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Joinings of three-interval exchange transformations

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Abstract. We show that among three-interval exchange transformations there exists a dichotomy: T has minimal self-joinings whenever the associated subshift is linearly recurrent, and is rigid otherwise. We also build a family of simple rigid three-interval exchange transformations, which is a step towards an old question of Veech, and a family of rigid three-interval exchange transformations, which includes Katok's rank-one map.

1. *Introduction*

One recurrent preoccupation of ergodic theorists in the last 20 years has been joinings: the notion of *self-joinings* of a system was introduced by Rudolph in [**Ru**], to generalize some useful invariants of measure-theoretic isomorphisms such as the factor algebra and the centralizer. A transformation which has *minimal self-joinings* (see Definition 4.2 later) has trivial centralizer and no non-trivial proper factor, and can be used to build a so-called *counter-example machine* with surprising properties. The first example of a transformation with minimal self-joinings was given in [**Ru**], and a little later the famous *Chacon map* was shown in [**dJRS**] to also have minimal self-joinings. However, both these examples may seem built on purpose, and they have no 'natural', i.e. geometric, realization. Geometric examples of transformations with minimal self-joinings were sought in the category of *interval exchange transformations* (Definition 2.1 later): these were introduced

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by Oseledec [**O**], following an idea attributed to Arnold [**Ad**]; see also Katok and Stepin [**KS**]. Indeed, in 1983 del Junco [**dJ**] built a one-parameter family of three-interval exchange transformations, depending on an irrational γ, and proved that whenever this γ has bounded partial quotients in its continued fraction expansion the system has minimal self-joinings (the interested reader is warned that he will *not* find the terms 'three-interval exchange transformation' or 'minimal self-joinings' in del Junco's paper; the systems which he describes as two-point extensions of rotations are indeed three-interval exchange transformations, and the notion of 'simplicity' he proves is only slightly weaker than the original notion of minimal self-joinings, and has been standing as the current definition of 'minimal self-joinings' since [**dJR1**]).

However, in the meantime, Veech had shown that almost all interval exchange transformations (in the sense, for a fixed permutation, for Lebesgue-almost all values of the lengths of the intervals) are *rigid* (see Definition 5.1 later), and hence have uncountable centralizers and cannot have minimal self-joinings (this is written in [**V2**]); thus he devised in [**V1**] a weakened notion of minimal self-joinings to allow for a non-trivial centralizer. The new notion, which Veech called 'property S' but which is now known as 'simplicity (in the sense of Veech)' (see Definition 6.1 later) is strong enough to keep many of the properties of systems with minimal self-joinings, although proving this required a lot of work [**dJR2, V1**]. Veech asked the following question [**V1**, Question 4.9].

QUESTION 1.1. *Are almost all interval exchange transformations simple?*

Note that if the answer is positive then almost all interval exchange transformations are simple and rigid; simplicity implies either discrete spectrum or weak mixing. Although there are non-trivial examples of interval exchange transformations with discrete spectrum [**Ax, FHZ3**], no one seriously suspects that these examples form a set a positive measure. Hence to give a positive answer to Veech's question, we need to know that almost all interval exchange transformations are weakly mixing, which has been known since [**KS**] for three intervals and has been proved for any number of intervals [**AF**] while the present paper was being processed.

However, while Veech's question stood unanswered, examples of simple transformations remained scarce: there were of course the systems with minimal self-joinings, and some systems without minimal self-joinings but naturally related to these systems (such as the time-one map of a flow which, as a flow, has minimal self-joinings). At last in [**dJR1**], a natural generalization of Chacon's map was (very cunningly!) shown to be simple and rigid. It remained to this day the only explicit simple rigid map; no 'natural' system was known to be simple and rigid, and not a single interval exchange transformation was there to give substance to Veech's conjecture, the only known simple interval exchange transformations being del Junco's ones [**dJ**], which have minimal self-joinings.

Among interval exchange transformations, the first non-trivial ones are the three-interval exchange transformations. Such a transformation involves two interval lengths α and β as parameters, but can also be viewed as an induced map of a rotation of angle α' on an interval of length β'. The properties of minimal self-joinings and rigidity are linked to the diophantine properties of α' and β'; in [**dJ**], for example $\beta' = 3\alpha'/2$, and it is shown

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that whenever α' has bounded partial quotients we have minimal self-joinings, and if α' has unbounded partial quotients with infinitely many even convergents we have rigidity, the remaining cases raising open questions.

Now, the structure of three-interval exchange transformations has been studied extensively in [**FHZ1, FHZ2**], where an expansion $(n_k, m_k, \epsilon_{k+1})_{k\in\mathbb{N}}$ reflects the diophantine properties of α' and β' and governs a representation of the system by Rokhlin stacks. This representation has already been used in [**FHZ3**] to derive new spectral properties of three-interval exchange transformations. In the present paper, which relies heavily on [**FHZ1, FHZ2**] but is essentially independent of [**FHZ3**], we use this representation to solve completely the problem of minimal self-joinings for three-interval exchange transformations, by a clear dichotomy: for an explicit set of expansions, corresponding to a subset of the α' with bounded partial quotients, and characterized by the word-combinatorial property of *linear recurrence* (this notion was defined independently in [**B**] and [**D**], see Definition 3.1 later), the three-interval exchange transformations have minimal self-joinings, and the proof is a non-trivial generalization of the proof of minimal self-joinings for Chacon's map. Outside this set, every three-interval exchange transformation is rigid, and these provide the first known examples of rigid systems built through a family of Rokhlin stacks, but where the rigidity sequence is not the sequence of heights of these stacks; this also puts the open cases in [**dJ**] on the side of rigidity. Among these rigid three-interval exchange transformations, we do have an uncountable family of examples which are weakly mixing and simple, as they are measure-theoretically isomorphic to variants of del Junco–Rudolph's map; this gives us a modest but first step towards a positive answer to Veech's question. Another interesting family of three-interval exchange transformations has a structure closely related to a well-known abstract example, the so-called Katok's map (in fact, a family of maps): we show the surprising result that the classic Katok's map is indeed a three-interval exchange transformation (up to measure-theoretic isomorphism), and that the unusual properties of its Cartesian square equipped with the product self-joining (loosely Bernoulli property, finite spectrum) are shared by the Cartesian square of several other three-interval exchange transformations. Thus we see that many people have been using three-interval exchange transformations without knowing it. The simplicity of Katok's map remains a tantalizing open problem.

Throughout the paper, every constant will be denoted by K.

2. *Generalities*

2.1. *Three-interval exchange transformations.*

DEFINITION 2.1. *A* k-interval exchange transformation *is given by a probability vector* $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ *together with a permutation* π *of* $\{1, 2, \ldots, k\}$. *The unit interval* $[0, 1)$ *is partitioned into* k *sub-intervals of lengths* $\alpha_1, \alpha_2, \ldots, \alpha_k$, *which are then rearranged according to the permutation* π.

In this paper we consider a *symmetric three-interval exchange transformation*, i.e. a three-interval exchange transformation T with probability vector $(\alpha, \beta, 1-(\alpha+\beta))$,

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$\alpha, \beta > 0$, and permutation $(3, 2, 1)$† defined by

$$Tx = \begin{cases} x + 1 - \alpha & \text{if } x \in [0, \alpha), \\ x + 1 - 2\alpha - \beta & \text{if } x \in [\alpha, \alpha + \beta), \\ x - \alpha - \beta & \text{if } x \in [\alpha + \beta, 1). \end{cases} \tag{1}$$

Throughout this paper, T will denote the symmetric three-interval exchange transformation on $I = [0, 1)$ defined in equation (1). It depends only on the two parameters $0 < \alpha < 1$ and $0 < \beta < 1 - \alpha$. We note that T is continuous except at the points α and $\alpha + \beta$.

Set

$$\alpha' = \frac{1 - \alpha}{1 + \beta} \quad \text{and} \quad \beta' = \frac{1}{1 + \beta}. \tag{2}$$

Then T is induced by a rotation on the circle by angle α'. More precisely, T is obtained from the two-interval exchange R on $[0, 1)$ given by

$$Rx = \begin{cases} x + \alpha' & \text{if } x \in [0, 1 - \alpha'), \\ x + \alpha' - 1 & \text{if } x \in [1 - \alpha', 1), \end{cases}$$

by inducing (according to the first return map) on the subinterval $[0, \beta')$, and then renormalizing by scaling by $1 + \beta$.

We say T satisfies the *infinite distinct orbit condition* (*i.d.o.c.*) of Keane [**K**] if the two negative trajectories $\{T^{-n}(\alpha)\}_{n \geq 0}$ and $\{T^{-n}(\alpha + \beta)\}_{n \geq 0}$ of the discontinuities are infinite disjoint sets. Under this hypothesis, T is both minimal and uniquely ergodic; the unique invariant probability measure is given by the Lebesgue measure μ on $[0, 1)$ (and hence (I, T, μ) is an ergodic system).

Because of the connection with the inducing rotation R, the exchange T does *not* satisfy the i.d.o.c. condition if and only if one of the following holds:

- α' is rational, or equivalently $p\alpha + q\beta = p - q$;
- $\beta' = p\alpha' - q$, or equivalently $p\alpha + q\beta = p - q - 1$;
- $\beta' = -p\alpha' + q$, or equivalently $p\alpha + q\beta = p - q + 1$,

for some non-negative integer p, and positive integer q. Therefore, the i.d.o.c. condition holds for almost every (α, β).

2.2. *Results of the arithmetic algorithm.* This subsection summarizes some results from [**FHZ1**] and [**FHZ2**].

Let I denote the open interval $(0, 1)$, $D_0 \subset \mathbb{R}^2$ the simplex bounded by the lines $y = 0$, $x = 0$, and $x + y = 1$, and D the triangular region bounded by the lines $x = \frac{1}{2}$, $x + y = 1$, and $2x + y = 1$. We define two mappings on $I \times I$,

$$F(x, y) = \left(\frac{2x - 1}{x}, \frac{y}{x} \right) \quad \text{and} \quad G(x, y) = (1 - x - y, y).$$

We check that if $(\alpha, \beta) \in D_0$ is not in D and is not on any of the rational lines $p\alpha + q\beta = p - q$, $p\alpha + q\beta = p - q + 1$, $p\alpha + q\beta = p - q - 1$, then there exists

† All other permutations on three letters reduce the transformation to an exchange of two intervals.

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a unique finite sequence of integers $l_0, l_1, \ldots, l_k$ such that (α, β) is in $H^{-1}D$, where H is a composition of the form $G^t \circ F^{l_0} \circ G \circ F^{l_1} \circ G \circ \cdots \circ G \circ F^{l_k} \circ G^s$, $s, t \in \{0, 1\}$.

The function $H(\alpha, \beta)$ is computed recursively as follows: we start with $\alpha^{(0)} = \alpha$, $\beta^{(0)} = \beta$. Then, given $(\alpha^{(k)}, \beta^{(k)})$, we have three mutually exclusive possibilities: if $(\alpha^{(k)}, \beta^{(k)})$ is in D, the algorithm stops; if $\alpha^{(k)} > \frac{1}{2}$, we apply G; if $2\alpha^{(k)} + \beta^{(k)} < 1$, we apply F.

Associated to each point $(\alpha, \beta) \in D_0$ is a sequence $(n_k, m_k, \epsilon_{k+1})_{k\geq 1}$, where n_k and m_k are positive integers, and ϵ_{k+1} is ± 1. This sequence, which we call the *three-interval expansion* of (α, β), is a variant of the *negative slope expansion* defined in [**FHZ1**]. It is constructed as follows:

- For (α, β) in D we put
$$x_0 = \frac{1-\alpha-\beta}{1-\alpha} \quad \text{and} \quad y_0 = \frac{1-2\alpha}{1-\alpha}$$
and define for $k \geq 0$
$$(x_{k+1}, y_{k+1}) = \begin{cases} \left(\left\{ \dfrac{y_k}{(x_k+y_k)-1} \right\}, \left\{ \dfrac{x_k}{(x_k+y_k)-1} \right\} \right) & \text{if } x_k + y_k > 1, \\ \left(\left\{ \dfrac{1-y_k}{1-(x_k+y_k)} \right\}, \left\{ \dfrac{1-x_k}{1-(x_k+y_k)} \right\} \right) & \text{if } x_k + y_k < 1, \end{cases}$$
$$(n_{k+1}, m_{k+1}) = \begin{cases} \left(\left\lfloor \dfrac{y_k}{(x_k+y_k)-1} \right\rfloor, \left\lfloor \dfrac{x_k}{(x_k+y_k)-1} \right\rfloor \right) & \text{if } x_k + y_k > 1, \\ \left(\left\lfloor \dfrac{1-y_k}{1-(x_k+y_k)} \right\rfloor, \left\lfloor \dfrac{1-x_k}{1-(x_k+y_k)} \right\rfloor \right) & \text{if } x_k + y_k < 1, \end{cases}$$
where $\{a\}$ and $\lfloor a \rfloor$ denote the fractional and integer part of a, respectively. For $k \geq 0$ set
$$\epsilon_{k+1} = \operatorname{sgn}(x_k + y_k - 1).$$
We note that ϵ_1 is always -1, hence we ignore it in the expansion.
- For $(\alpha, \beta) \notin D$ we let H be the function above for which $(\alpha, \beta) \in H^{-1}D$ and we put
$$(\bar{\alpha}, \bar{\beta}) = H(\alpha, \beta),$$
and define $(n_k, m_k, \epsilon_{k+1})$ as in the previous case, starting from $(\bar{\alpha}, \bar{\beta}) \in D$.

The following proposition sums up what we need of [**FHZ1**]. When (α, β) is in D, it is a translation (taking into account the fact that the initial conditions are slightly different) of results in [**FHZ1**]; in the general case, it comes from these results and the definition of $(\bar{\alpha}, \bar{\beta})$.

PROPOSITION 2.1.

(1) *If T satisfies the i.d.o.c. condition, then the three-interval expansion $(n_k, m_k, \epsilon_{k+1})$ of (α, β) is infinite.*

(2) *An infinite sequence $(n_k, m_k, \epsilon_{k+1})$ is the expansion of at least one pair (α, β) defining a symmetric three-interval exchange transformation satisfying the i.d.o.c. condition, if and only if n_k and m_k are positive integers, $\epsilon_{k+1} = \pm 1$, $(n_k, \epsilon_{k+1}) \neq (1, +1)$ for infinitely many k and $(m_k, \epsilon_{k+1}) \neq (1, +1)$ for infinitely many k.*

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(3) *Each infinite sequence* $(n_k, m_k, \epsilon_{k+1})$ *satisfying the conditions in (2) is the three-interval expansion of a countable family of pairs* (α, β)*, with exactly one pair in each of the disjoint triangles* $H^{-1}D$*, where* H *has any of the possible forms defined earlier in this section, including the identity.*

(4)
$$(\bar{\alpha})' = \frac{1-\bar{\alpha}}{1+\bar{\beta}} = \cfrac{1}{2+\cfrac{1}{m_1+n_1-\cfrac{\epsilon_2}{m_2+n_2-\cfrac{\epsilon_3}{m_3+n_3-\cdots}}}}.$$

(5) α' *has bounded partial quotients (in the usual continued fraction expansion) if and only if in the three-interval expansion of* (α, β) *the* $n_k + m_k$ *are bounded, as well as the lengths of strings of consecutive* $(1, 1, +1)$.

2.3. *Results of the combinatorial description.* This subsection summarizes some results from [**FHZ2**].

Let α, β, T be as in equation (1). We define the natural partition

$$P_1 = [0, \alpha),$$
$$P_2 = [\alpha, \alpha + \beta),$$
$$P_3 = [\alpha + \beta, 1).$$

For every point x in $[0, 1)$, we define an infinite sequence $(x_n)_{n\in\mathbb{N}}$ by putting $x_n = i$ if $T^n x \in P_i$, $i = 1, 2, 3$. This sequence, also denoted by x, is called the *trajectory* of x. If T satisfies the i.d.o.c. condition, the minimality of the system implies that all trajectories contain the same finite words as factors. The shift on the set of trajectories defines a symbolic dynamical system which is called the *natural coding* of T.

Let I' be a set of the form $\bigcap_{i=0}^{n-1} T^{-i} P_{k_i}$; we say I' has a *name* of length n given by $k_0, \ldots, k_{n-1}$. Note that I' is necessarily an interval, and $k_0, \ldots, k_{n-1}$ is the common beginning of trajectories of all points in I'.

For each interval J, it is known (see, for example, [**CFS**, p. 128, Lemma 2]) that the induced map of T on J is an exchange of three or four intervals. More precisely, there exists a partition J_i, $1 \le i \le t$, of J into subintervals (with $t = 3$ or $t = 4$), and t integers h_i, such that $T^{h_i} J_i \subset J$, and $\{T^j J_i\}$, $1 \le i \le t$, $0 \le j \le h_i - 1$, is a partition of $[0, 1[$ into intervals: this is the partition into *Rokhlin stacks* associated to T with respect to J. The intervals J_i have names of length h_i, called *return words* to J.

THEOREM 2.2. (Structure Theorem) *Let* T *be a symmetric three-interval exchange transformation as defined in equation (1), satisfying the i.d.o.c. condition, and let* $(n_k, m_k, \epsilon_{k+1})_{k\ge1}$ *be the three-interval expansion of* (α, β)*. Then there exists an infinite sequence of nested intervals* J_k, $k \ge 1$*, which have name* w_k *and exactly three return words*

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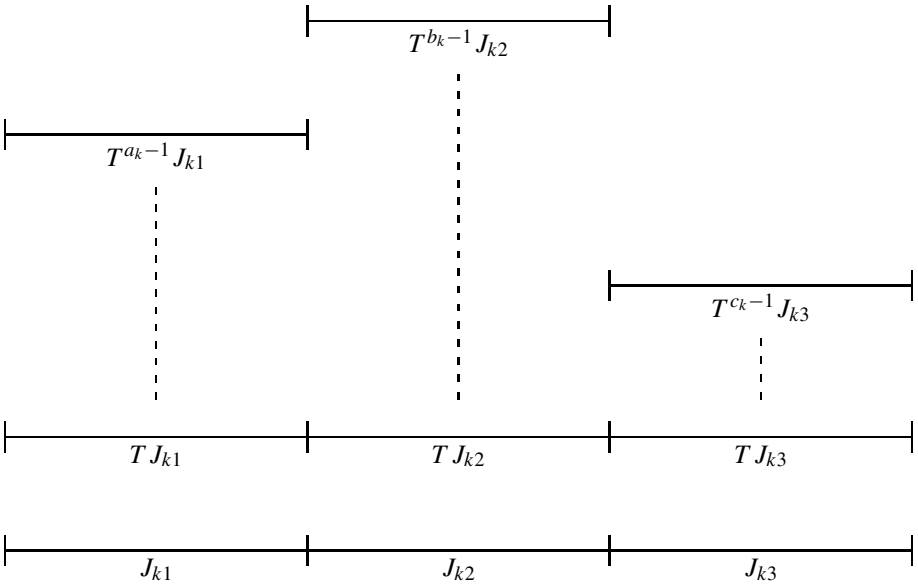


FIGURE 1. The three Rokhlin stacks.

(Figure 1), A_k, B_k, and C_k, given recursively for $k \geq 1$ by the following formulae:

$$A_k = A_{k-1}^{n_k-1} C_{k-1} B_{k-1}^{m_k-1} A_{k-1},$$
$$B_k = A_{k-1}^{n_k-1} C_{k-1} B_{k-1}^{m_k},$$
$$C_k = A_{k-1}^{n_k-1} C_{k-1} B_{k-1}^{m_k-1}$$

if $\epsilon_{k+1} = +1$, and

$$A_k = A_{k-1}^{n_k-1} C_{k-1} B_{k-1}^{m_k},$$
$$B_k = A_{k-1}^{n_k-1} C_{k-1} B_{k-1}^{m_k-1} A_{k-1},$$
$$C_k = A_{k-1}^{n_k-1} C_{k-1} B_{k-1}^{m_k} A_{k-1}$$

if $\epsilon_{k+1} = -1$, for initial words A_0, B_0, and C_0 such that the lengths of the first two differ by exactly one. The words A_k and B_k both contain w_k as a prefix.

The Structure Theorem gives an explicit construction of Rokhlin stacks.

Let w_k be the name of J_k. Every trajectory under T is a concatenation of words A_k, B_k, and C_k, which we call the *k-words*. We say that a k-word occurs at its *legal k-place* in a trajectory if it is immediately followed by the word w_k. A concatenation of k-words occurs at its legal k-place if each of its k-words occur at their legal k-place.

We define FA_k to be the set of $x \in X$ such that (in the trajectory of x) x_0 is the first letter of the k-word A_k in its legal k-place, and similarly for FB_k and FC_k. Note that $FA_k \cup FB_k \cup FC_k = J_k$, and these are exactly the three intervals of continuity of the induced map of T on J_k.

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Let a_k, b_k, and c_k be the lengths of A_k, B_k, and C_k, respectively; it follows from the Structure Theorem that $|a_k - b_k| = |a_0 - b_0| = 1$ and c_k is either $a_k - a_{k-1} = b_k - b_{k-1}$ or $a_k + a_{k-1} = b_k + b_{k-1}$. In particular, we have $c_k \leq 2a_k$. Then $X = [0, 1)$ is the disjoint union of $T^j FA_k$, $0 \leq j \leq a_k - 1$, $T^j FB_k$, $0 \leq j \leq b_k - 1$, $T^j FC_k$, $0 \leq j \leq c_k - 1$; we denote by τA_k the disjoint union of $T^j FA_k$, $0 \leq j \leq a_k - 1$, and define similarly for τB_k and τC_k. τA_k, τB_k, and τC_k are called the *k-stacks*, and the $T^j FA_k$, $0 \leq j \leq a_k - 1$, are the *levels* of the stack τA_k, and similarly for B and C. The levels are intervals of small diameter, as they have names of arbitrarily large length; hence any integrable function f can be approximated (in L_1, for example) by functions f_k which are constant on each level of each k-stack.

Hence, in the language of *finite rank systems*, see [**F**], the Structure Theorem implies the following.

COROLLARY 2.3. *T is of rank at most three, generated by the stacks τA_k, τB_k, and τC_k; the recursion formulas in the Structure Theorem give an explicit construction by the cutting and stacking of these stacks.*

The finite rank structure is particularly relevant in the measure-theoretic study of the system (I, T, μ), and in this framework the following lemma will be useful.

LEMMA 2.4. *For any trajectory x, the Lebesgue measure $\mu(\tau A_k)$ is the limit when n goes to infinity of $1/n$ times the total number of indices $0 \leq i \leq n - 1$ such that x_i belongs to a word A_k in its legal k-place. Similarly for B_k and C_k.*

Proof. This follows from the unique ergodicity of T. □

3. *Linear recurrence*

DEFINITION 3.1. *A symbolic system X is said to be* linearly recurrent *if there exists a constant K such that each word of length n occurring in a sequence of X occurs in every word of length at least Kn occurring in a sequence of X.*

We say that a three-interval exchange transformation is linearly recurrent whenever its natural coding is linearly recurrent.

PROPOSITION 3.1. *A symmetric three-interval exchange transformation satisfying the i.d.o.c. condition is linearly recurrent if and only if the m_k, n_k, the lengths of the strings of $(n_k = 1, \epsilon_{k+1} = +1)$ and the lengths of the strings of $(m_k = 1, \epsilon_{k+1} = +1)$, are bounded.*

Proof. If the condition is satisfied, then the ratios of the length of each k-word and the length of each $(k-1)$-word are bounded, while each k-word occurs in each $(k + K)$-word for a bounded K. As each word occurring in a sequence of X is a subword of some k-word, linear recurrence follows.

Conversely, if the condition is not satisfied: if m_k is not bounded, the word A_{k-1} in its legal place does not occur in $B_{k-1}^{m_k}$, whose length is at least $m_k(a_{k-1} - 1)$, and this means that $w_{k-1}A_{k-1}$, whose length is at most $2a_{k-1}$, does not occur in $w_{k-1}B_{k-1}^{m_k}$, and this prevents linear recurrence; if there are strings of length M of $(m_k = 1, \epsilon_{k+1} = +1)$, some word A_k in its legal place will not occur in B_{k+M} nor in C_{k+M}, the lengths of which are at

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least $M'a_k$ for some M' going to infinity with M, and we conclude similarly. The reasoning is similar if we replace m_k by n_k. □

PROPOSITION 3.2. *Let α' and β' be defined by equation (2). If α' has unbounded partial quotients (in the usual continued fraction expansion), T is not linearly recurrent. If α' has bounded partial quotients, the set of β' such that T is linearly recurrent is uncountable and of measure zero.*

Proof. If α' has unbounded partial quotients, assertion (5) of Proposition 2.1 implies the absence of linear recurrence.

Suppose now that α' has bounded partial quotients; we make the proof for (α, β) in the preferred triangle D, the situation in the other triangles $H^{-1}D$ being deduced by the transformation H which does not change the above properties. Then $\frac{1}{3} \leq \alpha' \leq \frac{1}{2}$; we expand it in the form

$$\alpha' = \cfrac{1}{2 + \cfrac{1}{r_1 - \cfrac{\epsilon_2}{r_2 - \cfrac{\epsilon_3}{\cdots}}}}$$

with $r_i \geq 2$ and $\epsilon_{i+1} = \pm 1$; there are many ways to do this, as at each stage we have two choices, one with $r_i = r$ and $\epsilon_i = -1$ and one with $r_i = r + 1$ and $\epsilon_i = +1$ (except if $r = 1$, where the only possible choice is $r_i = 2$, $\epsilon_{i+1} = +1$). Then, because of assertions (3) and (4) of Proposition 2.1, every possible sequence $(n_k, m_k, \epsilon_{k+1})$ with $n_k + m_k = r_k$ for every k is the three-interval expansion of some (α, β) such that $\alpha' = (1 - \alpha)/(1 + \beta)$. Hence, for any expansion (r_i, ϵ_{i+1}) and for any M such that there are at most M consecutive $(r_i, \epsilon_{i+1}) = (2, +1)$, which exists because α' has bounded partial quotients, we can find an uncountable number of expansions $(n_k, m_k, \epsilon_{k+1})$ with at most M consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$ or $(n_i, \epsilon_{i+1}) = (1, +1)$, and hence an uncountable number of β' such that T is linearly recurrent.

To prove the last assertion, we need to prove that for fixed $\alpha' = c$, where $\frac{1}{3} \leq c \leq \frac{1}{2}$ and the (usual) partial quotients of c are bounded by B, and for any M, the set of β' giving expansions with at most M consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$ or $(n_i, \epsilon_{i+1}) = (1, +1)$ is of measure zero. We use the notation of §2.2 and [**FHZ1**]; let S be the transformation sending (x_k, y_k) to (x_{k+}, y_{k+1}) in §2.2, and the Q_l be the Farey cells corresponding to the first l values of $(n_k, m_k, \epsilon_{k+1})$, see [**FHZ1**, §3]. Let Z_M be the set of (α, β) in the preferred triangle D having three-interval expansions with at least M consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$ or $(n_i, \epsilon_{i+1}) = (1, +1)$; we want to prove that each line $(1 - \alpha)/(1 + \beta) = c$ intersects Z_M on a set of full linear measure.

Let Z'_M be the set of (α, β) in the preferred triangle D having three-interval expansions beginning with M consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$; then, as the line $L_c = \{(1 - \alpha)/(1 + \beta) = c\}$ corresponds to $x_0 + y_0 = 3 - 1/c$, the set Z'_M intersects each L_c, where the partial quotients of c are bounded by B, on a set of linear Lebesgue measure of at least $s(B, M) > 0$. On the other hand, the intersection (assuming it is non-empty) of L_c with a kth level Farey quadrilateral is taken linearly by S^k to the intersection of a

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line L_d with the preferred triangle $D = Q_0$, where the partial quotients of d are bounded by $B + 1$, because of the formulas giving x_k and y_k. Hence, at least $s(B + 1, M)$ of the linear measure of the intersection of the line L_c with every kth level Farey quadrilateral Q_k is taken linearly by S^k to Z'_M; such points have a string of length M of consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$ beginning at place k in their expansions. By applying the argument to higher level quadrilaterals, neglecting boundary points, we can thus show that almost all the line L_c is in Z_M. □

Examples. The notion of linear recurrence is, for an irrational rotation, equivalent to the boundedness of the partial quotients of the angle. Thus, a linearly recurrent three-interval exchange transformation can be viewed as a three-interval exchange transformation with bounded partial quotients; as was noticed in [**B**], the three-interval exchange transformations for which α' has bounded partial quotients are not always linearly recurrent, but they always satisfy a weaker property called *predictable recurrence* (there exists a constant bounding the ratio of the highest and lowest return times of any word in their natural coding).

In [**dJ**], a one-parameter family of three-interval exchange transformations was built as follows: starting with an irrational $0 < \gamma < 1$, we put $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3} - \frac{2}{3}\gamma$ if $\gamma < \frac{1}{2}$, $\beta = 1 - \frac{2}{3}\gamma$ if $\gamma > \frac{1}{2}$. Then the three-interval expansion of (α, β) is such that $m_k = n_k$ for every $k \geq 2$: if $\gamma > \frac{1}{2}$, (α, β) is in the preferred triangle D with $\alpha' = \gamma$ and $\beta' = \frac{3}{2}\gamma$, and we check that $x_1 - y_1 = 1$, hence $m_1 = n_1 + 1$, $x_2 = y_2$, and $m_k = n_k$ for every $k \geq 2$; the case $\gamma < \frac{1}{2}$ splits into a countable number of subintervals according to the value of $(\bar{\alpha}, \bar{\beta})$, and we check that in each case $x_1 - y_1$ is an integer. Thus, among del Junco's examples, T is linearly recurrent whenever γ has bounded partial quotients, not linearly recurrent whenever γ has unbounded partial quotients.

It is proved in [**FHZ3**] that a weakly mixing three-interval exchange transformation can be linearly recurrent or not; for self-induced ones, we always have linear recurrence, as was already noticed in [**B**]. Note that linearly recurrent three-interval exchange transformations, when they satisfy the i.d.o.c. condition, are always weakly mixing: this comes from Proposition 3.1 and [**FHZ3**], or from Proposition 3.2 and [**BN**].

4. *Minimal self-joinings*

DEFINITION 4.1. *A* self-joining *(of order two) of a system (X, T, μ) is any measure ν on $X \times X$, invariant under $T \times T$, for which both marginals are μ.*

DEFINITION 4.2. *An ergodic system (X, T, μ) has* minimal self-joinings *(of order two) if any ergodic self-joining (of order two) ν is either the product measure $\mu \times \mu$ or a* diagonal *measure defined by $\nu(A \times B) = \mu(A \cap T^i B)$ for an integer i.*

The minimal self-joinings of order three or more would involve measures on higher-order Cartesian powers of X; we conjecture our results would generalize to higher orders. Throughout this paper, we use 'minimal self-joinings' for 'minimal self-joinings of order two'.

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THEOREM 4.1. *A linearly recurrent three-interval exchange transformation satisfying the i.d.o.c. condition has minimal self-joinings.*

In the remainder of this section, we shall prove this theorem, the proof being a natural generalization of the proof that Chacon's map has minimal self-joinings [**dJRS**]; a more involved generalization of that proof was made in [**Ra**] for the horocycle flow, and indeed our proof uses (implicitly) Ratner's famous *R-property*.

Let T be a linearly recurrent three-interval exchange transformation satisfying the i.d.o.c. condition. If we denote by h_k the smallest of the heights a_k, b_k, and c_k, we have $h_k \leq a_k \leq Kh_k$, $h_k \leq b_k \leq Kh_k$, $h_k \leq c_k \leq Kh_k$, and $h_k \leq h_{k+1} \leq Kh_k$.

For a point x, we say equivalently that x is in the ith level of the stack τW_k (W_k being A_k, B_k, or C_k), or that x_0 is in position i in the word W_k. This word W_k is a subword of a $(k+1)$-word occurring in its legal k-place according to the recursion formulas of the Structure Theorem; we say it has *order* j if it is the jth k-word in this $(k+1)$-word ; j is an integer between 1 and $m_{k+1}+n_{k+1}$, or $m_{k+1}+n_{k+1}+1$, or $m_{k+1}+n_{k+1}-1$. If x and y are in the same stack τW_k, we say x_0 and y_0 are in k-words of the same type (A, B, or C). If x and y are in the same stack and in the same column, then x_0 and y_0 are in k-words of the same type and of the same order inside $(k+1)$-words of the same type. If x and y are in the same stack but in different columns, then x_0 and y_0 are in k-words of the same type and with different orders, or else with the same order but inside $(k+1)$-words of different types.

If x_0 is at position i in its k-word which we denote by $X_k(0)$ and y_0 is at position j in the word $Y_k(0)$, the *overlap* between $X_k(0)$ and $y_k(0)$ is $\omega = i \wedge j + (|X_k(0)| - i) \wedge (|Y_k(0)| - j)$. We write then, for a given r, the $(r+1)$ first k-words in x and y above each other:

$$X_k(0)X_k(1)\dots X_k(r),$$
$$Y_k(0)Y_k(1)\dots Y_k(r).$$

We call $(X_k(i), Y_k(i))$ the pairs of *corresponding* k-words, the initial one being for $i = 0$. Note that there is not necessarily a positive overlap between $X_k(s)$ and $Y_k(s)$ for $1 \leq s \leq r$, in the following we use only the fact that the initial overlap is large enough.

DEFINITION 4.3. *For integers $k > 0$ and $t > 0$, x and y have a* k-forcing *of* shift t *if there exist integers $l \geq h_k/K$ and $-Kh_k \leq i_1 < i_2 \leq Kh_k - l$, and subwords u and v of x and y such that on (x, y) from i_1 to $i_1 + l$ we see $(u_0 \dots u_l, v_0 \dots v_l)$ while on (x, y) from i_2 to $i_2 + l - t$ we see $(u_0 \dots u_{l-t}, v_t \dots v_l)$ or $(u_t \dots u_l, v_0 \dots v_{l-t})$.*

LEMMA 4.2. *There exists a set E_0 of measure one such that if x and y are in E_0 and belong to different orbits, then, for infinitely many values of k, x_0 and y_0 are in k-words of different types or of different orders, with an overlap between these k-words larger than h_k/K.*

Proof. We check that, for x in a set E_0 of full measure, there exist infinitely many l such that x_0 is not in the first or the last h_l/K positions in its l-word (this come from the independence of these events when k varies). We start from x in X_0 and take such an l. If, for every $k \geq l$, x_0 and y_0 are in the same k-word, they are always in the same column of the k-stack, and their difference of level $i(x, y)$ is constant: then x and $T^i y$ are not

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separated by the partition into levels of the stacks and x and y are on the same orbit. So we take the first $k \geq l$ such that x_0 and y_0 are in different k-words. Then, if $k = l$, the position of x_0 in its l-word guarantees an overlap of at least h_l/K to the left or to the right of (x_0, y_0); if $k > l$, x_0 and y_0 are in a $(k-1)$-word of the same order, which guarantees an overlap of at least h_{k-1}, except in the case where a k-word is made with only one $(k-1)$-word, which happens only when $C_k = C_{k-1}$ because $(n_k, m_k, \epsilon_{k+1}) = (1, 1, +1)$, and in that case we wait until the first l such that $(n_l, m_l, \epsilon_{l+1}) \neq (1, 1, +1)$, which is at most $l + K$, and the overlap will be at least h_l/K. □

LEMMA 4.3. *If x and y are in E_0 and belong to different orbits, then there exist an integer $t > 0$ and infinitely many integers $k > 0$ such that x and y have a k-forcing of shift t.*

Proof. We take one of the k in the conclusion of Lemma 4.2. Take, for example, a k such that A_k is the longest of the two words A_k and B_k. We know that A_k, B_k, and C_k are return words of the word w_k; w_k is a prefix of A_k and B_k, and either w_k is a prefix of C_k or C_k is a prefix of w_k and w_k, A_k, B_k, C_k have lengths $h+2, h+1, h, c$, respectively, for some h and c. Each time A_k, B_k or C_k occur in their legal position in an orbit, they are followed by w_k. Occasionally, we shall write A for A_k and so on.

Case 1. x_0 and y_0 are in k-words of different type, x_0 is in A_k, y_0 is in B_k. Suppose, for example, $x_i \dots x_{i+h+1} = A_k$, and $y_{i+d} \dots y_{i+d+h} = B_k$ with $0 < d < h$. Then on (x, y) from $i + d$ to $i + h - 1$ we see $(w_d \dots w_{h-1}, w_0 \dots w_{h-d-1})$ while on (x, y) from $i + d + h + 1$ to $i + 2h$ we see $(w_{d-1} \dots w_{h-2}, w_0 \dots w_{h-d-1})$. $h - d$ is the overlap and is at least h_k/K, while $i + d + h + 1$ is between 0 and Kh_k, so we have a k-forcing with shift 1. Of course, the same is true for negative values of d and cases when A_k is shorter than B_k.

Case 2. x_0 and y_0 are in k-words of different type, x is in A_k, y is in C_k. Suppose first that $m_{k+1} \geq 2$; then after a C there is always a B, followed possibly by other Bs, then by a string of n_{k+1} or $n_{k+1} - 1$ As and a C, while, after an A, we see a string of length at most $n_{k+1} - 1$ of As then a C. Thus, in x from place i we see AA^rC and in y from place $i + d$ we see $CBD_1 \dots D_r$, where $0 \leq r \leq K$ and each D_i can be either an A or a B.

Suppose we see AC in x starting at place i, over CB in y, starting at place $i + d$; note that in fact we see ACw in x, CBw in y. Let h' be the smallest of h and c. Then on (x, y) from $i + d$ to $i + h' - 1$ we see $(w_d \dots w_{h'-1}, w_0 \dots w_{h'-d-1})$ while on (x, y) from $i + d + c + h + 1$ to $i + c + h + h'$ we see $(w_{d-1} \dots w_{h'-2}, w_0 \dots w_{h'-d-1})$; for the same reasons as in the last case, this gives a forcing of shift one. Suppose now we see AAC in x starting at place i, over CBB in y, starting at place $i + d$; note that in fact we see $AACw$ in x, $CBBw$ in y. Then on (x, y) from $i + d$ to $i + h' - 1$ we see $(w_d \dots w_{h'-1}, w_0 \dots w_{h'-d-1})$ while on (x, y) from $i+d+c+2h+2$ to $i+c+2h+h'+1$ we see $(w_{d-2} \dots w_{h'-3}, w_0 \dots w_{h'-d-1})$; here we have a forcing of shift 2. Similarly, the general case gives a forcing of shift $0 < r' < K$, the length l being at least the initial overlap minus r'.

Case 3. x is in A_k, y is in C_k, $m_{k+1} = 1$ and $\epsilon_{k+2} = -1$. The k-words occur in strings of the form $A^{n-1}CA$, $A^{n-1}CB$, and $A^{n-1}CBA$; we wait until we see a B in x or y, which will occur r words after the initial A or C, and this r is bounded as at least one $A^{n-1}CBA = C_{k+1}$ will occur in each $(k+2)$-word.

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Let us show that *we do not see the first B at the same time in x and y*. Suppose, for example, this B occurs in y. The first B in each of the two types of $(k+1)$-word where it may occur has the same order (considering the order of a k-word inside its $(k+1)$-word). Moreover, C_{k+1} contains B, hence none of the pairs of corresponding $(k+1)$-words between (strictly) the initial one and the one containing the first B can have a C_{k+1}; as for the initial one, it can have a C_{k+1}, but then it contains the first B, except in the exceptional case where the word A containing x_0 is the last k-word of its C_{k+1}. Now, the initial k-words A in x and C in y have different orders. We cannot overcome this difference of order unless we see one C_{k+1}, as A_{k+1} and B_{k+1} have the same number of k-words; hence, if there is any pair of corresponding $(k+1)$-words after (strictly) the initial one and before the one containing the first B, when we move to this second pair of corresponding $(k+1)$-words, two corresponding k-words also have different orders, either because we have not seen a C_{k+1}, or because we are in the exceptional case above and check that it must still be so. After that and until the pair of corresponding $(k+1)$-words containing the first B, we do not see a C_{k+1}; hence two corresponding k-words inside the $(k+1)$-words containing the first B cannot have the same order, so there cannot be another B above the first B.

As for the words C, as $(n_{k+1}, m_{k+1}, \epsilon_{k+2}) \neq (1, 1, +1)$ they are always isolated. Then, if the first B is in y, we write the words of (x, y) until the end of the first C in x after this B in y (note that this will not bring another B in x or y as the Bs are isolated and occur only after Cs); if this B is in x, we stop at its end. Because we have not seen a C_{k+1}, we have seen the same number of $(k+1)$-words above and below, and the number of C above and below until we stop is the same. So we have two strings of corresponding k-words, made with one B, and some words A and C with the same number of C above and below. We compare as in the previous case (x, y) from $i+d$ to $i+h'-1$ with (x, y) just after we stopped, and we get a k-forcing of shift 1, because the length of the string we compare is at least $\omega - K$ and the position of the furthest one is bounded by Kh_k.

Case 4. x is in A_k, y is in C_k, $\epsilon_{k+2} = +1$ and $m_{k+1} = 1$. This cannot be solved at the level of k-words. However, the strings of $(m_l = 1, \epsilon_{l+1} = +1)$ have bounded length: so let z be the first $l \geq k+1$ such that $(m_{l+1}, \epsilon_{l+2}) \neq (1, +1)$; z is smaller than $k+K$, B does not occur in any A_l or C_l for $k+1 \leq l \leq z$, but occurs in B_{z+1}, C_{z+1} and A_{z+1} if $m_{z+1} > 1$, in A_{z+1} and C_{z+1} otherwise. Hence we see a B in x or y after a bounded number of k-words. From the formulas giving the $(z+1)$-words as concatenations of k-words, we check that the first B in each of the two or three types of $(z+1)$-words where it may occur has the same order, considering orders of k-words inside $z+1$-words, and that the initial words A in x and C in y have different orders (if we see different types of words with the same order, it can only be an A and a B). Again, we cannot overcome this difference of order unless we see one C_{z+1}, and C_{z+1} contains B; hence none of the pairs of $(k+1)$-words we see in x and y between (strictly) the initial one and the one containing the first B can have a C_{k+1}. We can now show, as in Case 3, that there is no B under (or above) the first B we see, with only one problem: the case when the initial pair of $(z+1)$-words has a C_{z+1} but does not contain a B after the initial position. Note that this can only happen if $\epsilon_{z+2} = -1$ and x_0 or y_0 is in the last A_z of C_{z+1}; also, if this pair is made with two C_{z+1}, the problem is avoided. Otherwise, for example, x_0 is in the last A_z of C_{z+1}, while y_0 is in a B_z (and then we conclude by Case 1, as the overlap is a bounded proportion of the length of

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the z-words), or in a C_z (and then we conclude by Case 2 or 3, as $(m_{z+1}, \epsilon_{z+2}) \neq (1, +1)$, or in an A_z; in that last case, there will still be a difference of orders of the matched k-words in the next pair of $(z+1)$-words (considering the order of k-words inside $(z+1)$-words), and we can continue as in Case 3, unless (maybe) y_0 is in the last A_z of B_{z+1}; in that last subcase, $B_{z+1} = A_z^{t-1} C_z B_z^{u-1} A_z$ and $C_{z+1} = A_z^{t-1} C_z B_z^u A_z$ with $u \geq 1$, so we see a B_z above a C_z with $\epsilon_{z+2} = -1$, hence we conclude by the symmetric of Case 2 or 3. Then we finish Case 4 in the same way as Case 3, except that the C are not isolated; but, as we see the same number of C_{z+1} above and below, again we can stop in such a way that we see the same number of C above and below.

Symmetrics of Cases 2, 3, and 4. If x is in B_k, y is in C_k, a similar reasoning holds, replacing the m_k by the n_k: in Case 2S ($n_{k+1} \geq 2$), we go left instead of going right from (x_0, y_0), using that before a C there is an A. In Case 3S, we also go left, using that the first A we see on the left is the last k-word in any of the $(k+1)$-words where it may occur. In Case 4S, z is the first $l \geq k+1$ such that $(n_{l+1}, \epsilon_{l+2}) \neq (1, +1)$, and we go right if $n_{z+1} \geq 2$, and left if $n_{z+1} = 1$ and $\epsilon_{z+2} = +1$.

Of course, x and y play symmetric parts.

Other cases, when x and y are in the same k-stacks, in different columns. For example, x_0 is in A_k, y_0 is in A_k, but not in the same k-word inside their $(k+1)$-word. Then a translation sends us to one of the cases 2 to 4, or 2S to 4S.

Now, as we have infinitely many k-forcings with shift $0 < t_k < K$, there exists one t such that we have infinitely many k-forcings with shift t. □

LEMMA 4.4. [**dJRS**, Proposition 2, p. 278] *If (Y, S, ρ) is an ergodic transformation, and ν an ergodic measure on $Y \times Y$, with marginals ρ on each copy of Y, invariant under $S \times I$, where I is the identity, then $\nu = \rho \times \rho$.*

Proof of Theorem 4.2. Let ν be an ergodic joining; we choose a point (x, y) generic for ν, such that x and y are in E_0 (it is possible because the marginals are μ). If x and y are on the same orbit under T, we check that ν is diagonal. Henceforth, we suppose x and y are not on the same orbit. We then let t be given by Lemma 4.3.

Let P and Q be two arbitrary cylinders, of names p and q, and let us show that $\nu(P \times T^t Q) = \nu(P \times Q)$. If this holds for every cylinder, we shall deduce that ν is $I \times T^t$-invariant, and so is the product measure, as T^t is ergodic because T is weakly mixing.

We fix an ϵ, and take a k such that we have a k-forcing with shift t. If k is large enough, genericity ensures that on every segment $(x, y)_0, \ldots, (x, y)_l$ or $(x, y)_{-l}, \ldots, (x, y)_0$, where $l > \epsilon h_k / K$, the pair of words (p, q) appear with a frequency $(\epsilon / 2K^3)$-close to $\nu(P \times Q)$. We deduce that these pairs appear with a frequency (ϵ / K^3)-close to $\nu(P \times Q)$ in every segment of (x, y) of length at least h_k / K containing the origin.

Let us now take the string from i_1 to $i_1 + l$ given by the k-forcing: let ξ be the density of z in $(i_1, i_1 + l)$ for which $T^z x \in P$, $T^z y \in Q$; we have $|\xi - \nu(P \times Q)| < \epsilon/2$, for otherwise there would be a proportion bigger than $(\epsilon / 2K^2) - (\epsilon / K^3)$ of errors on the segment $(0 \wedge i_1, i_1 + l)$ or on the segment $(i_1, 0 \vee i_1 + l)$. Similarly we can ensure that, if ξ' is the density of z in $i_2, i_2 + l$ for which $T^z x \in P$, $T^z y \in T^t Q$, we have $|\xi' - \nu(P \times T^t Q)| < \epsilon/2$. But the forcing implies $\xi = \xi'$, so we deduce that $|\nu(P \times Q) - \nu(P \times T^t Q)| < \epsilon$. □

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5. *Rigidity*

DEFINITION 5.1. *A system* (X, T, μ) *is* rigid *if there exists a sequence* $s_n \to \infty$ *such that for any measurable set* A

$$\mu(T^{s_n} A \Delta A) \to 0.$$

THEOREM 5.1. *A nonlinearly recurrent three-interval exchange transformation satisfying the i.d.o.c. condition is rigid.*

Proof. It is enough to prove that for a given set E there exists a sequence $s_n \to \infty$ such that $\mu(T^{s_n} E \Delta E) \to 0$; such a sequence will be called a *rigidity sequence* for T.

Case 1. $\limsup(n_k/m_k) = +\infty$ or $\limsup(m_k/n_k) = +\infty$. Suppose, for example, that $n_k/m_k \to +\infty$ for k in a sequence S; then the measure of the stack τA_{k-1} is at least $(n_k - 1)/(m_k + n_k + 1)$, and hence tends to one on S, and, for a level L of this stack, $\mu(T^{a_{k-1}} L \Delta L) \leq \mu(L)/n_k$. Hence, by approximating A by levels of τA_{k-1}, we get the desired relation with the rigidity sequence a_{k-1}, $k \in S$.

Case 2. n_k/m_k and m_k/n_k are bounded, but $m_k + n_k$ are unbounded. Suppose, for example, $n_k \to +\infty$ for $k \in S$; we also have $m_k \to +\infty$ for $k \in S$, as $m_k \geq K n_k$; we can also suppose that for $k \in S$, $b_{k-1} = a_{k-1} + 1$. Note that, for a large enough $k \in S$, the proportion of the word A_{k-1} inside each of the three k-words is approximately the same, and hence close to $\mu(A_{k-1})$, and the same is true for B_{k-1} and C_{k-1}. Hence, for a large enough $k \in S$, each level of a k'-stack, $k' \leq k - 1$, is ϵ-independent of each k''-stack, $k'' \geq k$.

Suppose we have found some s and some $0 < t \leq 1$ such that $\mu(T^s A \Delta A) < t\mu(A)$, and fix ϵ. For $k \in S$ large enough, E is ϵ-close to a set $E_1 \cup E_2$, where E_1 is a union of levels of τA_{k-1} and E_2 is a union of levels of τB_{k-1} (the measure of τC_{k-1}, being at most $2/(n_k + m_k - 1)$, tends to 0 on S). By going to a larger $k \in S$ if necessary, we can ensure that for $i = 1, 2$, $\mu(E_i)$ is close to $K_i \mu(E)$, where $K_1 = \mu(\tau A_{k-1})$ is bounded away from 0 and 1 and $K_2 = 1 - K_1$. Similarly, we can ensure that $T^s E \cap E$ is cut into E_1' and E_2', unions of levels of the two stacks, with $\mu(E_i')$ close to $K_i \mu(T^s E \cap E)$; and, if s is small compared with a_{k-1}, E_1' is ϵ-close to $(T^s E_1 \cap E_1)$ and E_2' is ϵ-close to $(T^s E_2 \cap E_2)$. If k is large enough, s is also small compared with n_k and m_k.

Now let $s' = s a_{k-1} + s = s b_{k-1}$; by $T^{s'}$, each level of E_1, except the ones situated at a height in τA_{k-1} bigger than $a_{k-1} - s$, and except for a proportion of each level of at most s/n_k, is sent into a level of τA_{k-1} situated s levels above; by $T^{s'}$, each level of E_2, except for a proportion of each level of at most s/m_k, is sent into the same level of τB_{k-1}. Hence

$$\mu(T^{s'} E \cap E) \geq \mu(T^s E_1 \cap E_1) + \mu(E_2) - K\epsilon \geq K_1(1 - t)\mu(E) + K_2 \mu(E) - K\epsilon;$$

and $\mu(T^{s'} E \Delta E) \leq K_1 t - K\epsilon \leq Kt$ for a constant $K < 1$. So we get the required relation with a rigidity sequence of the form $b_{k_1 - 1} \ldots b_{k_n - 1}$, for some subsequence k_i of S.

Case 3. There are unbounded strings of $(n_k = 1, m_k = 1, \epsilon_{k+1} = +1)$. Then for k in a sequence S there exists $p(k) \to +\infty$ such that $A_{k+p} = C_k^p A$, $B_{k+p} = C_k^p B$, $C_{k+p} = C_k$. As in Case 1, the rigidity holds with the rigidity sequence $c_{p(k)}$, $k \in S$.

Case 4. The strings of $(n_k = 1, m_k = 1, \epsilon_{k+1} = +1)$ are bounded but there are unbounded strings of $(m_k = 1, \epsilon_{k+1} = +1)$ or of $(n_k = 1, \epsilon_{k+1} = +1)$. We take a long

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string of ($m_k = 1, \epsilon_{k+1} = +1$); thus $A_k = A_{k-1}^{n_k-1} C_{k-1} A_{k-1}$, $C_k = A_{k-1}^{n_k-1} C_{k-1}$, and B_k has small measure as long as we are far from the (upper) end of the string. We do not change anything (and in particular the measure of the stacks) if we replace these rules by $A_k = A_{k-1}^{n_k} C_{k-1}$, $C_k = A_{k-1}^{n_k-1} C_{k-1}$. As in Case 2, we take a set E and a k_0 large enough so that E is almost made of levels of τA_{k_0-1} and of τC_{k_0-1}, and we suppose k_0 is the beginning of a string of ($m_k = 1, \epsilon_{k+1} = +1$).

By $T^{a_{k_0-1}} E$, every level of E is sent on a level of E, except those which are in C_{k_0-1} and the part of those in A_{k_0-1} corresponding to the last A_{k_0-1} in a string of $A_{k_0-1}^{n_{k_0}-1}$ or $A_{k_0-1}^{n_{k_0}}$. By $T^{a_{k_0}} E$, every level of E is sent on a level of E, except those corresponding to the last A_{k_0-1} and the C_{k_0-1} in a string of $A_{k_0-1}^{n_{k_0}}$ followed by a string of $A_{k_0-1}^{n_{k_0}-1}$. By $T^{a_{k_0+1}} E$, every level of E is sent on a level of E, except those corresponding to the last A_{k_0-1} and the C_{k_0-1}, inside a string of $A_{k_0-1}^{n_{k_0}}$ followed by a string of $A_{k_0-1}^{n_{k_0}-1}$, which are themselves inside a string of $A_{k_0}^{n_{k_0+1}}$ followed by a string of $A_{k_0}^{n_{k_0+1}-1}$. By continuing, we see that $\mu(T^{a_{k_0+l}} E \Delta E)$ is small whenever we see between k_0 and $k_0 + l$ a large enough number of $n_i \geq 2$; but, as we are not in Case 3, this happens for infinitely many k_0 and l large enough. □

COROLLARY 5.2. *A three-interval exchange transformation satisfying the i.d.o.c. condition either has minimal self-joinings or is rigid. In particular, its centralizer is either trivial or uncountable.*

Examples. From §§2 and 4 we get that del Junco's examples are rigid whenever γ has unbounded partial quotients, which was conjectured but not proved in [**dJ**].

There exist systems which do not satisfy the dichotomy in the above corollary; for example, for the shift T associated to the usual *Morse sequence*, it is proved in [**L**] that the centralizer is generated by the powers T^n and the *flip* map ϕ, hence is countable but not trivial. Hence we get a new, completely different, proof of a result of [**BCF**].

COROLLARY 5.3. *A three-interval exchange transformation satisfying the i.d.o.c. condition cannot be measure-theoretically isomorphic with the shift associated to the usual Morse sequence.*

6. *Simplicity*

DEFINITION 6.1. *An ergodic system (X, T, μ) is* simple of order two *if any ergodic self-joining of order two ν is either the product measure $\mu \times \mu$ or a measure defined by $\nu(A \times B) = \mu(A \cap S^{-1} B)$ for some measurable transformation S commuting with T.*

Again, we use here 'simplicity' for 'simplicity of order two'.

THEOREM 6.1. *There exist uncountably many weakly mixing, simple, rigid three-interval exchange transformations.*

Proof. Build a three-interval exchange transformation such that, for every k, n_k is at least $2^k m_k$, and $a_k = (m_k + n_k) a_{k-1} + 1$. There are uncountably many ways to build such a system: for example, choose $\epsilon_{k+1} = +1$ for all k, $b_0 = a_0 + 1$ (by selecting a suitable triangle $H^{-1} D$ for the parameters), and choose for all k, $m_k = a_{k-2} + 2$ and some

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$n_k > 2^k m_k$; the required value of a_k then follows from the facts that $b_{k-1} = a_{k-1} + 1$ and $c_{k-1} = a_{k-1} - a_{k-2}$.

Thus the stacks τB_k and τC_k have measure at most 2^{-k}. The system (X, T, μ) is then of *rank one* as the sequence of stacks τA_k generate the whole space; see, for example, [**F**] for precise definitions. The formula

$$A_k = A_{k-1}^{n_k-1} C_{k-1} B_{k-1}^{m_k-1} A_{k-1},$$

together with those giving B_k and C_k, defines the system up to measure-theoretic isomorphism.

We define the rank-one system (X', T', μ') by the formula

$$A'_k = (A'_{k-1})^{n_k-1} s^{c_{k-1}+b_{k-1}(m_k-1)} A'_{k-1},$$

starting from $A'_0 = A_0$. We build a measure-theoretic isomorphism in between (X, T, μ) and (X', T', μ'), by sending the jth level of τA_k to the jth level of the kth stack $\tau A'_k$ for T': it is consistent by construction, and is defined almost everywhere because the total proportion of letters s ('spacers') in A'_k, and the total proportion of words B_{k-1} and C_{k-1} in A_k are general terms of a convergent series.

In the same way, the rank-one system (X'', T, μ'') defined by the formula

$$A''_k = (A''_{k-1})^{n_k} s (A''_{k-1})^{m_k}$$

is measure-theoretically isomorphic to (X, T, μ) and (X', T', μ'); and this last system is weakly-mixing, rigid and simple exactly in the same way as del Junco–Rudolph's map in [**dJR1**] (which has the same definition, but with n_k and m_k both replaced by 2^k). □

Note that there are many other possible families of parameters giving weakly mixing, simple and rigid three-interval exchange transformations: all that is needed to reproduce the above reasoning is that n_k is much larger than m_k, and that $c_{k-1} + b_{k-1}(m_k - 1) = p_k a_{k-1} + 1$ for an integer p_k. By modifying slightly the reasoning of [**dJR1**], we can get the same result while weakening this condition to $c_{k-1} + b_{k-1}(m_k - 1) = p_k a_{k-1} + q_k$, where q_k is a bounded integer, provided that a condition on the $a_k = (n_k + p_k)a_{k-1} + q_k$ ensures weak mixing, for example $q_k - q_{k-1} = \pm 1$ or $q_k - 2q_{k-1} + q_{k-2} = \pm 1$ for all k. Similar formulas ensure the same results if we choose m_k much larger than n_k.

However, we have only a small class of simple rigid maps to show, and thus we are quite far from a positive answer to Veech's question; the structure of maps in our class is always similar to that of del Junco–Rudolph's map. As for del Junco–Rudolph's map itself, we do not know whether it is actually measure-theoretically isomorphic to a three-interval exchange transformation, as the 2^k in the defining formula do not grow fast enough to allow us to prove it.

7. *Ubiquity of three-interval exchange transformations*

The fact that a version of the well-known del Junco–Rudolph's map is indeed a three-interval exchange transformation was completely unexpected, and even unsuspected. Another abstract construction which is apparently not connected with interval exchange

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transformations is *Katok's rank-one transformation*, described in [**G**]: this is in fact a family of rank-one systems, each one being described by the recursion formula

$$W_k = (W_{k-1}s)^{p_k}(W_{k-1})^{p_k}$$

for a given sequence (p_k) going to infinity fast enough.

The same reasoning as in Case 2 of Theorem 5.1 above proves the following result, which was noticed some time ago by one of the authors (in answer to a question of M. Lemańczyk) but did not appear in print.

PROPOSITION 7.1. *For any sequence (p_k) going to infinity, the corresponding Katok's map is rigid.*

Indeed, the abstract structure of Katok's map seems to be widely spread among rigid three-interval exchange transformations; in particular we have the following.

PROPOSITION 7.2. *If (p_k) is a sequence of integers growing fast enough, the corresponding Katok's map is measure-theoretically isomorphic to a three-interval exchange transformation.*

Proof. Let W_0 be the initial word of the rank-one map T', and h_k the length of W_k, and suppose

$$\sum_{k=1}^{+\infty} \frac{h_{k-2}}{p_k} < +\infty.$$

We build a three-interval exchange transformation T defined by a word B_0 of length h_0 and word A_0 of length $h_0 + 1$ (this is possible by selecting a particular triangle $H^{-1}D$ for the parameters), and then inductively by $\epsilon_{k+1} = +1$, $m_k = p_k - 1 - b_{k-1} + c_{k-1}$, $n_k = p_k + 1 + b_{k-1} - c_{k-1}$.

Thus we have $a_k = h_k + 1$, $b_k = h_k$ for all k, and $b_{k-1} - c_{k-1} = h_{k-2}$ for $k \geq 2$.

We modify T into T_1 by changing all the letters of each C_k and the last letter of each A_k into spacers s (remember that A_k deprived of its last letter is just B_k). T_1 is measure-theoretically isomorphic to T in the same way as in the proof of Theorem 6.1 (we have just changed a small part of A to a spacer), and is a rank-one map given by the formulas

$$B'_k = (B'_{k-1}s)^{n_k-1}s^{c_{k-1}}(B'_{k-1})^{m_k}.$$

Now, still with the same method of proof, both T' and T_1 are measure-theoretically isomorphic to the transformation T_2 defined by

$$B''_k = (B''_{k-1}s)^{p_k}s^{c_{k-1}+h_{k-2}(1+b_{k-1})}(B''_{k-1})^{m_k}. \qquad \square$$

The first interest of Katok's maps resides in the particular properties, shown by Katok and written in [**G**], of a specific self-joining, the Cartesian square $(T \times T, \mu \times \mu)$; among all transformations, this was the first known Cartesian square to be loosely Bernoulli (other ones are built in [**G**]), and, to this date, the only known Cartesian square to have a spectrum of finite multiplicity (namely, at most four), both these phenomena occurring whenever p_k is large compared to h_{k-1}, thus, in view of the previous result, for maps which are indeed three-interval exchange transformations. These properties both come from the stronger

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notion of *local rank one*, which is used unknowingly in [**G**], and for which we refer the reader to [**F**]. In fact these properties are shared by many other three-interval exchange transformations.

PROPOSITION 7.3. *If a three-interval exchange transformation T is such that, on an infinite set of k, $(m_k + n_k)/a_{k-1} \to +\infty$ and $1/K < m_k/n_k < K$, then its Cartesian square $(T \times T, \mu \times \mu)$ has local rank one, hence is loosely Bernoulli, and has a spectrum with multiplicity at most $\lfloor 2 + K + 1/K \rfloor$.*

Proof. Then we can reproduce the proof in [**G**]: for a k large enough, we can consider the stack for $T \times T$, of basis $FA_{k-1} \times FB_{k-1}$ and of height $a_{k-1}b_{k-1}$; it can be used to approximate every partition on a proportion ρ of the space close to $m_k n_k/(m_k + n_k)^2$. This implies local rank one, the loose Bernoulli property, and spectral multiplicity bounded by $[1/\rho]$. □

QUESTION 7.1. *Is Katok's map simple?*

This question seems difficult, but an answer (positive or negative) would certainly be a substantive step towards Veech's question.

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