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Some expansion formulae for Fox's H-function involving exponential functions

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1. *Introduction.* In this paper we have evaluated some integrals involving Fox's H-function and employed them to establish some expansion formulae for the H-function involving exponential functions. Some results recently given by MacRobert (7) and Kesarwani (5) have been shown as particular cases.

The H-function introduced by Fox ((4), p. 408), will be represented and defined as follows:

$$H_{p,q}^{k,l}\left[z\left|\begin{matrix}(a_1,e_1),\ldots,(a_p,e_p)\\(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{k}\Gamma(b_j-f_js)\prod_{j=1}^{l}\Gamma(1-a_j+e_js)z^s}{\prod_{j=k+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=l+1}^{p}\Gamma(a_j-e_js)}\,ds, \quad (1{\cdot}1)$$

where z is not equal to zero and an empty product is interpreted as unity, p, q, k and l are integers satisfying $0\leqslant k\leqslant q$, $0\leqslant l\leqslant p$; e_j $(j=1,2,\ldots,p)$, f_j $(j=1,2,\ldots,q)$ are positive numbers and a_j $(j=1,2,\ldots,p)$, b_j $(j=1,2,\ldots,q)$ are complex numbers such that no pole of $\Gamma(b_j-f_js)$ $(j=1,2,\ldots,k)$ coincides with pole of $\Gamma(1-a_i+e_is)$ $(i=1,2,\ldots,l)$, i.e.

(i) $e_i(b_j+\nu)\neq(a_i-\eta-1)f_j \quad (\nu,\eta=0,1,2,\ldots;\ j=1,2,\ldots,k;\ i=1,2,\ldots,l).$

Further, the contour L runs from $\sigma-i\infty$ to $\sigma+i\infty$ such that the points:

(ii) $s=\dfrac{b_j+\nu}{f_j} \quad (j=1,2,\ldots,k;\ \nu=0,1,2,\ldots),$

which are poles of $\Gamma(b_j-f_js)$ $(j=1,2,\ldots,k)$ lie on the right, and the points:

(iii) $s=\dfrac{a_i-\eta-1}{e_i} \quad (i=1,2,\ldots,l;\ \eta=0,1,2,\ldots),$

which are poles of $\Gamma(1-a_i+e_is)$ $(i=1,2,\ldots,l)$ lie to the left, of L.

Such a contour is possible on account of (i). These assumptions will be retained throughout this paper.

Recently Braaksma (1) has given

$$H_{p,q}^{k,l}\left[z\left|\begin{matrix}(a_1,e_1),\ldots,(a_p,e_p)\\(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right.\right]=O(|z|^{\alpha}) \quad \text{for small } z,$$

where $\sum_{j=1}^{p}e_j-\sum_{j=1}^{q}f_j\leqslant 0$ and $\alpha=\min\Re\,(b_j/f_j) \quad (j=1,2,\ldots,k),$

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and $$H_{p,q}^{k,l}\left[z\left|\begin{matrix}(a_1,e_1),\ldots,(a_p,e_p)\\(b_1,f_1),\ldots,(b_q,f_q)\end{matrix}\right.\right]=O(|z|^{\beta}) \quad \text{for large } z,$$

where $$\sum_{j=1}^{p}e_j-\sum_{j=1}^{q}f_j\leqslant 0,\quad \sum_{j=1}^{l}e_j-\sum_{j=l+1}^{p}e_j+\sum_{j=1}^{k}f_j-\sum_{j=k+1}^{q}f_j\equiv B>0,$$

$$|\arg z|<\tfrac{1}{2}B\pi \quad \text{and} \quad \beta=\max \Re\{(a_i-1)/e_i\} \quad (i=1,2,\ldots,l).$$

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\cos\theta)^{m+n-2}e^{i(m-n)\theta}\,d\theta=\frac{\pi\Gamma(m+n-1)}{2^{m+n-2}\Gamma(m)\,\Gamma(n)},\quad \Re(m+n)>1, \tag{1·2}$$

which is formula ((6), p. 340, (95)).

$$\left.\begin{aligned}\int_a^b \exp\left(\frac{2m\pi ix}{a-b}\right)\exp\left(\frac{2n\pi ix}{a-b}\right)dx&=0 \qquad \text{if} \quad m\neq -n,\\&=b-a \quad \text{if} \quad m=-n;\end{aligned}\right\} \tag{1·3}$$

which is formula ((2), p. 62).

In what follows for sake of brevity

$$\sum_{j=1}^{p}e_j-\sum_{j=1}^{q}f_j\equiv A,\quad \sum_{j=1}^{l}e_j-\sum_{j=l+1}^{p}e_j+\sum_{j=1}^{k}f_j-\sum_{j=k+1}^{q}f_j\equiv B \quad \text{and} \quad (a_p,e_p)$$

represents the set of parameters.

2. *The integrals.* The integrals to be established are

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\cos\theta)^{m+n-2}e^{i(m-n)\theta}H_{p,q}^{k,l}\left[z(e^{i\theta}\cos\theta)^{-h}\left|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right.\right]d\theta$$

$$=\frac{\pi}{2^{m+n-2}\Gamma(n)}H_{p+1,q+1}^{k+1,l}\left[2^h z\left|\begin{matrix}(a_p,e_p),(m,h)\\(m+n-1,h),(b_q,f_q)\end{matrix}\right.\right], \tag{2·1}$$

$\Re(m+n-ha_j/e_j)>1-h/e_j \quad (j=1,2,\ldots,l)$,

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\cos\theta)^{m+n-2}e^{i(m-n)\theta}H_{p,q}^{k,l}\left[z\,e^{ih\theta}(\cos\theta)^{-h}\left|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right.\right]d\theta$$

$$=\frac{\pi}{2^{m+n-2}\Gamma(m)}H_{p+1,q+1}^{k+1,l}\left[2^h z\left|\begin{matrix}(a_p,e_p),(n,h)\\(m+n-1,h),(b_q,f_q)\end{matrix}\right.\right], \tag{2·2}$$

$\Re(m+n-ha_j/e_j)>1-h/e_j \ (j=1,2,\ldots,l)$;

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\cos\theta)^{m+n-2}e^{i(m-n)\theta}H_{p,q}^{k,l}\left[z\,e^{2ih\theta}\left|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right.\right]d\theta$$

$$=\frac{\pi\Gamma(m+n-1)}{2^{m+n-2}}H_{p+1,q+1}^{k,l}\left[z\left|\begin{matrix}(a_p,e_p),(n,h)\\(b_q,f_q),(1-m,h)\end{matrix}\right.\right], \tag{2·3}$$

$\Re(m+n)>1$;

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\cos\theta)^{m+n-2}e^{i(m-n)\theta}H_{p,q}^{k,l}\left[z(\cos\theta)^{-2h}\left|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right.\right]d\theta$$

$$=\frac{\pi}{2^{m+n-2}}H_{p+2,q+1}^{k+1,l}\left[2^{2h}z\left|\begin{matrix}(a_p,e_p),(m,h),(n,h)\\(m+n-1,2h),(b_q,f_q)\end{matrix}\right.\right], \tag{2·4}$$

$\Re(m+n-2ha_j/e_j)>1-2h/e_j \ (j=1,2,\ldots,l)$;

where h is a positive number and $A\leqslant 0, B>0, |\arg z|<\frac{1}{2}B\pi$.

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Proof. To prove (2·1), expressing the H-function in the integrand as a Mellin–Barnes type integral (1·1), interchanging the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{k}\Gamma(b_j-f_js)\prod_{j=1}^{l}\Gamma(1-a_j+e_js)z^s}{\prod_{j=k+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=l+1}^{p}\Gamma(a_j-e_js)}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\cos\theta)^{m-hs+n-2}\,e^{i(m-hs-n)\theta}\,d\theta\,ds.$$

Evaluating the inner integral with the help of (1·2), we get

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{k}\Gamma(b_j-f_js)\prod_{j=1}^{l}\Gamma(1-a_j+e_js)z^s}{\prod_{j=k+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=l+1}^{p}\Gamma(a_j-e_js)}\cdot\frac{\pi\Gamma(m-hs+n-1)}{2^{m-hs+n-2}\Gamma(m-hs)\Gamma(n)}\,ds.$$

On applying (1·1) the formula (2·1) is established.

Formulae (2·2), (2·3) and (2·4) can be similarly established on applying the same procedure as above with the help of (1·2).

3. In this section the expansion formulae to be established are

$$(\cos\theta)^{2(m+n-1)}H_{p,q}^{k,l}\left[z(e^{i\theta}\cos\theta)^{-h}\middle|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]$$
$$=\sum_{r=-\infty}^{\infty}\frac{1}{\Gamma(2n)\,2^{2(r+n-1)}}H_{p+1,q+1}^{k+1,l}\left[2^hz\middle|\begin{matrix}(a_p,e_p),(2r,h)\\(2r+2n-1,h),(b_q,f_q)\end{matrix}\right]e^{-2i(r-n)\theta},\quad(3\cdot1)$$

$$(\cos\theta)^{2(m+n-1)}H_{p,q}^{k,l}\left[z\,e^{ih\theta}(\cos\theta)^{-h}\middle|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]$$
$$=\sum_{r=-\infty}^{\infty}\frac{1}{\Gamma(2r)\,2^{2(r+n-1)}}H_{p+1,q+1}^{k+1,l}\left[2^hz\middle|\begin{matrix}(a_p,e_p),(2n,h)\\(2r+2n-1,h),(b_q,f_q)\end{matrix}\right]e^{-2i(r-n)\theta},\quad(3\cdot2)$$

$$(\cos\theta)^{2(m+n-1)}H_{p,q}^{k,l}\left[ze^{2ih\theta}\middle|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]$$
$$=\sum_{r=-\infty}^{\infty}\frac{\Gamma(2r+2n-1)}{2^{2(r+n-1)}}H_{p+1,q+1}^{k,l}\left[z\middle|\begin{matrix}(a_p,e_p),(2n,h)\\(b_q,f_q),(1-2r,h)\end{matrix}\right]e^{-2i(r-n)\theta},\quad(3\cdot3)$$

$$(\cos\theta)^{2(m+n-1)}H_{p,q}^{k,l}\left[z(\cos\theta)^{-2h}\middle|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]$$
$$=\sum_{r=-\infty}^{\infty}\frac{1}{2^{2(r+n-1)}}H_{p+2,q+1}^{k+1,l}\left[2^{2h}z\middle|\begin{matrix}(a_p,e_p),(2r,h),(2n,h)\\(2r+2n-1,2h),(b_q,f_q)\end{matrix}\right]e^{-2i(r-n)\theta},\quad(3\cdot4)$$

where h is a positive number and

$$A\leqslant 0,\quad B>0,\quad |\arg z|<\tfrac{1}{2}B\pi,\quad(-\tfrac{1}{2}\pi<\theta<\tfrac{1}{2}\pi),\quad \Re(m+n)>1.$$

Proof. To prove (3·1), let

$$f(\theta)=(\cos\theta)^{2(m+n-1)}H_{p,q}^{k,l}\left[z(e^{i\theta}\cos\theta)^{-h}\middle|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]$$
$$=\sum_{r=-\infty}^{\infty}C_r\,e^{-2i(r-n)\theta},\ (-\tfrac{1}{2}\pi<\theta<\tfrac{1}{2}\pi),\ \Re(m+n)>1.\quad(3\cdot5)$$

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Equation (3·5) is valid, since $f(\theta)$ is continuous and of bounded variation in the open interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

Multiplying both sides of (3·5) by $e^{2(m-n)i\theta}$ and integrating with respect to θ from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, we get

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\cos\theta)^{2(m+n-1)} e^{2i(m-n)\theta} H_{p,q}^{k,l}\left[z(e^{i\theta}\cos\theta)^{-h}\middle|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] d\theta = \sum_{r=-\infty}^{\infty} C_r \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{2i(m-r)\theta}\, d\theta.$$

Now using (2·1) and the orthogonality property of exponential functions (1·3), we obtain

$$C_m = \frac{1}{\Gamma(2n)\, 2^{2(m+n-1)}} H_{p+1,q+1}^{k+1,l}\left[2^h z\middle|\begin{matrix}(a_p, e_p), (2m, h)\\(2m+2n-1, h), (b_q, f_q)\end{matrix}\right]. \tag{3·6}$$

With the help of (3·5) and (3·6), formula (3·1) is obtained.

Formulae (3·2), (3·3) and (3·4) have been established on applying the same procedure with the help of (2·2), (2·3) and (2·4) respectively.

4. (i) In this section the following expansion formula is to be established.

$$H_{p,q}^{k,l}\left[z(\cos\theta)^{-2h}\middle|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] = \sum_{r=-\infty}^{\infty} H_{p+2,q+1}^{k+1,l}\left[2^{2h} z\middle|\begin{matrix}(a_p, e_p), (1-r, h), (1+r, h)\\(1, 2h), (b_q, f_q)\end{matrix}\right] e^{2ri\theta}, \tag{4·1}$$

where h is a positive number and $A \leqslant 0$, $B > 0$, $|\arg z| < \frac{1}{2}B\pi$, $(-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi)$.

Proof. Let

$$f(\theta) = H_{p,q}^{k,l}\left[z(\cos\theta)^{-2h}\middle|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] = \sum_{r=-\infty}^{\infty} C_r e^{2ri\theta} \quad (-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi). \tag{4·2}$$

Multiplying both sides of (4·2) by $e^{-2si\theta}$ and integrating with respect to θ from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, we get

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-2si\theta} H_{p,q}^{k,l}\left[z(\cos\theta)^{-2h}\middle|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] d\theta = \sum_{r=-\infty}^{\infty} C_r \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{2i(r-s)\theta}\, d\theta.$$

Now using (2·4) [with $m+n=2$, $m=1-s$, $n=1+s$] and (1·3), we have

$$C_s = H_{p+2,q+1}^{k+1,l}\left[2^{2h} z\middle|\begin{matrix}(a_p, e_p), (1-s, h), (1+s, h)\\(1, 2h), (b_q, f_q)\end{matrix}\right]. \tag{4·3}$$

From (4·2), and (4·3), the formula (4·1) follows.

Proceeding as above three expansion formulae corresponding to (2·1), (2·2) and (2·3) can be obtained easily.

5. *Particular cases.* On specializing the parameters the H-function can be reduced to Meijer's G-function, MacRobert's E-function and other higher transcendental functions. Therefore, the results established in the paper lead to generalizations of many results. However some interesting particular cases are given below.

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The formula (4·1) can be written as

$$H^{k,l}_{p,q}\left[z(\cos\theta)^{-2h}\left|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right.\right]$$
$$= H^{k+1,l}_{p+2,q+1}\left[2^{2h}z\left|\begin{matrix}(a_p,e_p),(1,h),(1,h)\\(1,2h),(b_q,f_q)\end{matrix}\right.\right]$$
$$+2\sum_{r=1}^{\infty} H^{k+1,l}_{p+2,q+1}\left[2^{2h}z\left|\begin{matrix}(a_p,e_p),(1-r,h),(1+r,h)\\(1,2h),(b_q,f_q)\end{matrix}\right.\right]\cos 2r\theta, \quad (5\cdot1)$$

where the conditions are the same as in (4·1).

In (5·1), putting $\theta = \frac{1}{2}\pi - \phi$, $\cos(\pi - 2\phi) = (-1)^r \cos 2r\phi$ and using the following identity, which is apparent from the definition of the H-function, viz.

$$H^{k+1,l}_{p+2,q+1}\left[2^{2h}z\left|\begin{matrix}(a_p,e_p),(1-r,h),(1+r,h)\\(1,2h),(b_q,f_q)\end{matrix}\right.\right],$$
$$= \frac{(-1)^r}{\sqrt{(\pi)}} H^{k+1,l+1}_{p+2,q+2}\left[z\left|\begin{matrix}(1-r,h),(a_p,e_p),(1+r,h)\\(\frac{1}{2},h),(b_q,f_q),(1,h)\end{matrix}\right.\right],$$

it reduces to the form

$$H^{k,l}_{p,q}\left[z(\sin\phi)^{-2h}\left|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right.\right]$$
$$= \frac{1}{\sqrt{(\pi)}} H^{k+1,l}_{p+1,q+1}\left[z\left|\begin{matrix}(a_p,e_p),(1,h)\\(\frac{1}{2},h),(b_q,f_q)\end{matrix}\right.\right]$$
$$+\frac{2}{\sqrt{\pi}}\sum_{r=1}^{\infty} H^{k+1,l+1}_{p+2,q+2}\left[z\left|\begin{matrix}(1-r,h),(a_q,e_p),(1+r,h)\\(\frac{1}{2},h),(b_q,f_q),(1,h)\end{matrix}\right.\right]\cos 2r\phi, \quad (5\cdot2)$$

where the conditions are the same as in (4·1) except $(0 < \phi < \pi)$.

In (5·2), assuming h as a positive integer, putting

$$e_j = f_i = 1 \quad (j = 1, 2, \ldots, p;\ i = 1, 2, \ldots, q),$$

using the formula $$H^{k,l}_{p,q}\left[z\left|\begin{matrix}(a_p,1)\\(b_q,1)\end{matrix}\right.\right] = G^{k,l}_{p,q}\left[z\left|\begin{matrix}a_p\\b_q\end{matrix}\right.\right],$$

and simplifying with the help of (1·1) ((3), p. 4, (11)), ((3), p. 207, (1)), we get

$$G^{k,l}_{p,q}\left[z(\sin\phi)^{-2h}\left|\begin{matrix}a_p\\b_q\end{matrix}\right.\right]$$
$$= \frac{1}{\sqrt{(\pi h)}} G^{k+h,l}_{p+h,q+h}\left[z\left|\begin{matrix}a_p,\Delta(h,1)\\\Delta(h,\frac{1}{2}),b_q\end{matrix}\right.\right]$$
$$+\frac{2}{\sqrt{(\pi h)}}\sum_{r=1}^{\infty} G^{k+h,l+h}_{p+2h,q+2h}\left[z\left|\begin{matrix}\Delta(h,1-r),a_p,\Delta(h,1+r)\\\Delta(h,\frac{1}{2}),b_q,\Delta(h,1)\end{matrix}\right.\right]\cos 2r\phi, \quad (5\cdot3)$$

where h is a positive integer, $2(k+l) < p+q$, $|\arg z| < (k+l-\frac{1}{2}p-\frac{1}{2}q)\pi$ $(0 < \phi < \pi)$ and the symbol $\Delta(h,a)$ represents the set of parameters

$$\frac{a}{h}, \quad \frac{a+1}{h}, \ldots, \frac{a+h-1}{h}.$$

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In (5·3), putting $\phi = \frac{1}{2}\theta$, $h = 1$ it reduces to a known result recently given by Kesarwani ((5), p. 149, (1·2)), which itself is a generalization of a result given by MacRobert ((7), p. 79, (1)).

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