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# Weak precompactness, strong boundedness, and weak complete continuity

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*Abstract*

Weak precompactness in spaces of vector measures and in the space of Bochner integrable functions is studied. Uniform countable additivity and uniform integrability are characterized in terms of weak precompactness. Through this, a connection between strongly bounded operators and operators having weakly precompact adjoints on abstract continuous function spaces is established. These operators are compared with weakly completely continuous operators.

## 1. *Introduction*

In [1] Abbott studied the class of weakly precompact operators $T:C(H,X)\to Y$. She showed that if $T^*$ is weakly precompact, then $T$ is strongly bounded. More generally, in this paper we show that if $T:X\to Y$ is a bounded linear operator such that $T^*$ is weakly precompact, then $T$ is unconditionally converging. Further, if $\ell_1$ does not embed in $X^*$ and $X^*$ has RNP, then we show that the converse of Abbott's theorem holds. A key in obtaining this result is a brief study of $K$-weak precompactness in spaces of vector measures. As a consequence of this study, we characterize $L_\infty(\mu,X^*)$-weak precompactness in the Lebesgue–Bochner space $L_1(\mu,X)$ for arbitrary Banach spaces $X$ in terms of uniform integrability. Pisier [**21**], Bourgain [**8**], and Talagrand [**26**] have studied the relationship between weak precompactness and uniform integrability in $L_1(\mu,X)$ in terms of the presence of isomorphic copies of $\ell_1$ in $X$.

We also compare strongly bounded operators with both completely continuous operators (= Dunford–Pettis operators) and weakly completely continuous operators (= Dieudonné operators). We then characterize general weakly completely continuous operators in terms of the embedding of $X$ into $X^{**}$. We conclude our study by briefly returning to operators on $C(H,X)$ and present alternative (and perhaps more direct) proofs of some of the results of Bombal and Cembranos [**6**, **7**].

Our notation will be consistent with that developed by Brooks and Lewis in [**9**] and [**10**]. Specifically, $H$ will be a compact Hausdorff space, each of $X$ and $Y$ will be a real Banach space, $C(H,X)$ will be the Banach space (with sup norm) of all continuous $X$-valued functions defined on $H$, and $T:C(H,X)\to Y$ will be a continuous linear function (= operator) with representing measure $m:\Sigma\to B(X, Y^{**})$, where $\Sigma$ is

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the $\sigma$-algebra of Borel subsets of $H$, $Y^{**}$ is the bidual of $Y$ and $B(X, Y^{**})$ is the Banach space of all operators $L: X \to Y^{**}$. We denote the semivariation of $m$ by $\tilde{m}$ and we say that $m$ (or $T$) is strongly bounded (continuous at $\phi$) if $\tilde{m}(A_n) \to 0$ whenever $(A_n)$ is a pairwise disjoint sequence from $\Sigma$. If $(\Omega, \mathscr{A}, \mu)$ is a measure space, then $L_1(\mu, X)$ will be the space of $\mu$-measurable Bochner integrable functions $f: \Omega \to X$, and cabv $(\mathscr{A}, X, \mu)$ will be the Banach space (with total variation norm) of all countably additive measures $\nu: \mathscr{A} \to X$ so that $|\nu| \ll \mu$. A Banach space $X$ is said to have the Radon–Nikodym property (RNP) if for every finite measure space $(\Omega, \mathscr{A}, \mu)$ and bounded linear operator $T: L_1(\mu) \to X$, there exists a Bochner integrable function $g: \Omega \to X$ such that $T(f) = \int fg\, d\mu$ for every $f \in L_1(\mu)$. Similarly, $X$ is said to have the weak Radon–Nikodym property (weak RNP) if such operators can be represented by a Pettis integrable function. We refer the reader to Diestel [**11**] and Diestel and Uhl [**12**] for other standard notation and terminology.

## 2. *Weak precompactness*

The following two theorems set the tone for the first portion of this section.

THEOREM 2·1 ([**9**], p. 151). (*a*) *Suppose that $X$ and $X^*$ have RNP and $K \subset$* cabv $(\mathscr{A}, X)$. *If $K$ satisfies* (i) *$K$ is bounded*, (ii) $|K| = \{|\nu| : \nu \in K\}$ *is uniformly countably additive, and* (iii) $\{\nu(A) : \nu \in K\}$ *is relatively weakly compact in $X$ for each $A \in \mathscr{A}$, then $K$ is relatively weakly compact in* cabv $(\mathscr{A}, X)$.

(*b*) *Conversely, if $X$ is any Banach space and $K$ is a relatively weakly compact subset of* cabv $(\mathscr{A}, X)$, *then* (i), (ii), *and* (iii) *hold.*

THEOREM 2·2. (cf. [**16**]). *A Banach space $X$ and its dual $X^*$ have RNP if and only if for every finite measure space $(\Omega, \mathscr{A}, \mu)$ the subset $A$ of $L_1(\mu, X)$ is relatively weakly compact whenever* (i) *$A$ is bounded*, (ii) *$A$ is uniformly integrable, and* (iii) $\{\int_B f\, d\mu : f \in A\}$ *is relatively weakly compact in $X$ for each $B \in \mathscr{A}$.*

In this section, we study $K$-weak precompactness in cabv$(\mathscr{A}, X, \mu)$ (as well as in $L_1(\mu, X)$) in terms of conditions analogous to those in the preceding two theorems. We begin with the following definition.

*Definition* (cf. [**3**]). If $A \subset X$ and $K \subset X^*$, then we say that $A$ is $K$-weakly precompact if each bounded sequence $(x_n)$ from $A$ has a subsequence $(x_{n_i})$ so that $(x^*(x_{n_i}))$ converges for each $x^* \in K$. If $K = X^*$, then we say that $A$ is weakly precompact. An operator $T: X \to Y$ is said to be weakly precompact if $T(B_X)$ is weakly precompact in $Y$, where $B_X$ is the unit ball in $X$.

We remark that the fundamental result of Rosenthal [**23**] shows that a Banach space $X$ is weakly precompact if and only if $\ell_1$ does not embed isomorphically in $X$. Additionally, Riddle, Saab and Uhl [**22**] showed that an operator $L: X \to Y$ is weakly precompact if and only if it factors through a space not containing a copy of $\ell_1$. It is a theorem of Janika [**19**] and Musial [**20**] that $X$ does not contain an isomorphic copy of $\ell_1$ if and only if $X^*$ has the weak RNP.

We note that if $\nu \in$ cabv$(\mathscr{A}, X, \mu)$ and $g \in L_\infty(\mu, X^*)$, then we may use the general bilinear integral of Bartle [**2**] to define $\int g\, d\nu$. Specifically, if $(\Psi_n)$ is a uniformly bounded sequence of $X^*$-valued simple functions (defined over $\mathscr{A}$) so that $\Psi_n \to g$ $\mu$-a.e., then $(\int_\Omega \Psi_n\, d\nu)$ converges. Define $\int_\Omega g\, d\nu$ to be $\lim \int_\Omega \Psi_n\, d\nu$.

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LEMMA 2·3. *If* $(\Omega, \mathscr{A}, \mu)$ *is a finite measure space and* $(\nu_n)$ *is a sequence from* $\text{cabv}(\mathscr{A}, X, \mu)$ *so that* (i) $\sup_n \{|\nu_n|(\Omega) : n \in \mathbb{N}\} < \infty$, (ii) $|\nu_n| \ll \mu$ *uniformly in* $n$, *and* (iii) $(\nu_n(A))$ *is weakly Cauchy in* $X$ *for each* $A \in \mathscr{A}$, *then* $(\nu_n)$ *is* $K$*-weakly Cauchy, where* $K = L_\infty(\mu, X^*)$.

*Proof.* Suppose that $(\Omega, \mathscr{A}, \mu)$ is a finite measure space and $(\nu_n)$ is a sequence from $\text{cabv}(\mathscr{A}, X, \mu)$ which satisfies (i), (ii), and (iii). Let $g \in L_\infty(\mu, X^*)$, and let $(s_n)$ be a sequence of simple functions so that $(s_n) \to g$ $\mu$-a.e. and $\|s_n(t)\| \leqslant 2\|g(t)\|$ for each $n$ and each $t$.

Let $\epsilon > 0$. Since $(\nu_n)$ is uniformly absolutely continuous, there exists $\delta > 0$ so that if $\mu(A) < \delta$ for $A \in \mathscr{A}$, then $|\nu_n|(A) < \epsilon$ for all $n$. By Egorov's Theorem, there exists $A \in \mathscr{A}$ so that $\mu(A) < \delta$ and $(s_n) \to g$ uniformly on $\Omega \backslash A$. Let $A$ be one such element of $\mathscr{A}$. Pick $N \in \mathbb{N}$ so that $\|s_n(t) - g(t)\| < \epsilon$ for all $t \in \Omega \backslash A$ and $n \geqslant N$. Since $(\nu_n)$ satisfies (iii), we may pick $I \in \mathbb{N}$ so that if $i, j \geqslant I$, then $|\int_\Omega s_N \, d(\nu_i - \nu_j)| < \epsilon$. Let $i, j \geqslant I$; then

$$\begin{aligned}
\left| \int_\Omega g \, d\nu_i - \int_\Omega g \, d\nu_j \right| &\leqslant \left| \int_\Omega (s_N - g) \, d\nu_i \right| + \left| \int_\Omega s_N \, d(\nu_i - \nu_j) \right| + \left| \int_\Omega (s_N - g) \, d\nu_j \right| \\
&< \left| \int_{\Omega \backslash A} (s_N - g) \, d\nu_i \right| + \left| \int_A (s_N - g) \, d\nu_i \right| + \epsilon \\
&\quad + \left| \int_{\Omega \backslash A} (s_N - g) \, d\nu_j \right| + \left| \int_A (s_N - g) \, d\nu_j \right| \\
&< \epsilon \|\nu_i\| + 3\epsilon \|g\|_\infty + \epsilon + \epsilon \|\nu_j\| + 3\epsilon \|g\|_\infty .
\end{aligned}$$

Thus $(\nu_n)$ is $L_\infty(\mu, X^*)$-weakly Cauchy in $\text{cabv}(\mathscr{A}, X, \mu)$. ∎

The reader should compare the following proposition with Theorem 2·1. Note in particular that no restriction is placed on $X$ or $X^*$. In this proposition $\mathscr{B}$ denotes the Borel subsets of $[0, 1]$. If $K \subset \text{cabv}(\mathscr{B}, X)$, then the statement that $K$ is $L_\infty(\lambda, X^*)$-weakly precompact means that $K \subset \text{cabv}(\mathscr{B}, X, \lambda)$ and that every bounded sequence from $K$ has an $L_\infty(\lambda, X^*)$-weakly Cauchy subsequence.

THEOREM 2·4. *If* $X$ *is a Banach space and* $K \subset \text{cabv}(\mathscr{B}, X)$ *such that* (i) $|K| = \{|\mu| : \mu \in K\}$ *is uniformly countably additive and* (ii) $\{\nu(A) : \nu \in K\}$ *is weakly precompact for all* $A \in \mathscr{B}$, *then there is a non-negative countably additive* $\lambda : \mathscr{B} \to \mathbb{R}$ *such that* $K$ *is* $L_\infty(\lambda, X^*)$*-weakly precompact. In this case,* $|\mu| \ll \lambda$ *uniformly for* $\mu \in K$.

*Conversely, if* $K$ *is* $L_\infty(\lambda, X^*)$*-weakly precompact for some* $\lambda$, *then* $K$ *satisfies* (i). *In addition, if* $K$ *is bounded then* $K$ *satisfies* (ii).

*Proof.* Suppose that $K$ is a subset of $\text{cabv}(\mathscr{B}, X)$ which satisfies (i) and (ii). By the first portion of the proof of theorem 4, p. 11 of [**12**], we choose $0 \leqslant \lambda \in \text{ca}(\mathscr{B})$ so that $|\mu| \ll \lambda$ for each $\mu \in K$. By theorems 3·3 and 3·3*L* of [**5**], $|\mu| \ll \lambda$ uniformly for $\mu \in K$. Let $\mathscr{A}$ be a countable algebra of elements of $\mathscr{B}$ so that $\mathscr{A}$ is $\rho_\lambda$-dense in $\mathscr{B}$, where $\rho_\lambda(A, B) = \lambda(A \triangle B)$.

Let $(\nu_n)$ be a bounded sequence from $K$, and use a diagonal argument to select a subsequence $(\nu_{n_k})$ of $(\nu_n)$ so that $(\nu_{n_k}(A))$ is weakly Cauchy for each $A \in \mathscr{A}$. Since $\mathscr{A}$ is $\rho_\lambda$-dense in $\mathscr{B}$ and $|\nu_{n_k}| \ll \lambda$ uniformly in $k$, it follows that $(\nu_{n_k}(A))$ is weakly Cauchy for all $A \in \mathscr{B}$. Now apply Lemma 2·3 and conclude that $(\nu_{n_k})$ is $L_\infty(\lambda, X^*)$-weakly Cauchy. Therefore $K$ is $L_\infty(\lambda, X^*)$-weakly precompact.

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Conversely, suppose that $0 \leqslant \lambda \in \text{cabv}(\mathscr{B})$, $K$ is $L_\infty(\lambda, X^*)$-weakly precompact, and $K$ does not satisfy (i). Using Rosenthal's Lemma (see [12]), choose $\delta > 0$, let $(\nu_n)$ be a sequence in $K$, and let $(E_n)$ be a pairwise disjoint sequence from $\mathscr{B}$ so that

$$|\nu_n|(E_n) > \delta \quad \text{and} \quad |\nu_n|\left(\bigcup_{i \neq n} E_i\right) < \tfrac{1}{4}\delta \quad (n \in \mathbb{N}).$$

For each $n$, let $g_n$ be a simple function so that $g_n$ vanishes off $E_n$, $\|g_n\|_\infty \leqslant 1$, and $\int g_n \, d\nu_n > \delta$. Let $(\nu_{n_k})$ be any subsequence of $(\nu_n)$, and let $g = \sum_{n=1}^{\infty} g_{n_{2k}}$. Clearly $g \in L_\infty(\lambda, X^*)$, $\|g\|_\infty \leqslant 1$, $g = g_{n_{2k}}$ on $E_{n_{2k}}$, and $g = 0$ on $E_{n_{2k+1}}$, for $k \in \mathbb{N}$. Now

$$\int g \, d\nu_{n_{2k}} \geqslant \int_{E_{n_{2k}}} g \, d\nu_{n_{2k}} - |\nu_{n_{2k}}|\Big(\bigcup_{i \neq n_{2k}} E_i\Big) > \tfrac{3}{4}\delta.$$

However,
$$\int g \, d\nu_{n_{2k+1}} = \int_{\bigcup_i E_{n_{2i}}} g \, d\nu_{n_{2k+1}} \leqslant |\nu_{n_{2k+1}}|\Big(\bigcup E_{n_{2i}}\Big) < \tfrac{1}{4}\delta.$$

Thus $(\nu_{n_k})$ is not weakly Cauchy, and we have a contradiction. Consequently, (i) follows.

In addition, if $K$ is bounded then (ii) follows immediately since point evaluation defines a bounded linear map. ∎

If $(\Omega, \mathscr{A}, \mu)$ is an arbitrary finite measure space and $K$ is a bounded and weakly precompact subset of $L_1(\mu, X)$, then theorem IV·2·4 of [12] shows that $K$ must be uniformly integrable. In fact, essentially as a corollary of 2·3 and 2·4, we obtain the following characterization of $L_\infty(\mu, X^*)$-weak precompactness in $L_1(\mu, X)$.

THEOREM 2·5. (I) *If $(\Omega, \mathscr{A}, \mu)$ is a finite measure space and $(f_n)$ is a sequence from $L_1(\mu, X)$ so that* (i) $\sup_n \{\int \|f_n\| d\mu\} < \infty$, (ii) $\{f_n : n \in \mathbb{N}\}$ *is uniformly integrable, and* (iii) $(\int_A f_n \, d\mu)$ *is weakly Cauchy for all $A \in \mathscr{A}$, then $(f_n)$ is $L_\infty(\mu, X^*)$-weakly Cauchy.*

(II) *If $K$ is a bounded subset of $L_1(\mu, X)$, then $K$ is $L_\infty(\mu, X^*)$-weakly precompact if and only if* (a) *$K$ is uniformly integrable and* (b) *$\{\int_A f \, d\mu : f \in K\}$ is weakly precompact in $X$ for each $A \in \mathscr{A}$.*

*Proof.* Since the indefinite integral of an element in $L_1(\mu, X)$ clearly defines a linear isometric mapping of $L_1(\mu, X)$ into $\text{cabv}(\mathscr{A}, X, \mu)$, the conclusions of (I) follow immediately from 2·3. Further, with this mapping in mind, it is clear that the proof of Theorem 2·4 shows that if $K$ is an $L_\infty(\mu, X^*)$-weakly precompact subset of $L_1(\mu, X)$, then (a) and (b) must follow.

Now suppose that $K$ is a bounded subset of $L_1(\mu, X)$ such that (a) and (b) hold. Let $(f_n)$ be a sequence in $K$. Let $A$ be a countable algebra of subsets of $\mathscr{A}$ such that if $\mathscr{A}'$ is the $\sigma$-algebra generated by $A$, then $f_n$ is measurable with respect to $\mathscr{A}'$ for each $n$. If $g \in L_\infty(\mu, X^*)$ and $g' \in L_\infty(\mu_{|\mathscr{A}'}, X^*)$ is the conditional expectation of $g$ with respect to $\mathscr{A}'$, then $\int \langle g, f_n \rangle \, d\mu = \int \langle g', f_n \rangle \, d\mu_{|\mathscr{A}'}$ for all $n$. Since $\mathscr{A}'$ is separably generated, we see that the first portion of the proof of Theorem 2·4 finishes the argument. ∎

An immediate consequence of the above theorem is that if $X^*$ has RNP and $(\Omega, \mathscr{A}, \mu)$ is a finite measure space, then a bounded set $K \subset L_1(\mu, X)$ is weakly precompact if and only if $K$ is uniformly integrable, since $L_1(\mu, X)^* = L_\infty(\mu, X^*)$.

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However, using a deep theorem of M. Talagrand [**26**] and J. Bourgain [**8**], we get the following result.

THEOREM 2·6. *Let $X$ be a Banach space and let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space. Then the following are equivalent*:

(i) *$\ell_1$ does not embed in $X$*;

(ii) *a bounded subset $K \subset L_1(\mu, X)$ is weakly precompact if it is uniformly integrable*;

(iii) *a bounded subset $K \subset L_1(\mu, X)$ is weakly precompact if it is uniformly integrable and $\{\int_A f\,d\mu : f \in K\}$ is weakly precompact for every $A \in \mathscr{A}$.*

*Proof.* Theorem 6 of [**26**] gives the implication (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) is obvious. Hence we only need prove (iii) $\Rightarrow$ (i).

Suppose that $X$ is a Banach space so that $\ell_1$ embeds in $X$. Let $(x_n)$ be a copy of the $\ell_1$ basis in $X$. Without loss of generality, assume $\|x_n\| \leqslant 1$ for all $n \in \mathbb{N}$. Since $(x_n)$ is a copy of the $\ell_1$ basis, we may choose $\beta \in \mathbb{R}$ so that

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \geqslant \beta \sum_{i=1}^{n} |a_i|$$

for every $n \in \mathbb{N}$ and for every choice of scalars $\{a_i\}_{i=1}^n$.

Let $\Omega = [0, 1]$, $\mathscr{A} = \mathscr{B}([0, 1])$, and let $\mu$ be Lebesgue measure. Use $r_n$ to denote the $n$th Rademacher function on $[0, 1]$; that is, $r_n(t) = \operatorname{sgn}(\sin(2^n \pi t))$. (Take $\operatorname{sgn}(0) = 1$.) For $n \in \mathbb{N}$, define $f_n : [0, 1] \to X$ by $f_n = r_n x_n$. Let $K = \{f_n : n \in \mathbb{N}\}$.

Observe that

$$\|f_n\| = \int |f_n|\,d\mu \leqslant 1.$$

In addition

$$\lim_{n \to \infty} \left\| \int_E f_n\,d\mu \right\| = 0$$

for all $E \in \mathscr{A}$. Thus the set $K = \{f_n : n \in \mathbb{N}\}$ is bounded and $\{\int_E f\,d\mu;\ f \in K\}$ is weakly precompact for every $E \in \mathscr{A}$. Furthermore, as $\|f_n(t)\| \leqslant 1$ for all $n \in \mathbb{N}$ and $t \in [0, 1]$, $K$ is uniformly integrable. Let $\{a_i\}_{i=1}^n \subset \mathbb{R}$. Observe that

$$\left\| \sum_{i=1}^{n} a_i f_i \right\| = \int \left\| \sum_{i=1}^{n} a_i r_i x_i \right\| d\mu \geqslant \int \beta \sum_{i=1}^{n} |a_i r_i|\,d\mu = \beta \sum_{i=1}^{n} |a_i|.$$

Thus $(f_n)$ is a copy of the $\ell_1$ basis. Therefore $K$ is not weakly precompact. ∎

Next we discuss the relationship between unconditionally converging operators $T$ and those maps $T$ for which $T^*$ is weakly precompact. Recall that an operator is unconditionally converging provided that it maps weakly absolutely summable series to unconditionally converging series.

THEOREM 2·7. *If $T^* : Y^* \to X^*$ is weakly precompact, then $T : X \to Y$ is unconditionally converging.*

*Proof.* Suppose that $T^* : Y^* \to X^*$ is weakly precompact and $T : X \to Y$ is not unconditionally converging. Using Bessaga and Pelczynski [**4**] or problem 8, p. 54, of Diestel [**11**] one obtains an isomorphic copy of $c_0$ in $X$ upon which $T$ acts as an isomorphism. Denote this copy by $U$ and let $L : c_0 \to U \subset X$ be an isomorphic embedding. Thus $TL : c_0 \to Y$ is an isomorphism. Hence $(TL)^* : Y \to \ell_1$ is a surjection,

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$L^*(T^*B_{Y^*})$ contains a $\delta$-ball about the origin in $\ell_1$, and $T^*(B_{Y^*})$ cannot be weakly precompact. This contradiction finishes the proof. ∎

We remark that the converse of Theorem 2·7 is false. R. C. James [**18**] has constructed an example of a Banach space $X$ such that $X^{(2n)}$ (the $2n$-dual of $X$) contains $\ell_1$ and $X^{2n-1}$ does not contain $\ell_1$. Note that $X^{2n-1}$ does not contain $c_0$. For if it does, then $X^{2n-1}$ contains $\ell_1$ (by [**11**], theorem 10, p. 48), and thus $X^{2n-1}$ contains $\ell_1$. Therefore if $I: X^{2n-1} \to X^{2n-1}$ is the identity operator, then $I$ is unconditionally converging and $I^*$ is not weakly precompact.

We now discuss the relationship between strong boundedness and weak precompactness which was mentioned in the Introduction. We also give the connection between strong boundedness and Grothendieck's condition in [**17**] which was presented in Brooks and Lewis [**10**], since it is through Grothendieck's condition that the link between weak precompactness and weak complete continuity becomes clear.

THEOREM 2·8. *Suppose that $\ell_1$ does not embed in $X^*$ and $X^*$ has RNP. Suppose that $m \mapsto T: (H, X) \to Y$ is an operator. The following are equivalent:*

(sb) *$T$ is strongly bounded;*

(uc) *$T$ is unconditionally converging;*

(wpc*) *$T^*$ is weakly precompact;*

(*g*) *$T(f_i) \to 0$ whenever $(f_i)$ is a bounded sequence in $C(H, X)$ so that $(f_i(t)) \to 0$ for every $t \in H$.*

*Proof.* Suppose that $m \mapsto T$ and that $m$ is not strongly bounded. Therefore by a theorem of Grothendieck [**17**] and lemma 3·1 of [**9**], we may (and do) assume that there is a disjoint sequence $(O_i)$ of open subsets of $H$, a sequence $(y_i^*)$ in $B_{Y^*}$, and an $\epsilon > 0$ so that $|m_{y_i^*}|(O_i) > \epsilon$ for each $i$. Using the regularity properties of $m$, and we select a sequence $(f_i)$ from $C(H, X)$ so that support $(f_i) \subset O_i$, $\|T(f_i)\| > \epsilon$, and $\|f_i\| \leqslant 1$ for each $i$. Therefore $T$ does not satisfy (*g*), i.e. (*g*) implies (sb).

Conversely, suppose that $m$ is strongly bounded, and let

$$K = \{T^*(y^*) : \|y^*\| \leqslant 1\} = \{m_{y^*} : \|y^*\| \leqslant 1\}.$$

It follows from lemma 3·1 of [**9**] that $\{|\nu| : \nu \in K\}$ is uniformly countably additive. Let $0 \leqslant \lambda \in \mathrm{ca}(\Sigma)$ so that $\nu \ll \lambda$ uniformly for $\nu \in K$ (see [**12**], p. 11), i.e. let $\lambda$ be a control measure for $\tilde{m}$. Now suppose that $(f_i)$ satisfies the hypothesis of (*g*) and let $B$ be a uniform bound for $(f_i)$. By Egorov's theorem, $(f_i) \to 0$ almost uniformly $[\tilde{m}]$. Let $\epsilon > 0$, and choose $A \in \Sigma$ so that $\tilde{m}(A) < \epsilon$ and $(f_i) \to 0$ uniformly on $H \backslash A$. Suppose that $n_0 \in \mathbb{N}$ so that $\|f_i(t)\| < \epsilon/(\tilde{m}(H) + 1)$ for all $t \in H \backslash A$ and $i \geqslant n_0$. Thus

$$\|T(f_i)\| \leqslant (\epsilon/(\tilde{m}(H) + 1))\, \tilde{m}(H \backslash A) + B\tilde{m}(A) \quad \text{for } i \geqslant n_0,$$

and it follows that $T$ satisfies (*g*).

If $T^*$ is weakly precompact, then $T$ is unconditionally converging by Theorem 2·7. Swartz [**25**] has shown that every unconditionally converging map is strongly bounded.

Thus to complete the proof of the theorem, it suffices to show that (sb) implies (wpc*). Suppose then that $X^*$ has RNP, that $\ell_1$ does not embed in $X^*$, and that $T$ is strongly bounded. Let $K = \{m_{y^*} : y^* \in B_{Y^*}\}$, let $\lambda$ be a control measure for $\tilde{m}$, and, for

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each $y^* \in Y^*$, let $f_{y^*} \in L_1(\lambda, X^*)$ so that $m_{y^*}(A) = \int_A f_{y^*} d\lambda$ for all $A \in \Sigma$. The strong boundedness of $T$ (or $m$) ensures that $\{f_{y^*} : \|y^*\| \leqslant 1\}$ is uniformly integrable, and thus $\{f_{y^*} : \|y^*\| \leqslant 1\}$ is weakly precompact by Theorem 2·5. The isometry between cabv $(\Sigma, X^*, \lambda)$ and $L_1(\lambda, X^*)$ (provided by the Radon–Nikodym Theorem) provides the final step in the proof. ∎

We remark that if the conclusion of 2·8 holds, then it follows that $X^{**}$ must have the weak Radon–Nikodym property, i.e. $\ell_1$ must not embed in $X^*$. Otherwise, let $H$ be a singleton, $X = Y$, and $T : C(H, X) (\cong X) \to Y$ be the identity. Certainly $T$ is strongly bounded. However, since $\ell_1$ embeds in $X^*$, clearly $T^*$ cannot be weakly precompact. We do not know if the hypotheses involving the RNP can be removed from $X^*$ in the proof that (sb) implies (wpc*). We do note that the technique used so successfully by Saab [**24**] and Bombal and Cembranos [**6**] to demonstrate the necessity that $X^*$ have RNP in somewhat similar situations does not seem to apply in this case.

We also note that if $X^*$ has RNP and $\ell_1$ embeds in $X^*$, then (sb) need not imply (uc). However, if $\ell_1$ does not embed in $X^*$, then $c_0$ does not embed in $X$. Therefore (sb) implies (uc) in this case, by theorem 5·2 of [**9**].

## 3. *Weakly completely continuous operators*

In this section we begin by considering some obvious modifications of (*g*) of Theorem 2·8. Specifically, for operators $T : C(H, X) \to Y$ we focus on conditions

(cc) $(T(f_i))_i \to 0$ whenever $(f_i)$ is bounded and $(f_i(t)) \to 0$ weakly for each $t \in H$, and

(wcc) $(T(f_i))$ is weakly convergent whenever $(f_i)$ is bounded and $(f_i(t))_i$ is weakly Cauchy for every $t \in H$.

Before stating our first theorem, we recall that an operator $L : X \to Y$ is called completely continuous if $L$ maps weakly Cauchy sequences into norm convergent sequences and weakly completely continuous if $L$ maps weakly Cauchy sequences into weakly convergent sequences. (Since a weakly completely continuous operator is unconditionally converging, every weakly completely continuous operator $T : C(H, X) \to Y$ is strongly bounded.)

THEOREM 3·1. *Suppose that* $m \mapsto T : C(H, X) \to Y$ *is an operator.*

(*a*) *The operator* $T$ *is completely continuous if and only if* $T$ *satisfies condition* (cc).

(*b*) *The operator* $T$ *is weakly completely continuous if and only if* $T$ *satisfies condition* (wcc).

Before proceeding with the proof of Theorem 3·1, we need the following result.

LEMMA 3·2. *A bounded sequence* $(f_n)$ *in* $C(H, X)$ *converges weakly to* 0 *if and only if* $(f_n(t))_n \to 0$ *weakly for all* $t \in H$.

Although Lemma 3·2 seems to be well known (e.g. see [**6**], theorem 2 and [**14**] theorem 9), a simple proof does not appear readily available. For this reason and for the convenience of the reader, we give a short argument based on Rainwater's theorem ([**11**], p. 155).

*Proof of Lemma* 3·2. Suppose that $(f_n)$ is bounded in $C(H, X)$ and that $(f_n(t)) \to 0$ weakly for all $t \in H$. Let $\phi$ be an extreme point in the unit ball of $C(H, X)^*$. Using the

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fact that $C(H,X)^*$ is (isometrically) rcabv $(\Sigma, X^*)$ (see [9]), we see that $\phi$ must be a measure concentrated on a single point $t \in H$. Let $x^* \in X^*$ so that $\phi = \delta_t . x^*$. Therefore $\phi(f_n) = x^*(f_n(t))$, and it follows immediately that $(\phi(f_n))_n \to 0$. By Rainwater's theorem, $(f_n) \to 0$ weakly.

The converse is clear since $\langle t, x^* \rangle (f) = x^*(f(t))$

defines a continuous linear functional on $C(H,X)$ for each $t \in H$ and $x^* \in X^*$. ∎

*Proof of Theorem* 3·1. The proof of 3·1 $(a)$ follows from a rather straightforward application of Lemma 3·2.

Now suppose that $T$ satisfies (wcc) and that $(f_n)$ is a weakly Cauchy sequence in $C(H,X)$. Certainly it then follows that $(f_n(t))$ is weakly Cauchy for each $t \in H$. Therefore $(T(f_n))$ converges weakly, and $T$ is weakly completely continuous.

Conversely, suppose that $T$ is weakly completely continuous, $(f_n)$ is bounded, and $(f_n(t))$ is weakly Cauchy for each $t \in H$. Apply 3·2 again and conclude that $(f_n)$ is weakly Cauchy. Thus $T(f_n))$ is weakly convergent. ∎

Before we present our general characterization of weakly completely continuous operators $L : X \to Y$, we need some additional terminology. We let $\mathscr{B}_1(X)$ be the set of all points in $X^{**}$ which are *w*-sequential* limits of $X$. Further, if $K$ is a $w^*$-compact subset of $X^*$, then we say that $K$ has property (P1) if $x^{**}|K \in C(K, w^*)$ whenever $x^{**} \in \mathscr{B}_1(X)$, and we say that $K$ has property (P2) if $x^{**} \in \mathscr{B}_1(X)$ ensures that there is an element $x \in X$ so that $x|K = x^{**}|K$. (By $C(K, w^*)$ we mean the space of all real valued functions defined on $K$ which are continuous with respect to the $w^*$-topology on $K$.)

THEOREM 3·3. *Suppose that $T : X \to Y$ is an operator. The following are equivalent:*

(i) *$T$ is weakly completely continuous;*

(ii) *$T^*(B_{Y^*})$ has* (P1);

(iii) *$T^{**}(\mathscr{B}_1(X)) \subset Y$.*

*Proof.* (i) ⇒ (ii). Suppose that $T : X \to Y$ is weakly completely continuous, $x^{**} \in \mathscr{B}_1(X)$, and $(x_n)$ is a sequence in $X$ so that $(x_n) \overset{w^*}{\to} x^{**}$. Thus $(T(x_n))$ is weakly convergent. Suppose that $y \in Y$ so that $(T(x_n)) \to y$ weakly, i.e. $(T(x_n)) \overset{w}{\to} y$. Since $T^{**}$ is $(w^*, w^*)$-continuous, $(T^{**}(x_n)) \overset{w^*}{\to} T^{**}(x^{**})$, and therefore $T^{**}(x^{**}) = y$.

Now let $x^* \in K = T^*(B_{Y^*})$, and let $(x_\alpha^*)$ be a net in $K$ so that $(x_\alpha^*) \overset{w^*}{\to} x^*$. Suppose that $(x_\beta^*)$ is a subset of $(x_\alpha^*)$ and that $(y_\beta^*)$ is a net in $B_{Y^*}$ so that $T^*(y_\beta^*) = x_\beta^*$ for all $\beta$. Let $(y_\eta^*)$ be a subnet of $(y_\beta^*)$ so that $(y_\eta^*) \overset{w^*}{\to} y^* \in B_{Y^*}$; set $x_\eta^* = T^*(y_\eta^*)$ for each $\eta$. Thus

$$\langle x^{**}, x_\eta^* \rangle = \langle x^{**}, T^{**}(y_\eta^*) \rangle = \langle T^{**}(x^{**}), y_\eta^* \rangle$$

$$= \langle y, y_\eta^* \rangle \to \langle y, y^* \rangle = \langle T^{**}(x^{**}), y^* \rangle = \langle x^{**}, T^*(y^*) \rangle.$$

Now $(T^*(y_\eta^*)) \overset{w^*}{\to} T^*(y^*)$ since $(y_\eta^*) \overset{w^*}{\to} y^*$. Also $(T^*(y_\eta^*)) \overset{w^*}{\to} x^*$ since $(T^*(y_\eta^*))$ is a subnet of $(x_\beta^*)$, i.e. $T^*(y^*) = x^*$. Therefore $\langle x^{**}, x_\eta^* \rangle \to \langle x^{**}, x^* \rangle$, and it follows that $x^{**}|K \in C(K, w^*)$.

(ii) ⇒ (iii). Let $x^{**} \in \mathscr{B}_1(X)$. We assert that $T^{**}(x^{**})$ is $w^*$-continuous on $B_{Y^*}$. Suppose that $(y_\alpha^*) \overset{w^*}{\to} y^*$ in $B_{Y^*}$. Now $\langle T^{**}(x^{**}), y_\alpha^* \rangle = \langle x^{**}, T^*(y_\alpha^*) \rangle$. However, $x^{**} \in C(k, w^*)$ and $(T^*(y_\alpha^*)) \overset{w^*}{\to} T^*(y^*)$. Therefore

$$\langle x^{**}, T^*(y_\alpha^*) \rangle \to \langle x^{**}, T^*(y^*) \rangle = \langle T^{**}(x^{**}), y^{**} \rangle,$$

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and we have $T^{**}(x^{**})|B_{Y^*} \in C(B_{Y^*}, w^*)$. Thus by V·5·5 (or V·5·6) of [**15**], $T^{**}(x^{**})$ is $w^*$-continuous on $Y^*$, and consequently $T^{**}(x^{**}) \in Y$.

(iii) $\Rightarrow$ (i). Suppose that $T^{**}(\mathscr{B}_1(X)) \subset Y$ and $(x_n)$ is weakly Cauchy in $X$. Let $x^{**} \in X^{**}$ so that $(x_n) \overset{w^*}{\to} x^{**}$. Therefore $x^{**} \in \mathscr{B}_1(X)$, and $T^{**}(x^{**}) \in Y$. Since $T^{**}$ is $(w^*, w^*)$-continuous, $(T^{**}(x_n)) \overset{w^*}{\to} T^{**}(x^{**})$. Thus $(T(x_n)) \to T^{**}(x^{**})$ weakly, and $T$ is weakly completely continuous. ∎

COROLLARY 3·4. *If $T: X \to Y$ is an operator and $T(B_X)$ is closed in $Y$, then* (i)–(iii) *of the preceding theorem are equivalent to the following condition*:

(iv) $T^*(B_{Y^*})$ *has* (P2).

*Proof.* Clearly if $T^*(B_{Y^*})$ has (P2), then it must have (P1). Conversely, suppose that $x^{**} \in \mathscr{B}_1(X)$ and that $T: X \to Y$ is weakly completely continuous. Let $(x_n)$ be a sequence from $X$ so that $(x_n) \overset{w^*}{\to} x^{**}$, and let $y \in Y$ so that $(T(x_n)) \to y$ weakly. Since $T(B_X)$ is convex and norm closed, we see that $T(B_X)$ is weakly closed. Let $x \in B_X$ so that $T(x) = y$, and let $y^* \in B_{Y^*}$. Thus

$$x^{**}(T^*(y^*)) = \langle T^{**}(x^{**}), y^* \rangle = \lim_n \langle T^{**}(x_n), y^* \rangle$$

$$= \lim_n \langle T(x_n), y \rangle = \langle T(x), y \rangle = \langle x, T^*(y^*) \rangle,$$

and $x^{**}|K = x|K$. ∎

As a corollary of an extension theorem and an embedding theorem in [**6**], Bombal and Cembranos conclude that $m(A): X \to Y$ must be weakly completely continuous for each Borel set $A$ whenever $m \mapsto T: C(H, X) \to Y$ is a weakly completely continuous operator. We present an alternative proof of this fact by first demonstrating that the limit of weakly completely continuous operators is weakly completely continuous.

THEOREM 3·5. *If $T_n: X \to Y$ is a weakly completely continuous operator for each $n \in \mathbb{N}$ and $T: X \to Y$ is an operator so that $\|T_n - T\| \to 0$, then $T$ is weakly completely continuous.*

*Proof.* Suppose that $(T_n)_{n=1}^{\infty}$ and $T$ are as above, and let $(x_i)$ be a weakly Cauchy sequence in $X$. Let $y_n = \text{weak}\lim_i T_n(x_i)$ for each $n \in \mathbb{N}$, and consider the sequence $(y_n)$. If $y^* \in B_{Y^*}$, then

$$|y^*(y_n - y_k)| = \lim_i |\langle y^*, (T_n - T_k)x_i \rangle| = \lim_i |\langle (T_n^* - T_k^*)y^*, x_i \rangle|$$

$$\leqslant \|T_n - T_k\| \sup\{\|x_i\| : i \in \mathbb{N}\},$$

and this last product tends to 0 as $n, k \to \infty$. Therefore $(y_n)$ is Cauchy in $Y$; let $y_0 \in Y$ satisfy $\|y_n - y_0\| \to 0$.

We now assert that $(T(x_i)) \overset{w}{\to} y_0$. Suppose that $y^* \in B_{Y^*}$; then

$$|\langle y^*, T(x_i) - y_0 \rangle| \leqslant |\langle y^*, T(x_i) - T_n(x_i) \rangle| + |\langle y^*, y_n - y_0 \rangle| + |\langle y^*, T_n(x_i) - y_n|.$$

Consequently, if $\epsilon > 0$ and $n \in \mathbb{N}$ is fixed so that

$$\|y_n - y_0\| + \|T - T_n\| \sup\{\|x_i\| : i \in \mathbb{N}\} < \epsilon,$$

then we choose $i$ sufficiently large so that $|\langle y^*, T_n(x_i) - y_n \rangle| < \epsilon$. Thus $|\langle y^*, T(x_i) - y_0| < 2\epsilon$ for large $i$, and the assertion above follows. ∎

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Now suppose that $m \mapsto T: C(H, X) \to Y$ is weakly completely continuous and $f \in C(H)$. Let $T_f: X \to Y$ be defined by

$$T_f(x) = T(f.x) \quad (x \in X),$$

and let $A \in \Sigma$. Choose a decreasing sequence $(O_n)$ of open sets and an increasing sequence $(K_n)$ of compact sets so that $K_n \subset A \subset O_n$ for each $n$ and $\tilde{m}(O_n \backslash K_n) \to 0$. If $(f_n)$ is a sequence in $C(H)$ so that $\|f_n\| = 1$, support $(f_n) \subset O_n, f_n = 1$ on $K_n$, for $n \in \mathbb{N}$, and $T_n: X \to Y$ is defined by $T_n(x) = T(f_n.x)$, then $T_n$ is weakly completely continuous for each $n \in \mathbb{N}$. Let $T_A: X \to Y$ be defined by

$$T_A(x) = T^{**}(\chi_A.x) = m(A)(x) \quad (x \in X).$$

If $\|x\| \leqslant 1$ then

$$\|T_n(x) - T_A(x)\| = \|T(f_n.x) - T^{**}(\chi_A.x)\| = \|T^{**}(f_n.x) - T^{**}(\chi_A x)\| \leqslant \tilde{m}(O_n \backslash K_n),$$

and it follows that $\|T_n - T_A\| \to 0$. Therefore $m(A)$ is weakly completely continuous by 3·5. ∎

*Remark.* As a corollary of the preceding discussion and Theorem 3·3, we see that $m(A)^*(B_{Y^*})$ has (P1) for each $A \in \Sigma$ whenever $m \mapsto T: C(H, X) \to Y$ is weakly completely continuous.

We conclude with an alternative proof of the implication (i) ⇒ (ii) in the following theorem of Bombal and Cembranos [7]. Our argument seems to be somewhat simpler.

THEOREM. *Let $X$ be a Banach space whose dual has RNP. The following assertions are equivalent*:

(i) *$X^{**}$ has RNP*;

(ii) *an operator $m \mapsto T: C(H, X) \to Y$ is weakly completely continuous if and only if $m$ is strongly bounded and $m(A)$ is weakly completely continuous for $A \in \Sigma$.*

*Proof.* If $X^*$ has RNP, then $\ell_1$ does not embed in $X$, and thus every weakly completely continuous operator $L: X \to Y$ is weakly compact. Consequently if $X^{**}$ has RNP and $m$ satisfies the two requirements in $(b)$, then $T$ is automatically weakly compact by theorem 4·1 of [9]. Thus $T$ is weakly completely continuous. ∎

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