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ON SOME MULTIPLE CHARACTER SUMS

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Abstract. We improve a recent result of B. Hanson [Estimates for character sums with various convolutions. *Preprint*, 2015, arXiv:1509.04354] on multiplicative character sums with expressions of the type $a+b+cd$ and variables a, b, c, d from four distinct sets of a finite field. We also consider similar sums with $a+b(c+d)$. Our new bounds rely on some recent advances in additive combinatorics.

§1. *Introduction.*

1.1. *Motivation and previous results.*

Let p be a prime and let $\mathbb{F}_p$ be the finite field of p elements.

Given four sets $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \subseteq \mathbb{F}_p^*$, and two sequences of weights $\alpha = (\alpha_a)_{a\in\mathcal{A}}$ and $\beta = (\beta_{b,c,d})_{(b,c,d)\in\mathcal{B}\times\mathcal{C}\times\mathcal{D}}$ supported on $\mathcal{A}$ and $\mathcal{B}\times\mathcal{C}\times\mathcal{D}$, respectively, we consider the multilinear character sums

$$S_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta) = \sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\sum_{c\in\mathcal{C}}\sum_{d\in\mathcal{D}}\alpha_a\beta_{b,c,d}\chi(a+b+cd), \tag{1.1}$$

where χ is a fixed nontrivial multiplicative character of $\mathbb{F}_p^*$; see [**13**, Ch. 3] for a background on characters. We also consider related sums

$$T_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta) = \sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\sum_{c\in\mathcal{C}}\sum_{d\in\mathcal{D}}\alpha_a\beta_{b,c,d}\chi(a+b(c+d)). \tag{1.2}$$

In is easy to see that if the weight $\beta_{b,c,d}$ is multilinear, that is, of the form

$$\beta_{b,c,d} = \beta_b\gamma_c\delta_d, \tag{1.3}$$

the sums (1.2) can easily be reduced to the sums of the form (1.1) (with slightly modified weights).

First we recall that for the bilinear analogues of these sums, that is, for the sums

$$S_\chi(\mathcal{U},\mathcal{V};\varphi,\psi) = \sum_{u\in\mathcal{U}}\sum_{v\in\mathcal{V}}\varphi_u\psi_v\chi(u+v)$$

with sets $\mathcal{U}, \mathcal{V} \subseteq \mathbb{F}_p$ and weights $\varphi = (\varphi_u)_{u\in\mathcal{U}}$, $\psi = (\psi_v)_{v\in\mathcal{V}}$, we have the classical bound

$$|S_\chi(\mathcal{U},\mathcal{V};\varphi,\psi)| \leqslant \sqrt{p\Phi\Psi} \tag{1.4}$$

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with

$$\Psi = \sum_{u \in \mathcal{U}} |\varphi_u|^2 \quad \text{and} \quad \Psi = \sum_{v \in \mathcal{V}} |\psi_v|^2.$$

Indeed, using the trivial inequalities

$$\begin{aligned} |S_\chi(\mathcal{U}, \mathcal{V}; \varphi, \psi)|^2 &\leqslant \sum_{u \in \mathcal{U}} |\varphi_u|^2 \sum_{u \in \mathcal{U}} \left| \sum_{v \in \mathcal{V}} \psi_v \chi(u+v) \right|^2 \\ &\leqslant \sum_{u \in \mathcal{U}} |\varphi_u|^2 \sum_{u \in \mathbb{F}_p} \left| \sum_{v \in \mathcal{V}} \psi_v \chi(u+v) \right|^2, \end{aligned}$$

expanding the square, changing the order of summation and recalling the orthogonality of characters, we immediately obtain (1.4).

Although our results apply to more general weights, we always assume that the weights satisfy the inequalities

$$\max_{a \in \mathcal{A}} |\alpha_a| \leqslant 1 \quad \text{and} \quad \max_{(b,c,d) \in \mathcal{B} \times \mathcal{C} \times \mathcal{D}} |\beta_{b,c,d}| \leqslant 1. \tag{1.5}$$

We also recall that 0 is excluded from the sets $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ (it is trivial to adjust our bounds to include this case as well).

Thus, using A, B, C and D to denote the cardinalities of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$, respectively, under the condition (1.5), we easily derive

$$|S_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta)| \leqslant ABCD\sqrt{\frac{p}{AM}}, \tag{1.6}$$

where $M = \max\{B, C, D\}$ and a similar bound for $T_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta)$ (one only has to consider the contribution from the terms with $c + d = 0$ separately).

In the case of constant weights $\alpha_a = \beta_{b,c,d} = 1$, these sums, which we denote as $S_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, have been nontrivially estimated by Hanson [12] under various restrictions on A, B, C and D. For example, it is shown in [12] that for any fixed $\varepsilon > 0$ there exists some $\eta > 0$ such that if $A, B, C, D \geqslant p^\varepsilon$ and:

- either $C < p^{1/2}$ and $A^{56}B^{28}C^{33}D^4 \geqslant p^{60+\varepsilon}$;
- or $C \geqslant p^{1/2}$ and $A^{112}B^{56}D^8 \geqslant p^{87+\varepsilon}$,

then

$$S_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) = O(ABCDp^{-\eta}). \tag{1.7}$$

Balog and Wooley [2, §6] have suggested an alternative approach to bounding the sums $S_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, which may lead to more explicit statements with small, but explicit, values of η. However, our results seems to supersede these bounds as well.

Several more bounds of multiplicative character sums, which go beyond an immediate application of (1.4), have been given by Bourgain *et al* [6], Chang [8], Friedlander and Iwaniec [10], Karatsuba [14], Konyagin [15] and Shkredov and Volostnov [20]. However, despite the active interest in such sums

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of multiplicative characters, there is a large discrepancy between the strength and generality of the above results and the results available for similar exponential sums for which strong explicit bounds are known in very general scenarios; see [3–5, 7, 11, 16, 17].

1.2. *General notation.* Throughout the paper, the expressions $A \ll B$, $B \gg A$ and $A = O(B)$ are each equivalent to the statement that $|A| \leqslant cB$ for some positive constant c. Throughout the paper, the implied constants in these symbols may occasionally, where obvious, depend on the integral positive parameter ν, and are absolute otherwise.

1.3. *Main results.* It is convenient to assume that $BCD \leqslant p^2$. Clearly, this restriction is not important as if it fails we can use $M \geqslant p^{2/3}$ in the bound (1.6), getting a bound of the type (1.7) already for $A \geqslant p^{1/3+\varepsilon}$ with any fixed $\varepsilon > 0$.

We are now ready to present our main results.

THEOREM 1.1. *For any sets $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \subseteq \mathbb{F}_p^*$ of cardinalities A, B, C, D, respectively, and two sequences of weights*

$$\alpha = (\alpha_a)_{a\in\mathcal{A}} \quad \text{and} \quad \beta = (\beta_{b,c,d})_{(b,c,d)\in\mathcal{B}\times\mathcal{C}\times\mathcal{D}}$$

satisfying (1.5), for any fixed integer $\nu \geqslant 1$, we have

$$\begin{aligned} &S_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta), T_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta) \\ &\quad \ll ((BCD)^{1-1/(4\nu)} + (BCD)^{1-1/(2\nu)} M^{1/(2\nu)}) \\ &\qquad \times \begin{cases} A^{1/2} p^{1/2} & \text{if } \nu = 1, \\ (Ap^{1/(4\nu)} + A^{1/2} p^{1/(2\nu)}) & \text{if } \nu \geqslant 2, \end{cases} \end{aligned}$$

where $M = \max\{B, C, D\}$.

To understand the strength of Theorem 1.1, we assume that $A \geqslant p^\varepsilon$ for some $\varepsilon > 0$. Taking ν large enough (for example $\nu = \lfloor 2/\varepsilon \rfloor + 1$) and assuming that $M \leqslant (BCD)^{1/2}$ (which certainly holds in the most interesting case when the sets $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are of comparable sizes), we see that the bound of Theorem 1.1 takes the shape

$$S_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta), T_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta) \ll ABCD\left(\frac{p}{BCD}\right)^{1/(4\nu)},$$

which is nontrivial as long as $BCD > p^{1+\varepsilon}$ for some $\varepsilon > 0$.

In another interesting case of all sets of asymptotically the same size, that is, when $A \sim B \sim C \sim D$, taking $\nu = 1$, we obtain

$$S_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta), T_\chi(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta) \ll A^{11/4} p^{1/2},$$

which is nontrivial for $A \geqslant p^{2/5+\varepsilon}$.

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§2. *Preliminaries.*

2.1. *Background from arithmetic combinatorics.* For sets $\mathcal{B}, \mathcal{C}, \mathcal{D} \subseteq \mathbb{F}_p^*$, we denote by $I(\mathcal{B}, \mathcal{C}, \mathcal{D})$ the number of solutions to the equation

$$b_1 + c_1 d_1 = b_2 + c_2 d_2, \quad b_1, b_2 \in \mathcal{B},\ c_1, c_2 \in \mathcal{C},\ d_1, d_2 \in \mathcal{D}.$$

Roche-Newton *et al* [**18**, equation (4)] have shown that the points–planes incidence bound of Rudnev [**19**] yields the following estimate.

LEMMA 2.1. *Let $\mathcal{B}, \mathcal{C}, \mathcal{D} \subseteq \mathbb{F}_p^*$ be of cardinalities B, C, D, respectively, with $BCD \leqslant p^2$. Then we have*

$$I(\mathcal{B}, \mathcal{C}, \mathcal{D}) \ll B^{3/2} C^{3/2} D^{3/2} + BCDM,$$

where $M = \max\{B, C, D\}$.

Furthermore, for sets $\mathcal{B}, \mathcal{C}, \mathcal{D} \subseteq \mathbb{F}_p^*$, we denote by $J(\mathcal{B}, \mathcal{C}, \mathcal{D})$ the number of solutions to the equation

$$b_1(c_1 + d_1) = b_2(c_2 + d_2), \quad b_1, b_2 \in \mathcal{B},\ c_1, c_2 \in \mathcal{C},\ d_1, d_2 \in \mathcal{D}.$$

It is shown in [**17**, Lemma 2.3] that using some recent results of Aksoy Yazici *et al* [**1**, Theorem 19] (which are also based on the work of Rudnev [**19**]), one can derive the following analogue of Lemma 2.1.

LEMMA 2.2. *Let $\mathcal{B}, \mathcal{C}, \mathcal{D} \subseteq \mathbb{F}_p^*$ be of cardinalities B, C, D, respectively, with $BCD \leqslant p^2$. Then we have*

$$J(\mathcal{B}, \mathcal{C}, \mathcal{D}) \ll B^{3/2} C^{3/2} D^{3/2} + BCDM,$$

where $M = \max\{B, C, D\}$.

2.2. *Bounds of some character sums on average.* The following result is very well known and, in the case when $\mathcal{A}$ is an interval, it dates back to Davenport and Erdős [**9**]. The proof transfers to the case of general sets without any changes. Indeed, for $\nu = 1$, it is based on the elementary identity

$$\sum_{\lambda \in \mathbb{F}_p} \chi(u + \lambda)\overline{\chi}(v + \lambda) = \begin{cases} -1 & \text{if } u \neq v, \\ p - 1 & \text{if } u = v, \end{cases} \quad u, v \in \mathbb{F}_p,$$

where $\overline{\chi}$ is the complex conjugate character; see [**13**, equation (3.20)]. For $\nu \geqslant 2$, the proof is completely analogous but appeals to the Weil bound of multiplicative character sums, see [**13**, Theorem 11.23], to estimate "off-diagonal" terms.

LEMMA 2.3. *For any set $\mathcal{A} \subseteq \mathbb{F}_p^*$ of cardinality A and sequences of weights $\alpha = (\alpha_a)_{a \in \mathcal{A}}$ satisfying (1.5), for any fixed integer $\nu \geqslant 1$, we have*

$$\sum_{\lambda \in \mathbb{F}_p} \left| \sum_{a \in \mathcal{A}} \alpha_a \chi(\lambda + a) \right|^{2\nu} \ll \begin{cases} Ap & \text{if } \nu = 1, \\ A^{2\nu} p^{1/2} + A^{\nu} p & \text{if } \nu \geqslant 2. \end{cases}$$

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§3. *Proof of Theorem 1.1.*

3.1. *Bound on* $S_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)$. Using (1.5), we obtain

$$|S_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)| \leqslant \sum_{b\in\mathcal{B}}\sum_{c\in\mathcal{C}}\sum_{d\in\mathcal{D}}\left|\sum_{a\in\mathcal{A}}\alpha_a\chi(a+b+cd)\right|.$$

Now, for every $\lambda \in \mathbb{F}_p$ we collect together the terms with the same value of $b+cd=\lambda$, and write

$$|S_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)| \leqslant \sum_{\lambda\in\mathbb{F}_p} K(\mathcal{B},\mathcal{C},\mathcal{D};\lambda)\left|\sum_{a\in\mathcal{A}}\alpha_a\chi(a+\lambda)\right|, \tag{3.1}$$

where $K(\mathcal{B},\mathcal{C},\mathcal{D};\lambda)$ is the number of solutions to the equation

$$b+cd=\lambda, \qquad (b,c,d)\in\mathcal{B}\times\mathcal{C}\times\mathcal{D}.$$

Clearly, we have

$$\sum_{\lambda\in\mathbb{F}_p} K(\mathcal{B},\mathcal{C},\mathcal{D};\lambda) = BCD, \qquad \sum_{\lambda\in\mathbb{F}_p} K(\mathcal{B},\mathcal{C},\mathcal{D};\lambda)^2 = I(\mathcal{B},\mathcal{C},\mathcal{D}). \tag{3.2}$$

Therefore, by the Hölder inequality, for any integer $\nu \geqslant 1$, we have

$$\begin{aligned}
&|S_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)|^{2\nu}\\
&\quad\leqslant \left(\sum_{\lambda\in\mathbb{F}_p} K(\mathcal{B},\mathcal{C},\mathcal{D};\lambda)\right)^{2\nu-2}\sum_{\lambda\in\mathbb{F}_p} K(\mathcal{B},\mathcal{C},\mathcal{D};\lambda)^2\cdot\sum_{\lambda\in\mathbb{F}_p}\left|\sum_{a\in\mathcal{A}}\alpha_a\chi(a+\lambda)\right|^{2\nu}\\
&\quad= (BCD)^{2\nu-2}I(\mathcal{B},\mathcal{C},\mathcal{D})\sum_{\lambda\in\mathbb{F}_p}\left|\sum_{a\in\mathcal{A}}\alpha_a\chi(a+\lambda)\right|^{2\nu}.
\end{aligned}$$

Recalling Lemmas 2.1 and 2.3, we obtain

$$\begin{aligned}
|S_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)|^{2\nu} \ll{}& ((BCD)^{2\nu-1/2}+(BCD)^{2\nu-1}M)\\
&\times\begin{cases} Ap & \text{if } \nu=1,\\ A^{2\nu}p^{1/2}+A^\nu p & \text{if } \nu\geqslant 2\end{cases}
\end{aligned}$$

and the result follows.

3.2. *Bound on* $T_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)$. We proceed as in the case of the sums $S_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)$.

We define $L(\mathcal{B},\mathcal{C},\mathcal{D};\lambda)$ as the number of solutions to the equation

$$b(c+d)=\lambda, \qquad (b,c,d)\in\mathcal{B}\times\mathcal{C}\times\mathcal{D}$$

and then, instead of (3.1), write

$$|T_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)| \leqslant \sum_{\lambda\in\mathbb{F}_p} L(\mathcal{B},\mathcal{C},\mathcal{D};\lambda)\left|\sum_{a\in\mathcal{A}}\alpha_a\chi(a+\lambda)\right|.$$

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As before, by the Hölder inequality, for any integer $\nu \geqslant 1$, we have

$$|T_\chi(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)|^{2\nu} \leqslant (BCD)^{2\nu-2} J(\mathcal{B},\mathcal{C},\mathcal{D}) \sum_{\lambda\in\mathbb{F}_p} \left|\sum_{a\in\mathcal{A}} \alpha_a \chi(a+\lambda)\right|^{2\nu}.$$

Using Lemma 2.2 instead of Lemma 2.1 in the argument of §3.1, we obtain the desired estimate.

§4. *Comments.* We note that in the case of the multilinear weights of the form (1.3) (and in particular in the case of constant weights), the roles of the sets $\mathcal{A}$ and $\mathcal{B}$ can be interchanged. Furthermore, in this case, writing

$$\chi(a+b+cd) = \chi(c)\chi(d+(a+b)c^{-1}) = \chi(d)\chi(c+(a+b)d^{-1}),$$

one can obtain the bound of Theorem 1.1 with any permutation of the roles of A, B, C, D.

It is also easy to see that we can abandon the assumption (1.5) and obtain a more precise version of Theorem 1.1 with L^1 and L^2 norms of the weight sequences α and β.

Finally, using results from [**1**], one can obtain similar bounds for several other character sums in exactly in the same way; for example, for the sums

$$\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\sum_{c\in\mathcal{C}}\sum_{d\in\mathcal{D}} \alpha_a\beta_{b,c,d}\chi\left(a+\frac{b+c}{d+c}\right)$$

and

$$\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\sum_{c\in\mathcal{C}}\sum_{d\in\mathcal{D}} \alpha_a\beta_{b,c,d}\chi\left(a+\frac{b}{c+d}\right)$$

as well some other related sums.

We also note that the same approach (with $\nu = 1$) applies to the sums

$$U(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta) = \sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\sum_{c\in\mathcal{C}}\sum_{d\in\mathcal{D}} \alpha_a\beta_{b,c,d}\mathbf{e}_p(a(b+cd)),$$
$$V(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta) = \sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\sum_{c\in\mathcal{C}}\sum_{d\in\mathcal{D}} \alpha_a\beta_{b,c,d}\mathbf{e}_p(ab(c+d)),$$

where $\mathbf{e}_p(u) = \exp(2\pi \mathrm{i} u/p)$, which are very similar to those appearing in the proofs of [**17**, Theorems 1.1 and 1.3]. Indeed, by the Cauchy inequality, in the notation of §3.1, we obtain

$$|U(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D};\alpha,\beta)|^2 \leqslant \sum_{\lambda\in\mathbb{F}_p} K(\mathcal{B},\mathcal{C},\mathcal{D};\lambda)^2 \cdot \sum_{\lambda\in\mathbb{F}_p}\left|\sum_{a\in\mathcal{A}}\alpha_a\mathbf{e}_p(a\lambda)\right|^2 = I(\mathcal{B},\mathcal{C},\mathcal{D}) \sum_{\lambda\in\mathbb{F}_p}\left|\sum_{a\in\mathcal{A}}\alpha_a\mathbf{e}_p(a\lambda)\right|^2.$$

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Using the orthogonality of exponential functions, we derive

$$|U(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta)|^2 \leqslant pI(\mathcal{B}, \mathcal{C}, \mathcal{D})A$$

and, invoking Lemma 2.1, we derive

$$U(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta) \ll p^{1/2}((BCD)^{3/4} + (BCDM)^{1/2})A^{1/2},$$

where, as before, $M = \max\{B, C, D\}$. The same argument also gives

$$V(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \alpha, \beta) \ll p^{1/2}((BCD)^{3/4} + (BCDM)^{1/2})A^{1/2}.$$

Assuming that $M \leqslant (BCD)^{1/2}$, we see that these bounds are nontrivial as long as $A(BCD)^{1/2} > p^{1+\varepsilon}$ for some $\varepsilon > 0$. In particular, under the condition $A \sim B \sim C \sim D$, this holds for $A \geqslant p^{2/5+\varepsilon}$, which is consistent with the range of nontriviality of [**17**, Theorems 1.1 and 1.3].

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