L \subseteq \Sigma^* \times \Delta^*$ such that for all $(x, k) \in \Sigma^* \times \mathbb{N}$ we have that $(x, k) \in P \Leftrightarrow (x, \pi(k)) \in L$.

Notice that **paraP** = **FPT** and the two definitions of **paraNP** are equivalent.

A problem P is in the complexity class **W[P]**, if it can be decided by a NTM running in time $f(k) \cdot |x|^{O(1)}$ steps, with at most $g(k)$-many nondeterministic steps, where f, g are computable functions. Moreover, **W[P]** is contained in the intersection of **paraNP** and **XP** (for details see the textbook of Flum and Grohe 2006).

Let $c \in \mathbb{N}$ and $P \subseteq \Sigma^* \times \mathbb{N}$ be a PP, then the *c-slice of P*, written as P_c is defined as $P_c := \{ (x, k) \in \Sigma^* \times \mathbb{N} \mid k = c \}$. Notice that P_c is a classical problem then.

Definition 4. Let $P \subseteq \Sigma^* \times \mathbb{N}$, $Q \subseteq \Gamma^* \times \mathbb{N}$ be two PPs. One says that P is *fpt-reducible* to Q, $P \leq^{\mathbf{FPT}} Q$, if there exists an **FPT**-computable function $f : \Sigma^* \times \mathbb{N} \to \Gamma^* \times \mathbb{N}$ such that

- for all $(x, k) \in \Sigma^* \times \mathbb{N}$ we have that $(x, k) \in P \Leftrightarrow f(x, k) \in Q$,
- there exists a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all $(x, k) \in \Sigma^* \times \mathbb{N}$ and $f(x, k) = (x', k')$ we have that $k' \leq g(k)$.

Finally, in order to show that a problem P is **para**$\mathcal{C}$-hard for some complexity class $\mathcal{C}$, it is sufficient to prove that for some $c \in \mathbb{N}$, the slice P_c is $\mathcal{C}$-hard in the classical setting (Flum and Grohe 2006, Thm. 2.14).

To define the complexity classes in **W**-hierarchy, the parameterized version of the problem WD_φ is now defined as follows.

Problem:	p-WD_φ – parameterized weighted Fagin definability for fixed $\varphi \in \mathcal{FO}$
Input:	A τ-structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Is there a solution for φ of cardinality k?

The complexity classes of the **W**-hierarchy are characterized via the following definition.

Definition 5 (Flum and Grohe 2006, Def. 5.1)**.** For every $t \geq 1$, we let $\mathbf{W}[t] := [\text{p-WD-}\Pi_t]^{\mathbf{FPT}}$. The class $\mathbf{W}[t]$ forms the t-th level of the **W**-hierarchy.

Alternatively, the **W**-hierarchy can be defined via the weighted satisfiability problem for propositional formulas. Let I be a nonempty index set and $d \in \mathbb{N}$. Consider the following special subclasses of propositional formulas:

$$\Gamma_{0,d} = \{\ell_1 \wedge \cdots \wedge \ell_c \mid \ell_1, \ldots, \ell_c \text{ are literals and } c \leq d\},$$
$$\Delta_{0,d} = \{\ell_1 \vee \cdots \vee \ell_c \mid \ell_1, \ldots, \ell_c \text{ are literals and } c \leq d\},$$
$$\Gamma_{t,d} = \left\{ \bigwedge_{i \in I} \alpha_i \;\middle|\; \alpha_i \in \Delta_{t-1,d} \text{ for } i \in I \right\},$$
$$\Delta_{t,d} = \left\{ \bigvee_{i \in I} \alpha_i \;\middle|\; \alpha_i \in \Gamma_{t-1,d} \text{ for } i \in I \right\}.$$

Finally, $\Gamma^+_{t,d}$ (resp. $\Gamma^-_{t,d}$) denote the class of all positive (negative) formulas in $\Gamma_{t,d}$.

The parameterized weighted satisfiability problem (p-WSAT) for propositional formulas is defined as below. The parameter *weight* of an assignment s is the number of variables mapped to 1 by s.

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

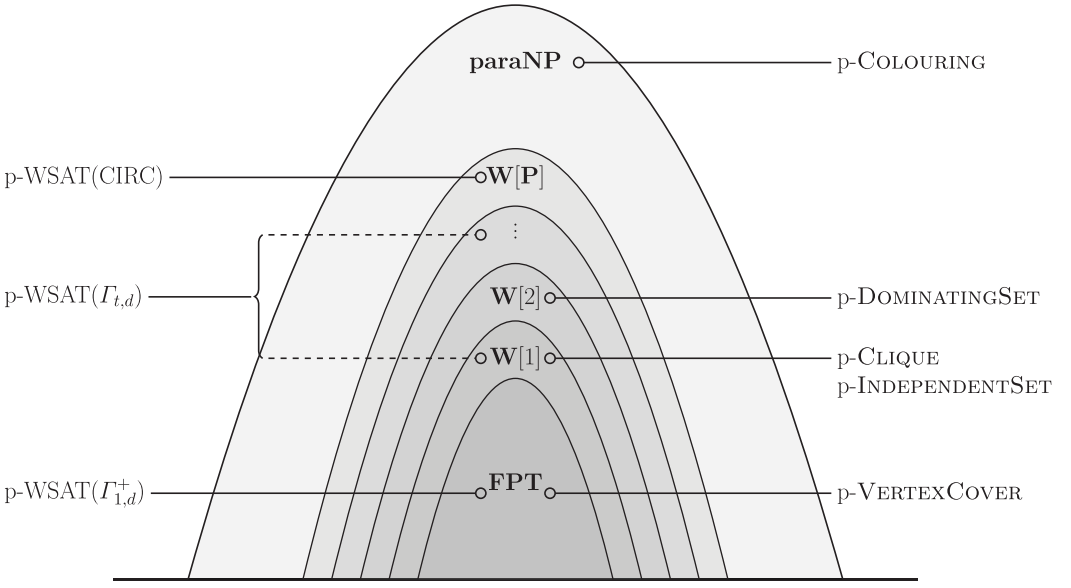


Figure 1. Landscape of relevant parameterized complexity classes with complete problems. The definition of several of these complete problems is mentioned in the relevant proofs.

Problem:	p-WSAT($\Gamma_{t,d}$) – parameterized weighted satisfiability
Input:	a $\Gamma_{t,d}$-formula α with $t, d \geq 1$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	is there a satisfying assignment for α of weight k?

The classes of the **W**-hierarchy are defined equivalently in terms of these problems.

Proposition 6 (Flum and Grohe 2006, Thm. 7.1)**.** *For every $t \geq 1$ the following problems are* **W**$[t]$*-complete under fpt-reductions.*

- *p*-WSAT($\Gamma^{+}_{t,1}$) *if t is even and p*-WSAT($\Gamma^{-}_{t,1}$) *if t is odd.*
- *p*-WSAT($\Gamma_{t,d}$) *for every $t, d \geq 1$.*

Fig. 1 draws the complexity landscape with complete problems in parameterized complexity that are relevant.

Team-based Logics. We assume basic familiarity with predicate logic (Ebbinghaus and Flum 1995). We consider first-order vocabularies τ that are sets of *function* symbols and *relation* symbols with an equality symbol $=$. Let VAR be a countably infinite set of *first-order variables*. Terms over τ are defined in the usual way, and the set of well-formed formulas of first-order logic ($\mathcal{FO}$) in negation normal form is defined by the following EBNF:

$$\psi ::= t_1 = t_2 \mid R(t_1, \ldots, t_k) \mid \neg R(t_1, \ldots, t_k) \mid \psi \wedge \psi \mid \psi \vee \psi \mid \exists x \psi \mid \forall x \psi,$$

where t_i are terms $1 \leq i \leq k$, R is a k-ary relation symbol from σ, $k \in \mathbb{N}$, and $x \in$ VAR. If ψ is a formula, then we use VAR(ψ) for its set of variables, and Fr(ψ) for its set of *free variables*. We evaluate $\mathcal{FO}$-formulas in τ-structures, which are pairs of the form $\mathcal{A} = (A, \tau^{\mathcal{A}})$, where A is

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

the *domain* of $\mathcal{A}$ (when clear from the context, we write A instead of $\mathrm{dom}(\mathcal{A})$), and $\tau^{\mathcal{A}}$ interprets the function and relational symbols in the usual way (e.g., $t^{\mathcal{A}}\langle s\rangle = s(x)$ if $t = x \in \mathrm{VAR}$). If $\mathbf{t} = (t_1, \ldots, t_n)$ is a tuple of terms for $n \in \mathbb{N}$, then we write $\mathbf{t}^{\mathcal{A}}\langle s\rangle$ for $(t_1^{\mathcal{A}}\langle s\rangle, \ldots, t_n^{\mathcal{A}}\langle s\rangle)$.

Dependence logic $\mathcal{FO}(\mathsf{dep})$ extends $\mathcal{FO}$ by dependence atoms of the form $\mathsf{dep}(\mathbf{t};u)$ where $\mathbf{t}$ and $\mathbf{u}$ are tuples of terms. Inclusion logic $\mathcal{FO}(\subseteq)$ is obtained by adding to $\mathcal{FO}$ the inclusion atoms of the form $\mathbf{t} \subseteq \mathbf{u}$ for tuples $\mathbf{t}$ and $\mathbf{u}$ of terms. Finally, independence logic $\mathcal{FO}(\perp)$ extends $\mathcal{FO}$ by independence atoms of the form $\mathbf{t}\perp_{\mathbf{v}}\mathbf{u}$ for tuples $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$ of terms. We call expressions of the kind $t_1 = t_2$, $R(\mathbf{t})$, $\mathsf{dep}(\mathbf{t};u)$, $\mathbf{t} \subseteq \mathbf{u}$ and $\mathbf{t}\perp_{\mathbf{v}}\mathbf{u}$ *atomic formulas*.

The semantics is defined through the concept of a team. Let $\mathcal{A}$ be a structure and $X \subseteq \mathrm{VAR}$, then an *assignment* s is a mapping $s\colon X \to A$.

Definition 7. Let $X \subseteq \mathrm{VAR}$. A *team* T *in* $\mathcal{A}$ *with domain* X is a set of assignments $s\colon X \to A$.

For a team T with domain $X \supseteq Y$ define its *restriction* to Y as $T\restriction Y := \{\, s\restriction Y \mid s \in T \,\}$. If $s\colon X \to A$ is an assignment and $x \in \mathrm{VAR}$ is a variable, then $s_a^x\colon X \cup \{x\} \to A$ is the assignment that maps x to a and $y \in X \setminus \{x\}$ to $s(y)$. Let T be a team in $\mathcal{A}$ with domain X. Then any function $f\colon T \to \mathcal{P}(A) \setminus \{\emptyset\}$ can be used as a *supplementing function* of T to extend or modify T to the *supplemented team* $T_f^x := \{\, s_a^x \mid s \in T, a \in f(s) \,\}$. For the case $f(s) = A$ is the constant function, we simply write $T_{\mathcal{A}}^x$ for T_f^x. The semantics of formulas is defined as follows.

Definition 8. Let τ be a vocabulary, $\mathcal{A}$ be a τ-structure and T be a team over $\mathcal{A}$ with domain $X \subseteq \mathrm{VAR}$. Then,

$$
\begin{aligned}
&\mathcal{A}, T \models t_1 = t_2 && \text{iff } \forall s \in T : t_1^{\mathcal{A}}\langle s\rangle = t_2^{\mathcal{A}}\langle s\rangle,\\
&\mathcal{A}, T \models t_1 \neq t_2 && \text{iff } \forall s \in T : t_1^{\mathcal{A}}\langle s\rangle \neq t_2^{\mathcal{A}}\langle s\rangle,\\
&\mathcal{A}, T \models R(t_1, \ldots, t_n) && \text{iff } \forall s \in T : (t_1^{\mathcal{A}}\langle s\rangle, \ldots, t_n^{\mathcal{A}}\langle s\rangle) \in R^{\mathcal{A}},\\
&\mathcal{A}, T \models \neg R(t_1, \ldots, t_n) && \text{iff } \forall s \in T : (t_1^{\mathcal{A}}\langle s\rangle, \ldots, t_n^{\mathcal{A}}\langle s\rangle) \notin R^{\mathcal{A}},\\
&\mathcal{A}, T \models \mathsf{dep}(\mathbf{t};u) && \text{iff } \forall s_1, s_2 \in T : \mathbf{t}^{\mathcal{A}}\langle s_1\rangle = \mathbf{t}^{\mathcal{A}}\langle s_2\rangle \implies \mathbf{u}^{\mathcal{A}}\langle s_1\rangle = \mathbf{u}^{\mathcal{A}}\langle s_2\rangle,\\
&\mathcal{A}, T \models \mathbf{t} \subseteq \mathbf{u} && \text{iff } \forall s_1 \in T, \exists s_2 \in T : \mathbf{t}^{\mathcal{A}}\langle s_1\rangle = \mathbf{u}^{\mathcal{A}}\langle s_2\rangle,\\
&\mathcal{A}, T \models \mathbf{t}\perp_{\mathbf{v}}\mathbf{u} && \text{iff } \forall s_1, s_2 \in T : \mathbf{v}^{\mathcal{A}}\langle s_1\rangle = \mathbf{v}^{\mathcal{A}}\langle s_2\rangle \text{ then } \exists s_3 \in T :\\
& && \quad \mathbf{v}\mathbf{t}^{\mathcal{A}}\langle s_3\rangle = \mathbf{v}\mathbf{t}^{\mathcal{A}}\langle s_1\rangle \text{ and } \mathbf{u}^{\mathcal{A}}\langle s_3\rangle = \mathbf{u}^{\mathcal{A}}\langle s_2\rangle,\\
&\mathcal{A}, T \models \varphi_0 \wedge \varphi_1 && \text{iff } \mathcal{A}, T \models \varphi_0 \text{ and } \mathcal{A}, T \models \varphi_1,\\
&\mathcal{A}, T \models \varphi_0 \vee \varphi_1 && \text{iff } \exists T_0 \exists T_1 : T_0 \cup T_1 = T \text{ and } \mathcal{A}, T_i \models \varphi_i \text{ for } i = 0, 1,\\
&\mathcal{A}, T \models \exists x\varphi && \text{iff } \mathcal{A}, T_f^x \models \varphi \text{ for some } f\colon T \to \mathcal{P}(A) \setminus \{\emptyset\},\\
&\mathcal{A}, T \models \forall x\varphi && \text{iff } \mathcal{A}, T_{\mathcal{A}}^x \models \varphi.
\end{aligned}
$$

For a structure $\mathcal{A}$ and a team T over X in $\mathcal{A}$, we let $\mathrm{rel}(T)$ denote the relation *defined* by T. That is, $\mathrm{rel}(T) := \{\, \mathbf{a} \mid s(\mathbf{x}) = \mathbf{a}, s \in T \,\}$. Moreover, we say that a formula φ is *flat* if for any team T over $\mathrm{Fr}(\varphi)$ we have that $\mathcal{A}, T \models \varphi$ if and only if $\mathcal{A}, \{s\} \models \varphi$ for every $s \in T$. The $\mathcal{FO}$-formulas satisfy this flatness property. Notice that, for $\mathcal{FO}$-formulas, by singleton equivalence, team semantics and classical Tarski semantics coincide, i.e., $\mathcal{A}, \{s\} \models \varphi$ if and only if $\mathcal{A} \models_s \varphi$. Furthermore, note that $\mathcal{A}, T \models \varphi$ for all φ when $T = \emptyset$ (this is also called the *empty team property*). Finally, $\mathcal{C}$-formulas for every $\mathcal{C} \in \{\mathcal{FO}(\mathsf{dep}), \mathcal{FO}(\subseteq), \mathcal{FO}(\perp)\}$ are *local*, that is, for a team T in $\mathcal{A}$ over domain X and a $\mathcal{FO}(\mathsf{dep})$-formula φ, we have that $\mathcal{A}, T \models \varphi$ if and only if $\mathcal{A}, T\restriction\mathrm{Fr}(\varphi) \models \varphi$.

We now extend the formulas classes (Σ_t and Π_t) to the logics under consideration. To this end, $\mathcal{FO}(\mathsf{dep})$-$\Pi_t \subseteq \mathcal{FO}(\mathsf{dep})$ (resp., $\mathcal{FO}(\mathsf{dep})$-$\Sigma_t$) denotes the collection of formulas φ of the form $\varphi := Q_1x_1Q_2x_2 \ldots Q_tx_t\psi$ such that ψ is a quantifier free $\mathcal{FO}(\mathsf{dep})$-formula, $Q_i \in \{\forall, \exists\}$ and $Q_1 = \forall$ ($Q_1 = \exists$). In other words, φ is a $\mathcal{FO}(\mathsf{dep})$-formula that starts with a $\forall$-quantifier (resp., $\exists$)

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

and has t-alternations of quantifiers. The classes $\mathcal{FO}(\subseteq)$-$\Pi_t \subseteq \mathcal{FO}(\subseteq)$ (resp., $\mathcal{FO}(\subseteq)$-Σ_t) for $\mathcal{FO}(\subseteq)$ and $\mathcal{FO}(\bot)$-$\Pi_t \subseteq \mathcal{FO}(\mathsf{dep})$ (resp., $\mathcal{FO}(\bot)$-Σ_t) for $\mathcal{FO}(\bot)$ are similarly defined.

Weighted Team Definability. Now we introduce a novel version of the weighted definability problem for formulas in team-based logics. Let $\mathcal{C} \in \{\mathcal{FO}(\mathsf{dep}), \mathcal{FO}(\subseteq), \mathcal{FO}(\bot)\}$, φ be a fixed $\mathcal{C}$-formula over free variables $\mathrm{Fr}(\varphi)$ and $k \in \mathbb{N}$. Then given a structure $\mathcal{A}$, the *weighted-team* definable problem WT_φ asks if there is a team of size k for φ over $\mathrm{Fr}(\varphi)$ in $\mathcal{A}$.

Problem:	WT_φ – weighted team definability for fixed φ
Input:	A τ-structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Question:	Is there a team T over $\mathrm{Fr}(\varphi)$ such that $\lvert T\rvert = k$ and $\mathcal{A}, T \models \varphi$?

Then the analogous parameterized version of WT_φ is defined as follows.

Problem:	p-WT_φ – parameterized weighted team definability for fixed φ
Input:	A τ-structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Is there a team T over $\mathrm{Fr}(\varphi)$ such that $\lvert T\rvert = k$ and $\mathcal{A}, T \models \varphi$?

Note that the problem WT_φ references the set of free variables $\mathrm{Fr}(\varphi)$ of the formula φ. As a consequence, our parameterization is trivial for sentences since there are only two teams $\emptyset$ and $\{\emptyset\}$ with the empty team domain. As before, for a set $\Theta \subseteq \mathcal{C}$ of formulas, we denote by WT-Θ the class of problems WT_φ such that $\varphi \in \Theta$.

3. Complexity Results for Weighted Team Definability

3.1 First-order formulas

We begin our study of the complexity for p-WT_φ in the case φ is a pure $\mathcal{FO}$-formula under team semantics. Notice that the consequence of disallowing free relation variables in φ is that p-WT_φ is different than the weighted Fagin definability p-WD_φ. The following theorem establishes that the two problems are also different from the classical complexity theoretic point of view. Here, we assume basic familiarity about the circuit complexity classes TC^0 and AC^0 (for an introduction into this area, see the textbook of Vollmer 1999).

Theorem 9. *For any $\mathcal{FO}$-formula φ the problem WT_φ is in DLOGTIME-uniform TC^0.*

Proof. The proof uses the flatness property of $\mathcal{FO}$-formulas under team semantics:

$$\mathcal{A}, T \models \varphi \Leftrightarrow \forall s \in T : \mathcal{A} \models_s \varphi.$$

It is well known that $\mathcal{A} \models_s \varphi$ can be decided by AC^0-circuits, whence the original question reduces to counting the number t of satisfying assignments of φ and checking whether $t \geq k$. This can be easily simulated by DLOGTIME-uniform TC^0 circuits as we can hardcode all possible assignments into the circuit. Here, notice that φ is fixed and thereby the number of free variables are fixed to some constant $c \in \mathbb{N}$. Then, the input is the structure $\mathcal{A}$ of size n yielding $O(n^c)$ many assignments. □

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

3.2 Inclusion logic

In this section, we relate the **W**-hierarchy and **W**[**P**] to weighted team definability for inclusion logic formulas. First, observe that if φ is an $\mathcal{FO}(\subseteq)$-sentence, then the problem p-WT$_\varphi$ is in **FPT**. This is due to the reason that the data complexity of fixed $\mathcal{FO}(\subseteq)$-sentences is in **P** (Galliani and Hella 2013).

Theorem 10. *Let φ be an $\mathcal{FO}(\subseteq)$-sentence, then p-WT$_\varphi$ is in* **FPT**.

Proof. Recall that an $\mathcal{FO}(\subseteq)$-sentence φ has a satisfying team T in $\mathcal{A}$ if and only if $\mathcal{A}, \{\emptyset\} \models \varphi$ due to the locality property. Then φ is true in $\mathcal{A}$ if and only if there is a team T such that $|T| = 1$ and $\mathcal{A}, T \models \varphi$. □

Now we prove, that p-WT$_\varphi$ can already be **W**[1]-hard when φ is a quantifier-free $\mathcal{FO}(\subseteq)$-formula with free variables.

Theorem 11. *There is a quantifier-free $\mathcal{FO}(\subseteq)$-formula φ such that the problem p-WT$_\varphi$ is* **W**[1]*-hard and in* **W**[2].

Proof. We present a reduction from the **W**[1]-complete problem p-CLIQUE to p-WT$_\varphi$ such that φ is a quantifier free $\mathcal{FO}(\subseteq)$-formula. Let $G := (V, E)$ be a graph and $k \in \mathbb{N}$. Then, we let $\varphi := E(x, y) \wedge x \neq y \wedge y \subseteq x \wedge x \subseteq y$. We claim that G has a clique of size k if and only if $G, T \models \varphi$ for a team T of size $(k^2 - k)$. It is easy to check that the existence of a k-clique is equivalent to φ having a satisfying team of cardinality $k(k-1)$. Clearly, a clique of size k contains $k(k-1)$-many edges. Then the variables x and y take same values under each assignment in the resulting team which form a clique.

For containment in **W**[2], it suffices to note that the formula φ can be expressed as an $\mathcal{FO}$-sentence $\psi(S)$ with a $\forall\exists$-quantifier prefix where the auxiliary binary predicate S encodes the team T. This gives an **FPT**-reduction between p-WT$_\varphi$ and p-WD$_\psi$. The result follows since **W**[2] := [p-WD-Π_2]. □

This result can be strengthened to more general formulas as witnessed by the following corollary.

Corollary 12. *For any quantifier-free $\mathcal{FO}(\subseteq)$-formula φ without $\vee$, the problem p-WT$_\varphi$ is in* **W**[2].

Proof. For containment in **W**[2], it suffices to note that the any quantifier-free formula without disjunction can be expressed as an $\mathcal{FO}$-sentence $\psi(S)$ with a $\forall\exists$-quantifier prefix where the auxiliary binary predicate S encodes the team T. □

Theorem 13. *There is an $\mathcal{FO}(\subseteq)$-formula φ with $\forall\exists$-quantifier prefix for which the problem p-WT$_\varphi$ is* **W**[2]*-complete.*

Proof. We present a reduction from the **W**[2]-complete problem p-DOMINATINGSET to p-WT$_\varphi$ such that φ is a $\mathcal{FO}(\subseteq)$-formula with $\forall\exists$-quantifier prefix. Let $G := (V, E)$ be a graph and $k \in \mathbb{N}$. Then we let, $\varphi := \forall x \exists y (y \subseteq z \wedge (E(x, y) \vee x = y))$. It is straightforward to check that G has a dominating set of size k if and only if $G, T \models \varphi$ for a team T with domain $\{z\}$ of size k.

For **W**[2]-membership, notice that for all graphs G and teams T:

$$G, T \models \varphi \Leftrightarrow (G, \mathrm{rel}(T)) \models \varphi_d(X),$$

where $\varphi_d(X)$ is the first-order sentence encoding the problem DOMINATINGSET (see Example 2). A formal proof for the above equivalence is similar to the one given in Theorem 19. □

The next lemma sets the stage for generalizing the two previous theorems to arbitrary levels of the **W**-hierarchy. To formulate the result, we assume an encoding of a formula $\psi \in \Gamma^+_{t,d}$ (and a truth assignment) by its syntax circuit defined as follows.

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

Definition 14. Let $\psi \in \Gamma^+_{t,d}$ be a propositional formula. We define the syntax circuit for ψ by $A_\psi = (A, E, I, o)$, where A is the set of subformulas of ψ, E is the immediate subformula relation, $I \subseteq A$ are the variables of ψ, and o is a constant symbol interpreted by ψ. Moreover, we use a free relation variable $S \subseteq I$ to represent a truth assignment for the variables of ψ.

Note that our encoding of ψ works for any $t \in \mathbb{N}$ but for the definability result below t has to be fixed.

Lemma 15. *Let $t \in \mathbb{N}$. Then there exists a fixed formula $\varphi_t \in \mathcal{FO}(\subseteq)$ with one free variable z such that for all $\psi \in \Gamma^+_{t,d}$ and $k \geq 1$: ψ has a satisfying assignment of weight k if and only if $A_\psi, T \models \varphi_t$, for some team T of cardinality k.*

Proof. Without loss of generality, we assume $d = 1$. For higher d-values, the presented proof easily generalizes via a conjunction/disjunction of arity d. By the results of Galliani and Hella (2013), it suffices to show that the required formula can be expressed by a first-order sentence $\theta(S)$ in which the relation symbol S occurs only positively. Then the existence of $\varphi_t(z)$ satisfying

$$A_\psi, T \models \varphi_t \Leftrightarrow A_\psi \models \theta(S), \tag{1}$$

for all nonempty T and $\mathrm{rel}(T) = S$ follows. Note that $\theta(S)$ is not true under the assignment setting all the variables to false, but on the other hand φ_t is always satisfied for $T = \emptyset$ by the empty team property. It is easy to check that, for an even t, $\theta(S)$ can be expressed as follows:

$$\theta(S) \coloneqq \forall x_1 \big(\neg E(o, x_1) \vee \exists x_2 (E(x_1, x_2) \wedge \cdots Q x_t (E(x_{t-1}, x_t) \wedge I(x_t) \wedge S(x_t)) \cdots \big).$$

In the case t is odd the conjunction after $E(x_{t-1}, x_t)$ has to be replaced by an implication. The relation symbol S has only one occurrence in the formula and it is positive. Now by Proposition 20 of Galliani and Hella (2013), there exists an $\mathcal{FO}(\subseteq)$-formula φ_t such (1) holds for the sentence $\forall \vec{x}(S(\vec{x}) \to \theta(S))$ for all $\mathcal{A}$ and all T. It is easy to see that $\theta(S)$ is equivalent with $\forall \vec{x}(S(\vec{x}) \to \theta(S))$ modulo the cases when $S = \emptyset$. In fact, it is straightforward to show that φ_t can be obtained from $\theta(S)$ simply by replacing $S(x_t)$ by the inclusion atom $x_t \subseteq z$. The proof then is analogous to the proof of Theorem 19. □

Notice further that the translation of the formula θ to an $\mathcal{FO}(\subseteq)$-formula only introduces inclusion atoms and, in particular, does not require any further quantification. Therefore, the following corollary follows immediately from the proof in Lemma 15.

Corollary 16. *Let $t \geq 2$ be even. Then there is an $\mathcal{FO}(\subseteq)$-Π_t-formula φ_t for which the problem p-WT_{φ_t} is $\mathbf{W}[t]$-complete. Moreover, $\mathbf{W}[t] \subseteq [p\text{-WT-}\mathcal{FO}(\subseteq)\text{-}\Pi_t]^{\mathbf{FPT}}$ for all even $t \geq 1$ and $\bigcup_{t \geq 1} \mathbf{W}[t] \subseteq [p\text{-WT-}\mathcal{FO}(\subseteq)]^{\mathbf{FPT}}$.*

Proof. For the $\mathbf{W}[t]$-membership of p-WT_{φ_t}, notice that the translation between θ and the $\mathcal{FO}(\subseteq)$-formula φ_t in the proof of Lemma 15 preserves a one-to-one correspondence between the solutions S for θ and satisfying teams T for φ_t. In other words, θ has a solution of size k if and only if φ_t has a satisfying team of size k. This yields $\mathbf{W}[t]$-membership since $\theta \in \Pi_t$ for each $t \geq 1$ (see Def. 5). The $\mathbf{W}[t]$-hardness and the containment $\mathbf{W}[t] \subseteq [\text{p-WT-}\mathcal{FO}(\subseteq)\text{-}\Pi_t]^{\mathbf{FPT}}$ for all even $t \geq 1$ follows from Proposition 6. □

We conclude this section by presenting the upper bounds for WT_φ when φ is an arbitrary $\mathcal{FO}(\subseteq)$-formula.

Theorem 17. $[p\text{-WT-}\mathcal{FO}(\subseteq)]^{\mathbf{FPT}} \subseteq \mathbf{W}[\mathbf{P}]$.

Proof. We prove this via the machine characterization of the class $\mathbf{W}[\mathbf{P}]$, analogous to the proof for $\mathcal{FO}$-formulas (Flum and Grohe 2006, Prop. 5.3). Let φ be a $\mathcal{FO}(\subseteq)$-formula with s free variables. An algorithm for the problem p-WT_φ proceeds as follows: Given a structure $\mathcal{A}$ and a k, nondeterministically guess k times an assignment (i.e., an s-tuple of elements of $\mathcal{A}$), then

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

deterministically verify that the team T has cardinality k and $\mathcal{A}, T \models \varphi$. Guessing T requires $s \cdot k \cdot \log |A|$ nondeterministic bits, and the verification that $\mathcal{A}, T \models \varphi$ can be done in deterministic polynomial time in $|A|$ (Galliani and Hella 2013). Thus, p-WT$_\varphi$ is in $\mathbf{W}[\mathbf{P}]$ because the formula φ is fixed and s is a constant. Moreover, the containment [p-WT-$\mathcal{FO}(\subseteq)$]$^{\mathbf{FPT}} \subseteq \mathbf{W}[\mathbf{P}]$ holds since p-WT$_\varphi \in \mathbf{W}[\mathbf{P}]$ for an arbitrary but fixed $\mathcal{FO}(\subseteq)$-formula φ. □

3.3 Dependence logic

First observe that if φ is a $\mathcal{FO}(\mathsf{dep})$-sentence, then the problem p-WT$_\varphi$ is **paraNP**-complete. This is due to the reason that the data complexity of certain $\mathcal{FO}(\mathsf{dep})$-sentences is already **NP**-complete (Väänänen 2007).

Theorem 18. *There is a $\mathcal{FO}(\mathsf{dep})$-sentence φ, such that the problem p-WT$_\varphi$ is* **paraNP***-complete.*

Proof. Recall that by the locality property a $\mathcal{FO}(\mathsf{dep})$-sentence φ is satisfied by all teams T over $\mathcal{A}$ if and only if $\mathcal{A}, T^* \models \varphi$ for some nonempty team T^*. Let now φ be a sentence of dependence logic expressing an **NP**-complete problem. Then, for any fixed $k > 0$, the problem: given a model $\mathcal{A}$, determine whether there exists a team with k elements satisfying φ in $\mathcal{A}$ is **NP**-complete. This immediately implies that the problem p-WT$_\varphi$ is **paraNP**-complete. □

Now, we relate the **W**-hierarchy to the weighted definability for dependence logic. This also settles the complexity of p-WT for $\mathcal{FO}(\mathsf{dep})$-formulas. In the following, we prove that already one universal quantifier is enough in $\mathcal{FO}(\mathsf{dep})$ to define $\mathbf{W}[1]$-complete problems.

Theorem 19. *There is a $\mathcal{FO}(\mathsf{dep})$-formula φ with only one universal quantifier such that the problem p-WT$_\varphi$ is $\mathbf{W}[1]$-complete.*

Proof. We present a reduction from the $\mathbf{W}[1]$-complete problem p-IndependentSet to p-WT$_\varphi$ such that φ is $\mathcal{FO}(\mathsf{dep})$-formula with only one universal quantifier. An input to IndependentSet is a graph $\mathcal{G} := (V, E)$ and a number $k \in \mathbb{N}$. The question is whether there is a set S of size k in $\mathcal{G}$ such that $(a, b) \notin E$ for every $a, b \in S$. We let $\tau := \{N^1, P^1, I^2\}$ as our vocabulary where N, P are unary relations and I is a binary relation symbol. Moreover, the τ-structure $\mathcal{A}$ is such that: $\mathrm{dom}(\mathcal{A}) := V \cup E$, $N^{\mathcal{A}} := V$, $P^{\mathcal{A}} := E$ and $I^{\mathcal{A}}$ simulates the edge relation $E^{\mathcal{G}}$. That is, $I := \{\, (a, b), (c, b) \mid a, c \in V, \text{ and } b \in P \text{ denotes the edge } (a, c) \in E \,\}$. Finally, we define a $\mathcal{FO}(\mathsf{dep})$-formula φ over a single free variable x as in the following.

$$\varphi(x) := \forall y\big(N(x) \wedge (\neg P(y) \vee \neg I(x, y) \vee \mathsf{dep}(y;x))\big)$$

The correctness of our reduction is established via the following claim and also shows that the formula φ is, in fact, equivalent to the familiar definition of independent sets via a Π_1-formula; hence, p-WT$_\varphi$ is $\mathbf{W}[1]$-complete.

Claim 20. There is a team T over x in $\mathcal{A}$ such that $|T| = k$ and $\mathcal{A}, T \models \varphi$ if and only if there is an independent set in $\mathcal{G}$ of size k.

It remains to prove the claim. Suppose that $T = \{\, s_i \mid i \leq k \,\}$ is a team over x for φ such that $s_i(x) = a_i$ for $a_i \in A$. Moreover, let $T' = \{\, s_{i,j} \mid i \leq k, j \leq |\mathcal{A}| \,\}$ denote the supplemented team, that is, $s_{i,j}(x) = a_i$ and $s_{i,j}(y) = a_j$ for every $a_j \in \mathcal{A}$. We prove that $S = \{a_i \mid \exists s \in T, s(x) = a_i\}$ constitutes an independent set in $\mathcal{G}$. Let $a_i, a_j \in S$, then there are $s_i, s_j \in T$ such that $s_i(x) = a_i$, $s_j(x) = a_j$. Suppose further that $(a_i, a_j) = e \in E^{\mathcal{G}}$. Then, by our construction, there are $s_{i,j}, s_{j,j} \in T'$ such that $s_{i,j}(xy) = a_i a_j$ and $s_{j,j} = a_j a_j$. Moreover, the pair $s_{i,j}, s_{j,j}$ cannot be in the subteam for the first disjunct since e is an edge and $T' \models P(e)$; it can also not belong to the subteam for the second disjunct since both a_i and a_j appear in the edge e. As a result, $s_{i,j}, s_{j,j}$ must be in the subteam for the third disjunct, but then it cannot satisfy the atom $\mathsf{dep}(y; x)$ since $s_{i,j}(y) = s_{j,j}(y)$ and $s_{i,j}(x) \neq s_{j,j}(x)$. Consequently, $T' \not\models (\neg P(y) \vee \neg I(x, y) \vee \mathsf{dep}(y;x))$ and $T \not\models \varphi$, which is a contradiction.

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

Conversely, if there is an independent set S of size k in $\mathcal{G}$ then we prove that $T \models \varphi(x)$ for $T = \{s_i \mid i \leq k, s_i(x) \in S\}$. Clearly, the supplemented team T' over $\{x, y\}$ has the following effect: for every y that corresponds to an edge e between elements $a_i, a_j \in A$, at most one of its endpoint a_i or a_j is *in* T, which is the case if and only if S is in independent set. □

Once again, we prove the next lemma that generalizes the previous theorem to arbitrary levels of the **W**-hierarchy using the circuit representation of a formula (Def. 14).

Lemma 21. *Let $t \in \mathbb{N}$. Then there exists a fixed formula $\varphi_t \in \mathcal{FO}(\mathsf{dep})$ with one free variable z such that for all $\psi \in \Gamma^-_{t,d}$ and $k \geq 1$: ψ has a satisfying assignment of weight k if and only if $A_\psi, T \models \varphi_t$, for some team T of cardinality k.*

Proof. Without loss of generality, we assume that $d = 1$. Otherwise, the presented proof will easily generalize to larger values of d by a disjunction/conjunction of arity d. By the results of Kontinen and Väänänen (2009), it suffices to show that the required formula can be expressed by a first-order sentence $\theta(S)$ in which the relation symbol S occurs only negatively. Then the existence of $\varphi_t(z)$ satisfying

$$A_\psi, T \models \varphi_t \Leftrightarrow A_\psi \models \theta(S), \tag{2}$$

for all nonempty T and $\mathrm{rel}(T) = S$ follows. Now (for t even) it is easy to check that $\theta(S)$ can be expressed as follows:

$$\theta(S) := \forall x_1 \big(\neg E(o, x_1) \vee \exists x_2 (E(x_1, x_2) \wedge \cdots Q x_t (E(x_{t-1}, x_t) \wedge I(x_t) \wedge \neg S(x_t)) \cdots \big).$$

In the case t is odd the conjunction after $E(x_{t-1}, x_t)$ has to be replaced by an implication. The relation symbol S appears only once in the formula and this appearance is negative. □

Notice further that the translation of the formula θ to a $\mathcal{FO}(\mathsf{dep})$-formula only introduces dependence atoms and, in particular, does not require any further quantification. Therefore, the following corollary (with proof analogous to Corollary 16) follows. Recall that every dependence logic formula can be put into the $\forall\exists$-normal form. As a result, tracking the quantifier prefix in Lemma 21 is not useful and we get the much stronger statement that the whole **W**-hierarchy is already contained in $\mathcal{FO}(\mathsf{dep})$-$\Pi_2$. Yet, there is a trade-off one has to keep in mind while reaching this normal form: each existential quantifier (beyond the first) then induces a dependence atom in that case.

Corollary 22. *Let $t \geq 1$ be odd. Then there is an $\mathcal{FO}(\mathsf{dep})$-$\Pi_2$-formula φ_t for which the problem p-WT$_{\varphi_t}$ is $\mathbf{W}[t]$-complete. Moreover, $\bigcup_{t \geq 1} \mathbf{W}[t] \subseteq [p\text{-WT-}\mathcal{FO}(\mathsf{dep})\text{-}\Pi_2]^{\mathbf{FPT}}$.*

Finally, $\mathcal{FO}(\mathsf{dep})$ captures the class **paraNP** as established below.

Theorem 23. $[p\text{-WT-}\mathcal{FO}(\mathsf{dep})]^{\mathbf{FPT}} = \mathbf{paraNP}$.

Proof. Hardness follows from Theorem 18. For membership, we present the following nondeterministic algorithm that runs in polynomial time in the size of $\mathrm{dom}(\mathcal{A})$. Notice that since the formula is fixed, we have fixed many connectives including disjunctions and quantifiers. The idea of the algorithm is that it guesses a team T of size k, as well as, a sequence T_i for $i \in \mathbb{N}$ of teams which corresponds to the operations of supplementation and splits according to the formula φ. In other words, let $\varphi = Q_1 x_1 Q_2 x_2 \ldots Q_\ell x_\ell \psi$ be an $\mathcal{FO}(\mathsf{dep})$-formula, where $Q \in \{\forall, \exists\}$ and ψ is quantifier free. Then the algorithm performs the following steps.

- Guess a team T of size k over $\mathrm{Fr}(\varphi)$ in $\mathcal{A}$.
- For each $i \in \{\ell, \ldots, 1\}$, guess a team T_i over $\mathrm{Fr}(\varphi) \cup \{x_\ell, \ldots, x_i\}$ in an *inside-out* order of quantification, such that: if $Q_i = \forall$, then $T_i = P^x_{\mathcal{A}}$ and if $Q_i = \exists$, then $T_i = P^x_f$ where $f \colon P \to \mathcal{P}(A) \setminus \emptyset$ and $P = T_{i+1}$ (we let $T_{\ell+1} = T$).

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

Notice that $|T_i| = |T_{i+1}|$ if $Q_i = \exists$ (because $\mathcal{FO}(\mathsf{dep})$ is downward closed) and $|T_i| = |T_{i+1}| \cdot |A|$ otherwise. As a result, we have that $|T_i| \leq k \cdot |A|^{\ell - i}$ and the size of the team $|T_1|$ is bounded by $k \cdot |A|^{\ell}$.

Once the team T_1 has been guessed, the algorithm needs to determine whether $T_1 \models \psi$. Since the data complexity of $\mathcal{FO}(\mathsf{dep})$ is still **NP**-complete for quantifier free formulas, this step is also nontrivial. Nevertheless, we can list recursively all the subformulas of ψ in terms of its syntax tree. This helps in labeling a subteams of T_1 according to the connectives of ψ.

- Guess subteams for subformulas in a *top-down* order of the syntax tree of ψ. To be precise, let $T_\psi := T_1$. Then, for each subformula $\alpha = \beta \wedge \gamma$ of ψ, let $T_\beta = T_\gamma = T_\alpha$, and for each $\alpha = \beta \vee \gamma$, let $T_\beta \cup T_\gamma = T_\alpha$.

  Clearly, the size of the subteam T_α for each α is atmost $k \cdot |A|^{\ell}$.
- Accept if $T_\alpha \models \alpha$ for each atomic subformula α of φ.

Notice that for atomic formulas the truth evaluation $T_\alpha \models \alpha$ can be determined in polynomial time. Moreover, the intermediate steps including the verification of the team supplementation can also be performed in polynomial time. Finally, the correctness follows from the fact that the algorithm simulates the truth evaluation (see Def. 8) for φ. This results in **paraNP**-membership of WT_φ for a fixed $\mathcal{FO}(\mathsf{dep})$-formula φ. □

3.4 Independence logic

In this section, we turn to independence logic. The following theorem is obtained from the results in the previous sections and the fact that any $\mathcal{ESO}$-sentence $\psi(S)$ (with an extra relation encoding the team) can be represented by an independence logic formula (Galliani 2012). It is worth remarking that $\mathcal{FO}(\perp)$ has the empty team property and can therefore only represent an $\mathcal{ESO}$-sentence $\psi(S)$ if ψ is also true for the empty relation. This is unproblematic since we implicitly assume that $k \geq 1$.

Theorem 24. (1) *For all $t \in \mathbb{N}$ there is an $\mathcal{FO}(\perp)$-formula φ_t such that p-WT_{φ_t} is $\mathbf{W}[t]$-complete.*
(2) *There is an $\mathcal{FO}(\perp)$-formula φ_w such that p-WT_{φ_w} is $\mathbf{W}[\mathbf{P}]$-complete.*
(3) *There is a $\mathcal{FO}(\perp)$-sentence φ, such that the problem p-WT_φ is **paraNP**-complete.*
(4) $[p\text{-WT-}\mathcal{FO}(\perp)]^{\mathbf{FPT}} = \mathbf{paraNP}$.

Proof. The first and third claim follow immediately from the fact that the logics $\mathcal{FO}(\mathsf{dep})$ and $\mathcal{FO}(\subseteq)$ are sublogics of $\mathcal{FO}(\perp)$ (Galliani 2012) together with Theorem 23. The containment of $\mathcal{FO}(\mathsf{dep})$ (resp., $\mathcal{FO}(\subseteq)$) inside $\mathcal{FO}(\perp)$ yields the results for odd (even) $t \in \mathbb{N}$ in conjunction with Corollary 22 (Cor. 16), thereby proving the first claim for all $t \geq 1$.

For the second claim, we use the fact that p-WSAT(CIRC$^+$) is $\mathbf{W}[\mathbf{P}]$-complete (Flum and Grohe 2006, Thm. 3.14), where CIRC$^+$ is the class of negation-free propositional formulas encoded as monotone Boolean circuits. Note that the circuit value problem can be readily expressed by an $\mathcal{ESO}$-sentence $\psi(S)$, where S represents an input for the circuit. More precisely, assume we a given a DAG (A, E, D, K, I, o) encoding a Boolean circuit. Here A is the set of nodes/gates, E is the edge relation, $I \subseteq A$ are the input gates of the circuit, o is the unique output, $D \subseteq A$ is the set of OR-gates, and $K \subseteq A$ is the set of AND-gates. A Boolean input for the circuit is represented by a subset $S \subseteq I$, i.e., a gate g gets input 1 if and only if $g \in S$. Now in $\mathcal{ESO}$ we can existentially quantify a proof tree witnessing the circuit accepting the input S. In other words, we quantify a subset $P \subseteq A$ such that

- $o \in A$,
- $P \cap I = S$,

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

- for all $g \in P \cap D$ there exists at least one $g' \in P$ such that $E(g', g)$,
- for all $g \in P \cap K$ and all g', if $E(g', g)$ then $g' \in P$.

It is straightforward to check that the above conditions can be expressed in first-order logic.

Finally, the hardness for the last claim follows again from the fact that $\mathcal{FO}(\mathsf{dep})$ is a sublogic of $\mathcal{FO}(\perp)$ and the membership proof is analogous to that of Theorem 23. □

4. Conclusion

We have defined and studied the parameterized complexity of weighted team definability with respect to formulas of several team-based logics. Our results show that for plain first-order formulas weighted team definability differs greatly from weighted Fagin definability; the former being computationally much simpler. For dependence, independence, and inclusion logic formulas, the complexity of weighted team definability ranges between the classes $\mathbf{W}[t]$ and **paraNP**. Now, these results provide a wide range of natural complete problems for the aforementioned complexity classes enriching the landscape in a nontrivial way. Interestingly, the sentences in the considered logics depict different complexities: namely, membership in **FPT** for $\mathcal{FO}(\subseteq)$ and **paraNP**-completeness for $\mathcal{FO}(\mathsf{dep})$ and $\mathcal{FO}(\perp)$. The main open question is whether the converse directions of Corollary 16 or Theorem 17 can be proven, i.e., if one of the inclusions $\bigcup_{t\in\mathbb{N}} \mathbf{W}[t] \subseteq [\text{p-WT-}\mathcal{FO}(\subseteq)]^{\mathbf{FPT}} \subseteq \mathbf{W}[\mathbf{P}]$ is in fact an equality.

Funding. *Juha Kontinen:* Partially funded by Academy of Finland grant 338259 *Yasir Mahmood:* Partially funded by the EU's Horizon Europe research and innovation programme within project ENEXA (101070305) *Arne Meier:* Partially funded by DFG grant ME 4279/3-1 *Heribert Vollmer:* Partially funded by DAAD Project-ID 57570031

References

Downey, R. G. and Fellows, M. R. (2013). *Fundamentals of Parameterized Complexity*, Texts in Computer Science, Springer. URL: https://doi.org/10.1007/978-1-4471-5559-1, doi:10.1007/978-1-4471-5559-1.

Downey, R. G., Fellows, M. R. and Regan, K. W. (1998). *Descriptive Complexity and the W Hierarchy*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 39, AMS, 119–134.

Durand, A., Hannula, M., Kontinen, J., Meier, A. and Virtema, J. (2018). Probabilistic team semantics. In: Ferrarotti, F. and Woltran, S. (eds.) *Foundations of Information and Knowledge Systems - 10th International Symposium, FoIKS 2018, Budapest, Hungary, May 14–18, 2018, Proceedings*, Lecture Notes in Computer Science, vol. 10833, Springer, 186–206. URL: https://doi.org/10.1007/978-3-319-90050-6_11, doi:10.1007/978-3-319-90050-6_11.

Durand, A., Kontinen, J., de Rugy-Altherre, N. and Väänänen, J. (2022). Tractability frontier of data complexity in team semantics. *ACM Transactions on Computational Logic* **23** (1) 3:1–3:21. URL: https://doi.org/10.1145/3471618, doi:10.1145/3471618.

Ebbinghaus, H.-D. and Flum, J. (1995). *Finite Model Theory*, Perspectives in Mathematical Logic, Berlin, Springer.

Flum, J. and Grohe, M. (2006). *Parameterized Complexity Theory*, Texts in Theoretical Computer Science. An EATCS Series, Springer. URL: https://doi.org/10.1007/3-540-29953-X, doi:10.1007/3-540-29953-X.

Galliani, P. (2012). Inclusion and exclusion dependencies in team semantics: On some logics of imperfect information. *Annals of Pure and Applied Logic* **163** (1) 68–84. doi:10.1016/j.apal.2011.08.005.

Galliani, P. and Hella, L. (2013). Inclusion logic and fixed point logic. In: Della Rocca, S. R. (ed.) *Computer Science Logic 2013 (CSL 2013)*, Leibniz International Proceedings in Informatics (LIPIcs), vol. 23, Dagstuhl, Germany, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 281–295. URL: http://drops.dagstuhl.de/opus/volltexte/2013/4203, doi:10.4230/LIPIcs.CSL.2013.281.

Grädel, E. and Väänänen, J. (2013). Dependence and independence. *Studia Logica* **101** (2) 399–410. doi:10.1007/s11225-013-9479-2.

Gutsfeld, J. O., Meier, A., Ohrem, C. and Virtema, J. (2022). Temporal team semantics revisited. In: Baier, C. and Fisman, D. (eds.) *LICS'22: 37th Annual ACM/IEEE Symposium on Logic in Computer Science, Haifa, Israel, August 2–5, 2022*, ACM, 44:1–44:13. URL: https://doi.org/10.1145/3531130.3533360, doi:10.1145/3531130.3533360.

Haak, A., Kontinen, J., Müller, F., Vollmer, H. and Yang, F. (2019). Counting of teams in first-order team logics. In: *MFCS*, LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, vol. 138, 19:1–19:15.

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press

Haak, A., Meier, A., Müller, F. and Vollmer, H. (2022). Enumerating teams in first-order team logics. *Annals of Pure and Applied Logic* **173** (10) 103163.

Hannula, M., Kontinen, J., Van den Bussche, J. and Virtema, J. (2020). Descriptive complexity of real computation and probabilistic independence logic. In: *LICS*, ACM, 550–563.

Hannula, M., Kontinen, J., Virtema, J. and Vollmer, H. (2018). Complexity of propositional logics in team semantic. *ACM Transactions on Computational Logic* **19** (1) 2:1–2:14.

Hella, L., Kuusisto, A., Meier, A. and Virtema, J. (2019). Model checking and validity in propositional and modal inclusion logics. *Journal of Logic and Computation* **29** (5) 605–630. URL: https://doi.org/10.1093/logcom/exz008, doi:10.1093/logcom/exz008.

Hella, L., Kuusisto, A., Meier, A. and Vollmer, H. (2020). Satisfiability of modal inclusion logic: Lax and strict semantics. *ACM Transactions on Computational Logic* **21** (1) 7:1–7:18. URL: https://doi.org/10.1145/3356043, doi:10.1145/3356043.

Kontinen, J. Meier, A. and Mahmood, Y. (2022). A parameterized view on the complexity of dependence and independence logic. *Journal of Logic and Computation* **32** (8) 1624–1644. URL: https://doi.org/10.1093/logcom/exac070, doi:10.1093/logcom/exac070.

Kontinen, J. and Väänänen, J. A. (2009). On definability in dependence logic. *Journal of Logic, Language and Information* **18** (3) 317–332. URL: https://doi.org/10.1007/s10849-009-9082-0, doi:10.1007/s10849-009-9082-0.

Lück, M. (2019). Canonical models and the complexity of modal team logic. *Logical Methods in Computer Science* **15** (2). URL: https://doi.org/10.23638/LMCS-15(2:2)2019, doi:10.23638/LMCS-15(2:2)2019.

Mahmood, Y. and Meier, A. (2022). Parameterised complexity of model checking and satisfiability in propositional dependence logic. *Annals of Mathematics and Artificial Intelligence* **90** (2-3) 271–296.

Mahmood, Y. and Virtema, J. (2021). Parameterised complexity of propositional logic in team semantics. *CoRR*, abs/2105.14887.

Meier, A. and Reinbold, C. (2018). Enumeration complexity of poor man's propositional dependence logic. In: *FoIKS*, Lecture Notes in Computer Science, vol. 10833, Springer, 303–321.

Papadimitriou, C. H. (1994). *Computational Complexity*, Addison-Wesley.

Väänänen, J. A. (2007). *Dependence Logic - A New Approach to Independence Friendly Logic*, London Mathematical Society Student Texts, vol. 70, Cambridge University Press. URL: http://www.cambridge.org/de/knowledge/isbn/item1164246/?site_locale=de_DE.

Vollmer, H. (1999). *Introduction to Circuit Complexity - A Uniform Approach*, Texts in Theoretical Computer Science. An EATCS Series, Springer. URL: https://doi.org/10.1007/978-3-662-03927-4, doi:10.1007/978-3-662-03927-4.

Yang, F. and Väänänen, J. (2017). Propositional team logics. *Annals of Pure and Applied Logic* **168** (7) 1406–1441. URL: https://doi.org/10.1016/j.apal.2017.01.007, doi:10.1016/j.apal.2017.01.007.

Cite this article: Kontinen J, Mahmood Y, Meier A and Vollmer H (2024). Parameterized complexity of weighted team definability. *Mathematical Structures in Computer Science* **34**, 375–389. https://doi.org/10.1017/S0960129524000033

https://doi.org/10.1017/S0960129524000033 Published online by Cambridge University Press