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HIGHER-ORDER ESSENCES: LOGIC AND SEMANTICS

ANDREAS DITTER
Department of Philosophy, University College London

Abstract. This paper develops a logic of essence (HLE) in the framework of higher-order logic. The theory aims to provide a general framework for theorizing about the essences of objects, properties, propositions, and logical operations like conjunction, negation, quantification, etc. The first part of the paper presents the formal language and axiom system of HLE. After that, some theorems of the system are proved and it is shown how the logic of metaphysical necessity can be developed within the framework of HLE. The second part of the paper develops a possible worlds semantics for HLE, gives a proof of soundness, and provides examples of models that demonstrate the consistency of some simple essentialist theories.

§1. Introduction. This paper develops a logic of essence (HLE) in the framework of higher-order logic. The theory aims to provide a general framework for theorizing about the essences of objects, properties, propositions, and logical operations like conjunction, negation, etc. A Hilbert-style axiom system of HLE was first presented in Ditter [15]. The present paper complements the presentation in Ditter [15] in two key respects: first, by proving some theorems of HLE and discussing their philosophical ramifications; and second, by developing a sound model-theoretic semantics for HLE.

The notion of essence HLE is concerned with is the notion that Fine [20] influentially argued to be not definable in terms of metaphysical necessity. Fine's counterexamples to the definability of essence in terms of metaphysical necessity have convinced many philosophers that the right approach to take in theorizing about essence is to work with a primitive notion of essence not explained in terms of metaphysical modality or anything else. This conceptually basic notion has since been used and discussed in a variety of different theoretical contexts.[1] Yet the logic of this notion remains relatively little explored. Notably, Fine's logic of essence (LE) [21, 22], the most fully developed logic of essence in the literature, is not concerned with the *basic* notion of essence, but with what Fine calls the 'constrained consequential notion' of essence—an idealized notion that, roughly speaking, results from the basic notion by closure under (a restricted form of) logical consequence.[2] It therefore leaves open what the correct logic of the basic notion might be.

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[1] See, e.g., Correia [6, 8], Dasgupta [11, 12], Glazier [26], Kment [30], Koslicki [31], Leary [32], Rosen [38, 39], and Wedgwood [43].

[2] Some authors have questioned whether the consequential notion should be called a notion of *essence* at all. See, e.g., Kment [30, p. 158 n. 15] and Dasgupta [11, p. 589 n. 46].

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Perhaps the most central logical difference between the constrained consequential and the basic notion of essence that LE and HLE are, respectively, concerned with is the question of logical closure. For the constrained consequential notion formalized in LE, the propositions true in virtue of the nature of given objects are closed under logical consequence, subject to the constraint that the consequences in question are not about objects that are not involved in the nature of the given objects. Thus given that Socrates is essentially human, it will also be true that Socrates is essentially human or a set. Moreover, any "purely qualitative" logical truth, such as 'Everything is self-identical' or 'Everything is either red or not red', will be true in virtue of Socrates' nature, and indeed true in virtue of the nature of any given objects. By contrast, if Socrates' nature does not involve his singleton set, then it won't be true in virtue of the nature of Socrates that his singleton set is self-identical, since the proposition that his singleton set is self-identical is about the singleton.

All of the positive instances of closure mentioned in the previous paragraph are highly controversial for the basic notion of essence. For the basic notion formalized in HLE, it is plausible to deny, for example, that Socrates is essentially human or a set, even if he is essentially human, on the grounds that, intuitively, his nature does not involve, or "knows nothing about," sethood. Likewise, it is natural to deny that it lies in the nature of Socrates that everything is either red or not red, since intuitively, redness has no part in Socrates' nature. Unlike the consequential notion, the basic notion is not only sensitive to the "objectual content" of propositions, but also to the content expressed by predicates like 'set' or 'red', and terms of other grammatical categories. In order to capture the above failures of logical closure we therefore need to generalize the notion of essential involvement from a relation restricted to objects to a relation whose relata may be entities expressed by all kinds of grammatical categories. This is achieved in HLE by the adoption of a higher-order language. The higher-order regimentation of essentialist statements employed here is a generalization of the first-order regimentation of LE. It enables us to represent not only talk about the essences of *objects*, such as Socrates or his singleton set, but also talk about the essences of properties, propositions, and logical operations—or more generally, the essences of what is expressed by terms of arbitrary grammatical categories—in a natural and uniform way.[3] The language of HLE allows us to provide a simple formalization of claims like 'It lies in the nature of being red that all red things are colored' or 'It lies in the collective nature of identity and universality that everything is self-identical',[4]

[3] I will often pronounce sentences of our higher-order language by using terms, such as 'entities', 'properties' and 'propositions'. This talk of 'entities', 'properties' and 'propositions' should only be taken to be a rough ordinary language approximation of what is more perspicuously expressed in our higher-order language, since it is often the case that sentences in a higher-order language cannot be precisely translated into natural language. For influential defenses of the intelligibility of higher-order languages, see Prior [37] and Williamson [45]. Higher-order methods have proved to be a powerful tool in recent metaphysics. See, e.g., Bacon [1, 2], Dorr [17], Dorr et al. [18], Fine [19], Fritz [23], Fritz and Jones [25], Fritz and Goodman [24], Goodman [27], Litland [34], and Williamson [46].

[4] I will be using all of the following expressions interchangeably: (i) 'the collective nature of ...'; (ii) 'the nature of the collection of ...'; and (iii) 'the nature of ... taken together'. I will also use 'essence' and 'nature' interchangeably throughout.

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whereas it is difficult to express such claims in the first-order setting of LE.[5]

The expressive generality of the higher-order framework of HLE is key to the solution of the problem of logical closure for the basic notion of essence. Apart from allowing for a consistent denial of the problematic instances of closure mentioned above, HLE also allows us to account for the plausible *positive* instances of closure. It is plausible to assume that the essences of at least *some* entities manifest some logical closure. For example, while we might think that it is not true in virtue of Socrates' nature that everything is self-identical, it is very natural to assume that it *is* true in virtue of the nature of identity and universality that everything is self-identical. More generally, it is natural to think that any logical truth that is expressed by purely logical vocabulary is true in virtue of the collective nature of the logical operations expressed by this vocabulary. HLE predicts just that. In general, according to HLE, any type of logical closure manifested by the essence of a collection of entities derives from the logical operations involved in the essence of that collection. So, for example, the essence of a collection of entities will be closed under conjunction elimination and introduction just in case the essence of the collection involves conjunction, and it will be closed under modus ponens just in case it involves the conditional. This leads to a natural and systematic account of the closure of essences under logical consequence on which the way in which the essence of a given collection is closed under logical consequence derives from the natures of the logical operations which are naturally regarded as the source of the closure in question, while at the same time ensuring that no "extraneous entities"—i.e., entities not involved in the nature of the given entities—are introduced into the essence of the collection via logical closure.

The resulting picture requires a very fine-grained individuation of propositions; in particular, it requires propositions to be individuated more finely than by metaphysically necessary equivalence. However, it is worth stressing that it does not require propositions to be structured in anything like the way in which sentences are structured—a view that is known to lead to inconsistency given some natural background assumptions, due to the Russell–Myhill paradox.[6] This is demonstrated by the semantics for HLE developed here, which shows how we can model a wide range of fine-grained essentialist distinctions in a semantic framework that is in many respects similar to the possible worlds frameworks employed in discussions of higher-order modal logic.

The differences between LE and HLE mentioned above notwithstanding, it is worth noting that there are some important continuities between the systems and their general approach to the logic of essence. As mentioned above, the higher-order language of HLE is essentially a higher-order generalization of the language of LE. Furthermore some, but not all, of the axioms and rules of LE are not specific to the notion of essence LE deals with and remain correct for the basic notion of essence which forms the

[5] One can express such claims in the first-order setting of LE, if one adds a first-order theory of properties to LE. But any first-order theory of properties must deal with the property-theoretic paradoxes that arise in the context of such a theory. The higher-order approach allows us to sidestep this issue.

[6] The argument was originally discovered by Russell [40, Appendix B] and independently rediscovered by Myhill [36]; for recent discussions of its implications for theories of structured propositions, see Deutsch [13], Goodman [27], Dorr [17], Klement [28, 29], and Uzquiano [42].

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subject matter of HLE. Finally, the semantics developed here shares some important structural features with the first-order semantics for LE in Fine [22]. For example, just as in Fine's semantics for LE, the condition for a statement to be true in virtue of the nature of certain entities is that it should be true in all those worlds that contain those entities.[7]

The paper is structured as follows. §2 introduces the language of HLE. The next three sections are concerned with the development of the logic. After presenting the proof system of HLE (§3), some important theorems of the system are proved (§4) and it is shown how to develop the logic of metaphysical necessity within the setting of HLE (§5). The next four sections deal with the semantics of HLE. §6 develops a model-theoretic semantics for HLE and §7 provides example models that show the consistency of some simple essentialist theories. §8 proves the soundness of HLE with respect to this semantics. §9 ties up some loose ends from §5 by giving model-theoretic proofs of some results announced there. Finally, §10 concludes with some directions for future research.

§2. Language. The formal language of HLE is a higher-order generalization of the first-order language developed in Fine's LE [21]. In LE, essentialist statements are regimented by means of operators subscripted with a one-place predicate, written $\Box_F$. On its intended meaning, a sentence of the form $\Box_F\phi$ says that it lies in the nature of the things which F that ϕ.[8] Thus the subject of the essentialist attribution is the collection of objects which F. However, Fine's formalization only allows us to pick out (collections of) *objects* as the subjects of essentialist attributions: LE uses a first-order language in which predicates can only combine with terms in name position to form sentences. But as pointed out in §1, some essentialist statements, such as 'It lies in the collective nature of identity and universality that everything is self-identical' cannot be straightforwardly expressed in such a language. We therefore need to extend our expressive resources. The basic idea is to formalize such sentences by picking out what is expressed by words and phrases of different grammatical categories—such as the categories of predicates, sentences or operators—as the subjects of essentialist attributions. The most straightforward way to achieve this is to use a higher-order language in which predicates can combine with terms of different grammatical categories to form sentences. For example, if we have at our disposal a predicate that is satisfied by conjunction in the same way in which 'is identical to Socrates' is satisfied by Socrates, we could state what lies in the nature of conjunction in the same way as we can state what lies in the nature of Socrates.

Let me make this a bit more precise. The language we will be working with is a relationally typed language with lambda abstraction.

DEFINITION 1. *The set $\mathcal{T}$ of types is the smallest set such that*:

(i) $e \in \mathcal{T}$;
(ii) *if* $\tau_1, \ldots, \tau_n \in \mathcal{T}$, $n \geq 0$, *then* $\langle \tau_1, \ldots, \tau_n \rangle \in \mathcal{T}$;
(iii) *if* $\tau \in \mathcal{T}$, *then* $[\tau] \in \mathcal{T}$.

[7] Though as we will see, the precise condition in the semantic clause differs slightly from Fine's semantic clause.

[8] Equivalently: It is essential to the things which F that ϕ. Or: It is true in virtue of the nature of the things which F that ϕ.

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Clauses (i) and (ii) are the usual clauses for relational types: e is the type of objects (individuals), $\langle \tau_1, \ldots, \tau_n \rangle$ is the type of ordinary relations between entities of type $\tau_1, \ldots, \tau_n$ with the special case $\langle\rangle$ being the type of propositions, and clause (iii) concerns the type of *rigid properties*. Intuitively, a rigid property is a property of being one of $x_1, x_2, \ldots$, for certain specific entities $x_1, x_2, \ldots$. We can thus think of rigid properties as predicative analogs of pluralities.

DEFINITION 2. *Let $\mathcal{T}$ be the set of all types. A* typed family *is a set $\mathcal{D} = \{\mathcal{D}_\tau : \tau \in \mathcal{T}\}$ of pairwise disjoint non-empty sets.*

We assume given, for each $\tau \in \mathcal{T}$, a denumerably infinite set $\mathcal{V}_\tau$ of variables with unique type τ. The set of all variables is $\mathcal{V} = \bigcup_{\tau \in \mathcal{T}} \mathcal{V}_\tau$. A *signature* is a set of uniquely typed constants. If Σ is a signature, we write Σ_τ for the set of constants from Σ that have type τ. We will generally assume that Σ is disjoint from $\mathcal{V}$. Note that $\{\mathcal{V}_\tau : \tau \in \mathcal{T}\}$ is a typed family. To define our language, we first define a more general kind of language, a λK-language, in which vacuous λ-abstracts are allowed.

DEFINITION 3. *For any signature Σ, we define sets of* ΣK-terms $\mathcal{L}K^\Sigma_\tau$ of type τ, *for each $\tau \in \mathcal{T}$, to be the smallest sets satisfying the following conditions*:

(i) $\mathcal{V}_\tau \subseteq \mathcal{L}K^\Sigma_\tau$ *and* $\Sigma_\tau \subseteq \mathcal{L}K^\Sigma_\tau$, *for each* $\tau \in \mathcal{T}$;
(ii) *if* $A \in \mathcal{L}K^\Sigma_{\langle \tau_1, \ldots, \tau_n \rangle}$ *and* $B_1 \in \mathcal{L}K^\Sigma_{\tau_1}, \ldots, B_n \in \mathcal{L}K^\Sigma_{\tau_n}$, $n \geq 1$, *then* $A(B_1, \ldots, B_n) \in \mathcal{L}K^\Sigma_{\langle\rangle}$;
(iii) *if* $A \in \mathcal{L}K^\Sigma_{[\tau]}$ *and* $B \in \mathcal{L}K^\Sigma_\tau$, *then* $A(B) \in \mathcal{L}K^\Sigma_{\langle\rangle}$;
(iv) *if* $\phi \in \mathcal{L}K^\Sigma_{\langle\rangle}$ *and* $x_1 \in \mathcal{V}_{\tau_1}, \ldots, x_n \in \mathcal{V}_{\tau_n}$ *are pairwise distinct variables*, $n \geq 1$, *then* $(\lambda x_1 \ldots x_n . \phi) \in \mathcal{L}K^\Sigma_{\langle \tau_1, \ldots, \tau_n \rangle}$.

The notions of *free* and *bound* occurrences of a variable are defined in the standard way. The definition of the sets of Σ-*terms* $\mathcal{L}^\Sigma_\tau$ of type τ, for each type τ, is the same as the definition of $\mathcal{L}K^\Sigma_\tau$, except that clause (iv) is replaced by the following clause:

(iv′) if $\phi \in \mathcal{L}^\Sigma_{\langle\rangle}$ and $x_1 \in \mathcal{V}_{\tau_1}, \ldots, x_n \in \mathcal{V}_{\tau_n}$ are pairwise distinct variables all of which are free in ϕ, $n \geq 1$, then $(\lambda x_1 \ldots x_n . \phi) \in \mathcal{L}^\Sigma_{\langle \tau_1, \ldots, \tau_n \rangle}$.

The set of all Σ-terms $\bigcup_{\tau \in \mathcal{T}} \mathcal{L}^\Sigma_\tau$ is denoted by $\mathcal{L}^\Sigma$ and constitutes the *λI-language of* Σ. Note that $\{\mathcal{L}^\Sigma_\tau : \tau \in \mathcal{T}\}$ is a typed family. In what follows, we will be exclusively concerned with such λI-languages in which vacuous λ-abstracts are not well-formed. A justification for this choice of language is provided in §4. We call terms of type $\langle\rangle$ *formulas* (or *sentences* if they don't contain free variables), terms of type $\langle \tau_1, \ldots, \tau_n \rangle$ (ordinary) *predicates*, and terms of type $[\tau]$ *rigid predicates*. Our use of rigid predicates is a higher-order generalization of the category of rigid predicates in LE.[9] We call types of the form $\langle \sigma \rangle$ or $[\sigma]$ *one-place predicate types* and terms of this type *one-place predicate terms*. A *combinator* is a term without free variables or constants, such as $\lambda p . p$. If not otherwise specified, we assume that each signature Σ contains the following logical constants:

[9] We could in principle allow for n-ary rigid predicates as well, but the generalization to the n-ary case involves some complexities in the axiomatics and semantics that I have preferred to avoid here.

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$\neg \in \Sigma_{\langle\langle\rangle\rangle}$;
$\wedge, \vee, \rightarrow, \leftrightarrow \in \Sigma_{\langle\langle\rangle,\langle\rangle\rangle}$;
$\forall_\sigma, \exists_\sigma \in \Sigma_{\langle\langle\sigma\rangle\rangle}$ for all $\sigma \in \mathcal{T}$;
$=_\sigma \in \Sigma_{\langle\sigma,\sigma\rangle}$ for all $\sigma \in \Sigma$;
$\Box_{\tau_1,\tau_2,\ldots,\tau_k} \in \Sigma_{\langle\tau_1,\ldots,\tau_k,\langle\rangle\rangle}$ for any $k \geq 0$ and any one-place predicate types $\tau_i \in \mathcal{T}$.

Constants of this last kind are *essentialist operators*. The case where $k = 0$ is written $\Box_\varnothing$. When $F_1, F_2, \ldots, F_n$ are one-place predicate terms and ϕ is a formula, we write $\Box_{F_1,F_2,\ldots,F_n}\phi$ for $\Box_{\tau_1,\ldots,\tau_n}(F_1, \ldots, F_n, \phi)$. We also write $\forall x^\tau(\phi)$ for $\forall_\tau((\lambda x^\tau.\phi))$, and $\phi \wedge \psi$ for $\wedge(\phi, \psi)$, and similarly for other logical constants. Note that in this setting, quantifiers don't bind variables, and they combine with predicates to make sentences. For instance, if F is a predicate of type $\langle\sigma\rangle$, then $\forall_\sigma(F)$ states that F applies to every entity of type σ (or, equivalently, that F is universal). Thus, to express that everything of type e is self-identical, we can write $\forall_e((\lambda x.x =_e x))$, which our convention above allows us to abbreviate to the more usual $\forall x^e(x =_e x)$. I will generally omit parentheses and type annotations whenever no confusion is likely to arise and follow other standard conventions. For example, we may write $\forall x(x = x)$ instead of $\forall x^e(x =_e x)$ whenever it is clear from the context or it doesn't matter what the appropriate types are.

When A is a term, $x_1, \ldots, x_n$ are pairwise distinct variables, and $B_1, \ldots, B_n$ are terms, we write $A[B_i/x_i]$ for the term that results from replacing each free occurrence of x_1 in A with B_1, and each free occurrence of x_2 with B_2, and so on, replacing bound variables in such a way that no free variables in any B_i become bound. We also adopt the following abbreviations:

(i) $F \subseteq G$ for $\forall x(F(x) \rightarrow G(x))$, where x is the first variable of the relevant type not free in either F or G;
(ii) $F \approx G$ for $(F \subseteq G \wedge G \subseteq F)$;
(iii) $[A]$ for $\lambda x.x = A$, where x is the first variable of the same type as A not free in A;
(iv) $(F_1, \ldots, F_n) \succeq_{\sigma_1,\ldots,\sigma_n,\tau} A$ for $\exists X^{\langle\tau\rangle}\Box_{F_1,\ldots,F_n}X(A)$, where $X^{\langle\tau\rangle}$ is the first variable not free in $F_1, \ldots, F_n, A$;
(v) $c_\tau(F_1, \ldots, F_n)$ for $\lambda x^\tau.(F_1, \ldots, F_n) \succeq_{\sigma_1,\ldots,\sigma_n,\tau} x^\tau$, where x^τ is the first variable not free in $F_1, \ldots, F_n$;
(vi) $A \geq_{\sigma,\tau} B$ for $([A]) \succeq_{\langle\sigma\rangle,\tau} B$.

The predicate $\succeq_{\sigma_1,\ldots,\sigma_n,\tau}$ expresses the relation of *essential involvement* between a collection of entities and a single entity. We can pronounce a statement of the form $(F_1, \ldots, F_n) \succeq_{\sigma_1,\ldots,\sigma_n,\tau} A$ as 'the nature of the collection of entities picked out by $F_1, \ldots, F_n$ involves A', where the collection of entities picked out by $F_1, \ldots, F_n$ consists of all the entities in the extensions of $F_1, \ldots, F_n$. By (iv) this means that there is some property $X^{\langle\tau\rangle}$ such that it lies in the nature of the collection picked out by $F_1, \ldots, F_n$ that A has X. We will sometimes suppress the types of the predicates on the left-hand side and simply write $\succeq_\tau$ for $\succeq_{\tau_1,\ldots,\tau_n,\tau}$. The predicate $c_\tau(F_1, \ldots, F_n)$ applies to all and only those entities of type τ that are involved in the nature of the collection of entities picked out by $F_1, \ldots, F_n$. For example, $c_e([s])$ applies to all and only the *objects* involved in Socrates' nature and $c_{\langle\rangle}([\neg])$ applies to all and only the *propositions* involved in the nature of negation. The predicate $\geq_{\sigma,\tau}$ expresses the relation of essential involvement between a single entity of type σ and a single entity of type τ. Thus a statement of the form $A \geq_{\sigma,\tau} B$ expresses that the nature of A involves B. We will sometimes omit type subscripts and write $\geq$ instead of $\geq_{\sigma,\tau}$.

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Here are some examples of formalized essentialist statements:

1. (a) $\Box_{[s]}H(s)$.
   (It is essential to Socrates that he is human.)
   (b) $\neg\Box_{[H]}H(s)$.
   (It is not in the nature of being human that Socrates is human.)
   (c) $\Box_N\forall x(N(x) \rightarrow (E(x) \vee O(x))$.
   (It lies in the collective nature of the natural numbers that every natural number is either even or odd.)
   (d) $\Box_{[\forall],[=]}\forall x(x = x)$.
   (It lies in the nature of identity and universality taken together that everything is self-identical.)
   (e) $\Box_{\lambda p\langle\rangle .p=p}\exists p\ p$.
   (It is true in virtue of the collective nature of all propositions that there is a true proposition.)

Note the difference between $\Box_{[F]}$ and $\Box_F$: The former expresses truth in virtue of the nature of *being F*, so the property of *being F* is the subject of the essentialist attribution, whereas the latter expresses truth in virtue of the nature of the entities which instantiate *F*, so the entities in the extension of *F* are the (collective) subject of the essentialist attribution.

§3. The system. In this section, we present the proof system of HLE. The system HLE is constituted by the following axioms and rules.

(i) *Background logic*:
(PC) ϕ, whenever ϕ is a tautology.
(UI) $\forall_\tau(F) \rightarrow F(A)$.
(EG) $F(A) \rightarrow \exists_\tau(F)$.
(MP) If $\vdash \phi$ and $\vdash \phi \rightarrow \psi$, then $\vdash \psi$.
(GEN) If $\vdash \phi \rightarrow \psi$ and x is free in ψ but not ϕ, then $\vdash \phi \rightarrow \forall_\tau x(\psi)$.
(INST) If $\vdash \phi \rightarrow \psi$ and x is free in ϕ but not ψ, then $\vdash \exists_\tau x(\phi) \rightarrow \psi$.
(Ref) $F =_\sigma F$.
(LL) $F =_\sigma G \rightarrow (\phi[G/x] \rightarrow \phi[F/x])$.
(β-conversion) $\phi \leftrightarrow \phi^*$, where ϕ^* results from ϕ by replacing some constituent of the form $(\lambda x_1 \ldots x_n.\psi)(t_1, \ldots, t_n)$ with $\psi[t_i/x_i]$.
(η-conversion) $\phi \leftrightarrow \phi^*$, where ϕ^* results from ϕ by replacing some constituent of the form $(\lambda x_1 \ldots x_k.F(x_1, \ldots, x_k))$, where none of $x_1, \ldots, x_k$ is free in F, with F.

(ii) *Background essentialist axioms*
(Permutation) $\Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{F_{\pi(1)},\ldots,F_{\pi(n)}}\phi$, where π is a permutation of $1, \ldots, n$.
(Idempotence) $\Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{F_1,\ldots,F_{n-1}}\phi$, if $F_n = F_i$, for some $i = 1, \ldots, n-1$.
(Separation) $\Box_{F_1,\ldots,F_n,\lambda x.(G(x)\vee H(x))}\phi \rightarrow \Box_{F_1,\ldots,F_n,G,H}\phi$.
(Subtraction) $\forall x^\sigma \neg F_{n+1}(x) \rightarrow (\Box_{F_1,\ldots,F_{n+1}}\phi \leftrightarrow \Box_{F_1,\ldots,F_n}\phi)$.
(MON) $\forall x^\sigma(F_1(x) \rightarrow G(x)) \rightarrow (\Box_{F_1,\ldots F_k}\phi \rightarrow \Box_{G,F_2,\ldots,F_k}\phi)$, $1 \leq k$.

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(Decomposition) $\Box_{F_1,\ldots,F_k,[B]}\phi \rightarrow \Box_{F_1,\ldots,F_k,[A_1],\ldots,[A_n]}\phi$, where $A_1, \ldots, A_n$ are all the constants and free variables of B.

(iii) *Axioms for rigidity*

(R-Comp) $\forall X^{\langle\sigma\rangle}\exists Y^{[\sigma]}\forall x^{\sigma}(X(x) \leftrightarrow Y(x))$.

(R-Ext) $\forall X^{[\sigma]}\forall Y^{[\sigma]}(\forall x^{\sigma}(X(x) \leftrightarrow Y(x)) \rightarrow X =_{[\sigma]} Y)$.

(Rigidity) $R^{[\sigma]}(x) \rightarrow \Box_R R(x)$.

(R-Equiv) $\Box_{R,F_1,\ldots,F_n}\phi \leftrightarrow \Box_{[R],F_1,\ldots,F_n}\phi$, where R is a rigid predicate.

(iv) *Core essentialist axioms and rules*

(CH) $\Box_{G_1,\ldots,G_n,c_\sigma(F_1,\ldots,F_k)}\phi \rightarrow \Box_{G_1,\ldots,G_n,F_1,\ldots,F_k}\phi$.

(RC) If $\vdash \phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi$, then $\vdash \Box_{F_1,\ldots,F_k}\phi_1 \wedge \cdots \wedge \Box_{F_1,\ldots,F_k}\phi_n \rightarrow \Box_{F_1,\ldots,F_k,[A_1],\ldots,[A_m]}\psi$, where $A_1, \ldots, A_m$ are all the constants and free variables occurring in ψ but not any of $\phi_1, \ldots, \phi_n$.

(T) $\Box_{F_1,\ldots,F_n}\phi \rightarrow \phi$.

(4) $\Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{F_1,\ldots,F_n,[\Box_{\sigma_1,\ldots,\sigma_n}]}\Box_{F_1,\ldots,F_n}\phi$, whenever $F_1^{\sigma_1}, \ldots, F_n^{\sigma_n}$ are rigid predicates.

(5) $\neg\Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{F_1,\ldots,F_n,[\neg],[\Box_{\sigma_1,\ldots,\sigma_n}],[\phi]}\neg\Box_{F_1,\ldots,F_n}\phi$, whenever $F_1^{\sigma_1}, \ldots, F_n^{\sigma_n}$ are rigid predicates.

I will only briefly comment on the rationale for the different axioms and rules. For a more detailed justification, the reader is referred to Ditter [15]. The background logic consists of the principles of classical higher-order logic with identity together with the standard principles of β- and η-conversion. The first five axioms in group (ii) capture the principle of *Monotonicity*: If X is a subcollection of Y, then if it lies in the nature of X that ϕ then it lies in the nature of Y that ϕ. They do so by making sure that if the predicates $F_1, \ldots, F_n$ pick out some entities that form a (proper or improper) subcollection of the entities picked out by $G_1, \ldots, G_m$, then $\Box_{F_1,\ldots,F_n}\phi$ materially implies $\Box_{G_1,\ldots,G_m}\phi$.[10] In writing formal proofs, I will often refer to one or more of these axioms simply by 'Monotonicity'.

The principle of Decomposition says that whatever lies in the nature of an entity expressed by a term B also lies in the collective nature of the entities denoted by the free variables and constants occurring in B. For instance, Decomposition allows us to infer $\Box_{[\forall],[=]}\forall x(x = x)$ from $\Box_{[\forall x(x=x)]}\forall x(x = x)$.[11] Model-theoretically, Decomposition corresponds to the condition that the denotation of a term exists at a world if the denotations of all of its constants and free variables exist at that world. This condition is crucial not only for the validity of Decomposition but for virtually every principle of the system, as the proof of the Soundness Theorem (Theorem 1) will illustrate.

The axioms in group (iii) state characteristic principles for rigid properties. R-Comp says that for any property of any type there is a rigid property coextensive with it.[12] R-Ext is an extensionality principle for rigid properties. It says that coextensiveness suffices for the identity of rigid properties. This is in line with our thinking of rigid

[10] Monotonicity is higher-order generalization of the analogous first-order principle in Fine's LE. The first-order analog of Monotonicity can be expressed by a single axiom; see Fine [21, p. 247].

[11] Recall that $\forall x(x = x)$ is short for $\forall((\lambda x.x = x))$.

[12] R-Comp is analogous to a comprehension principle for pluralities, according to which for every property, there is a plurality coextensive with it. The relevant principle for pluralities is often restricted to properties with a non-empty extension, a restriction we don't impose for rigid properties. Compare Linnebo [33] and Burgess [4].

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properties as predicative analogs of pluralities. The axiom of Rigidity says that if x is one of some entities, then it lies in the nature of those entities that x is one of them. This axiom is a higher-order generalization of Fine's axiom of Rigidity in LE. R-Equiv concerns the relation between a rigid property and its extension. It says that whatever lies in the nature of a rigid property lies in the collective nature of the entities in its extension, and *vice versa*. The nature of a rigid property is thus exhausted by the collective nature of the entities that it applies to. R-Equiv and Rigidity together entail that if x is one of some entities, then it lies in the nature of the *property* of being one of those entities that x is one of them. This is analogous to the essentialist principle about sets and pluralities that says that sets and pluralities have their members essentially.

The last three axiom schemas in group (iv) are essentialist analogs of the modal T-, 4- and 5-schemas. The T-schema says that whatever is true in virtue of the nature of a collection of entities is true. The 4-principle says that if ϕ is true in virtue of the nature of a collection X, then it is true in virtue of the nature of X together with the nature of essentiality (i.e., what is expressed by the essentialist operator) that it is true in virtue of the nature of X that ϕ. In conjunction with the widely accepted principle that essence implies necessity (i.e., if ϕ is true in virtue of the nature of some entities then ϕ is necessary), the 4-principle entails the plausible claim that whatever is true in virtue of the nature of a collection X is necessarily true in virtue of the nature of X—entities have their essence necessarily.

In stating the essentialist 4-schema in our formal language it is crucial that the predicates $F_1, \ldots, F_n$ be rigid, since the inner essentialist operator in the consequent should have the same subject as the operator in the antecedent. The schema does not in general hold for non-rigid predicates. For example, suppose that Mary is wearing a hat and that it is essential to the hat wearers that Mary is human. Then it may be false that it is essential to the hat wearers, taken together with essentiality, that it is essential to the hat wearers that Mary is human, since it may not be the case that it is essential to Mary to be wearing a hat, and so it may not be essential to whoever is wearing a hat that Mary is human. Such a case is averted if the predicates in (4) rigidly pick out Mary.

The same restriction to rigid predicates is required in the statement of the essentialist 5-principle. This principle says that if it is not true in virtue of the nature of a collection X that ϕ, then it is true in virtue of the nature of X together with negation, essentiality and ϕ that it is not true in virtue of the nature of X that ϕ. Together with the principle that essence implies necessity, (5) entails that if ϕ is not true in virtue of the nature of X, then this is necessarily so—the nature of X could not have been richer.[13]

The first two principles in group (iv) are the key components of the account of logical closure provided by HLE. The principle CH is a generalization of the 'chaining axiom' from LE to our higher-order framework. Recall that $c_\sigma(F_1, \ldots, F_k)$ picks out the collection of entities of type σ involved in the nature of the entities picked out by $F_1, \ldots, F_k$. Let p be some proposition true in virtue of the collective nature of the entities in $c_\sigma(F_1, \ldots, F_k)$. Then CH says that p is also true in virtue of the collective nature of the entities picked out by $F_1, \ldots, F_k$. Thus every collection of entities inherits the collective natures of the entities involved in its nature. RC is a closure principle for essences. In HLE, any type of logical closure manifested by the essence of a

[13] The existence of iterated essentialist truths has been disputed by Dasgupta [11] and Glazier [26]. Their arguments are discussed and rebutted in Ditter [16].

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collection of entities derives from the logical operations involved in the essence of the collection. So, for example, the essence of a collection of entities will be closed under conjunction elimination and introduction just in case the essence of the collection involves conjunction, and it will be closed under modus ponens just in case it involves the conditional. For a concrete example, suppose it lies in the nature of the number two that it is a number and even ($\Box_{[2]}(N(2) \wedge E(2))$). RC then immediately implies that it lies in the nature of the number two that it is a number ($\Box_{[2]}N(2)$). The rationale for closure under conjunction elimination is that the hypothesis implies that the nature of the number two involves conjunction (in the sense of involvement defined above), since $\Box_{[2]}(N(2) \wedge E(2))$ implies $\Box_{[2]}\lambda X.(X(N(2), E(2))(\wedge)$ by β-conversion, which implies $2 \geq \wedge$ by β-conversion and EG. The converse implication is only valid under the additional assumption that the number two involves conjunction. By RC alone we have that $\Box_{[2]}N(2)$ and $\Box_{[2]}E(2)$ together imply $(*)$ $\Box_{[2],[\wedge]}(N(2) \wedge E(2))$; if we further assume $2 \geq \wedge$, then $c_{\langle\langle\rangle,\langle\rangle\rangle}([2])(\wedge)$ and thus $\Box_{[2],c_{\langle\langle\rangle,\langle\rangle\rangle}([2])}(N(2) \wedge E(2))$ by Monotonicity and $(*)$, whence $\Box_{[2]}(N(2) \wedge E(2))$ follows by CH and Monotonicity. RC and CH together imply that the nature of a collection of entities is closed under precisely those consequences that are stated solely in terms of vocabulary expressing entities that are involved in the nature of the collection.

The special case of RC where $n = 0$ gives us the following essentialist "necessitation-rule":

(RC^0) If $\vdash \psi$, then $\vdash \Box_{[A_1],\ldots,[A_m]}\psi$, where $A_1, \ldots, A_m$ are all the constants and free variables occurring in ψ.

RC^0 implies that every logical truth whose only constants are logical constants is true in virtue of the nature of the logical operations expressed by these constants. So, for example, since $\forall x(x = x)$ is a logical truth whose only constants are $\forall$ and $=$, RC^0 implies $\Box_{[\forall],[=]}\forall x(x = x)$. Note, however, that RC^0 does not imply that every logical truth is true *solely* in virtue of some logical operations. This would have the implausible consequence that the collective nature of the logical operations involves absolutely every entity of any type, but plausibly the nature of identity does not involve Socrates, for example.

Correia [8, 9] has proposed an account of the way in which essences are closed under logical consequence which differs in important ways from the account proposed here. A brief comparison between HLE and Correia's account will be instructive. Correia's target notion of essence, which he calls the 'Finean notion of essence', is characterized in terms of a primitive notion of a proposition being *basically essential* to a plurality (the logic of which is not specified), and a relation of *relative logical consequence*, defined as follows:[14]

> Given any proposition α, plurality of propositions Δ, and set of logical concepts S, say that α is a logical consequence of Δ relative to S iff there is a proof of α from Δ, such that given any rule concerning a logical concept which appears in that proof, that concept is a member of S. (See Correia [8, p. 647])

[14] Correia speaks of the logical *concepts* instead of the logical *operations*. In discussing his view, I will use his terminology.

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After providing some general structural principles guiding the relation of relative logical consequence (the details of which don't matter for present purposes), Correia goes on to define what he calls the 'Finean notion of essence', thus:

> [A proposition] α is true in virtue of the nature of [a plurality] X just in case α is a logical consequence, relative to the logical concepts in X, of the basic nature of X.[15] (Ibid., p. 648)

I will refer to the propositions that are true in virtue of the nature of X so defined as just the *essence* or *nature* of X. According to Correia, this notion of essence differs importantly from Fine's consequential notion, since, e.g., 'the propositions which are true in virtue of the nature of Socrates are, on the proposed account, just the propositions which belong to his basic nature.' (ibid.) More generally, he maintains, 'in case X comprises no logical concepts, the propositions which are true in virtue of the nature of X are just those which belong to the basic nature of X.' (ibid.) Thus, in contrast to HLE, the logical closure of the essence of X does not depend on which logical concepts are *involved in the nature* of X, but only on whether X *comprises* any logical concepts. The notion of essential involvement plays no role in Correia's account.

This leads to some interestingly different predictions about the logical closure properties of essences. It is consistent with both accounts that, say, Socrates' nature is "logically simple" in the sense of not being closed under logical consequence at all. However, if it is essential to Socrates that he is human and mortal ($\Box_{[s]}(H(s) \wedge M(s))$), then HLE entails that he is essentially human ($\Box_{[s}H(s)$) and essentially mortal ($\Box_{[s]}M(s)$), whereas this doesn't follow on Correia's account. Furthermore, Correia's account predicts that it lies in the nature of identity *alone* that Socrates is self-identical, because the proposition that Socrates is self-identical can be proved using only an instance of (Ref), whereas in HLE this can be consistently denied, because we can consistently deny that the nature of identity involves Socrates.

However, there are some important details about Correia's account that need to be spelled out before we can offer a more systematic comparison between it and HLE. As we have seen above, and as Correia himself stresses, his proposal crucially rests on taking the relation of relative logical consequence, as well as the notions of proof and rule of inference in terms of which it is defined, not to be relations between *sentences* (as logical consequence is usually defined) but to be relations between *propositions* (ibid. p. 647 fn. 10). (Otherwise the account would be problematically meta-linguistic.) But the usual characterizations of the notions of proof and rule of inference crucially presuppose various syntactic notions which do not obviously have propositional analogs. For example, in order to state an inference rule governing a given connective, we need to be able to identify the *occurrence* of the connective in a certain *syntactic position* of a formula. But unless propositions are structured in something like the way sentences are structured—an assumption that we have reason to question, given the Russell–Myhill paradox[16] —it is not obvious that there are propositional analogs of these notions which work in the intended way. Without further details, it thus remains an open question how Correia's central notion of relative logical consequence as well

[15] The basic nature of X is defined as the plurality of propositions α such that for some Y which is part of X, α is basically essential to Y.

[16] See note 6. Analogs of the Russell–Myhill paradox also arise in the context of first-order theories of properties and propositions.

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as the corresponding notion of essence are to be understood, and whether they can ultimately be developed in a consistent way.

§4. Some theorems. This section establishes a number of theorems of HLE and discusses their philosophical ramifications. Principles of classical logic will be referred to by the label 'CL'. Theoremhood is defined in the usual way. We begin with a useful consequence of the Monotonicity axioms.

PROPOSITION 1.

(i) $\vdash \Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{G,F_1,\ldots,F_n}\phi$.
(ii) $\vdash \Box_{F_1,\ldots,F_k}\phi \rightarrow \Box_{F_1,\ldots,F_n}\phi$, $k \leq n$.

Proof. (i) By Subtraction, Permutation and CL, $\vdash \Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{\lambda x.\neg(x=_\sigma x),F_1,\ldots,F_n}\phi$. This yields $\vdash \Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{G,F_1,\ldots,F_n}\phi$ by MON and CL.
(ii) From (i). □

From now on, I will normally use the label 'Monotonicity' to refer to either Proposition 1 or one or more of the Monotonicity axioms. The following result concerns some important properties of the relation $\geq$ of essential involvement between particular entities.

PROPOSITION 2.

(i) $\vdash A \geq_{\sigma,\sigma} A$.
(ii) $\vdash A \geq_{\tau,\sigma} B \wedge B \geq_{\sigma,\rho} C \rightarrow A \geq_{\tau,\rho} C$.
(iii) $\vdash B[A/x] \geq_{\sigma,\tau} A$, *provided* x *is free in* B.
(iv) $\vdash \forall x^\sigma \exists p^{\langle\rangle}(p \geq x \wedge x \geq p)$.

Proof. (i) By Monotonicity, (1) $\vdash \forall y(X^{[\sigma]}(y) \leftrightarrow (\lambda z.z =_\sigma A)(y)) \rightarrow ((\Box_X\phi \leftrightarrow \Box_{[A]}\phi)$. By CL, (2) $\vdash \forall y(X^{[\sigma]}(y) \leftrightarrow (\lambda z.z =_\sigma A)(y)) \rightarrow X(A)$. From this, (3) $\vdash \forall y(X^{[\sigma]}(y) \leftrightarrow (\lambda z.z =_\sigma A)(y)) \rightarrow \Box_X X(A)$ by the instance $X^{[\sigma]}(A) \rightarrow \Box_X X(A)$ of Rigidity. Combining (1) and (3) we obtain (4) $\vdash \forall y(X^{[\sigma]}(y) \leftrightarrow (\lambda z.z =_\sigma A)(y)) \rightarrow \exists X^{[\sigma]}\Box_{[A]}X(A)$. Hence, by CL, (5) $\vdash \exists X^{[\sigma]}\forall y(X^{[\sigma]}(y) \leftrightarrow (\lambda z.z =_\sigma A)(y)) \rightarrow \exists X^{[\sigma]}\Box_{[A]}X(A)$. But the antecedent of (5) is an instance of R-Comp, so (6) $\vdash \exists X^{[\sigma]}\Box_{[A]}X(A)$. By β-conversion, (7) $\vdash \Box_{[A]}X(A) \leftrightarrow \Box_{[A]}(\lambda x.X(x))(A)$. From this, $A \geq_{\sigma,\sigma} A$ follows by (6) and CL.

(ii) By β-conversion, (1) $\vdash A \geq_{\tau,\sigma} B \rightarrow c_\sigma([A])(B)$. By Monotonicity, (2) $\vdash \forall z^\sigma(\lambda x.(x = B)(z) \rightarrow c_\sigma([A])(z)) \rightarrow (\Box_{[B]}\phi \rightarrow \Box_{c_\sigma([A])}\phi)$. By CH, (3) $\vdash \Box_{c_\sigma([A])}\phi \rightarrow \Box_{[A]}\phi$. From (1), (2), (3) and CL, (4) $\vdash A \geq_{\tau,\sigma} B \wedge \Box_{[B]}X(C) \rightarrow \exists Y\Box_{[A]}Y(C)$. From this, the claim follows by CL and the definition of $\geq$.

(iii) By β-conversion, (1) $\vdash \Box_{[B[A/x]]}Y(B[A/x]) \rightarrow \Box_{[B[A/x]]}(\lambda x.Y(B))(A)$, where Y is a variable distinct from x not free in B or A. From (1), we obtain (2) $\vdash \Box_{[B[A/x]]}Y(B[A/x]) \rightarrow \exists X\Box_{[B[A/x]]}X(A)$ by CL. By item (i) we have (3) $\vdash \exists Y\Box_{[B[A/x]]}Y(B[A/x])$, which together with (2) entails $\vdash B[A/x] \geq A$ by CL.

(iv) The claim immediately follows from R-Comp, β-conversion and $\vdash \forall y(X^{[\sigma]}(y) \leftrightarrow (\lambda z.z =_\sigma x)(y)) \rightarrow (\Box_{[x]}X(x) \wedge (X(x) \geq X(x)))$, which follows from Rigidity, Monotonicity and part (i). □

Proposition 2(i) and (ii) concern the reflexivity and transitivity of $\geq$, respectively. Part (iv) shows that the relation $\geq$ is not asymmetric across types: For every entity of any type, there is a proposition such that the nature of the entity involves the

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proposition and the nature of the proposition involves the entity. Let us call the rigid property that applies to x and only x the *haecceity* of x. (Uniqueness is justified by R-Ext.) When F is the haecceity of x, we say that $F(x)$ is the *haecceity proposition* of x. For any x, the haecceity proposition of x is a canonical example of a proposition whose nature involves x and which is involved in the nature of x. Although Proposition 2(iv) crucially relies on Rigidity, it is worth noting that the same result would follow without Rigidity if we posited that for any entity x, there is a proposition true in virtue of the nature of x that predicates something of x.

We now turn to some important consequences of CH.

PROPOSITION 3.

(i) $\vdash (F_1, \dots, F_n) \succeq_\sigma A \wedge \Box_{G_1,\dots,G_k,[A]}\phi \rightarrow \Box_{G_1,\dots,G_k,F_1,\dots,F_n}\phi$;
(ii) $\vdash \Box_{F_1,\dots,F_n}\phi \wedge \Box_{[\phi]}\psi \rightarrow \Box_{F_1,\dots,F_n}\psi$;
(iii) $\vdash \Box_{F_1,\dots,F_n}\phi \wedge \phi \geq_{\langle\rangle,\sigma} A \rightarrow (F_1, \dots, F_n) \succeq_\sigma A$;
(iv) $\vdash \Box_{F_1,\dots,F_k,[A_1],\dots,[A_n]}\phi \rightarrow \Box_{F_1,\dots,F_k,[B]}\phi$, *where* $A_1, \dots, A_n$ *are all the constants and free variables of* B.

Proof. (i) By β-conversion, $\vdash (F_1, \dots, F_n) \succeq_\sigma A \rightarrow c_\sigma(F_1, \dots, F_n)(A)$. By Monotonicity, $\vdash \forall z^\sigma(\lambda x.(x = A)(z) \rightarrow c_\sigma(F_1, \dots, F_n)(z)) \rightarrow (\Box_{G_1,\dots,G_k,[A]}\phi \rightarrow \Box_{G_1,\dots,G_k,c_\sigma(F_1,\dots,F_n)}\phi)$. By CH and CL we obtain $\vdash (F_1, \dots, F_n) \succeq_\sigma A \wedge \Box_{G_1,\dots,G_k,[A]}\phi \rightarrow \Box_{G_1,\dots,G_k,F_1,\dots,F_n}\phi$.

(ii) From (i).

(iii) From (ii).

(iv) Suppose $A_1, \dots, A_n$ are all the constants and free variables of B. By Proposition 2(iii), $\vdash B \geq A_i$ for $1 \geq i \geq n$. The claim then follows by n applications of part (i) and Monotonicity. □

Note that Proposition 3(iv) is the converse of Decomposition. Proposition 3(iii) bears resemblance to, but is importantly different from, the axiom of Localization in LE. The latter says that if ϕ is true in virtue of the nature of a collection of objects, then any object x which 'occurs in' ϕ is such that the nature of one of the members of the collection must involve x. Formally (in the language of HLE):[17]

(Localization) $\Box_F\phi \wedge x\eta\phi \rightarrow \exists y^e(F(y) \wedge y \geq_{e,e} x)$.

We can state a higher-order generalization of Localization as follows, replacing $x\eta\phi$ with $\phi \geq_{\langle\rangle,\sigma} x$:[18]

(Localization*) $\Box_{F_1,F_2,\dots,F_n}\phi \wedge \phi \geq x^\sigma \rightarrow (\exists x_1^{\sigma_1}(F_1(x_1) \wedge x_1 \geq x) \vee \dots \vee \exists x_n^{\sigma_n}(F_n(x_n) \wedge x_n \geq x))$.

Localization* says that if ϕ is true in virtue of the nature of some collection of entities and the nature of ϕ involves some entity x, then the nature of some member of the

[17] $x\eta\phi$ is Fine's abbreviation for $(x = x_1) \vee \dots \vee (x = x_m) \vee P_1(x) \vee \dots \vee P_n(x)$, where $x_1, \dots, x_m$ are the free variables of ϕ in order of appearance and that $P_1, \dots, P_n$ are the rigid predicate symbols of ϕ, likewise in order of appearance. Informally, $x\eta\phi$ is to be read 'x occurs in the proposition expressed by ϕ' or 'ϕ is about x'.

[18] Morvarid [35] argues for a version of Localization on which not only the *objectual* but also the *logical* constituents of a proposition are taken into account. This goes some way towards motivating a full higher-order version of Localization, although Morvarid does not propose it in a higher-order context but as an amendment to Fine's first-order version in LE.

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collection involves x.[19] The difference between Localization* and Proposition 3(iii) is that the former requires the nature of a *particular* member of the collection to involve x, whereas the latter only requires the nature of the *collection* to involve x. Localization* immediately yields problematic results in HLE. For example, since both $\Box_{[\forall],[\neg],[\vee]}\forall p(p \vee \neg p)$ and $\forall p(p \vee \neg p) \geq \forall p(p \vee \neg p)$ are theorems of HLE (by RC^0 and Proposition 2(i), respectively), Localization* implies that the nature of one of $\forall$, $\neg$ or $\vee$ involves $\forall p(p \vee \neg p)$, and therefore each of $\forall$, $\neg$ and $\vee$ since the nature of $\forall p(p \vee \neg p)$ involves each of them (by Proposition 2(ii) and (iii). Similarly, if it is true in virtue of the nature of x and y that they are distinct, then it follows from Localization* and the theoremhood of $\neg(x = y) \geq \neg(x = y)$ that either x or y involves the proposition $\neg(x = y)$; hence, it follows that either x involves y or vice versa, since the nature of the proposition $\neg(x = y)$ involves both x and y.[20]

Although the failure of the higher-order generalization already presents at least a *prima facie* reason to reject Localization, the above problems for Localization* make crucial use of higher-order resources and therefore do not directly challenge Fine's first-order principle of Localization. However, potential counterexamples to Localization are not hard to come by. Fine [21, p. 249 f.] himself notes that potential counterexamples to Localization arise from cases 'in which the nature of several objects is simultaneously understood in terms of one another.' He discusses the following kind of case.[21] Suppose it lies in the nature of a mind and body taken together to comprise a given person. Localization then entails that either the nature of the mind or the nature of the body involves the person. But it is natural to think that the person is not involved in the nature of either of them, taken individually.[22] However, Fine does not take this case to be a counterexample to Localization. But the reason is not that he argues that such a case is impossible; rather, he holds that

> a decision needs to be made concerning objects whose nature is understood in terms of one another (this is the objectual counterpart of simultaneous definition). Perhaps it lies in the nature of all the points of Euclidean space to enter into certain geometric relationships with one another. What then do we say of the individual points? Does every other point pertain to its nature, or do none? We can go either way on this question, but my preference, at least for the purpose of the

[19] Note that it makes no difference whether we use $x\eta\phi$ or $\phi \geq_\sigma x$ in the formulation of Localization*, since the two principles are equivalent in HLE. Let Localization** be like Localization* except that the second conjunct $\phi \geq_\sigma x$ in the antecedent is replaced by $x\eta\phi$. Localization** immediately follows from Localization* by Proposition 3(iii). For the proof of the converse, assume Localization** and suppose $\Box_{F_1,\ldots,F_n}\phi \wedge \exists z^{\langle\sigma\rangle}\Box_{[\phi]}z(x)$. By Proposition 3(iii), $\exists z^{\langle\sigma\rangle}\Box_{F_1,\ldots,F_n}z(x)$. The consequent of Localization* then follows by CL.

[20] Note that, in fact, adding Localization* to HLE leads to inconsistency. For by the reflexivity of $\geq$, $\lambda p.p \geq \lambda p.p$, whence by Decomposition, $\exists z\Box_{\varnothing}z(\lambda p.p)$. By Localization* this implies a contradiction. However, the problems described in the text do not depend on Decomposition and are of interest even if we rejected Decomposition.

[21] I have modified Fine's case slightly in order to make it a straightforward counterexample to Localization. The case Fine describes is only a potential counterexample to Localization if we add some further substantial premises.

[22] One can generate similar potential counterexamples by taking a mereologically complex object that has (some of) its parts essentially.

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> present paper, is to allow each point to pertain to the nature of every other point. Thus every collectively understood nature will resolve into a network of individually understood natures. [21, p. 242 f.]

Fine resolves the potential counterexample above in accordance with this decision: just as each point of Euclidean space is involved in the nature of every other point, so too do the nature of the mind and that of the body individually involve the person as well as each other (ibid., p. 249 f.).[23]

In my view, the question of whether Localization is true is a substantive question that cannot be resolved in the way Fine suggests. I also think that the potential counterexample to Localization discussed above is an actual counterexample. Indeed, the case of a body and a mind essentially comprising a person is in an important way analogous to the problems I have pointed out for Localization* but *disanalogous* to the case of points of Euclidean space. Both the problems for Localization* and the case of the body and the mind essentially comprising a person represent cases where some entity that is not involved in the nature of any *individual* member of a given collection *is* involved in the nature of the *collection*. By contrast, the case of the points of Euclidean space Fine describes is not of this sort: since each of the points is involved in its own nature, there will be, for any point, a point whose nature involves it. Therefore, the case of the points of Euclidean space essentially entering into certain geometric relationships with one another does not pose a threat to Localization.[24] But this disanalogy between the case of the mind and the body on the one hand and the points of Euclidean space on the other further calls into question Fine's response to the potential counterexample concerning a person being comprised by a mind and body.

As mentioned above, counterexamples to Localization* (and Localization) arise whenever the nature of a collection involves new entities, in the sense that those entities are not involved in the nature of any individual members of the collection. The proposition $\forall p(p \vee \neg p)$, for example, is involved in the collective nature of $\forall$, $\vee$ and $\neg$, but arguably not in the nature of any of them individually. The following consequence of Localization* is therefore subject to counterexamples:

$$\text{(F-Collective)}\ \ (F_1, \ldots, F_n) \succeq_\sigma x \rightarrow (\exists x_1^{\sigma_1}(F_1(x_1) \wedge x_1 \geq x) \vee \cdots \vee \exists x_n^{\sigma_n}(F_n(x_n) \wedge x_n \geq x)).$$

F-Collective says that whenever the nature of some collection involves some x, then the nature of some member of the collection involves x. The failure of F-Collective entails that we cannot define collective essential involvement in terms of individual essential involvement, as Fine does in LE. Individual essential involvement ($\geq$) is a

[23] It is worth noting that Localization does not imply the stronger claim that if it is in the nature of all the points of Euclidean space to enter in certain geometric relationships with one another, then each point pertains to the nature of every other point. Indeed, it is consistent in LE that all points of Euclidean space pertain only to themselves, and not to any other points, even on the assumption that it lies in the nature of all the points to enter into certain geometric relationships with one another. It is thus not generally true in LE that every collectively understood nature will resolve into a network of individually understood natures.

[24] Note that I'm not suggesting that Fine intended it to be a potential counterexample. We could however turn this into a potential counterexample by adding the claim that it lies in the nature of all the points of Euclidean space to comprise Euclidean space, where the latter is considered to be a new entity composed of all the points of Euclidean space.

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special case of collective essential involvement ($\succeq$), and not *vice versa*. As we will see later, the failure of Localization* requires a modification (which turns out to be a simplification) of Fine's semantic clause for the essentialist operator.

Our next result concerns some consequences of RC.

PROPOSITION 4.

(i) *If* $\vdash \phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi$, *then* $\vdash \Box_{F_1,\ldots,F_k}\phi_1 \wedge \cdots \wedge \Box_{F_1,\ldots,F_k}\phi_n \rightarrow \Box_{F_1,\ldots,F_k,[\psi]}\psi$.
(ii) $\vdash \phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi$, *then* $\vdash \Box_{F_1,\ldots,F_k}\phi_1 \wedge \cdots \wedge \Box_{F_1,\ldots,F_k}\phi_n \wedge (F_1, \ldots, F_k) \succeq_{\langle\rangle} \psi \rightarrow \Box_{F_1,\ldots,F_k}\psi$.
(iii) *If* $\vdash \phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi$, *then* $\vdash \Box_{F_1,\ldots,F_k}\phi_1 \wedge \cdots \wedge \Box_{F_1,\ldots,F_k}\phi_n \rightarrow \Box_{F_1,\ldots,F_k}\psi$, *if every constant or free variable occurring in* ψ *occurs in some of* $\phi_1, \ldots, \phi_n$.
(iv) *If* $\vdash \phi \leftrightarrow \psi$, *then* $\vdash \Box_{F_1,\ldots,F_n}\phi \leftrightarrow \Box_{F_1,\ldots,F_n}\psi$, *provided* ϕ *and* ψ *contain the same constants and free variables.*
(v) *If* $\vdash \psi$, *then* $\vdash \Box_{[\psi]}\psi$.
(vi) *If* $\vdash \psi$, *then* $\vdash (F_1, \ldots, F_n) \succeq_{\langle\rangle} \psi \rightarrow \Box_{F_1,\ldots,F_n}\psi$.

Proof. (i) From RC and Proposition 3(iv).
(ii) From part (i) and Proposition 3(i).
(iii) and (iv) are immediate consequences of (RC).
(v) and (vi) are the special cases of parts (i) and (ii) in which $n = 0$. □

By Decomposition, part (i) of Proposition 4 is in fact equivalent to RC. Note that part (ii) immediately implies that the essence of any collection of entities whose nature involves every proposition is closed under unrestricted consequence.

Next we state some consequences of Decomposition.

PROPOSITION 5.

(i) $\vdash ([A_1], \ldots, [A_n]) \succeq B$, *where* $A_1, \ldots, A_n$ *are all the constants and free variables of* B.
(ii) $\vdash \Box_{F_1,\ldots,F_n,[B]}\phi \rightarrow \Box_{F_1,\ldots,F_n}\phi$, *if* B *is a combinator.*
(iii) $\vdash (F_1, \ldots, F_n) \succeq B$, *if* B *is a combinator.*

Proof. (i) We have $\vdash \exists X \Box_{[B]} X(B)$ by Proposition 2(i). Thus, by Decomposition and CL, $\vdash \exists X \Box_{[A_1],\ldots,[A_n]} X(B)$.
(ii) is an instance of Decomposition.
(iii) By Proposition 2(i), $\vdash \exists X \Box_{[B]} X(B)$. So by Monotonicity and CL, $\vdash \exists X \Box_{F_1,\ldots,F_n,[B]} X(B)$. It follows by part (ii) that $\vdash \exists X \Box_{F_1,\ldots,F_n} X(B)$. □

Part (iii) of the above result illustrates the special status of the combinators in HLE: The nature of the operations expressed by the combinators is involved in the nature of any collection of entities. The combinators are naturally regarded as expressing broadly "grammatical" operations presupposed by the nature of any entity. Part (i) is equivalent to Decomposition given the other principles of HLE. It is a kind of non-linguistic counterpart of the idea that for any linguistic expressions of a given grammar, the complex expressions that can be formed from them by the grammatical operations are part of the collective nature of the linguistic expressions. For example, it is natural to think that it lies in the nature of the name 'Socrates' and the predicate 'is human' to form such sentences as 'Socrates is human'. And just as it lies in the collective nature of the *name* 'Socrates' and the *predicate* 'is human' to form the *sentence* 'Socrates is human', so too is it natural to think that it lies in the nature of Socrates and *being*

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human to form the *proposition* that Socrates is human. More generally, the entities that are formed from Socrates and *being human* by broadly "grammatical" operations like predication and (the metaphysical counterpart of) predicate abstraction are involved in the collective nature of Socrates and *being human*.[25]

Decomposition also implies that whatever is true in virtue of the nature of the converse of a relation is true in virtue of the relation itself. For let L stand for the relation *loves*, for example. Then its converse $\lambda xy.L(y,x)$ stands for *being loved by*. By Decomposition, we can infer $\Box_{[L]}\phi$ from $\Box_{[\lambda xy.L(y,x)]}\phi$. The same is true for the "reflexivization" of L, $\lambda x.L(x,x)$, which represents the relation of *loving oneself*.

The previous results allow us to illustrate why we are working in a λI-language in which vacuous abstraction is not allowed. Suppose that we allowed for vacuous abstraction, so that we can form lambda-abstracts like $\lambda x.\phi$, where x is not free in ϕ. Let A and B be any terms of arbitrary type. By Proposition 2(i), $\vdash \exists X \Box_{[A]} X(A)$. Now let z be a variable not free in $X(A)$. Then $\vdash X(A) = (\lambda z.X(A))(B)$ by a vacuous instance of β-conversion. By CL and LL, this implies $\vdash \exists Y \Box_{[A]} Y(B)$, and thus $\vdash A \geq B$ for arbitrary A and B. It follows that all subscripted essentialist operators are logically coextensive: $\vdash \Box_{F_1,\ldots,F_n}\phi \leftrightarrow \Box_{G_1,\ldots,G_k}\phi$. Instead of avoiding this result as we do here by working with a language in which vacuous instances of β-conversion are not well-formed we could instead work with a λK-language and restrict the principle of β-conversion to its non-vacuous instances. The present choice of language is primarily a matter of convenience, although there is something to be said for taking the λI-language to be metaphysically more perspicuous than the λK-language in a context in which vacuous β-conversion is rejected.[26]

We now turn to some important results about identity and distinctness.

PROPOSITION 6.

(i) $\vdash x =_\sigma y \rightarrow \Box_{[x],[=_\sigma]} x =_\sigma y$;
(ii) $\vdash \neg(x =_\sigma y) \rightarrow \Box_{[x],[y],[=_\sigma],[\neg],[\Box_{[\sigma],[\sigma],[\langle\sigma,\sigma\rangle]}]}\neg(x =_\sigma y)$;
(iii) $\vdash \neg R^{[\sigma]}(x) \rightarrow \Box_{R,[x],[\neg],[\Box_{[\sigma],\langle\sigma\rangle]}]}\neg R(x)$.

Proof. (i) By RC^0, $\vdash \Box_{[x],[=]} x = x$. Hence, by CL and LL, $\vdash x = y \rightarrow \Box_{[x],[=]} x = y$.
(ii) Let X, Y, Z be variables of types $[\sigma]$, $[\sigma]$ and $[\langle\sigma,\sigma\rangle]$, respectively. By part (i) and Monotonicity, we have

$$\vdash X(x) \wedge Y(y) \wedge Z(=) \rightarrow (x = y \rightarrow \Box_{X,Y,Z} x = y). \tag{1}$$

Hence, by CL,

$$\vdash X(x) \wedge Y(y) \wedge Z(=) \rightarrow (\neg\Box_{X,Y,Z} x = y \rightarrow \neg(x = y)). \tag{2}$$

[25] This principle is closely related to a plausible principle about *mereological* compounds. For example, a mereological universalist might be inclined to accept the essentialist principle that for any two material objects x and y, it is essential to x and y (and perhaps the composition relation) taken together to compose some further object z if both x and y are concrete, from which it follows that z is involved in the collective nature of x and y. This is an analog for the composition of material objects of the principle described in the text, although the principle in the text does of course not commit us to any such principle for material objects, since it is not concerned with mereological composition.

[26] See Dorr [17, p. 57 f.] for further discussion.

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In what follows, we sometimes abbreviate $\Box_{X,Y,Z,[\neg],[=],[\Box_{[\sigma],[\sigma],[\langle\sigma,\sigma\rangle]}]}$ by $\Box_*$ and use $\Box$ to abbreviate $\Box_{[\sigma],[\sigma],[\langle\sigma,\sigma\rangle]}$ in subscript position. From (2), it follows by RC that

$$\vdash \Box_* X(x) \wedge \Box_* Y(y) \wedge \Box_* Z(=) \rightarrow (\Box_* \neg \Box_{X,Y,Z} x = y \rightarrow \Box_* \neg (x = y)). \quad (3)$$

From (3) by CL,

$$\vdash \Box_* X(x) \wedge \Box_* Y(y) \wedge \Box_* Z(=) \rightarrow (\neg \Box_* \neg (x = y) \rightarrow \neg \Box_* \neg \Box_{X,Y,Z} x = y). \quad (4)$$

By Monotonicity, the essentialist 5-schema, and Decomposition:

$$\vdash [x] \approx X \wedge [y] \approx Y \rightarrow (\neg \Box_{X,Y,Z,[\neg],[=],[\Box]} \neg \Box_{X,Y,Z} x = y \rightarrow \Box_{X,Y,Z} x = y). \quad (5)$$

By Rigidity and Monotonicity:

$$\vdash [x] \approx X \wedge [y] \approx Y \wedge [=] \approx Z \rightarrow (\Box_* X(x) \wedge \Box_* Y(y) \wedge \Box_* Z(=)). \quad (6)$$

So by (4)–(6), (T), and CL:

$$\vdash [x] \approx X \wedge [y] \approx Y \wedge [=] \approx Z \rightarrow (\neg \Box_{X,Y,Z,[\neg],[=],[\Box]} \neg (x = y) \rightarrow x = y)). \quad (7)$$

Monotonicity then yields:

$$\vdash [x] \approx X \wedge [y] \approx Y \wedge [=] \approx Z \rightarrow (\neg \Box_{[x],[y],[=],[\neg],[\Box]} \neg (x = y) \rightarrow x = y)). \quad (8)$$

By R-Comp and CL, we then obtain

$$\vdash \neg (x = y) \rightarrow \Box_{[x],[y],[=],[\neg],[\Box]} \neg (x = y). \quad (9)$$

(iii) Similar to the proof of part (ii), using Rigidity in place of part (i) in the first step. □

Parts (i) and (ii) are essentialist analogs of the necessity of identity and distinctness. A stronger form of (i) to the effect that if x and y are identical, then it lies in the nature of x (or y) that they are identical is not derivable in HLE, as can be shown by the semantic methods developed in §6. The strengthenings of parts (ii) and (iii) where the essentialist operator is deleted from the subject of the essentialist operator in the consequent do not seem to be provable in HLE, although they are valid in the models developed in later sections. These strengthenings do strike me as very plausible, at least if the essentialist 5-schema is adopted. However, I have preferred not to take them as axioms here, in part because it is instructive to see that the weaker principles are already theorems of HLE. We could in principle have adopted a similar attitude with regard to the essentialist 4-schema, a weaker version of which is also derivable from the other principles of the system.[27] But it seems preferable to adopt (4) as an axiom, both because there is a direct and compelling justification for it and because it is more evident than the 5-schema which is needed for the derivation of the weaker version.

Our next result states that every essentialist statement is equivalent to an essentialist statement whose subject is a collection of propositions. To state the result, we introduce the following abbreviation. Let $F_1, \ldots, F_n$ be one-place predicates. Then,

$$\mathfrak{h}(F_1, \ldots, F_n) =_{\mathrm{df}} \lambda p^{\langle\rangle}.(\exists x(F_1(x) \wedge \exists X^{[\sigma_1]}(\forall z(X(z) \leftrightarrow x =_{\sigma_1} z) \wedge p = X(x))) \vee \cdots \vee (\exists x(F_n(x) \wedge \exists X^{[\sigma_n]}(\forall z(X(z) \leftrightarrow x =_{\sigma_n} z) \wedge p = X(x))).$$

[27] The derivable weaker version of (4) is $\Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{F_1,\ldots,F_n,[\Box],[\neg]}\Box_{F_1,\ldots,F_n}\phi$, where $F_1, \ldots, F_n$ are of rigid type.

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The predicate $\mathfrak{h}(F_1, \ldots, F_n)$ picks out the *haecceity propositions* of the entities picked out by $F_1, \ldots, F_n$.

PROPOSITION 7 (Haec). $\vdash \Box_{F_1,\ldots,F_n}\phi \leftrightarrow \Box_{\mathfrak{h}(F_1,\ldots,F_n)}\phi$.

Proof. Left-to-right direction: it follows at once from Proposition 2(i), the definition of $\mathfrak{h}(F_1, \ldots, F_n)$ and Monotonicity that for every F_i, $\vdash \forall x(F_i(x) \rightarrow (\mathfrak{h}(F_1, \ldots, F_n) \geq_{\sigma_i} x))$. Thus by Monotonicity, $\vdash \Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{c_{\sigma_i}(\mathfrak{h}(F_1,\ldots,F_n)),\ldots,c_{\sigma_n}(\mathfrak{h}(F_1,\ldots,F_n))}\phi$. From this, by CH and Monotonicity, $\vdash \Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{\mathfrak{h}(F_1,\ldots,F_n)}\phi$. For the right-to-left direction, we have $\vdash \forall y(X^{[\sigma]}(y) \leftrightarrow (\lambda z.z =_\sigma x)(y)) \wedge (X(x) =_{\langle\rangle} p) \rightarrow \Box_{[x]}p$ by Rigidity, Monotonicity and LL. From this, we get $\vdash \forall p(\mathfrak{h}(F_1, \ldots, F_n)(p) \rightarrow c_{\langle\rangle}(F_1, \ldots, F_n)(p))$, whence by Monotonicity, CL and CH, $\vdash \Box_{\mathfrak{h}(F_1,\ldots,F_n)} \rightarrow \Box_{F_1,\ldots,F_n}\phi$. □

The following are immediate consequences of Proposition 7.

PROPOSITION 8.

(i) $\vdash \forall p(c_{\langle\rangle}(F_1, \ldots, F_n)(p) \leftrightarrow c_{\langle\rangle}(G_1, \ldots, G_m)(p)) \rightarrow (\Box_{F_1,\ldots,F_n}\phi \leftrightarrow \Box_{G_1,\ldots,G_m}\phi)$.
(ii) $\vdash \forall p(c_{\langle\rangle}(F_1, \ldots, F_n)(p) \leftrightarrow c_{\langle\rangle}(G_1, \ldots, G_m)(p)) \rightarrow \forall x(c_\sigma(F_1, \ldots, F_n)(x) \leftrightarrow c_\sigma(G_1, \ldots, G_m)(x))$.

Proof. (i) By Proposition 7, CH and Monotonicity, $\vdash \Box_{F_1,\ldots,F_n}\phi \leftrightarrow \Box_{c_{\langle\rangle}(F_1,\ldots,F_n)}\phi$. Thus, by CH and Monotonicity, $\vdash \forall p(c_{\langle\rangle}(F_1, \ldots, F_n)(p) \leftrightarrow c_{\langle\rangle}(G_1, \ldots, G_m)(p)) \rightarrow (\Box_{F_1,\ldots,F_n}\phi \leftrightarrow \Box_{G_1,\ldots,G_m}\phi)$.
(ii) From (i). □

Proposition 8(i) says that for the essences of two collections of entities to be the same, it suffices that their collective natures involve the same propositions. Proposition 8(ii) says that if the natures of two collections involve the same *propositions*, then for any type σ, they involve the same entities of that type; or more briefly, for the natures of two collections to involve exactly the same entities (of any type), it is sufficient that they involve the same propositions. It is worth noting that the converse of Proposition 8(ii) is not provable for every type σ. For example, the nature of the collection of all objects and the nature of the collection of all propositions each involve all objects but may fail to involve the same propositions, because there may be no objects whose nature involves negation, for instance. (The consistency of this can be easily demonstrated using the semantics developed in §6.) This counterexample to the converse of Proposition 8(ii) is specific to the case where σ is e, however. It is less clear whether there are counterexamples to the instances of the converse of Proposition 8(ii) in which σ is not e or $\langle\rangle$. These instances do not seem to be provable either, although they are valid on the semantics developed in §6.

§5. The logic of necessity—Part I. This section shows how we can develop the logic of metaphysical necessity within HLE. Fine [20] proposed that metaphysical necessity is a special case of essence. His proposal is that for a proposition to be metaphysically necessary is for it to be true in virtue of the nature of all objects. We can use the operator $\Box_{\lambda x^e.x=_e x}$ to formally express truth in virtue of the nature of all objects. In the system E5 of Fine's LE, the propositional logic of this operator (more precisely, its analog in the language of LE) is exactly S4.[28] Fine also shows how to extend the

[28] See Ditter [14] for a proof.

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system E5 in such a way that the logic of $\Box_{\lambda x^e.x=_e x}$ is S5.[29] By contrast, in HLE the operator does not even obey S4 without adding further principles. The reason is that HLE does not require the collective nature of all objects to involve any of the logical operations, or any other entities expressed by constants of type $\neq e$. This implies that in HLE, $\Box_{\lambda x^e.x=_e x}$ is not even closed under unrestricted consequence. However, there is another way of reducing metaphysical necessity to essence that suggests itself in our higher-order setting. Instead of taking metaphysical necessity to be truth in virtue of the nature of all objects, we may identify it with truth in virtue of the nature of all propositions. We can use the operator $\Box_{\lambda p^{\langle\rangle}.p=_{\langle\rangle} p}$ to formally express truth in virtue of the nature of all propositions. It turns out that this operator *does* obey exactly the principles of S4 already in HLE. This section first demonstrates this fact and then goes on to show how the system can be extended so that $\Box_{\lambda p^{\langle\rangle}.p=_{\langle\rangle} p}$ obeys exactly S5. After that, we will consider ways of extending HLE so as to make $\Box_{\lambda x^e.x=_e x}$ obey S4 and S5, respectively. A detailed discussion of the philosophical ramifications of the results in this section can be found in Ditter [15].

In what follows, we will use the following abbreviations: $\Omega := \lambda x^e.x =_e x$; $\Pi := \lambda p^{\langle\rangle}.p =_{\langle\rangle} p$. We start by observing that the following essentialist analog of the K-principle from modal logic is an immediate consequence of RC.

PROPOSITION 9. $(\mathrm{K}) \vdash \Box_{F_1,\ldots,F_n}(\phi \to \psi) \to (\Box_{F_1,\ldots,F_n}\phi \to \Box_{F_1,\ldots,F_n}\psi)$.

Proof. Immediate from RC. □

Next we show that the propositional logic of $\Box_\Pi$ is at least S4 in HLE. We first note that if $\Box_\Pi$ is taken to express metaphysical necessity, then the principle that essence implies necessity is a theorem of HLE.

PROPOSITION 10 $(\mathrm{EN}^{\Box_\Pi})$. $\vdash \Box_{F_1,\ldots,F_n}\phi \to \Box_\Pi\phi$.

Proof. Immediate from Proposition 7 and Monotonicity. □

PROPOSITION 11 $(\mathrm{RN}^{\Box_\Pi})$. *If* $\vdash \phi$ *then* $\vdash \Box_\Pi\phi$.

Proof. From RC^0 and Proposition 10. □

PROPOSITION 12 $(4^{\Box_\Pi})$. $\vdash \Box_\Pi\phi \to \Box_\Pi\Box_\Pi\phi$.

Proof. Let $X^{[\langle\rangle]}$ be a variable of rigid type that does not occur in ϕ. By Monotonicity, the essentialist 4-schema, (T), and CL, $(1) \vdash X \approx \Pi \to (\Box_\Pi\phi \leftrightarrow \Box_{X,[\Box_{[\langle\rangle]}]}\Box_X\phi)$. From this, $(2) \vdash \Box_\Pi\phi \wedge X \approx \Pi \to \Box_{X,[\Box_{[\langle\rangle],\langle\rangle}]}\Box_X\phi$ follows by CL. By $\mathrm{EN}^{\Box_\Pi}$, $(3) \vdash \Box_\Pi\phi \wedge X \approx \Pi \to \Box_\Pi\Box_X\phi$. By CL and $\mathrm{RN}^{\Box_\Pi}$ we have $(4) \vdash \Box_\Pi(X \subseteq \Pi)$ and by Monotonicity and $\mathrm{RN}^{\Box_\Pi}$, $(5) \vdash \Box_\Pi(X \subseteq \Pi \to (\Box_X\phi \to \Box_\Pi\phi))$. From (4), (5) and (K) we get $(6) \vdash \Box_\Pi\Box_X\phi \to \Box_\Pi\Box_\Pi\phi$. So by (3), (6) and CL, $(7) \vdash X \approx \Pi \to (\Box_\Pi\phi \to \Box_\Pi\Box_\Pi\phi)$. But X is not free in ϕ, so $(8) \vdash \exists X(X \approx \Pi) \to (\Box_\Pi\phi \to \Box_\Pi\Box_\Pi\phi)$. But $\exists X(X \approx \Pi)$ is an instance of R-Comp, therefore $(9) \vdash \Box_\Pi\phi \to \Box_\Pi\Box_\Pi\phi$. □

COROLLARY 1. *In* HLE, *the logic of* $\Box_\Pi$ *is at least* S4.

[29] The extension is potentially problematic from a philosophical point of view, though. See Ditter [14] for discussion.

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The principles of S5 become provable once we add an axiom to the effect that it is true in virtue of the nature of all propositions that they are all the propositions:

$(\mathrm{DOM}^{\langle\rangle})\ \ \forall pX^{[\langle\rangle]}(p) \rightarrow \Box_X \forall pX(p).$

The same principle restricted to objects (entities of type e) is invoked by Fine [21] in LE to show that the operator that expresses truth in virtue of the nature of all objects satisfies all the principles of S5 in his system $\mathsf{E5}^+$ (see Theorem 4 in Fine [21]). We write $\vdash_+$ to indicate theoremhood in the system $\mathsf{HLE} + \mathsf{DOM}^{\langle\rangle}$.

Proposition 13 ($\mathrm{B}^{\Box_\Pi}$). $\vdash_+ \neg\phi \rightarrow \Box_\Pi \neg \Box_\Pi \phi$.

Proof. Let $X^{[\langle\rangle]}$ be a variable of rigid type that does not occur in ϕ. By $\mathrm{DOM}^{\langle\rangle}$ we have (1) $\vdash_+ \forall pX(p) \rightarrow \Box_X \forall pX(p)$. By the essentialist 5-schema, (2) $\vdash_+ \neg\Box_X\phi \rightarrow \Box_{X,[\neg],[\Box_{[\langle\rangle]}],[\phi]} \neg \Box_X \phi$. From (2), Monotonicity and $\mathrm{EN}^{\Box_\Pi}$, (3) $\vdash_+ \neg\Box_\Pi\phi \wedge \forall pX(p) \rightarrow \Box_\Pi \neg \Box_X \phi$. By Monotonicity and $\mathrm{RN}^{\Box_\Pi}$, (4) $\vdash_+ \Box_\Pi(\forall pX(p) \rightarrow (\neg\Box_\Pi\phi \leftrightarrow \neg\Box_X\phi))$. From (4), (1) and (K), we can infer (5) $\vdash_+ \forall pX(p) \rightarrow (\Box_\Pi\neg\Box_\Pi\phi \leftrightarrow \Box_\Pi\neg\Box_X\phi)$. From (3) and (5), we obtain (6) $\vdash_+ \neg\Box_\Pi\phi \wedge \forall pX(p) \rightarrow \Box_\Pi\neg\Box_\Pi\phi$. So by CL, (7) $\vdash_+ \exists X \forall pX(p) \rightarrow (\neg\Box_\Pi\phi \rightarrow \Box_\Pi\neg\Box_\Pi\phi)$, because X is not free in ϕ. But the antecedent of (7) is equivalent to an instance of R-Comp, which gives us (8) $\vdash_+ \neg\Box_\Pi\phi \rightarrow \Box_\Pi\neg\Box_\Pi\phi$. By the contrapositive of (T), (8) implies $\vdash_+ \neg\phi \rightarrow \Box_\Pi\neg\Box_\Pi\phi$. □

Corollary 2. *In* $\mathsf{HLE} + \mathsf{DOM}^{\langle\rangle}$, *the logic of* $\Box_\Pi$ *is at least* S5.

In §9.1 we will show by semantic techniques that the propositional logic of $\Box_\Pi$ is exactly S4 (S5) in HLE ($\mathsf{HLE} + \mathsf{DOM}^{\langle\rangle}$).

We now turn to the logic of $\Box_\Omega$. As mentioned above, $\Box_\Omega$ does not obey S4 in HLE, since it is not even closed under unrestricted logical consequence. One way to ensure that $\Box_\Omega$ obeys S4 is to add a principle to HLE to the effect that every entity of type $\neq e$ is involved in the nature of some objects (entities of type e). There are several ways to motivate such a principle, though perhaps the most obvious option is to endorse some form of Platonism, positing for every entity of type $\neq e$ some objectual surrogate or *Form* of type e that stands in a specific relation to the higher-order entity. Here I will focus on a fairly minimalist, "Propositional Platonist" theory that posits objectual surrogates for propositions:

$(\text{P-Truth})\ \ \forall p^{\langle\rangle} \exists x^e \Box_{[x]}(\mathit{true}(x) =_{\langle\rangle} p).$

It turns out that this suffices for the purpose at hand. Supplementing HLE with P-Truth allows us to infer $\mathrm{EN}^{\Box_\Omega}$. Once we have $\mathrm{EN}^{\Box_\Omega}$, the analogs for $\Box_\Omega$ of the preceeding results are proved exactly like those for $\Box_\Pi$, except that we use $\mathrm{EN}^{\Box_\Omega}$ and $\mathrm{RN}^{\Box_\Omega}$ instead of $\mathrm{EN}^{\Box_\Pi}$ and $\mathrm{RN}^{\Box_\Pi}$ in the relevant places. We write $\vdash_*$ for provability in HLE+P-Truth.

Proposition 14 ($\mathrm{EN}^{\Box_\Omega}$). $\vdash_* \Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_\Omega\phi$.

Proof. By P-Truth, (1) $\vdash_* \exists y^e \Box_{[y]}(\mathit{true}(y) =_{\langle\rangle} (x =_\tau x))$, whence by β-conversion, (2) $\vdash_* \exists y^e (y \geq x \wedge y \geq =_\tau)$. By RC^0, (3) $\vdash_* \Box_{[x],[=_\tau]} x =_\tau x$, and so by (2), Proposition 3(i), and Monotonicity, (4) $\vdash_* \exists y^e \Box_{[y]} x =_\tau x$. Thus, (5) $\vdash_* \exists X^{\langle e\rangle} \Box_X x =_\tau x$ and (6) $\vdash_* \forall x^\tau \exists X^{\langle e\rangle} \Box_X x =_\tau x$ by CL. From (6) and CL, (7) $\vdash_* \forall x^{\sigma_i}(F(x) \rightarrow \exists X^{\langle e\rangle} \Box_X x =_{\sigma_i} x)$. From (7) we get (8) $\vdash_* \forall x^{\sigma_i}(F(x) \rightarrow \Box_\Omega x =_{\sigma_i} x)$ by Monotonicity and CL. Hence, (9) $\vdash_* \forall x^{\sigma_i}(F(x) \rightarrow c_{\sigma_i}(\Omega)(x))$, whence (10) $\vdash_* \Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_{c_{\sigma_1}(\Omega),\ldots c_{\sigma_n}(\Omega)}\phi$ follows

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by Monotonicity. From (10), CH, and Monotonicity, we finally obtain $\vdash_* \Box_{F_1,\ldots,F_n}\phi \rightarrow \Box_\Omega\phi$. □

PROPOSITION 15 ($\mathrm{RN}^{\Box_\Omega}$). *If* $\vdash_* \phi$ *then* $\vdash_* \Box_\Omega\phi$.

PROPOSITION 16 ($4^{\Box_\Omega}$). $\vdash_* \Box_\Omega\phi \rightarrow \Box_\Omega\Box_\Omega\phi$.

COROLLARY 3. *In* HLE+P-Truth, *the logic of* $\Box_\Omega$ *is at least* S4.

As in the case of $\Box_\Pi$, we obtain S5 for $\Box_\Omega$ if we add an axiom to the effect that it is true in virtue of the nature of all objects that they are all the objects:

(DOM) $\forall x X^{[e]}(x) \rightarrow \Box_X \forall x X(x)$.

We write $\vdash_{*+}$ for provability in HLE+P-Truth+DOM.

PROPOSITION 17 ($\mathrm{B}^{\Box_\Omega}$). $\vdash_{*+} \neg\phi \rightarrow \Box_\Omega\neg\Box_\Omega\phi$.

COROLLARY 4. *In* HLE+P-Truth+DOM, *the logic of* $\Box_\Omega$ *is at least* S5.

We will show in §9.2 that the propositional logic of $\Box_\Omega$ is exactly S4 (S5) in HLE+P-Truth (HLE+P-Truth+DOM).

§6. Semantics. We now proceed to the development of a sound model-theoretic semantics for HLE. The basic structure of the model theory bears some similarity to the structure of the model theory of intensional type theory with varying domains developed in Fine [19] and Fritz and Goodman [24]. We start with a *frame*, an ordered pair $\langle W, D\rangle$ consisting of a non-empty set W and a domain function D taking each member w of W and τ of the set of types $\mathcal{T}$ into some non-empty set D_τ^w. The set W can informally be thought of as the set of *worlds*, and the sets D_τ^w as the domains of entities of type τ in w. A characteristic feature of the model theory developed here concerns the structure of the elements of the domains D_τ^w. The frames we are interested in are ones in which every element of a domain is an ordered pair whose first component can be informally thought of as the "worldly" content and whose second component as the "intensional" content of the entity. In the case of entities of type $\langle\rangle$ the second component is just a subset of W, whereas for entities of type $\langle\tau\rangle$ it is a function from entities of type τ to subsets of W; there are no constraints on what the second component is for entities of type e. The worldly content of each ordered pair in the domain is provided by a complete lattice, allowing the worldly content of each entity of any type to freely combine with the worldly content of any entity of arbitrary type. For example, the worldly content of the denotation of the sentence 'Socrates is human' will consist of the (lattice-theoretic) join of the worldly contents of the denotations of 'Socrates' and 'human'.

An essentialist statement $\Box_{F_1,\ldots,F_n}\phi$ is true at a world w just in case ϕ is true at every world whose domain contains the union of the extensions of $F_1, \ldots, F_n$ at w. For example, the sentence $\Box_{[s],[\wedge]}\forall x(x =_e x)$ ('It is essential to Socrates and conjunction that everything is self-identical') is true at w just in case $\forall x(x =_e x)$ is true at every world whose domain contains the denotations of s and $\wedge$. Crucially, the models allow that the denotation of a constant of any type can in principle fail to exist at any world. Thus, for example, the denotations of s and $\wedge$ need not exist in the same worlds. The denotation of a syntactically complex term A is in the domain of a world just in case all of the denotations of its constants and free variables (relative to an assignment)

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are in the domain of that world. Thus, the denotation of $\forall x(x =_e x)$ exists in exactly those worlds in which both the denotations of $\forall_e$ and $=_e$ exist, though it fails to exist in a world in which only one of them exists.

An important consequence of the semantics is that a collection of entities $\mathbf{x}_1, \mathbf{x}_2, \ldots$ *essentially involves* an entity $\mathbf{y}$ if and only if $\mathbf{y}$ exists in every world in which all of $\mathbf{x}_1, \mathbf{x}_2, \ldots$ exist. Thus the relation of essential involvement expressed by the defined predicates $\succeq_{\sigma_1,\ldots,\sigma_n,\tau}$ is modeled by a relation between the existence sets of entities: $(F_1, \ldots, F_n) \succeq_{\sigma_1,\ldots,\sigma_n,\tau} x$ is true at w just in case the denotation of x exists in every world whose domain contains the union of the extensions of $F_1, \ldots, F_n$ at w. So, for example, $([s]) \succeq \neg$ (or, equivalently, $s \geq \neg$) is true at w just in case the denotation of $\neg$ exists in every world in which the denotation of s exists.

Another distinctive feature of the semantics is that formulas are only evaluated at those worlds at which their denotations exist. This is compatible with classical logic given the notion of validity adopted here, according to which a formula ϕ is valid if and only if for every model, ϕ is true at every world at which its denotation exists.[30] The present semantics is similar in this respect to the first-order semantics for LE in Fine [22], in which a formula is evaluated only at those worlds at which the *objectual content* of the formula exists, where the objectual content of a formula consists, intuitively, of the *objects* that the formula is about. Our notion of worldly content is a generalization of the notion of objectual content within a higher-order setting. One consequence of Fine's semantics is that a sentence like $\forall x(\mathrm{set}(x) \vee \neg\mathrm{set}(x))$, which has no objectual content on Fine's semantics, is true at every world in every model, which, given Fine's interpretation of the essentialist operator, entails that $\forall x(\mathrm{set}(x) \vee \neg\mathrm{set}(x))$ is true in virtue of the nature of any particular object. The present semantics, by contrast, allows us to avoid this, because the denotation of $\forall x(\mathrm{set}(x) \vee \neg\mathrm{set}(x))$ may fail to be in the domain of some world, since all of the denotations of $\forall_e$, set, $\vee$ and $\neg$ may fail to be in the domain of some world, as explained above. I will now turn to the details.

DEFINITION 4. *When $\mathcal{C}$ and $\mathcal{D}$ are typed families, we say that f is a* typed function *from $\bigcup_{\tau\in\mathcal{T}} \mathcal{C}_\tau$ to $\bigcup_{\tau\in\mathcal{T}} \mathcal{D}_\tau$ just in case for all $\tau \in \mathcal{T}$, if $x \in \mathcal{C}_\tau$, then $f(x) \in \mathcal{D}_\tau$.*

DEFINITION 5. *A* frame *$\mathfrak{F}$ is a pair $\langle W, D\rangle$, where W is a non-empty set and D is a function taking each $w \in W$ and $\tau \in \mathcal{T}$ into some non-empty set, written D^w_τ, such that $\{\bigcup_{w\in W} D^w_\tau : \tau \in \mathcal{T}\}$ is a typed family. We say that D^w_τ is the domain of entities of type τ at* w. *We further define $D^w := \bigcup_{\tau\in\mathcal{T}} D^w_\tau$, the* domain *of* w, *and $D_\tau := \bigcup_{w\in W} D^w_\tau$. We call $\bigcup D := \bigcup_{\tau\in\mathcal{T}}^{w\in W} D^w_\tau$ the* outer domain *of the frame $\mathfrak{F}$.*

DEFINITION 6. *Let $\langle W, D\rangle$ be a frame. If $G \subseteq \bigcup D$, the* existence set *$E(G)$ of G is $\{w : \forall a(a \in G \to a \in D^w)\}$ $(= \bigcap_{a\in G} E(\{a\}))$. We say that a subset $G \subseteq \bigcup D$* depends *on $H \subseteq \bigcup D$ just in case $E(G) \subseteq E(H)$. In the case where H is a singleton set $\{a\}$, $a \in \bigcup D$, we simply say that G depends on* a.

DEFINITION 7. *A partially ordered set $\langle N, \sqsubseteq\rangle$ is a* complete lattice *iff every subset of N has both a least upper bound (called the* join*) and a greatest lower bound (called the* meet*) relative to $\sqsubseteq$. When $G \subseteq N$, its join is denoted by $\bigsqcup G$ and its meet by $\bigsqcap G$. The least element of N is denoted by $\bot$ $(= \bigsqcup \emptyset)$ and the greatest element by $\top$ $(= \bigsqcap \emptyset)$. When G is a two-element set consisting of elements m, n, we sometimes write $m \sqcup n$ instead of $\bigsqcup\{m, n\}$, and $m \sqcap n$ instead of $\bigsqcap\{m, n\}$.*

[30] See Definition 12 for a more precise statement of the definition of validity.

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When $\mathcal{N} = \langle N, \sqsubseteq \rangle$ is a complete lattice, we usually indicate membership in the lattice by simply writing $x \in \mathcal{N}$ instead of $x \in N$. A canonical example of a complete lattice is the power set of a given set, ordered by inclusion, with the join and meet given by union and intersection of subsets, respectively. We will make use of such power set lattices in the construction of our example models in §7. We now define the "worldly content structures" that provide the basis of the worldly contents of the elements of the frames we are primarily interested in.

DEFINITION 8. *A* worldly content structure *is a quintuple* $\langle \mathcal{N}, W, I, \iota, \overline{E} \rangle$, *where*:

(i) $\mathcal{N} = \langle N, \sqsubseteq \rangle$ *is a complete lattice*;
(ii) W *is a non-empty set*;
(iii) I *is a function taking each* $w \in W$ *into a non-empty set* I^w;
(iv) ι *is a function from* $\bigcup_{w \in W} I^w$ *to* $\mathcal{N}$;
(v) $\overline{E}$ *is a function from* $\mathcal{N}$ *to* $\wp(W)$ *such that* $\overline{E}(\bigsqcup G) = \bigcap_{n \in G} \overline{E}(n)$, $\overline{E}(\bot) = W$, *and for every* $o \in \bigcup_{w \in W} I^w$: $\overline{E}(\iota(o)) = \{w \in W : o \in I^w\}$.

Elements of the set $\bigcup_{w \in W} I^w$ will serve as the second components of the members of the set of individuals in the frames defined below. The function $\overline{E}$ assigns every element of the lattice to a set of worlds which can be thought of as the "existence set" of the element. By the last part of clause (v), we have for every $o \in \bigcup_{w \in W} I^w : o \in I^w$ if and only if $w \in \overline{E}(\iota(o))$. In the frames defined below, this means that the existence set of an individual $\langle \iota(o), o \rangle$ is just $\{w \in W : o \in I^w\}$. If a lattice element n is below a lattice element m in the lattice ordering, then the existence set of n is a superset of the existence set of m. We record this useful fact in the following lemma.

LEMMA 1. *Let* $\langle \mathcal{N}, W, I, \iota, \overline{E} \rangle$ *be a worldly content structure. Then for all* $n, m \in \mathcal{N}$: *if* $n \sqsubseteq m$, *then* $\overline{E}(m) \subseteq \overline{E}(n)$.

Proof. Let $n, m \in \mathcal{N}$ and suppose $n \sqsubseteq m$. Since $n \sqsubseteq m$ iff $m = n \sqcup m$, we have $\overline{E}(m) = \overline{E}(\bigsqcup\{n, m\}) = \overline{E}(n) \cap \overline{E}(m)$ and thus $\overline{E}(m) \subseteq \overline{E}(n)$. The last equality holds by Definition 8. □

Here are two simple examples of worldly content structures. We will use variants of these structures in our model constructions in §7.

EXAMPLE 1. Let $\langle \mathcal{N}, W, I, \iota, \overline{E} \rangle$ be the worldly content structure defined as follows: $W = \{w_1, w_2\}$, $\mathcal{N} = \langle \wp(W), \supseteq \rangle$, with join given by intersection of subsets and meet by union, $I : w \mapsto \mathbb{N}$, $\iota : n \mapsto W$, and $\overline{E}$ is the identity function on $\wp(W)$. (Note that $\langle \wp(W), \supseteq \rangle$ is just the dual lattice of $\langle \wp(W), \subseteq \rangle$.) In this worldly content structure, the existence set of a lattice element is just the lattice element itself.

EXAMPLE 2. Let Σ be a signature. We define a worldly content structure $\mathfrak{N} = \langle \mathcal{N}, W, I, \iota, \overline{E} \rangle$ by letting $\mathcal{N} = \langle \wp(\Sigma), \subseteq \rangle$, with join given by union and meet given by intersection of subsets, $W = \wp(\Sigma) \setminus \{\emptyset\}$, $I : w \mapsto \{c \in \Sigma : c \in w\}$, $\iota : c \mapsto \{c\}$, and $\overline{E} : n \mapsto \{w \in W : n \subseteq w\}$. In this worldly content structure, both the worlds and the lattice elements are sets of constants, and a lattice element exists at a world if and only if it is a subset of that world.

As mentioned above, the frames we are primarily interested in are ones in which the elements of domains of all types are ordered pairs. Where x is an ordered pair, we denote its first component by $\pi_0(x)$ and its second component by $\pi_1(x)$. In the frames

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we define below, every element of the outer domain is an ordered pair whose first component is an element of a complete lattice drawn from a worldly content structure underlying the frame. We will later show that the existence set of an element of the outer domain of a frame is just the value of $\overline{E}$ applied to its first component.

DEFINITION 9. *Let* $\langle \mathcal{N}, W, I, \imath, \overline{E} \rangle$ *be a worldly content structure. The* dual-content frame *induced by* $\langle \mathcal{N}, W, I, \imath, \overline{E} \rangle$ *is the frame* $\langle W, D \rangle$ *such that for all* $w \in W$:

(i) $D_e^w = \{\langle \imath(o), o \rangle : o \in I^w\}$.
(ii) $D_{\langle\rangle}^w = \{\langle n, l \rangle : n \in \mathcal{N}, w \in \overline{E}(n)$ *and* $l \subseteq \overline{E}(n)\}$.
(iii) $D_{\langle \tau_1, \ldots, \tau_k \rangle}^w = \{\langle n, f \rangle : n \in \mathcal{N}, w \in \overline{E}(n)$ *and* $f : D_{\tau_1} \times \cdots \times D_{\tau_k} \to \wp(W)$ *such that* $f(x_1, \ldots, x_k) \subseteq \overline{E}(\bigsqcup\{\pi_0(x_1), \ldots, \pi_0(x_k), n\})\}$.
(iv) $D_{[\tau]}^w = \{\langle n, r \rangle : n \in \mathcal{N}, w \in \overline{E}(n)$ *and* $r \subseteq D_\tau$ *such that* $n = \bigsqcup\{\pi_0(x) : x \in r\}\}$.

A frame $\langle W, D \rangle$ *is a* dual-content frame *if it is induced by a worldly content structure. We define the* extension of $\mathbf{X} \in D_\sigma$ at $w \in E(\{\mathbf{X}\})$ *to be* $Ext^w(\mathbf{X}) = \{\mathbf{x} \in D_\tau^w : w \in \pi_1(\mathbf{X})(\mathbf{x})\}$, *if* $\sigma = \langle\tau\rangle$, *and* $Ext^w(\mathbf{X}) = \pi_1(\mathbf{X})$, *if* $\sigma = [\tau]$. $Ext^w(\mathbf{X})$ *is only defined if* $\mathbf{X} \in D_\sigma$ *for* σ *a one-place predicate type and* $w \in E(\{\mathbf{X}\})$.

Conditions (iii) and (v) of Definition 8 jointly ensure that for any world w in a frame, the domain of individuals D_e^w is non-empty. For entities of type $\neq e$ this is guaranteed by the condition that $\overline{E}(\bot) = W$. As the next lemma shows, the existence set $E(\{\mathbf{x}\})$ of an element of the outer domain of a frame is identical to the existence set $\overline{E}(\pi_0(\mathbf{x}))$ of its first component; more generally, we have $E(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} = \overline{E}(\bigsqcup\{\pi_0(\mathbf{x}_1), \ldots, \pi_0(\mathbf{x}_n)\})$. Thus the existence set of an element of the outer domain of a frame is uniquely determined by the worldly content structure underlying the frame.

An important feature of clause (ii) is that the intensional content $\pi_1(\mathbf{x})$ of an element of the domain of propositions $D_{\langle\rangle}$ is a subset of its existence set $E(\{\mathbf{x}\})$. We can think of the intensional content of a proposition as the set of worlds at which the proposition is true. The condition that the intensional content of a proposition should be a subset of its existence set therefore entails that a proposition can only be true at those worlds at which it exists. Similarly, we can think of the intensional content $\pi_1(\mathbf{X})$ of a non-rigid property or relation (i.e., an element of some $D_{\langle \tau_1, \ldots, \tau_k \rangle}$) specified in clause (iii) as the function that takes entities of appropriate type to the set of worlds at which the entities have the property or relation in question. For example, if $\mathbf{X} \in D_{\langle\tau\rangle}$ and $\mathbf{x} \in D_\tau$, then $\pi_1(\mathbf{X})(\mathbf{x})$ represents the set of worlds at which $\mathbf{x}$ has property $\mathbf{X}$, or in other words, the intensional content of the proposition that $\mathbf{x}$ has $\mathbf{X}$. Clause (iii) guarantees that the set of worlds $\pi_1(\mathbf{X})(\mathbf{x})$ is a subset of the set $E(\{\mathbf{X}, \mathbf{x}\})$, the set of worlds at which both $\mathbf{X}$ and $\mathbf{x}$ exist. Thus an entity can only instantiate a property at those worlds at which both the entity and the property exist.

Unlike the intensional content $\pi_1(\mathbf{X})$ of a non-rigid property, the intensional content of a rigid property (i.e., an element of some $D_{[\tau]}$) is simply its extension, i.e., a subset of the domain of entities of appropriate type. For example, if $\mathbf{X} \in D_{[\tau]}$, then $\pi_1(\mathbf{X})$ is a subset of D_τ, and an entity $\mathbf{x} \in D_\tau$ has $\mathbf{X}$ if and only if $\mathbf{x} \in \pi_1(\mathbf{X})$. One difference between rigid and non-rigid properties is that the extension of a rigid property always remains fixed across worlds (see Lemma 4 (iv)), whereas the extension of a non-rigid property may vary across worlds. Another difference is that the worldly content $\pi_0(\mathbf{X})$ of a rigid property—but not that of a non-rigid property—is determined by its

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intensional content $\pi_1(\mathbf{X})$, and thus by the entities in its extension. In fact, the worldly content $\pi_0(\mathbf{X})$ of a rigid property $\mathbf{X}$ is just the lattice join of the set of worldly contents of the entities in its extension. So a rigid property exists in exactly those worlds at which the entities in its extension exist (see Lemma 4 (i)).

LEMMA 2. *Let* $\langle W, D\rangle$ *be a dual-content frame. Then for all* $\mathbf{x} \in \bigcup D$: $\overline{E}(\pi_0(\mathbf{x})) = E(\{\mathbf{x}\})$.

Proof. Immediate from Definition 9. □

LEMMA 3. *Let* $\langle W, D\rangle$ *be a dual-content frame. Then for all* $\mathbf{x}, \mathbf{y} \in \bigcup D$: *if* $\pi_0(\mathbf{x}) \sqsubseteq \pi_0(\mathbf{y})$ *then* $E(\{\mathbf{y}\}) \subseteq E(\{\mathbf{x}\})$.

Proof. From Lemmas 1 and 2. □

Next we need to check that the rigid properties behave in the intended way.

LEMMA 4. *Let* $\langle W, D\rangle$ *be a dual-content frame. Then the following conditions hold*:

1. *for every* $\mathbf{X} \in D_{[\sigma]}$, $E(\{\mathbf{X}\}) = E(Ext^w(\mathbf{X}))$ *for all* $w \in E(\{\mathbf{X}\})$;
2. *for every* $\mathbf{X} \in D^w_{\langle\sigma\rangle}$ *there is a* $\mathbf{Y} \in D^w_{[\sigma]}$ *such that* $Ext^w(\mathbf{X}) = Ext^w(\mathbf{Y})$;
3. *for any* $\mathbf{X}, \mathbf{Y} \in D_{[\sigma]}$ *if* $Ext^w(\mathbf{X}) = Ext^w(\mathbf{Y})$ *for some* $w \in W$, *then* $\mathbf{X} = \mathbf{Y}$;
4. *for every* $\mathbf{X} \in D_{[\sigma]}$, $Ext^w(\mathbf{X}) = Ext^v(\mathbf{X})$ *for all* $w, v \in W$.

Proof. (i) Let $\mathbf{X} \in D_{[\sigma]}$. By Lemma 2 and Definition 9: $E(\{\mathbf{X}\}) = \overline{E}(\pi_0(\mathbf{X})) = \overline{E}(\bigsqcup\{\pi_0(\mathbf{Y}) : \mathbf{Y} \in \pi_1(\mathbf{X})\}) = \bigcap_{\mathbf{Y}\in\pi_1(\mathbf{X})} \overline{E}(\pi_0(\mathbf{Y})) = \bigcap_{\mathbf{Y}\in\pi_1(\mathbf{X})} E(\{\mathbf{Y}\}) = E(\pi_1(\mathbf{X})) = E(Ext^w(\mathbf{X}))$, for all $w \in E(\{\mathbf{X}\})$. (ii) follows from the fact that for every $w \in W$ and $\tau \in \mathcal{T}$, $\{\pi_1(\mathbf{X}) : \mathbf{X} \in D^w_{[\tau]}\} = \wp(D^w_\tau)$. (iii) Let $\mathbf{X}, \mathbf{Y} \in D_{[\sigma]}$ and suppose $Ext^w(\mathbf{X}) = Ext^w(\mathbf{Y})$ for some $w \in W$. Then $\pi_1(\mathbf{X}) = \pi_1(\mathbf{Y})$ and so $\pi_0(\mathbf{X}) = \bigsqcup\{\pi_0(\mathbf{Z}) : \mathbf{Z} \in \pi_1(\mathbf{X})\} = \bigsqcup\{\pi_0(\mathbf{Z}) : \mathbf{Z} \in \pi_1(\mathbf{Y})\} = \pi_0(\mathbf{Y})$. Hence, $\mathbf{X} = \mathbf{Y}$. (iv) is an immediate consequence of Definition 9. □

We now specify how terms of our formal language are interpreted in a dual-content frame.

DEFINITION 10. *When* $\mathfrak{F} = \langle W, D\rangle$ *is a frame, an* assignment *for* $\mathfrak{F}$ *is a typed function from* $\mathcal{V}$ *to* $\bigcup D$. *Given an assignment* g, *pairwise distinct variables* $x_1, \ldots, x_k$ *and* $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \bigcup D$, *we use* $g[x_i \mapsto \mathbf{x}_i]$ *to denote the assignment with* $g[x_i \mapsto \mathbf{x}_i](x_i) = \mathbf{x}_i$ *for* $1 \leq i \leq k$, *and* $g[x_i \mapsto \mathbf{x}_i](y) = g(y)$ *for variables* y *other than* $x_1, \ldots, x_k$.

DEFINITION 11. *Let* Σ *be a signature and* $\mathfrak{N} = \langle \mathcal{N}, W, I, \iota, \overline{E}\rangle$ *be a worldly content structure. A* dual-content $\mathcal{L}^\Sigma$-model based on $\mathfrak{N}$ *is a triple* $\langle W, D, [\![\cdot]\!]\rangle$, *where* $\langle W, D\rangle$ *is the dual-content frame induced by* $\mathfrak{N}$, *and* $[\![\cdot]\!]$ *is a function which maps every assignment* g *for* $\langle W, D\rangle$ *to a typed partial function* $[\![\cdot]\!]^g$ *from* $\mathcal{L}^\Sigma$ *to* $\bigcup D$, *satisfying the following conditions*:

Compositional clauses:

1. $[\![x]\!]^g = g(x)$;
2. *if* $c \in \Sigma$, *then* $[\![c]\!]^g$ *is defined, and for all assignments* g, h: $[\![c]\!]^g = [\![c]\!]^h$;
3. $[\![A]\!]^g$ *is defined iff* $\overline{E}(\bigsqcup\{\pi_0([\![A_1]\!]^g), \ldots, \pi_0([\![A_n]\!]^g)\}) \neq \emptyset$, *where* $A_1, \ldots, A_n$ *are all the free variables and constants occurring in* A;
4. $\pi_0([\![A]\!]^g) = \bigsqcup\{\pi_0([\![A_1]\!]^g), \ldots, \pi_0([\![A_n]\!]^g)\}$, *where* $A_1, \ldots, A_n$ *are all the free variables and constants occurring in* A;

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5. *if* A *is of type* $\langle\sigma_1, \ldots, \sigma_n\rangle$, *then* $\pi_1([\![A(B_1, \ldots, B_n)]\!]^g) = \pi_1([\![A]\!]^g)([\![B_1]\!]^g, \ldots, [\![B_n]\!]^g)$;
6. *if* A *is of type* $[\sigma]$, *then* $\pi_1([\![A(B)]\!]^g) = \overline{E}(\pi_0([\![A]\!]^g)$ *if* $[\![B]\!]^g \in \pi_1([\![A]\!]^g)$, *and* $\pi_1([\![A(B)]\!]^g) = \emptyset$ *otherwise*;
7. $\pi_1([\![\lambda x_1 \ldots x_k.\phi]\!]^g)(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \pi_1([\![\phi]\!]^{g[x_i \mapsto \mathbf{x}_i]})$ *if* $[\![\phi]\!]^{g[x_i \mapsto \mathbf{x}_i]}$ *is defined*; *and* $\emptyset$ *otherwise.*

Clauses for logical constants:[31]

1. $\pi_1([\![\neg]\!])(\mathbf{p}) = \overline{E}(\bigsqcup\{\pi_0(\mathbf{p}), \pi_0([\![\neg]\!])\}) \setminus \pi_1(\mathbf{p})$;
2. $\pi_1([\![\wedge]\!])(\mathbf{p}, \mathbf{q}) = \overline{E}(\pi_0([\![\wedge]\!])) \cap \pi_1(\mathbf{p}) \cap \pi_1(\mathbf{q})$;[32]
3. $\pi_1([\![\exists_\sigma]\!])(\mathbf{X}) = \overline{E}(\pi_0([\![\exists_\sigma]\!])) \cap \{w \in W : w \in \pi_1(\mathbf{X})(\mathbf{y})$ *for some* $\mathbf{y} \in D_\sigma^w\}$;
4. $\pi_1([\![\Box_{\sigma_1,\ldots,\sigma_n}]\!])(\mathbf{X}_1, \ldots, \mathbf{X}_n, \mathbf{p}) = \overline{E}(\bigsqcup\{\pi_0(\mathbf{X}_1), \ldots, \pi_0(\mathbf{X}_n), \pi_0(\mathbf{p}), \pi_0([\![\Box_{\sigma_0,\ldots,\sigma_n}]\!])\}) \cap \{w \in W :$ *for all* $v \in W$, *if* $\bigcup_{i=1}^n Ext^w(\mathbf{X}_i) \subseteq D^v$ *then* $v \in \pi_1(\mathbf{p})\}$;
5. $\pi_1([\![=_\sigma]\!])(\mathbf{x}, \mathbf{y}) = \overline{E}(\bigsqcup\{\pi_0(\mathbf{x}), \pi_0(\mathbf{y}), \pi_0([\![=_\sigma]\!])\}$ *if* $\mathbf{x} = \mathbf{y}$; *and* $\emptyset$ *otherwise.*

The clauses for $\vee, \rightarrow, \leftrightarrow$ *and* $\forall_\sigma$ *with their classical interpretations are defined analogously.*

An easy induction on the complexity of terms shows that the function $[\![\cdot]\!]^g$ is well-defined, such that whenever defined, $[\![A]\!]^g$ is indeed in $\bigcup D$. To construct a dual content model based on a given worldly content model we only have to assign appropriate worldly and intensional contents to every non-logical constant of our signature; in the case of logical constants we only have to assign a worldly content, because the intensional contents of the logical constants are fixed by the clauses for the logical constants. Example models will be given in §7. Note that by compositional clause 3, the denotation of a term A is defined if and only if the denotations of its free variables and constants all exist together at some world. Formulas are only evaluated at worlds at which their denotations exist. A formula ϕ is said to be *true* at $w \in E(\{[\![\phi]\!]^g\})$ relative to a model $\mathfrak{M}$ and assignment g just in case $w \in \pi_1([\![\phi]\!]^g)$; otherwise it is *false* at w. We sometimes abbreviate this by $\mathfrak{M}, w, g \vDash \phi$ and $\mathfrak{M}, w, g \nvDash \phi$, respectively. A formula is neither true nor false at a world at which its denotation does not exist. In what follows, we generally omit the assignment and write $[\![A]\!]$ instead of $[\![A]\!]^g$ when A is a term without free variables. Using Definition 11, we can derive the familiar truth-clauses for the logical constants. For example, the clauses for negation, conjunction, existential quantification and the essentialist operator are as follows. Let $\mathfrak{M} = \langle W, D, [\![\cdot]\!]\rangle$ be a dual-content model. Then:

For $w \in E(\{[\![\neg\phi]\!]^g\} : \mathfrak{M}, w, g \vDash \neg\phi$ iff $\mathfrak{M}, w, g \nvDash \phi$.
For $w \in E(\{[\![\phi \wedge \psi]\!]^g\}) : \mathfrak{M}, w, g \vDash \phi \wedge \psi$ iff $\mathfrak{M}, w, g \vDash \phi$ and $\mathfrak{M}, w, g \vDash \psi$.
For $w \in E(\{[\![\exists_\tau(F)]\!]^g\}) : \mathfrak{M}, w, g \vDash \exists_\tau(F)$ iff for some $\mathbf{x} \in D_\tau^w : \mathbf{x} \in Ext^w([\![F]\!]^g)$.
For $w \in E(\{[\![\Box_{F_1,\ldots,F_n}\phi]\!]^g\}) : \mathfrak{M}, w, g \vDash \Box_{F_1,\ldots,F_n}\phi$ iff for all $v \in W$: if $\bigcup_{i=1}^n Ext^w([\![F_i]\!]^g) \subseteq D^v$, then $\mathfrak{M}, v, g \vDash \phi$.

In words: if $[\![\Box_{F_1,\ldots,F_n}\phi]\!]^g$ is in the domain of w, then $\Box_{F_1,\ldots,F_n}\phi$ is true at w (relative to g) just in case ϕ ist true at every world v (relative to g) whose domain contains the

[31] Because of condition (ii) we will generally omit the assignment and write $[\![c]\!]$ instead $[\![c]\!]^g$ when c is a constant.
[32] Recall that for all $\mathbf{p} \in D_{\langle\rangle}, \pi_1(\mathbf{p}) \subseteq \overline{E}(\pi_0(\mathbf{p}))$.

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union of the extensions of $F_1, \ldots, F_n$ at w. For example, $\Box_{\lambda p.p=_{\langle\rangle} p} H(s)$ ('It is true in virtue of the collective nature of all propositions that Socrates is human') is true at w just in case $H(s)$ is true at every world that contains all the propositions in w, or in other words, at every world that contains $Ext^w([\![\lambda p.p =_{\langle\rangle} p]\!])$. For another example, $\Box_{[s]} H(s)$ ('It is true in virtue of the nature of Socrates that Socrates is human') is true at w just in case $H(s)$ is true at every world at which $[\![s]\!]$ exists.[33] So $\Box_{[s]} H(s)$ is false at w just in case either $[\![H(s)]\!]$ is not in the domain of every world whose domain contains $[\![s]\!]$, or $H(s)$ is false at some world whose domain contains $[\![s]\!]$. It follows from Lemma 6 below that the denotation of a term (relative to an assignment g) is in the domain of a world w just in case the denotations (relative to g) of the constants and free variables occurring in it are in the domain of w. So, for example, $[\![H(s)]\!]$ is in the domain of w just in case both $[\![H]\!]$ and $[\![s]\!]$ are in the domain of w. As we will see below, it is straightforward to construct models where the denotations of different constants have different existence sets, so that even, e.g., $[\![\wedge]\!]$ and $[\![\neg]\!]$ may be in the domains of some, none, or all of the same worlds.

The following derived clause for $\Box_\Pi$ deserves special mention:

> For $w \in E(\{[\![\Box_\Pi \phi]\!]^g\}) : \mathfrak{M}, w, g \vDash \Box_\Pi \phi$ iff for all $v \in W$: if $D^w \subseteq D^v$, then $\mathfrak{M}, v, g \vDash \phi$

If, as discussed in §5, $\Box_\Pi$ expresses metaphysical necessity, then the clause says that ϕ is metaphysically necessary at w if and only if it is true at every world that contains D^w, i.e., the domain of all entities that exist at w. A world v is thus metaphysically accessible from a world w if and only if $D^w \subseteq D^v$. But even if $\Box_\Pi$ does not express metaphysical necessity, any candidate metaphysical necessity operator would have to obey the same semantic clause, provided that essence implies necessity and the logic of metaphysical necessity is at least S4.

Our semantic clause for the essentialist operator is a higher-order generalization of the clause for the essentialist operator in Fine [22] except for one small but crucial difference. On Fine's semantics, $\Box_F \phi$ is true at a world at which the formula is defined just in case (a) for every object x in the objectual content of ϕ, there is some element y in the extension of F that depends on x,[34] and (b) ϕ is true at every world that contains the extension of F.[35] Part (a) of Fine's clause ensures the validity of Localization, a principle that we have seen reason to reject. Dropping part (a) constitutes a welcome simplification of the semantic clause for the essentialist operator.[36]

As mentioned previously, we also have the following derived clause for the relation of essential involvement, the proof of which is supplied in Lemma 11:

> For $w \in E(\{[\![(F_1, \ldots, F_n) \succeq_\sigma A]\!]^g\}) : \mathfrak{M}, w, g \vDash (F_1, \ldots, F_n) \succeq_\sigma A$ iff $\bigcup_{i=1}^n Ext^w([\![F_i]\!]^g)$ depends on $[\![A]\!]^g$.

---

[33] Recall that $[s]$ is short for $\lambda x.x =_e s$.

[34] Unlike the models for HLE defined here, Fine's models for LE contain a primitive dependence relation between objects of the domain. Unlike the notion of dependence defined in Definition 6, Fine's dependence relation cannot be defined in terms of the existence sets of its relata.

[35] Here the *objectual content* of a formula ϕ in a model is to be understood as the union of the extensions of the rigid predicates and the denotations of the free variables and constants in ϕ.

[36] The semantic clause for essentialist statements in Correia [5, pp. 302–303] is in line with the clause adopted here on this point. This is also essentially the approach followed in Correia [7], even though there essentialist statements are first analyzed in terms of a strict conditional. Thanks to an anonymous referee here.

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The relation of essential involvement can thus be modeled as a relation between existence sets of entities. This is another important difference to Fine's first-order semantics, in which essential involvement (Fine's dependence relation) is interpreted by a primitive dependence relation in the structures which cannot be defined in terms of existence sets.[37]

Before we move on to some example models, we supply the definition of validity.

DEFINITION 12 (Validity). *A formula ϕ is* valid in a class of dual-content models $\mathfrak{S}$ *iff for every dual-content model $\langle W, D, [\![\cdot]\!]\rangle \in \mathfrak{S}$ and assignment* g *for which $[\![\phi]\!]^g$ is defined*: $E(\{[\![\phi]\!]^g\}) \subseteq \pi_1([\![\phi]\!]^g)$. *$\phi$ is* valid *iff it is valid in the class of all dual-content models.*

Note that since we always have $\pi_1([\![\phi]\!]^g) \subseteq E(\{[\![\phi]\!]^g\})$, because a proposition can only be true at those worlds at which it exists, $E(\{[\![\phi]\!]^g\}) \subseteq \pi_1([\![\phi]\!]^g)$ entails $E(\{[\![\phi]\!]^g\}) = \pi_1([\![\phi]\!]^g)$.

§7. Example models. We will now construct some examples of models that show the consistency of some simple HLE-theories. The models constructed here illustrate some general methods for constructing models of more complicated theories.

EXAMPLE 3. We construct a model that shows the joint consistency of the sentences $\Box_{[s]}H(s)$ and $\neg\Box_{[H]}H(s)$, which we may read 'Socrates is essentially human' and 'It is not in the nature of being human that Socrates is human', respectively.

Let Σ be a signature whose only non-logical constants are s (of type e) and H (of type $\langle e\rangle$). We first define a worldly content structure $\mathfrak{N} = \langle \mathcal{N}, W, I, \iota, \overline{E}\rangle$ similar to the one defined in Example 1. Let $W = \{w, v\}$, $\mathcal{N} = \langle \wp(W), \supseteq\rangle$, with join being given by intersection of subsets and meet by union, $I^w = \mathbb{N} \cup \{s\}$, $I^v = \mathbb{N}$, $\iota(s) = \{w\}$, and $\iota(n) = W$ for all $n \in \mathbb{N}$, and let $\overline{E}$ be the identity function on $\wp(W)$. Thus the existence set of a lattice element is just the lattice element itself. We can now define a dual-content $\mathcal{L}^\Sigma$-model $\mathfrak{M} = \langle W, I, [\![\cdot]\!]\rangle$ based on $\mathfrak{N}$. Let $[\![s]\!] = \langle \iota(s), s\rangle$, $\pi_0([\![c]\!]) = W$ for all logical constants c, $\pi_0([\![H]\!]) = W$, and $\pi_1([\![H]\!])(\mathbf{x}) = \{w\}$ if $\mathbf{x} = [\![s]\!]$, and $\emptyset$ otherwise. It is readily checked that $\mathfrak{M}, w, g \vDash \Box_{[s]}H(s)$ and $\mathfrak{M}, w, g \vDash \neg\Box_{[H]}H(s)$.

In the model constructed in Example 3, neither world is metaphysically accessible from the other, because the domain of neither is a subset of the other (no element whose first component is $\{v\}$ is in D^w and no element whose first component is $\{w\}$ is in D^v). We may sometimes want to construct models based on a particular metaphysical accessibility relation on the set of worlds. The next example provides a template for constructing dual-content frames given a particular metaphysical accessibility relation.

EXAMPLE 4. Let W be a non-empty set and R be a reflexive and transitive relation on W. A subset $G \subseteq W$ is *R-closed* iff for all $w \in G$ and $v \in W$, wRv implies $v \in G$. Let $\wp^R(W) = \{G \subseteq W \mid G \text{ is} R\text{-closed}\}$. Then $\mathcal{N} = \langle \wp^R(W), \supseteq\rangle$ is a complete lattice, with join given by intersection and meet by union. Let $\overline{E}$ be the identity function on $\wp^R(W)$, and I be a function mapping each $w \in W$ to some non-empty set I^w such that for all $w, v \in W$, wRv implies $I^w \subseteq I^v$; and let $\iota : o \mapsto \{w : o \in I^w\}$. Then $\langle \mathcal{N}, W, I, \iota, \overline{E}\rangle$ is a worldly content structure. Let $\langle W, D\rangle$ be the dual-content frame induced by $\langle \mathcal{N}, W, I, \iota, \overline{E}\rangle$. We then have for all $w, v \in W$: wRv if and only if $D^w \subseteq D^v$.

[37] See Ditter [14, p. 376] for discussion of a condition on structures that does allow Fine's dependence relation to be defined in terms of existence sets of individuals in the structure.

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Thus the accessibility relation R is indeed the metaphysical accessibility relation. If R is also symmetric, then wRv if and only if $D^w = D^v$.

Our next example shows the consistency of the schema $\neg(A \geq_{\sigma,\tau} B)$, where A and B are distinct constants of a given signature. The consistency of this schema implies in particular that it is consistent to assume that the nature of no logical operation involves any other logical operation, where by 'logical operation' we mean 'denotation of a logical constant' of a given signature.

Example 5. Let Σ be a signature. We define a worldly content structure $\mathfrak{N} = \langle \mathcal{N}, W, I, \iota, \overline{E} \rangle$ as in Example 2: $\mathcal{N} = \langle \wp(\Sigma), \subseteq \rangle$, with join given by union and meet by intersection of subsets, $W = \wp(\Sigma) \setminus \{\emptyset\}$, $I^w = \{c \in \Sigma : c \in w\}$ for all $w \in W$, ι is the function from Σ to $\wp(\Sigma)$ such that $\iota(x) = \{x\}$, and $\overline{E}$ is the function from $\wp(\Sigma)$ to $\wp(W)$ such that $\overline{E} : n \mapsto \{w \in W : n \subseteq w\}$. Let $\langle W, D, [\![\cdot]\!] \rangle$ be a dual-content $\mathcal{L}^\Sigma$-model based on $\mathfrak{N}$, where $\pi_0([\![c]\!]) = \{c\}$ for all $c \in \Sigma$. For the purpose of demonstrating the consistency of the schema above, the intensional contents of the non-logical constants don't matter, so we leave them unspecified; the intensional contents of the logical constants are determined by Definition 11. Note that in this model, the worlds are sets of constants and by construction, the denotation of a constant exists at a world if and only if it is an element of that world. Thus $w := \Sigma$ is the only world in the model in which all denotations of the constants of Σ exist together. It is readily checked that each instance of the schema $\neg(A \geq_{\sigma,\tau} B)$ in which A and B are distinct constants from Σ, for any $\tau \in \mathcal{T}$, is true at w. The reason is that for each constant c, there is some world, namely, $\{c\}$, at which $[\![c]\!]$ but no other denotation of a constant exists.

The next examples illustrate that the denotations of logically equivalent formulas may fail to be identical in a model.

Example 6. Let Σ be the signature that contains only logical constants. Let $\mathfrak{N}$ be defined as in the previous example except that $I^w = \Sigma$ for all $w \in W$ and $\overline{E}$ is the constant function that maps every element of $\mathcal{N}$ to W. Let $\mathfrak{M} = \langle W, D, [\![\cdot]\!] \rangle$ be the dual-content $\mathcal{L}^\Sigma$-model based on $\mathfrak{N}$. Now consider the formulas p and $\neg\neg p$ and let g be an assignment such that $g(p) = \langle \emptyset, W \rangle$. The formulas p and $\neg\neg p$ are logically equivalent and true in the same worlds, but their denotations are distinct: $[\![p]\!]^g = \langle \emptyset, W \rangle \neq \langle \{\neg\}, W \rangle = [\![\neg\neg p]\!]^g$.

The model in the previous example is one in which the domain function D is such that for all $\sigma \in \mathcal{T}$, $D^w_\sigma = D^v_\sigma$ for all $w, v \in W$; the typed domains are constant across worlds. Let us call models of this kind *constant domain models*. Constant domain models validate the schema $A \geq_{\sigma,\tau} B$, which, as discussed in §4, leads to the consequence that all subscripted essentialist operators become logically coextensive. The example also shows that two formulas ϕ and ψ may be equivalent in the class of constant domain models without $\phi = \psi$ being valid in that class. However, any constant domain model in which $\mathcal{N}$ is a trivial one-element lattice is such that if $\pi_1([\![\phi]\!]^g) = \pi_1([\![\psi]\!]^g)$, then $[\![\phi]\!]^g = [\![\psi]\!]^g$.

There is also a natural class of models without constant domain in which sameness of intensional content and existing at the same worlds *does* entail sameness of denotation. Call a dual-content model *injective* if it is based on a worldly content model in which the function $\overline{E}$ is injective. The models constructed in Examples 3 and 5 are examples of injective models. In an injective model, any elements of the domain that exist in the same worlds have the same worldly content.

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Injective models validate some interesting interaction principles between essence and identity that are not valid in the class of all dual-content models. Among the injective dual-content models, those in which the denotations of $=$ and $\wedge$ exist at all worlds are of special interest because they validate an especially tight connection between essence and identity. For example, suppose $\Box_{[x]}p$ is true at some world w for some assignment g in such a model. Then $[\![x = x]\!]^g = [\![(x = x) \wedge p]\!]^g$, and so $(x = x) = ((x = x) \wedge p)$ is true on g at w as well. Conversely, if $(x = x) = ((x = x) \wedge p)$ is true at w on some assignment g in such a model, then $\Box_{[x]}p$ is true on g at w too, if the denotation of $\Box_\sigma$ exists at w. Thus, the formula $\Box_{[x]}p \leftrightarrow ((x = x) = (x = x) \wedge p)$ is valid in this class of models. The argument above can be easily generalized to show that $\Box_{[x_1],\ldots,[x_n]}p \leftrightarrow ((x_1 = x_1) \wedge \cdots \wedge (x_n = x_n) = ((x_1 = x_1) \wedge \cdots \wedge (x_n = x_n) \wedge p)$ is also valid in this class of models. Whether or not these connections are plausible is a question that deserves more careful investigation elsewhere.

§8. Soundness. In this section, we prove the soundness of HLE with respect to the semantics developed in §6. We start with a few auxiliary lemmas.

LEMMA 5. *Let* $\langle W, D, [\![\cdot]\!]\rangle$ *be a dual-content* $\mathcal{L}^\Sigma$*-model. Then* $\overline{E}(\pi_0([\![A]\!]^g)) = \overline{E}(\bigsqcup\{\pi_0([\![A_1]\!]^g), \ldots, \pi_0([\![A_n]\!]^g)\})$, *where* $A_1, \ldots, A_n$ *are all the free variables and constants occurring in* A.

Proof. Immediate from Definition 11 4. □

LEMMA 6. *Let* $\langle W, D, [\![\cdot]\!]\rangle$ *be a dual-content* $\mathcal{L}^\Sigma$*-model. Then* $E(\{[\![A]\!]^g\}) = E(\{[\![A_1]\!]^g, \ldots, [\![A_n]\!]^g\})$, *where* $A_1, \ldots, A_n$ *are all the constants and free variables occurring in* A.

Proof. $E(\{[\![A]\!]^g\}) \overset{(1)}{=} \overline{E}(\pi_0([\![A]\!]^g)) \overset{(2)}{=} \overline{E}\bigsqcup\{\pi_0([\![A_1]\!]^g), \ldots, \pi_0([\![A_n]\!]^g)\} \overset{(3)}{=} \overline{E}(\pi_0([\![A_1]\!]^g) \cap \cdots \cap \overline{E}(\pi_0([\![A_n]\!]^g) \overset{(4)}{=} E(\{[\![A]\!]^g\}) \cap \cdots \cap E(\{[\![A_n]\!]^g\}) \overset{(5)}{=} E(\{[\![A_1]\!]^g, \ldots, [\![A_n]\!]^g\})$. Equalities (1) and (4) hold by Lemma 2, equality (2) by Lemma 5. The other equalities hold by the definitions of $\overline{E}$ and E, respectively. □

Note that, by Lemmas 2, 5 and 6, a formula ϕ is defined in $\mathfrak{M}$ just in case $[\![\phi]\!]^g$ exists at some world in $\mathfrak{M}$, for some assignment g.

LEMMA 7 (Coincidence lemma). *Let* $\langle W, D\rangle$ *be a dual-content frame. If* $A \in \mathcal{L}^\Sigma$ *and* $\langle W, D, [\![\cdot]\!]\rangle$ *and* $\langle W, D, [\cdot]\rangle$ *are dual-content* $\mathcal{L}^\Sigma$*-models such that for all constants* c *occurring in* A, $[\![c]\!] = [c]$, *then for all assignments* g, h *that agree on all the free variables in* A: $[\![A]\!]^g = [A]^h$.

Proof. By a straightforward induction on the complexity of A. □

LEMMA 8. *Let* $\langle W, D, [\![\cdot]\!]\rangle$ *be a dual-content* $\mathcal{L}^\Sigma$*-model. Then if* $[\![A]\!]^g = [\![C]\!]^h$ *and* $[\![B_i]\!]^g = [\![D_i]\!]^h$, *for all* $i, 1 \leq i \leq n$, *then* $[\![A(B_1, \ldots, B_n)]\!]^g = [\![C(D_1, \ldots, D_n)]\!]^h$.

Proof. Immediate from Definition 11. □

LEMMA 9 (Substitution lemma). *Let* $\langle W, D, [\![\cdot]\!]\rangle$ *be a dual-content* $\mathcal{L}^\Sigma$*-model and* g *be an assignment for* $\langle W, D\rangle$. *For all types* σ, τ, *variables* x^τ, *and terms* A^σ, B^τ, *we have* $[\![A]\!]^{g[x \mapsto [\![B]\!]^g]} = [\![A[B/x]]\!]^g$.

Proof. By a straightforward induction on the complexity of A. □

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DEFINITION 13. A immediately $\beta\eta$-reduces *to B iff either A is a term of the form* $(\lambda x_1 \dots x_n.\phi)(A_1, \dots, A_n)$ *and B is* $\phi[A_i/x_i]$, *or A is a term of the form* $(\lambda x_1 \dots x_k.F(x_1, \dots, x_k))$, *where none of* $x_1, \dots, x_k$ *is free in F, B is F, and F is of non-rigid type.*

A one-step $\beta\eta$-reduces *to B iff B results from A by replacing one constituent with something to which it immediately* $\beta\eta$*-reduces*; *i.e., iff there are two finite sequences* $\langle A_1, \dots, A_n\rangle$, $\langle B_1, \dots, B_n\rangle$ *such that* A_1 *immediately* $\beta\eta$*-reduces to* B_1, $A_n = A$, $B_n = B$, *and either for some* $C_0, \dots, C_m$, A_{i+1} *is* $A_i(C_0, \dots, C_m)$ *and* B_{i+1} *is* $B_i(C_0, \dots, C_m)$, *or for some* $0 \leq k \leq m$, A_{i+1} *is* $C_0(C_1, \dots, C_k, A_i, C_{k+1}, \dots, C_m)$ *and* B_{i+1} *is* $C_0(C_1, \dots, C_k, B_i, C_{k+1}, \dots, C_m)$, *or for some* $x_1, \dots, x_m$, A_{i+1} *is* $\lambda x_1 \dots x_m.A_i$ *and* B_{i+1} *is* $\lambda x_1 \dots x_m.B_i$.

A is one-step $\beta\eta$-equivalent *to B iff either A one-step* $\beta\eta$*-reduces to B or B one-step* $\beta\eta$*-reduces to A.*

A is $\beta\eta$-equivalent *to B iff there is a finite sequence of terms* $\langle C_1, \dots, C_n\rangle$ *such that* $A = C_1$, $B = C_n$ *and whenever* $0 < i < n$, C_1 *is one-step* $\beta\eta$*-equivalent to* C_{i+1}.

We will now prove that any two $\beta\eta$-equivalent terms have the same denotation in a dual-content model.

LEMMA 10. *Let* $\langle W, D, [\![\cdot]\!]\rangle$ *be a dual-content model. Then for all terms* A, B *and assignments* g *for* $\langle W, I\rangle$: $[\![A]\!]^g = [\![B]\!]^g$, *if A and B are* $\beta\eta$*-equivalent.*

Proof. It suffices to show that $[\![A]\!]^g = [\![B]\!]^g$ whenever A one-step $\beta\eta$-reduces to B. The proof is by induction using the definition of one-step $\beta\eta$-reduction. We only show $\pi_1([\![A]\!]^g) = \pi_1([\![B]\!]^g)$, the case of $\pi_0([\![A]\!]^g) = \pi_0([\![B]\!]^g)$ being an immediate consequence of Definition 114. *Base case*: A immediately $\beta\eta$-reduces to B. (i) A is of the form $\lambda x_1 \dots x_n.C(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are not free in C, and B is C. Let g be an arbitrary assignment. Then,

$$\begin{aligned}\pi_1([\![\lambda x_1 \dots x_n.C(x_1, \dots, x_n)]\!]^g)(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \pi_1([\![C(x_1, \dots, x_n)]\!]^{g[x_i \mapsto \mathbf{x}_i]}) \\ &= \pi_1([\![C]\!]^{g[x_i \mapsto \mathbf{x}_i]})(\mathbf{x}_1, \dots, \mathbf{x}_n) \\ &= \pi_1([\![C]\!]^g)(\mathbf{x}_1, \dots, \mathbf{x}_n).\end{aligned}$$

The last equation holds by Lemma 7 because none of $x_1, \dots, x_n$ is free in C. (ii) A is of the form $(\lambda x_1 \dots x_n.\phi)(A_1, \dots, A_n)$ and B is $\phi[A_i/x_i]$. Let g be an arbitrary assignment. Then,

$$\begin{aligned}\pi_1([\![(\lambda x_1 \dots x_n.\phi)(A_1, \dots, A_n)]\!]^g) &= \pi_1([\![\lambda x_1 \dots x_n.\phi]\!]^g)([\![A_1]\!]^g, \dots, [\![A_n]\!]^g) \\ &= \pi_1([\![\phi]\!]^{g[x_i \mapsto [\![A_i]\!]^g]}) \\ &= \pi_1([\![\phi[A_i/x_i]]\!]^g). \qquad \text{(By Lemma 9)}\end{aligned}$$

Induction step: Suppose that $A, B \in \mathcal{L}^{\Sigma}_{\sigma}$ are such that $[\![A]\!]^g = [\![B]\!]^g$ for all assignments g. Then if σ is of the form $\langle \sigma_0, \dots, \sigma_m\rangle$ (and similarly if σ is a rigid type) and $C_0, \dots, C_m$ are of appropriate types, $\pi_1([\![A(C_0, \dots, C_m)]\!]^h) = \pi_1([\![B(C_0, \dots, C_m)]\!]^h)$, for any assignment h, by Lemma 8. Moreover, if $C_0, \dots, C_m$ are of appropriate types, then $\pi_1([\![C_0(C_1, \dots, C_k, A, C_{k+1}, \dots, C_m)]\!]^h) = \pi_1([\![C_0(C_1, \dots, C_k, B, C_{k+1}, \dots, C_m)]\!]^h)$, $0 \leq k \leq m$ by Lemma 8. Finally, if σ is $\langle\rangle$ and A and B contain all of $x_1, \dots, x_n$ free, then for all assignments g,

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$$\begin{aligned}\pi_1([\![\lambda x_1 \ldots x_n.A]\!]^g)(\mathbf{x}_1, \ldots, \mathbf{x}_n) &= \pi_1([\![A]\!]^{g[x_i \mapsto \mathbf{x}_i]}) \\ &= \pi_1([\![B]\!]^{g[x_i \mapsto \mathbf{x}_i]}) \\ &= \pi_1([\![\lambda x_1 \ldots x_n.B]\!]^g)(\mathbf{x}_1, \ldots, \mathbf{x}_n). \qquad \square\end{aligned}$$

LEMMA 11. *Let* $\mathfrak{M} = \langle W, D, [\![\cdot]\!] \rangle$ *be a dual-content model. Then* $\mathfrak{M}, w, g \vDash (F_1, \ldots, F_n) \succeq_\sigma A$ *iff* $\bigcup_{i=1}^n Ext^w([\![F_i]\!]^g)$ *depends on* $[\![A]\!]^g$.

Proof. The left-to-right direction is an immediate consequence of Lemma 6 and the semantic clauses in Definition 11. For the converse, observe that by Definition 9, for any $\mathbf{x} \in D_\sigma$ there is an $\mathbf{X} \in D_{\langle\sigma\rangle}$ with $\pi_0(\mathbf{x}) = \pi_0(\mathbf{X})$ and $Ext^w(\mathbf{X}) = \{\mathbf{x}\}$ for all $w \in E(\{\mathbf{X}\})(= E(\{\mathbf{x}\}))$. The claim then follows from Lemma 6 and the semantic clauses in Definition 11. $\square$

THEOREM 1 (Soundness). *Every theorem of* HLE *is valid in the class of dual-content models.*

Proof. (i) *Background logic*: The validity of the tautologies and the quantificational axioms and rules is straightforward. But the case of modus ponens is instructive. To show that the rule of modus ponens preserves validity, assume that ϕ and $\phi \to \psi$ are valid, and suppose for *reductio* that ψ is not valid. Then there is a model $\mathfrak{M} = \langle W, D, [\![\cdot]\!] \rangle$ such that $w \in E(\{[\![\psi]\!]^g\})$ and $\mathfrak{M}, w, g \nvDash \psi$ for some $w \in W$ and assignment g. Let $\mathfrak{M}' = \langle W, D, [\![\cdot]\!]' \rangle$ be a model that is such that for all constants c that occur in ψ, $[\![c]\!]' = [\![c]\!]$, and for all constants c not occurring in ψ: (i) if c is a non-logical constant of type σ, then c is assigned to an arbitrary element of D_σ^w (this is legitimate since D_σ^w is non-empty for all $w \in W$ and $\sigma \in \mathcal{T}$) and (ii) if c is a logical constant, then $\pi_0([\![c]\!]') = \bot$. Let h be an assignment that agrees with g on all free variables of ψ and that assigns every free variable v^σ of ϕ to an element of D_σ^w. This guarantees that $w \in E(\{[\![\phi]\!]'^h, [\![\phi \to \psi]\!]'^h\})$ by Lemma 6. By the validity of ϕ and $\phi \to \psi$, we have $\mathfrak{M}', w, h \vDash \phi$ and $\mathfrak{M}', w, h \vDash \phi \to \psi$, and thus $\mathfrak{M}', w, h \vDash \psi$. But by Lemma 7, $[\![\psi]\!]'^h = [\![\psi]\!]^g$, and so, $\mathfrak{M}, w, g \vDash \psi$, in contradiction to the hypothesis.

Ref is trivial and LL immediately follows from the semantic clause for $=_\sigma$ and Lemma 9. β- and η-conversion follow from Lemma 10.

(ii) *Background essentialist axioms*: Permutation, Idempotence, Separation, Subtraction and MON are straightforward.

(Decomposition) Suppose $\mathfrak{M}, w, g \vDash \Box_{F_1,\ldots,F_k,[B]}\phi$ and let $A_1, \ldots, A_n$ be all the constants and free variables of B. Let $v \in W$ be such that $\bigcup_{i=1}^k Ext^w([\![F_i]\!]^g) \subseteq D^v$ and $\{[\![A_i]\!]^g : 1 \leq i \leq n\} \subseteq D^v$. Then by Lemma 6, $[\![B]\!]^g \in D^v$, and thus $\mathfrak{M}, v, g \vDash \phi$ by the hypothesis, whence $\mathfrak{M}, w, g \vDash \Box_{F_1,\ldots,F_k,[A_1],\ldots,[A_n]}\phi$.

(iii) *Axioms for Rigidity*:

(R-Comp): From Lemma 4(ii).

(R-Ext): From Lemma 4(iii).

(Rigidity): Suppose $\mathfrak{M}, w, g \vDash F^{[\sigma]}(x)$. Suppose $v \in W$ is such that $Ext^w([\![F]\!]^g) \subseteq I_\sigma^v$. Then by Lemma 4(i), $[\![F]\!]^g \in I_{[\sigma]}^v$. Moreover, since $Ext^v([\![F]\!]^g) = Ext^w([\![F]\!]^g)$, it follows that $[\![x]\!]^g \in Ext^v([\![F]\!]^g)$. Now since $[\![F(x)]\!]^g \in D_{\langle\rangle}^v$ by Lemma 6, we have $\mathfrak{M}, v, g \vDash F(x)$.

(R-Equiv): From Lemma 4(i).

(iv) *Core essentialist axioms and rules*:

(RC): We first show that the special case RC^0 preserves validity. Suppose that ϕ is valid and suppose for *reductio* that there is a dual-content model $\mathfrak{M} = \langle W, D, [\![\cdot]\!] \rangle$ such

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that for some $w \in W$ and assignment g: $\mathfrak{M}, w, g \not\models \Box_{[A_1],\dots,[A_n]}\phi$, where $A_1, \dots, A_n$ are all the free variables and constants in ϕ. By Lemma 6, there is a $v \in W$ such that $[\![\phi]\!]^g \in D^v$ and $\mathfrak{M}, v, g \not\models \phi$, in contradiction to the assumption that ϕ is valid.

Now let $n \geq 1$. Suppose $\phi_1 \wedge \dots \wedge \phi_n \to \psi$ is valid and suppose for *reductio* that there is a dual-content model $\mathfrak{M} = \langle W, D, [\![\cdot]\!]\rangle$ such that for some $w \in W$ and assignment g: $(*)$ $\mathfrak{M}, w, g \not\models \Box_{F_1,\dots,F_k}\phi_1 \wedge \dots \wedge \Box_{F_1,\dots,F_k}\phi_n \to \Box_{F_1,\dots,F_k,[A_1],\dots,[A_l]}\psi$, where $A_1, \dots, A_l$ are all the free variables and constants in ψ but not any of $\phi_1, \dots, \phi_n$. Let $B_1, \dots, B_m$ be all of the constants and free variables occurring both in ψ and some ϕ_i, so that $A_1, \dots, A_l, B_1, \dots, B_m$ are all of the free variables and constants in ψ. It follows from $(*)$ and Lemma 6 that $\bigcup_{i=1}^k Ext^w([\![F_i]\!]^g)$ depends on $\{[\![B_1]\!]^g, \dots, [\![B_m]\!]^g\}$. So by Lemma 6, for every $v \in W$ such that $\bigcup_{i=1}^k Ext^w([\![F_i]\!]^g) \cup \{[\![A_1]\!]^g, \dots, [\![A_l]\!]^g\} \subseteq D^v$, we have $[\![\psi]\!]^g \in D^v$, since the denotations of all free variables and constants in ψ are in D^v. It follows from this together with $(*)$ that there is a $v \in W$ such that $\bigcup_{i=1}^k Ext^w([\![F_i]\!]^g) \subseteq D^v$ and $[\![\psi]\!]^g \in D^v$, where (1) $\mathfrak{M}, v, g \models \phi_i$, $1 \leq i \leq n$ and (2) $\mathfrak{M}, v, g \not\models \psi$. If $\wedge$ and $\to$ occur in any of $\phi_1, \dots, \phi_n, \psi$, then $[\![\wedge]\!], [\![\to]\!] \in D^v$ and so, by Lemma 6, $[\![\phi_1 \wedge \dots \wedge \phi_n \to \psi]\!]^g \in D^v$. The validity of $\phi_1 \wedge \dots \wedge \phi_n \to \psi$ together with (1) imply that $\mathfrak{M}, v, g \models \psi$, which contradicts (2). Let us now suppose that neither $\wedge$ nor $\to$ occurs in any of $\phi_1, \dots, \phi_n, \psi$ (the cases where only one of them occurs in them are treated similarly). Let $\mathfrak{M}' = \langle W, D, [\![\cdot]\!]'\rangle$ be a dual-content model whose first two components are identical to the first two components of $\mathfrak{M}$, and whose denotation function $[\![\cdot]\!]'$ agrees with $[\![\cdot]\!]$ on all constants except for $\wedge$ and $\to$, whose denotations are defined as follows: $\pi_0([\![\wedge]\!]') = \pi_0([\![\to]\!]') = \bot$. Since none of the formulas $\phi_1, \dots, \phi_n, \psi$, contains any of $\wedge$ or $\to$, it follows from (1), (2), and Lemma 7 that (1') $\mathfrak{M}', v, g \models \phi_i$, $1 \leq i \leq n$ and (2') $\mathfrak{M}', v, g \not\models \psi$. It follows from $E(\{[\![\wedge]\!]', [\![\to]\!]'\}) = W$, (1'), (2') and Lemma 6 that $[\![\phi_1 \wedge \dots \wedge \phi_n \to \psi]\!]'^g \in D^v$. By (1') and the validity of $\phi_1 \wedge \dots \wedge \phi_n \to \psi$, this implies $\mathfrak{M}', v, g \models \psi$, in contradiction to (2').

(CH) Suppose $\mathfrak{M}, w, g \models \Box_{G_1,\dots,G_n,c_\sigma(F_1,\dots,F_k)}\phi$. By Lemma 11, for any $\mathbf{x} \in D_\sigma^w$, $\mathbf{x} \in Ext^w([\![c_\sigma(F_1, \dots, F_k)]\!]^g)$ iff $\bigcup_{i=1}^k Ext^w([\![F_i]\!]^g)$ depends on $\mathbf{x}$. Thus, $\bigcup_{i=1}^k Ext^w([\![F_i]\!]^g)$ depends on $Ext^w([\![c_\sigma(F_1, \dots, F_k)]\!]^g)$. So let $v \in W$ be such that $\bigcup_{i=1}^k Ext^w([\![F_i]\!]^g) \cup \bigcup_{i=1}^n Ext^w([\![G_i]\!]^g) \subseteq D^v$. Then since $\bigcup_{i=1}^k Ext^w([\![F_i]\!]^g)$ depends on $Ext^w([\![c_\sigma(F_1, \dots, F_k)]\!]^g)$, the hypothesis entails that $\mathfrak{M}, v, g \models \phi$ and thus $\mathfrak{M}, w, g \models \Box_{G_1,\dots,G_n,F_1,\dots,F_k}\phi$.

(T): Suppose $\mathfrak{M}, w, g \models \Box_{F_1,\dots,F_n}\phi$. Then since $\bigcup_{i=1}^n Ext^w([\![F_i]\!]^g) \subseteq D^w$ and $[\![\phi]\!]^g \in D^w_{\langle\rangle}$, we have $\mathfrak{M}, w, g \models \phi$.

(4): Suppose $\mathfrak{M}, w, g \models \Box_{F_1,\dots,F_n}\phi$, where $F_1^{\sigma_1}, \dots, F_n^{\sigma_n}$ are rigid predicates. Let $v \in W$ be such that $\bigcup_{i=1}^n Ext^w([\![F_i]\!]^g) \subseteq D^v$ and $[\![\Box_{\sigma_1,\dots,\sigma_n}]\!] \in D^v$. Then by the hypothesis and Lemma 4(i), $[\![F_i]\!]^g \in D^v$, for each $i \leq n$, and $[\![\phi]\!]^g \in D^v$. So by Lemma 6, $[\![\Box_{F_1,\dots,F_n}\phi]\!]^g \in D^v$. Now let $z \in W$ be such that $\bigcup_{i=1}^n Ext^v([\![F_i]\!]^g) \subseteq D^z$. Then since by the rigidity of the F_i and Lemma 4(iv), $Ext^v([\![F_i]\!]^g) = Ext^w([\![F_i]\!]^g)$, for every $i \leq n$, we get $\bigcup_{i=1}^n Ext^w([\![F_i]\!]^g) \subseteq D^z$. Hence, by the hypothesis, $\mathfrak{M}, z, g \models \phi$, and thus $\mathfrak{M}, v, g \models \Box_{F_1,\dots,F_n}\phi$, from which it follows that $\mathfrak{M}, w, g \models \Box_{F_1,\dots,F_n,[\Box_{\sigma_1,\dots,\sigma_n}]}\Box_{F_1,\dots,F_n}\phi$.

(5) Suppose $\mathfrak{M}, w, g \models \neg\Box_{F_1,\dots,F_n}\phi$, where $F_1^{\sigma_1}, \dots, F_n^{\sigma_n}$ are rigid predicates. Let $v \in W$ be such that $\bigcup_{i=1}^n Ext^w([\![F_i]\!]^g) \subseteq D^v$, $[\![\neg]\!] \in D^v$, $[\![\Box_{\sigma_1,\dots,\sigma_n}]\!] \in D^v$ and $[\![\phi]\!]^g \in D^v$. So by Lemma 6, $[\![\neg\Box_{F_1,\dots,F_n}\phi]\!]^g \in D^v$. But $\mathfrak{M}, v, g \models \Box_{F_1,\dots,F_n}\phi$ would contradict our hypothesis, since by Lemma 4(iv), $Ext^v([\![F_i]\!]^g) = Ext^w([\![F_i]\!]^g)$, for every $i \leq n$. So, $\mathfrak{M}, v, g \models \neg\Box_{F_1,\dots,F_n}\phi$, and thus $\mathfrak{M}, w, g \models \Box_{F_1,\dots,F_n,[\neg],[\Box_{\sigma_1,\dots,\sigma_n}],[\phi]}\neg\Box_{F_1,\dots,F_n}\phi$. □

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It is evident that the formulas that are valid in the class of dual-content models do not coincide with the provable formulas of HLE, because among the validities are all validities of full second-order logic. So there cannot be a completeness result with respect to this class of models. For present purposes, the model theory primarily serves as a tool for checking the consistency of various essentialist theories. Still, it would be interesting to develop a model theory for HLE (or some extension of it) for which the theory is complete. It is quite clear that, at least if the model theory is to take a similar form as the one proposed here, the theory HLE would have to be strengthened in order to prove completeness. For example, we would plausibly have to adopt the strengthened version of Proposition 6(ii) and (iii) discussed above as axioms, since any class of models that validates the weaker versions would plausibly also validate the stronger versions of these principles.

Apart from the question of completeness, the class of dual-content models validates some principles that are not only plausibly unprovable in HLE but are also intuitively false on their intended interpretation. For example, the models validate the claim that the nature of any entity whatsoever involves, for any given property, some property that is necessarily coextensive with it. For instance, let $\langle n, f \rangle$ be the representation of the property of being human in some model. Then for any entity, the property $\langle \bot, f \rangle$ exists in every world in which the entity exists, because $\overline{E}(\bot) = W$. This seems undesirable. Since it is the "fullness" of the domains that is responsible for these validities, it would seem that they would be invalidated in appropriately generalized *Henkin* models, in which the typed domains of a model are subsets of the domains D^w_σ, as defined in Definition 9, rich enough to validate R-Comp and to provide an interpretation $[\![A]\!]^g$, relative to an assignment g, for every term A for which $\overline{E}(\bigsqcup\{\pi_0([\![A_1]\!]^g), \ldots, \pi_0([\![A_n]\!]^g)\}) \neq \emptyset$, where $A_1, \ldots, A_n$ are all the free variables and constants occurring in A. Such generalized Henkin models remain to be further explored.

Among the most important extensions of HLE that remain to be explored are those which add further principles concerning the interaction between identity and other the logical constants. In HLE the only axioms that specifically concern identity $(=)$ are Ref, LL and R-Ext. The system is deliberately neutral about more substantive principles concerning identity, such as, for example, the principle that conjunction is commutative $((p \wedge q) = (q \wedge p))$ or more substantive interaction principles between essence and identity. While a fuller investigation of this matter is left for future work, it is interesting to observe that the following rule, which generates a number of interesting identities, preserves validity in the class of dual-content models:

(RI) If $\vdash \forall x_1 \ldots \forall x_n (R^{\langle \sigma_1, \ldots, \sigma_n \rangle}(x_1, \ldots, x_n) \leftrightarrow S^{\langle \sigma_1, \ldots, \sigma_n \rangle}(x_1, \ldots, x_n))$ and R and S contain exactly the same free variables and constants, then $\vdash R = S$.

THEOREM 2. *RI preserves validity in the class of dual-content models.*

Proof. Suppose first that $n = 0$. Suppose that $\phi \leftrightarrow \psi$ is valid and ϕ and ψ contain exactly the same constants and free variables. Suppose for *reductio* that there is a dual-content model $\mathfrak{M} = \langle W, D, [\![\cdot]\!] \rangle$ such that for some $w \in W$ and assignment g, $\mathfrak{M}, w, g \not\models \phi =_{\langle\rangle} \psi$. Since ϕ and ψ contain exactly the same constants and free variables, $\pi_0([\![\phi]\!]^g) = \pi_0([\![\psi]\!]^g)$ by Definition 11 Item 4. So $\pi_1([\![\phi]\!]^g) \neq \pi_1([\![\psi]\!]^g)$ and thus for some $v \in W$, $v \in \pi_1([\![\phi]\!]^g)$ and $v \notin \pi_1([\![\psi]\!]^g)$. By the validity of $\phi \leftrightarrow \psi$ we can infer that $[\![\leftrightarrow]\!] \notin D^v$. Now let $\mathfrak{M}' = \langle W, D, [\![\cdot]\!]' \rangle$ be a model that is just like $\mathfrak{M}$ but in which $\pi_0([\![\leftrightarrow]\!]') = \bot$. By Lemma 7, $(*)$ $[\![\phi]\!]^g = [\![\phi]\!]'^g$ and $[\![\psi]\!]^g = [\![\psi]\!]'^g$. But since $E(\{[\![\leftrightarrow]\!]'\}) = W$,

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it follows from Lemma 6 and the validity of $\phi \leftrightarrow \psi$ that $\pi_1([\![\phi]\!]'^g) = \pi_1([\![\psi]\!]'^g)$ and thus, by $(*)$, $[\![\phi]\!]^g = [\![\psi]\!]^g$. Contradiction. The case where $n \geq 1$ is proved similarly. □

RI immediately delivers such principles as the commutativity of conjunction and disjunction. It is interesting to observe, however, that it does not imply idempotence $(\phi = (\phi \wedge \phi))$ for conjunction or disjunction, and it does not imply involution $(\phi = \neg\neg\phi)$. However, we do get weakened versions of these: each of $(\phi \wedge \phi) = (\phi \wedge (\phi \wedge \phi))$, $(\phi \vee \phi) = (\phi \vee (\phi \vee \phi))$ and $\neg\phi = \neg\neg\neg\phi$ are implied by RI given the other principles of HLE. The weakened versions of involution and the commutativity of conjunction and disjunction seem rather natural from an essentialist standpoint.

§9. The logic of necessity—Part II. This section provides model-theoretic proofs of some results announced in §5 regarding the logics of the operators $\Box_\Pi$ and $\Box_\Omega$.

9.1. The logic of $\Box_\Pi$**.** We first show that the propositional logic of $\Box_\Pi$ is at most S4 in HLE and at most S5 in HLE+DOM$^{\langle\rangle}$. Together with Corollaries 1 and 2 from §5, this implies that its logic is exactly S4 and S5 in the respective systems.

DEFINITION 14. *An* S4-frame *is a pair* $\langle W, R\rangle$*, where* W *is a non-empty set and* R *is reflexive and transitive relation on* W*. A* propositional S4-model *is a triple* $\langle W, R, V\rangle$*, where* $\langle W, R\rangle$ *is an* S4*-frame, and* V *is a* valuation function *mapping each propositional variable into a subset of* W*.*

LEMMA 12. *Every* S4*-frame* $\langle W, R\rangle$ *can be extended to a dual-content frame* $\langle W, D\rangle$ *in which for all* $w, v \in W : wRv$ *iff* $D^w_{\langle\rangle} \subseteq D^v_{\langle\rangle}$*.*

Proof. Let $\langle W, R\rangle$ be an S4-frame. We first construct a worldly content structure on which we base the frame to be constructed. Let $\mathfrak{N} = \langle \mathcal{N}, W, I, \iota, \overline{E}\rangle$, where for all $w \in W : I^w = \{v \in W : vRw\}$, $\mathcal{N} = \langle \wp(W), \subseteq\rangle$, for all $w \in W : \iota(w) = \{w\}$, and for all $x \in \mathcal{N} : \overline{E}(x) = \{w \in W : \forall v \in x : vRw\}$. Let $\langle W, D\rangle$ be the dual-content frame induced by $\mathfrak{N}$. We show that for all $w, v \in W : wRv$ iff $D^w_{\langle\rangle} \subseteq D^v_{\langle\rangle}$. Let $w, v \in W$ be arbitrary. Left-to-right: Suppose wRv and let $\mathbf{p} \in D^w_{\langle\rangle}$. Then $w \in \overline{E}(\pi_0(\mathbf{p}))$ and so, by the definition of $\overline{E}$, for every $z \in \pi_0(\mathbf{p})$, zRw and hence, by the transitivity of R, zRv. Thus again by the definition of $\overline{E}$, $v \in \overline{E}(\pi_0(\mathbf{p}))$ and thus $\mathbf{p} \in D^v_{\langle\rangle}$. Right-to-left: Suppose $D^w_{\langle\rangle} \subseteq D^v_{\langle\rangle}$. By the reflexivity of R and the definition of $\overline{E}$, there is a $\mathbf{p} \in D^w_{\langle\rangle}$ such that $w \in \pi_0(\mathbf{p})$. By the hypothesis, this entails that $\mathbf{p} \in D^v_{\langle\rangle}$. So $v \in \overline{E}(\pi_0(\mathbf{p}))$ and hence, by the definition of $\overline{E}$, wRv, because $w \in \pi_0(\mathbf{p})$. □

We call the frame constructed in the proof above the *corresponding dual-content frame for* $\langle W, R\rangle$. Next we define a mapping between a modal language $\mathcal{L}_M$ and $\mathcal{L}^\Sigma$, for any signature Σ. We assume that $\mathcal{L}_M$ and $\mathcal{L}^\Sigma$ have a shared set of propositional variables $\mathcal{V}_{\langle\rangle}$. We recursively define a translation $'$ from the formulas of $\mathcal{L}_M$ to the formulas of $\mathcal{L}^\Sigma$ as follows:

(i) $p' = p$, for all $p \in \mathcal{V}^\Sigma_{\langle\rangle}$;
(ii) $(\neg\phi)' = \neg\phi'$;
(iii) $(\phi \vee \psi)' = (\phi' \vee \psi')$;
(iv) $(\Box\phi)' = \Box_\Pi\phi'$.

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LEMMA 13 *Let* $\langle W, R\rangle$ *be an* S4*-frame and* $\langle W, D\rangle$ *its corresponding dual-content frame. If* $\mathfrak{M} = \langle W, R, V\rangle$ *is an* S4*-model and* $\mathfrak{E} = \langle W, D, [\![\cdot]\!]\rangle$ *is a dual-content* $\mathcal{L}^{\Sigma}$*-model such that for all logical constants* c, $\pi_0([\![c]\!]) = \emptyset$, *then for any assignment* g *such that for all* $p \in \mathcal{V}_{\langle\rangle} : g(p) = \langle\emptyset, V(p)\rangle$, *for any formula* $\phi \in \mathcal{L}_M$, *and* $w \in W$: $\mathfrak{M}, w \vDash \phi$ *iff* $\mathfrak{E}, w, g \vDash \phi'$.

Proof. We first note that it is easily established by an induction on the complexity of $\phi \in \mathcal{L}_M$ that $\pi_o([\![\phi']\!]^g) = \emptyset$ and hence $E(\{[\![\phi']\!]^g\}) = W$. This fact will be tacitly presupposed in the proof below.

The proof is by induction on the complexity of ϕ. *Base case*: ϕ is $p \in \mathcal{V}_{\langle\rangle}$:

$$\mathfrak{M}, w \vDash p \Leftrightarrow w \in V(p) \Leftrightarrow w \in \pi_1(g(p)) \Leftrightarrow \mathfrak{E}, w, g \vDash p.$$

Induction step: The only interesting case is where ϕ is of the form $\Box\psi$:

$$\begin{aligned}
\mathfrak{M}, w \vDash \Box\psi &\Leftrightarrow \forall v \in W : wRv \Rightarrow \mathfrak{M}, v \vDash \psi \\
&\Leftrightarrow^{IH} \forall v \in W : wRv \Rightarrow \mathfrak{E}, v, g \vDash \psi' \\
&\Leftrightarrow \forall v \in W : D^w_{\langle\rangle} \subseteq D^v_{\langle\rangle} \Rightarrow \mathfrak{E}, v, g \vDash \psi' && \text{(By Lemma 12)} \\
&\Leftrightarrow \mathfrak{E}, w, g \vDash \Box_{\Pi}\psi'. && \square
\end{aligned}$$

THEOREM 3. *Let* ϕ *be a propositional modal schema that is not an* S4*-schema. Then* ϕ *has an* $\mathcal{L}^{\Sigma}$*-instance that is not an* HLE*-theorem.*

Proof. Suppose that ϕ is an invalid schema of propositional S4. Then ϕ has an $\mathcal{L}_M$-instance ψ that fails on some S4-frame $\langle W, R\rangle$. So there is an S4-model $\mathcal{M} = \langle W, R, V\rangle$ and some $w \in W$ such that $\mathcal{M}, w \nvDash \psi$. Let $\mathfrak{E} = \langle W, D\rangle$ be the corresponding dual-content frame, $\langle W, D, [\![\cdot]\!]\rangle$ be a model satisfying the conditions of the previous lemma and g be an assignment as in the previous lemma. Then $\mathfrak{E}, w, g \nvDash \psi'$, by Lemma 13. But ψ' is an $\mathcal{L}^{\Sigma}$-instance of ϕ. The claim then follows from Theorem 1. $\square$

COROLLARY 5. *In* HLE, *the propositional logic of* $\Box_{\Pi}$ *is exactly* S4.

Proof. From Corollary 1 and Theorem 3. $\square$

We can similarly show that the propositional logic of $\Box_{\Pi}$ is at most S5 in HLE+DOM$^{\langle\rangle}$. The proof for this is similar to, but simpler than, the above result and will be omitted here. We merely mention that the principles of HLE+DOM$^{\langle\rangle}$ are valid in the class of *equivalence models*, defined as follows.

DEFINITION 15. *A dual-content model* $\langle W, D, [\![\cdot]\!]\rangle$ *is an* equivalence model *just in case* $\forall w, v \in W$: *if* $D^w_{\langle\rangle} \subseteq D^v_{\langle\rangle}$, *then* $D^w_{\langle\rangle} = D^v_{\langle\rangle}$.

COROLLARY 6. *In* HLE+DOM$^{\langle\rangle}$, *the propositional logic of* $\Box_{\Pi}$ *is exactly* S5.

9.2. Models for propositional platonism. This section shows how to construct models for the Propositional Platonist theory HLE+P-Truth from §5 and uses these models to demonstrate that the propositional logic of $\Box_{\Omega}$ is at most S4 in HLE+P-Truth and at most S5 in HLE+P-Truth+DOM. Together with the results proved in §5, this implies that its logic is exactly S4 and S5 in the respective systems.

DEFINITION 16. *A* Platonist worldly content structure *is a worldly content structure* $\mathfrak{N} = \langle \mathcal{N}, W, I, \imath, \overline{E}\rangle$, *where for all* $w \in W$: $I^w \supseteq P^w_e := \{\langle n, l\rangle : n \in \mathcal{N}, w \in \overline{E}(n)$ *and* $l \subseteq \overline{E}(n)\}$ *and* $\imath | P^w_e : \langle n, l\rangle \mapsto n$.

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DEFINITION 17. *A* Platonist frame *is a dual-content frame induced by a Platonist worldly content structure.*

DEFINITION 18. *Let* Σ *be a signature including the constant true of type* $\langle e\rangle$. *A* Platonist worldly content $\mathcal{L}^{\Sigma}$-model *is a worldly content* $\mathcal{L}^{\Sigma}$*-model based on a Platonist worldly content structure such that* $\pi_0([\![true]\!]) = \bot$ *and for all* $\sigma \in \mathcal{T}$, $\pi_0([\![=_\sigma]\!]) = \bot$.

DEFINITION 19. *A* Platonist $\mathcal{L}^{\Sigma}$-model *is a model based on a Platonist worldly content* $\mathcal{L}^{\Sigma}$*-model where for all* $\mathbf{x} \in D_e$:

$$\pi_1([\![true]\!])(\mathbf{x}) = \begin{cases} \pi_1(\pi_1(\mathbf{x})), \text{ if } \pi_1(\mathbf{x}) \in P_e^w \text{ for some } w \in W \\ \emptyset \text{ otherwise.} \end{cases}$$

THEOREM 4. *P-Truth is valid in the class of Platonist models.*

Proof. Let $\langle W, D, [\![\cdot]\!]\rangle$ be a Platonist $\mathcal{L}^{\Sigma}$-model. Let $w \in W$ be such that $w \in E(\{[\![\forall p^{\langle\rangle}\exists x^e \Box_{[x]} true(x) = p]\!]\})$. We show that $w \in \pi_1([\![\forall p^{\langle\rangle}\exists x^e \Box_{[x]} true(x) = p]\!])$. Let $\mathbf{p} = \langle n, l\rangle \in D^w_{\langle\rangle}$. Then there is some $\mathbf{x} \in D_e^w$ such that $\mathbf{x} = \langle n, \langle n, l\rangle\rangle$, since $\langle n, l\rangle \in P_e^w \subseteq I^w$. Now let $v \in W$ be such that $\mathbf{x} \in D_e^v$. Then since $\pi_0([\![true]\!]) = \pi_0([\![=_{\langle\rangle}]\!]) = \bot$ and $\pi_0([\![\mathbf{x}]\!]) = \pi_0([\![\mathbf{p}]\!])$, it follows that $v \in E(\{[\![true(\mathbf{x}) = \mathbf{p}]\!]\})$ by Lemma 6. But $[\![true(\mathbf{x})]\!] = \langle n, \pi_1(\pi_1(\mathbf{x}))\rangle = \langle n, l\rangle = \mathbf{p}$. Hence, $v \in \pi_1([\![true(\mathbf{x}) = \mathbf{p}]\!])$. □

We will now use Theorem 4 to show that the logic of $\Box_\Omega$ is at most S4 in HLE+P-Truth. We first show the analog of Lemma 12.

LEMMA 14. *Every* S4*-frame* $\langle W, R\rangle$ *can be extended to a Propositional dual-content frame* $\langle W, I\rangle$ *in which for all* $w, v \in W : wRv$ *iff* $D_e^w \subseteq D_e^v$.

Proof. Let $\langle W, R\rangle$ be an S4-frame. We first construct a Platonist worldly content structure as follows. Let $\mathfrak{N} = \langle \mathcal{N}, W, I, \iota, \overline{E}\rangle$, where $\mathcal{N} = \langle \wp(W), \subseteq\rangle$, for all $x \in \mathcal{N}$: $\overline{E}(x) = \{w \in W : \forall v \in x : vRw\}$, for all $w \in W : I^w = \{\langle n, l\rangle : n \in \mathcal{N}, w \in \overline{E}(n),$ and $l \subseteq \overline{E}(n)\}$, and for all $x \in \bigcup I^w : \iota(x) = \pi_0(x)$. It is readily verified that $\mathfrak{N}$ is indeed a Platonist worldly content structure. Let $\langle W, D\rangle$ be the dual-content frame induced by $\mathfrak{N}$. $\langle W, D\rangle$ is a Platonist dual-content frame. The proof that for all $w, v \in W : wRv$ iff $D_e^w \subseteq D_e^v$ is exactly analogous to the related proof in Lemma 12. □

We call the frame constructed in the proof of the previous lemma the *corresponding Platonist frame* for $\langle W, R\rangle$. We now obtain the analogs of Lemma 13 and Theorem 3, with $(\Box\phi)' = \Box_\Omega\phi'$.

LEMMA 15. *Let* $\langle W, R\rangle$ *be an* S4*-frame and* $\langle W, D\rangle$ *its corresponding Platonist frame and let* $\mathfrak{M} = \langle W, R, V\rangle$ *be an* S4*-model and* $\mathfrak{E} = \langle W, D, [\![\cdot]\!]\rangle$ *a Platonist* $\mathcal{L}^{\Sigma}$*-model such that for all logical constants* c, $\pi_0([\![c]\!]) = \emptyset$. *Then for any assignment* g *such that for all* $p \in \mathcal{V}_{\langle\rangle} : g(p) = \langle\emptyset, V(p)\rangle$, *for any formula* $\phi \in \mathcal{L}_M$, *and* $w \in W$: $\mathfrak{M}, w \vDash \phi$ *iff* $\mathfrak{E}, w, g \vDash \phi'$.

Proof. Similar to the proof of Lemma 13. □

THEOREM 5. *Let* ϕ *be a propositional modal schema that is not an* S4*-schema. Then* ϕ *has an* $\mathcal{L}^{\Sigma}$*-instance that is not a theorem of* HLE+Truth.

Proof. The proof is analogous to the proof of Theorem 3, using Lemma 15 and Theorems 1 and 4. □

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COROLLARY 7. *The logic of* $\Box_\Omega$ *is exactly* S4 *in* HLE+Truth.

Proof. From Corollary 3 and Theorem 5. □

As in the case of HLE+DOM$^{\langle\rangle}$ we can show that there is a class of models with respect to which HLE+P-Truth+DOM is sound and that the logic of $\Box_\Omega$ is exactly S5 in HLE+P-Truth+DOM. The details are again omitted here. We merely mention that HLE+P-Truth+DOM is valid in the following class of models.

DEFINITION 20. *A Platonist model* $\langle W, D, [\![\cdot]\!]\rangle$ *is a* Platonist type-e equivalence model *iff* $\forall w, v \in W$: *if* $D_e^w \subseteq D_e^v$, *then* $D_e^w = D_e^v$.

COROLLARY 8. *In* HLE+Truth+DOM, *the logic of* $\Box_\Omega$ *is exactly* S5.

§10. Concluding remarks. I conclude with some questions for future research. The system HLE is based on a classical higher-order logic. The reductions of necessity to essence investigated in §5 therefore all validate *necessitism*, the view that necessarily everything is necessarily something. It would be interesting to develop versions of HLE that are based on a free logic. It is an open question whether we can understand necessity in terms of essence in such a free version of HLE.[38] Moreover, it would be desirable to find a semantics for which HLE is complete. There are also various extensions of HLE that would be worth investigating. One important type of extension concerns the interaction between essence and ground;[39] another one concerns the addition of interaction principles for essence and identity, such as those discussed in §7 and §8.[40] Finally, it is worth noting that variants of the model theory developed here may have applications that do not concern essence. For example, it is straightforward to interpret a higher-order language containing no essentialist operators but a primitive necessity operator in a suitably modified class of dual-content models. One could, for example, take the class of dual-content constant domain frames and interpret a necessity operator as ranging over all worlds in the model. This generates a version of classical higher-order modal logic where properties and propositions are not individuated by necessary coextensiveness and equivalence, respectively.

It is also possible to use dual-content frames to model various kinds of free higher-order modal logics. For example, one could model a positive free higher-order modal logic by letting all formulas be defined at every world in a model, letting the quantifiers be restricted to the relevant domains D_σ^w—interpreted as the inner domains of a world—and allowing the extensions of predicates and propositions to be subsets of the outer domain of the frame. $D_{\langle\rangle}^w$ would be $\{\langle n, l\rangle : n \in \mathcal{N}, w \in \overline{E}(n) \text{ and } l \subseteq W\}$, for example. So a proposition that doesn't exist in the inner domain of a world may still be true at that world. The denotation of a term then exists in the inner domain of a world just in case the denotations of all of its constants and free variables exist

[38] Teitel [41] has argued that there is a tension between the reduction of necessity to essence on the one hand and contingentism (the negation of necessitism) on the other. See Werner [44] and Yan [47] for replies.

[39] For an informal exploration of some interaction principles for essence and ground, see Bliss and Trogdon [3, sec. 4.2] and the references therein.

[40] For a unified account of ground and essence in terms of identity, see Correia and Skiles [10]. It would be interesting to extend their theory to a higher-order setting and compare the resulting logic of essence with HLE.

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in the inner domain of that world. This allows us to model a version of higher-order contingentism on which (a) properties and propositions are not individuated by necessary coextensiveness or equivalence and (b) even such qualitative properties as *being red* can fail to have necessary existence. Whether or not such a view is philosophically motivated and defensible remains to be investigated.

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DEPARTMENT OF PHILOSOPHY
UNIVERSITY COLLEGE LONDON
UK
E-mail: a.ditter@ucl.ac.uk

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