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On a functional equation

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1. *Introduction.* The object of this note is to give a proof of the following theorem on the solution of an important functional equation.

THEOREM A. *Let $f(z)$ be an entire function of order ρ $(0<\rho<\infty)$, and type σ $(0\leqslant\sigma<\infty)$. Let ω, λ be non-zero constants (real or complex). Then there is an entire function $g(z)$, of order ρ and type σ, such that*

$$g(z+\omega)-\lambda g(z)=f(z). \tag{1}$$

A brief history of results leading up to Theorem A, which may be gathered from the references given at the end ((6), page 22; (3), page 305; (5), page 543; (1), pages 70–1), is as follows. First, C. Guichard proved Theorem A for $\lambda=1$ and $f(z), g(z)$ of unspecified order and type, and then Hurwitz extended Guichard's result to $\lambda\neq 1$. Later, Whittaker proved Theorem A for $\lambda=1$ and $f(z)$, $g(z)$ of unspecified type but order ρ, including the values $\rho=0$ and $\rho=\infty$, of which the latter is covered by Guichard's result (see e.g. (6), page 22, Theorem 3). Subsequently Sheila Scott proved Theorem A for $\lambda=1$ ((5), Theorem I), by refining Whittaker's arguments. Lastly Buck imposed on $f(z)$ the condition that it is of finite exponential type and has its type on the whole imaginary axis, i.e. its Phragmén–Lindelöf function $h(\pm\frac{1}{2}\pi)$, less than π, proving that, for any λ not real and negative, the functional equation (1) has always a solution $g(z)$ which satisfies the same condition as $f(z)$. To this historical background of Theorem A may be added Krishnamurthy's attempt to prove Theorem A in (4) by employing functional analysis, an attempt which, however, is not valid, as stated, for instance, by Ganapathy Iyer ((2)). The proof of Theorem A in this note uses only classical analysis; for $0<\rho<1$, it is based on an idea kindly suggested by Prof. Ganapathy Iyer, while for $\rho=1$ it is essentially a simplification of Sheila Scott's proof of Theorem A in the case $\lambda=1$, and for $\rho>1$ it is just an argument (due to the referee) for deducing the case $\lambda\neq 1$ from the known case $\lambda=1$. However, it may be mentioned that, for $\rho>1$, as for $\rho=1$, there is a direct proof of Theorem A which is a simplification of Sheila Scott's in the case $\lambda=1$.

In the following proof of Theorem A, we suppose that $\omega=1$, this supposition involving no loss of generality, as we may see from the following argument (also due to the referee). If $\omega\neq 1$ and the theorem has been proved for $\omega=1$, then $f(\omega z)$ is an entire function, of order ρ and type $|\omega|^{\rho}\sigma$, corresponding to which there exists an entire function $G(z)$, of the same order and type, such that

$$G(z+1)-\lambda G(z)=f(\omega z), \quad \text{or} \quad G\left(\frac{z}{\omega}+1\right)-\lambda G\left(\frac{z}{\omega}\right)=f(z).$$

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Since $g(z) = G(z/\omega)$ is an entire function of order ρ and type $|\omega|^\rho \sigma/|\omega|^\rho$, satisfying (1), we see that Theorem A for $\omega \neq 1$ follows from Theorem A for $\omega = 1$ and that it is sufficient to prove the latter.

2. *Proof of Theorem A.* Since $f(z)$ is of order ρ and type σ, we have

$$\limsup_{|z|=R\to\infty} \frac{\log \max_{|z|=R} |f(z)|}{R^\rho} = \sigma.$$

Supposing that $\omega = 1$ *in* (1), we need only prove the existence of a solution $g(z)$ of (1), analytic for all finite z and such that

$$\limsup_{|z|=R\to\infty} \frac{\log \max_{|z|=R} |g(z)|}{R^\rho} \leqslant \sigma. \tag{2}$$

For (2) ensures that the order and the type of $g(z)$ shall not exceed ρ and σ, respectively, while (1) ensures that the order and the type of $g(z)$ shall not be less than ρ and σ, respectively.

Case $\lambda \neq 1$, *subcase* $0 < \rho < 1$. *We first suppose that* $\sigma > 0$, and note that the integral

$$g_n(z) = \frac{n!}{2\pi i} \int_{|t|=1/R'} \frac{e^{zt}}{(e^t - \lambda)\, t^{n+1}}\, dt \quad (R' > R_0'), \tag{3}$$

where R_0' is suitably large, is an analytic function of z. We further note that

$$g_n(z+1) - \lambda g_n(z) = z^n, \tag{4}$$

this relation being, in fact, true even if the circle $|t| = 1/R'$ in (3) is replaced by any simple closed curve enclosing $t = 0$ and not containing any zero of $e^t - \lambda$. And so, if we write

$$f(z) = \sum_0^\infty a_n z^n, \tag{5}$$

the series

$$g(z) = \sum_0^\infty a_n g_n(z), \tag{6}$$

whenever it is convergent, will satisfy (1) with $\omega = 1$. We proceed to show that the series in (6) is indeed uniformly convergent in every circle $|z| \leqslant R$ and that its sum-function satisfies (2).

In (3), the factor $(e^t - \lambda)^{-1}$ of the integrand is bounded as $t \to 0$ and hence we get, from (6) and (3),

$$|g(z)| \leqslant \sum_0^\infty |a_n|\, |g_n(z)| < K\, e^{R|R'} \sum_0^\infty |a_n|\, n!\, R'^n \quad (|z| \leqslant R). \tag{7}$$

In (7), K is a constant independent of R' and the series on the extreme right is convergent, since (as will be shown presently)

$$h(z) = \sum_0^\infty a_n\, n!\, z^n \tag{8}$$

is an entire function. Therefore $g(z)$ in (6) is a series of analytic functions uniformly (and absolutely) convergent in any circle $|z| \leqslant R$, i.e. $g(z)$ is analytic for all finite z.

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To show that $h(z)$ in (8) is an entire function, we appeal to a well-known theorem by which the order ρ of $f(z)$ as defined in (5), is given by

$$\liminf_{n\to\infty} \frac{\log |a_n|^{-1}}{n \log n} = \frac{1}{\rho}.$$

This relation in conjunction with Stirling's approximation, $n! \sim \sqrt{(2\pi)}\, e^{-n} n^{n+\frac{1}{2}}$, gives us

$$\liminf_{n\to\infty} \frac{\log (|a_n|\, n!)^{-1}}{n \log n} = \frac{1}{\rho} - 1.$$

Hence, by the converse of the well-known theorem just referred to, $h(z)$ in (8) is an entire function of order ρ' given by

$$\frac{1}{\rho'} = \frac{1}{\rho} - 1. \tag{9}$$

Next, by another well-known theorem, the type σ of $f(z)$ and the type σ' of $h(z)$ are given by

$$\left.\begin{aligned} \limsup_{n\to\infty} |a_n|^{1/n} n^{1/\rho} &= (\sigma e \rho)^{1/\rho}, \\ \limsup_{n\to\infty} (|a_n|\, n!)^{1/n} n^{1/\rho'} &= (\sigma' e \rho')^{1/\rho'}. \end{aligned}\right\} \tag{10}$$

Now Stirling's approximation leads at once to the relation

$$(n!)^{1/n} \sim e^{-1} n$$

which, together with (9), used in the second formula of (10), gives us

$$\begin{aligned} e^{1/\rho'} (\sigma' \rho')^{1/\rho'} &= e^{-1} \limsup_{n\to\infty} |a_n|^{1/n} n^{1+1/\rho'} \\ &= e^{-1} \limsup_{n\to\infty} |a_n|^{1/n} n^{1/\rho} = e^{-1+1/\rho} (\sigma\rho)^{1/\rho}, \end{aligned}$$

in virtue of the first formula of (10). And so

$$(\sigma' \rho')^{1/\rho'} = (\sigma\rho)^{1/\rho}. \tag{11}$$

Thus ρ', given by (9), defines the order of the entire power series $h(z)$ of (8) in terms of the moduli of its coefficients, while σ', given by (11), defines the type of $h(z)$. Consequently, from (7),

$$|g(z)| < K \exp\{(R/R') + (\sigma' + \epsilon')\,(R')^{\rho'}\} \quad (R = |z|,\ R' > R_1' > R_0'), \tag{12}$$

where we may suppose that $\epsilon' > 0$ is defined in terms of any small $\epsilon > 0$ by the relation

$$\{(\sigma' + \epsilon')\,\rho'\}^{1/\rho'} = \{(\sigma + \epsilon)\,\rho\}^{1/\rho}, \tag{11'}$$

thus ensuring, by (11), that ϵ' shall be as small as we please through a small enough choice of ϵ. Now, given R, the most advantageous form of the inequality (12) corresponds to the absolute minimum of the right-hand member of (12) as a function of R', and this minimum in its turn corresponds to the positive value of R' got from the equation

$$-(R/R'^2) + (\sigma' + \epsilon')\,\rho'(R')^{\rho'-1} = 0.$$

Using this equation and (11') in (12), we can rewrite (12) in terms of R alone as

$$|g(z)| < K \exp\{(\sigma + \epsilon)\, R^{\rho}\} \quad (|z| = R > R_1), \tag{13}$$

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where $R = R_1$ corresponds to $R' = R_1'$. (13) leads at once to (2) and so completes the proof.

If it is given that $\sigma = 0$, it will follow from (11) that $\sigma' = 0$, so that (11'), (12) and (13) will emerge with $\sigma = \sigma' = 0$ and finally (2) with $\sigma = 0$ as required.

In the proof of the next subcase $\rho = 1$ (case $\lambda \neq 1$) of Theorem A, we require the following lemmas.

LEMMA 1. *If* $f(z) = \Sigma a_n z^n$ *is an entire function of order* $\rho = 1$ *and type* σ, $0 \leqslant \sigma < \infty$, *then* $\Sigma a_n n!/z^n$ *converges (absolutely) for* $|z| > \sigma$.

For, by Cauchy's root-test, $\Sigma a_n n!/z^n$ converges absolutely for any $z \neq 0$ such that

$$|z|^{-1} \limsup_{n\to\infty} (|a_n| n!)^{1/n} < 1,$$

where
$$\limsup_{n\to\infty} (|a_n| n!)^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n} e^{-1} n = \sigma$$

by the first formula of (10) with $\rho = 1$.

LEMMA 2. *The function* $(e^t - 1)^{-1}$ *has simple poles at* $t = 2n\pi i$ (*n zero or an integer*). *And if non-overlapping squares are drawn with centres at these poles and sides of length* $2\delta > 0$ *parallel to the real and imaginary t-axes, then in the part of the entire t-plane from which all such squares are excluded we have*

$$|e^t - 1|^{-1} < M(\delta), \tag{14}$$

where $M(\delta)$ *is a constant depending only on* δ.

The proof of this lemma is omitted, being direct and simple.

Case $\lambda \neq 1$, *subcase* $\rho = 1$. For convenience, we first choose the principal value of the logarithm in the definition $e^{\operatorname{Log}\lambda} = \lambda$, and then draw non-overlapping squares $S_n(\delta)$ with centres at $t = \log\lambda + 2n\pi i$ (n zero or an integer) and sides of length $2\delta > 0$ parallel to the real and imaginary t-axes, ensuring that in the portion of the entire t-plane from which $S_n(\delta)$ are excluded we shall have

$$|e^t - \lambda|^{-1} = |\lambda^{-1}(e^{t-\log\lambda} - 1)^{-1}| < |\lambda|^{-1} M(\delta) \tag{14'}$$

by Lemma 2. Whether the circle $|t| = \sigma$ does, or does not, pass through the centres of any of the squares $S_n(\delta)$, we can choose $\eta > 0$ so that the circles $|t| = \sigma + \eta$ and $|t| = \sigma + \epsilon$, for every ϵ such that $0 < \epsilon < \eta$, are boundaries of an annulus within and on which $(e^t - \lambda)^{-1}$ is analytic. Furthermore, if, corresponding to any such ϵ, we choose $\delta = \frac{1}{2}\epsilon$, the function $(e^t - \lambda)^{-1}$ will obey (14') with $\delta = \frac{1}{2}\epsilon$ on the circle $|t| = \sigma + \epsilon$.

It is now clear that, when the definition of $g_n(z)$ in (3) is changed to the following,

$$g_n(z) \equiv \frac{n!}{2\pi i}\int_{|t|=\sigma+\eta} \frac{e^{zt}}{(e^t - \lambda)\, t^{n+1}}\, dt = \frac{n!}{2\pi i}\int_{|t|=\sigma+\epsilon} \frac{e^{zt}}{(e^t - \lambda)\, t^{n+1}}\, dt, \tag{15}$$

$g_n(z)$ still is analytic for all finite z and satisfies (4), so that

$$f(z) = \Sigma a_n z^n \quad \text{and} \quad g(z) = \Sigma a_n g_n(z),$$

if convergent, will be related by (1) with $\omega = 1$. Moreover,

$$|g(z)| \leqslant \Sigma |a_n g_n(z)| < |\lambda|^{-1} M(\epsilon/2)\, e^{(\sigma+\epsilon)R} \Sigma \frac{|a_n| n!}{(\sigma+\epsilon)^n} \quad (|z| \leqslant R), \tag{16}$$

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since (14′) holds with $\delta = \frac{1}{2}\epsilon$ on $|t| = \sigma + \epsilon$ and the series on the extreme right of (16) converges by Lemma 1 (irrespective of whether $\sigma > 0$ or $\sigma = 0$), thus ensuring that $g(z)$ shall converge uniformly (and absolutely) in any circle $|z| \leqslant R$ and be analytic for all finite z. Inequality (16) also shows that

$$|g(z)| < K e^{(\sigma+\epsilon)R} \quad (|z| = R),$$

where K is a constant depending on ϵ alone. Keeping ϵ fixed and letting $R \to \infty$, we deduce from the last inequality

$$\limsup_{R\to\infty} \frac{\log \max_{|z|=R} |g(z)|}{R} \leqslant \sigma + \epsilon,$$

which reduces to (2) with $\rho = 1$ since ϵ is arbitrary. The proof is now complete.

Case $\lambda \neq 1$, *subcase* $\rho > 1$. Assuming the case $\lambda = 1$, subcase $\rho > 1$, of Theorem A, proved by Sheila Scott, we note that $\lambda^{-z-1}f(z)$ is an entire function, of order ρ and type σ, corresponding to which there exists an entire function $H(z)$, of the same order and type, such that

$$H(z+1) - H(z) = \lambda^{-z-1}f(z).$$

This shows that $g(z) = \lambda^z H(z)$ is an entire function, of order ρ and type σ, such that

$$g(z+1) - \lambda g(z) = \lambda^{z+1}\{H(z+1) - H(z)\} = f(z),$$

and so completes the proof.

Case $\lambda = 1$. In this case we change as follows the definitions of $g_n(z)$ in (3) and (15),

$$g_n(z) = \frac{n!}{2\pi i}\int \frac{e^{zt}}{\{(e^t-1)/t\}t^{n+1}}\,dt,$$

but (as before) taking the integral along $|z| = 1/R'$ in subcase $\rho < 1$, and along $|z| = \sigma + \eta$ in subcase $\rho = 1$. At the same time we change our former definition of $g(z)$ to the following

$$g(z) = \sum_0^\infty \frac{a_n}{n+1} g_{n+1}(z).$$

With these changes, the proof of Theorem A for the case $\lambda \neq 1$, subcases $\rho \leqslant 1$, can be readily adapted to the case $\lambda = 1$, subcases $\rho \leqslant 1$. As for the case $\lambda = 1$, subcase $\rho > 1$, we may define (as above) $g(z)$ in terms of $g_n(z)$ $(n = 1, 2, 3, \ldots)$, and $g_n(z)$ as an integral taken along a suitable closed curve, using $g_n(z)$ thus defined in a simplified version of Sheila Scott's proof (excluded from this note in accordance with the referee's suggestion).

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