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The H-function and its application to a problem in semi-circular electron focusing in metals

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Abstract. In this paper, a method is given for solving certain types of integral equations. It involves a technique of building up the solution of the given integral equation from that of a much simpler auxiliary equation and a consideration of this auxiliary equation forms a large section of the work. The method of solution is applied to a particular problem in semi-circular electron focusing in metals.

1. *Introduction.* In this paper we are concerned with the solution $\phi(\eta)$ of the integral equation

$$\phi(\eta) = \int_0^\infty \phi(\xi) K_1(|\xi - \eta|)\, d\xi + \frac{1}{2\pi}\int_0^\pi \{\epsilon(\theta, \beta) + \phi(0)\} \exp(-\eta \operatorname{cosec}\theta)\, d\theta, \tag{1}$$

where
$$\epsilon(\theta, \beta) + \phi(0) = \operatorname{cosec}\theta \exp(-i\beta \sin\theta) \int_0^\infty \phi(\xi) \exp(-\xi \operatorname{cosec}\theta)\, d\xi, \tag{2}$$

and
$$K_1(u) = \frac{1}{\pi}\int_1^\infty \{\exp(-ux)/\sqrt{(x^2-1)}\}\, dx. \tag{3}$$

These equations arose in connexion with a proposed experiment in semi-circular electron focusing in metals. Professor A. B. Pippard wished to evaluate the constant term in the asymptotic expansion of $\phi(\eta)$ in order to ascertain whether the experiment would yield measurable results. His account of the proposed experiment and of the physical theory involved is given in the Appendix.

The mathematics of the problem is of real interest and, further a method of solution is employed which it should be possible to generalize for use in similar problems. The solution provides a good example of a technique involving the building up of the solution of the given integral equation from the solution of a much simpler auxiliary equation. This method necessitates the investigation of the H-function of Chandrasekhar for a characteristic function Ψ which is not a polynomial (see (2), chapter 5). Much of the paper is devoted to this investigation and to the solution of the appropriate auxiliary equation. After this the remainder of the solution is fairly simple though somewhat tedious. It is hoped that the work will provide an outline of a method which can be used for many physical problems.

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2. *The equations of the problem.* The relevant equations (see Appendix) are (1), (2) and (3) of section 1, where, in addition, it is known that

$$\epsilon(\theta,\beta) = \epsilon(\pi-\theta,\beta) \tag{4}$$

and it is assumed that

$$\int_0^\pi |\epsilon(\theta,\beta)+\phi(0)|\,d\theta \tag{5}$$

exists.

If we now write (see (1), section 13)

$$\Psi(x) = 1/\{\pi\sqrt{(1-x^2)}\}, \tag{6}$$

$$K_1(u) = \int_1^\infty \Psi(x^{-1})\,x^{-1}\exp(-ux)\,dx, \tag{7}$$

$$L_\eta\{\phi(\xi)\} = \int_0^\infty \phi(\xi)\,K_1(|\xi-\eta|)\,d\xi, \tag{8}$$

then equations (1) and (2) become

$$\phi(\eta)-L_\eta\{\phi(\xi)\} = \frac{1}{2\pi}\int_0^\pi \{\epsilon(\theta,\beta)+\phi(0)\}\exp(-\eta\operatorname{cosec}\theta)\,d\theta, \tag{9}$$

$$\epsilon(\theta,\beta)+\phi(0) = \operatorname{cosec}\theta\exp(-i\beta\sin\theta)\int_0^\infty \phi(\xi)\exp(-\xi\operatorname{cosec}\theta)\,d\xi. \tag{10}$$

From the form of the equations, it can be seen that the Ambartsumian technique should be applicable, with auxiliary equation

$$J(\eta,\theta)-L_\eta\{J(\xi,\theta)\} = \exp(-\eta\operatorname{cosec}\theta).$$

In order to make use of this method, we need to consider a function $H(z)$ satisfying the equation

$$\frac{1}{H(z)} = 1-z\int_0^1 \frac{\Psi(x)\,H(x)}{z+x}\,dx. \tag{11}$$

In this case

$$\psi_0 = \int_0^1 \Psi(x)\,dx = \frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = \frac{1}{2}, \tag{12}$$

and so we are dealing with the conservative case.

Dr Busbridge, in her book ((1)) considers the case in which

(i) Ψ is real and non-negative in the closed interval $[0, 1]$,

(ii) every point of the closed interval $[-1, +1]$ lies in a domain D for which Ψ is regular,

(iii) $\psi_0 \leqslant \frac{1}{2}$.

Quite clearly the function Ψ occurring in our problem does not belong to the class of functions discussed by Dr Busbridge. It is possible, however, to adapt many of her methods to the present problem. In the cases in which the required modifications are slight the results will be stated without proof.

3. *The function $T(z)$.* Let $\Psi(x)$ be given by (6) and let $T(z)$ be defined by

$$T(z) = 1-2z^2\int_0^1 \frac{\Psi(x)}{z^2-x^2}\,dx, \tag{13}$$

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where we use $T(z)$ to denote the branch of the many-valued function, for which $T(z)$ is real when z is real and $z > 1$.

THEOREM 1. *In the z-plane cut along the real axis from* $[-1, +1]$

$$T(z) = 1 - z/\sqrt{(z^2-1)},$$

where $\sqrt{(z^2-1)}$ *is used for the branch which is positive in* $(1, \infty)$.

Proof. If we write, $F(z,x) = \Psi(x)/(z^2-x^2)$, then in the cut plane,

(i) $F(z,x)$ is a continuous function of both variables for $0 \leqslant x \leqslant 1-\delta$ $(0 < \delta < 1)$ for every such δ,

(ii) for each such value of x, $F(z,x)$ is a regular function of z in the cut plane,

(iii) $\int_0^1 F(z,x)\,dx$ is uniformly convergent in any compact set in the cut z-plane.

Hence $\int_0^1 F(z,x)\,dx$ is a regular function in the cut plane and, therefore, $T(z)$ is a regular function in the cut plane.

From the definition, we have,

$$T(z) = 1 - \frac{2z^2}{\pi}\int_0^1 \frac{dx}{\sqrt{(1-x^2)\,(z^2-x^2)}}.$$

Now for z real and $z > 1$, the substitution $x = \cos\theta$ yields

$$T(z) = 1 - \frac{2z^2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{d(\tan\theta)}{z^2(1+\tan^2\theta)-1} = 1 - z/\sqrt{(z^2-1)},$$

where $\sqrt{(z^2-1)}$ denotes the branch which is positive in $(1, \infty)$, i.e.

$$T(z) = 1 - z/\sqrt{(z^2-1)}$$

in the interval $(1, \infty)$ on the real axis. Since $T(z)$ is regular in the cut plane the theorem follows by analytic continuation.

4. *The H-function.*

THEOREM 2. *Let* $\Psi(x)$ *be given by equation* (6). *If* $H(z)$ *is any solution of the equation*

$$\frac{1}{H(z)} = 1 - z\int_0^1 \frac{H(x)\,\Psi(x)}{z+x}\,dx \tag{14}$$

which is continuous in the interval $0 \leqslant z \leqslant 1$ *on the real axis, then* $1/H(z)$ *is regular in the z-plane cut along* $[-1, 0]$; *and in the z-plane cut along* $[-1, +1]$, $H(z)$ *satisfies the functional relation*

$$H(z)\,H(-z) = 1/T(z).$$

In the z-plane cut along $[-1, 0]$, $H(z)$ *is regular except for a simple pole at infinity.*

Proof. Using the same method as in the proof of Theorem 1, we see that

$$\int_0^1 \frac{H(x)\,\Psi(x)}{z+x}\,dx$$

represents a regular function in the z-plane cut along $[-1, 0]$.

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The proof of the functional relation follows as in ((1), pages 16 and 17) except that Fubini's theorem is employed to justify the change of order of integration since $\Psi(x)$ is infinite at $x = 1$.

Moreover, in the z-plane cut along $[-1, +1]$ the only zero of $T(z)$ is one of order two at infinity and

$$1/\{H(z)\,H(-z)\} = T(z).$$

Hence in the z-plane cut along $[-1, 0]$, $1/H(z)$ has a simple zero at infinity and possibly zeros in the interval $[0, 1]$. The latter do not exist as $H(z)$ is continuous in the closed interval $[0, 1]$ on the real axis and so $H(z)$ is regular in the z-plane cut along $[-1, 0]$ except for a simple pole at infinity.

THEOREM 3. *The integral equation* (14) *has a solution* $H(z)$ *which is regular in the* z*-plane cut along* $[-1, 0]$ *except for a simple pole at infinity. For* $\Re z > 0$, $H(z)$ *is given by*

$$H(z) = \exp\left\{\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \tan^{-1}(z\tan\theta)\cot\tfrac{1}{2}\theta\,d\theta\right\}, \tag{15}$$

and this is the only solution of (14) *which is continuous in the closed interval* $[0, +1]$ *on the real axis.*

Proof. Except for the expression for $H(z)$, Theorem 3 is the same as Theorem 11·1 (case (i) of (1)) and the proof needs only slight modifications.

The expression for $H(z)$ is given by Chandrasekhar's integral

$$H(z) = \exp\left\{\frac{z}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\log T(\mu)}{\mu^2 - z^2}\,d\mu\right\} \quad (\Re z > 0)$$

(see (2), page 117 and (1), theorem 11·1). In the present problem

$$T(z) = 1 - z/\sqrt{(z^2 - 1)},$$

and using the fact that $T(z)$ is an even function we have, on putting $\mu = iy$,

$$H(z) = \exp\left\{\frac{z}{\pi}\int_0^{\infty} \frac{\log\{1 - y/\sqrt{(1+y^2)}\}}{-y^2 - z^2}\,dy\right\}$$

and the result follows on putting $y = \cot\theta$ and integrating by parts.

The expression for $H(z)$ is similar in form to that used by Stibbs and Weir ((3)) in their work on the H-functions for isotropic scattering. Their methods, should, therefore, be applicable and tables of H-functions could be prepared using an electronic computer.

The moments of the H-functions. We define the moments h_n by

$$h_n = \int_0^1 \Psi(x)\,H(x)\,x^n\,dx \quad (n = 0, 1, 2, 3, \ldots). \tag{16}$$

If we write

$$\psi_n = \int_0^1 \Psi(x)\,x^n\,dx \quad (n = 0, 1, 2, 3, \ldots),$$

we have

$$h_0 = 1 - (1 - 2\psi_0)^{\frac{1}{2}},$$

$$h_{2n}(1 - 2\psi_0)^{\frac{1}{2}} = \psi_{2n} - \tfrac{1}{2}(h_1 h_{2n-1} - h_2 h_{2n-2} + \ldots + h_{2n-1} h_1),$$

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the proofs given in ((1), section 12) being unaltered. Since $\psi_0 = \frac{1}{2}$, we have

$$h_0 = 1, \tag{17}$$

$$h_1^2 = 2\psi_2 = \tfrac{1}{2}. \tag{18}$$

5. *The functions* K_n. The functions K_n are defined by

$$K_n(\xi) = \int_1^\infty \Psi(x^{-1})\, x^{-n} \exp(-x\xi)\, dx \quad (n = 1, 2, \ldots). \tag{19}$$

They have the following properties

$$\text{(i)} \quad K_n'(\xi) = -K_{n-1}(\xi) \quad (n \geqslant 2, \xi > 0), \tag{20}$$

$$\text{(ii)} \quad K_n(0) = \psi_{n-2} \quad (n \geqslant 2), \tag{21}$$

$$\text{(iii)} \quad \pi K_1(\xi) = \ln \xi^{-1} + O(1) \quad (\text{as } \xi \to 0+), \tag{22}$$

$$\text{(iv)} \quad K_n(\xi) = O[\exp(-\xi)] \tag{23}$$

for large ξ.

In order to prove (20), we must justify differentiating under the integral sign in the expression for $K_n(\xi)$. This can be done by the standard method, although care must be taken as $\Psi(x^{-1})$ becomes infinite at the lower limit. The proof of (21) is trivial.

From the definition

$$\pi K_1(\xi) = \int_1^\infty \frac{1}{\sqrt{(x^2-1)}} \exp(-x\xi)\, dx$$

and on putting $u = x\xi$, we have

$$\pi K_1(\xi) - \cosh^{-1}(1/\xi) = \int_1^\infty \frac{1}{\sqrt{(1-\xi^2/u^2)}} \exp(-u) \frac{du}{u} + \int_\xi^1 \frac{1}{\sqrt{(1-\xi^2/u^2)}} \exp(-u) \frac{du}{u} - \int_\xi^1 \frac{1}{\sqrt{(u^2-\xi^2)}}\, du.$$

The first integral is clearly bounded as $\xi \to 0+$ and also

$$\left| \int_\xi^1 \frac{1}{\sqrt{(u^2-\xi^2)}} \exp(-u)\, du - \int_\xi^1 \frac{1}{\sqrt{(u^2-\xi^2)}}\, du \right| = \left| \int_\xi^1 \frac{1}{\sqrt{(u^2-\xi^2)}} \{\exp(-u) - 1\}\, du \right| \leqslant \int_\xi^1 \frac{u}{\sqrt{(u^2-\xi^2)}}\, du = \sqrt{(1-\xi^2)}.$$

Hence as $\xi \to 0+$ $\qquad \pi K_1(\xi) - \cosh^{-1}(1/\xi) = O(1),$

and using the fact that $\qquad \cosh^{-1}(1/\xi) = \ln(\xi^{-1}) + O(1)$

as $\xi \to 0+$, we have equation (22).

Moreover, we have, for $n \geqslant 1$,

$$\pi K_n(\xi) = \int_1^\infty \frac{x^{-(n-1)}}{\sqrt{(x^2-1)}} \exp(-x\xi)\, dx \leqslant \int_1^\infty \frac{\exp(-x\xi)}{\sqrt{(x^2-1)}}\, dx.$$

Now

$$\int_1^2 \frac{\exp(-x\xi)}{\sqrt{(x^2-1)}}\, dx \leqslant \exp(-\xi) \int_1^2 \frac{dx}{\sqrt{(x^2-1)}}$$

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and on putting $x = 1+y$, we have

$$\int_2^\infty \frac{\exp(-x\xi)}{\sqrt{(x^2-1)}}dx = \int_1^\infty \frac{\exp\{-\xi(1+y)\}}{\sqrt{(y^2+2y)}}dy \leqslant \frac{\exp(-\xi)}{\sqrt{3}}\int_1^\infty \exp(-\xi y)\,dy.$$

Hence $$\pi K_n(\xi) = O\{\exp(-\xi)\}$$

for large ξ.

Thus the functions K_n have the properties which are used by Dr Busbridge for the corresponding functions in her work ((1), chapter 3) and, therefore, many of her theorems need no further proof.

6. *The homogeneous Milne equation.*

Notation. If $B(\eta)$ is continuous for $\eta > 0$, $B(\eta) = O(\ln \eta^{-1})$ as $\eta \to 0+$, and $B(\eta) = O(\eta^\alpha)$ as $\eta \to \infty$, we shall write $B(\eta) \in C(\ln \eta^{-1}, \eta^\alpha)$.

THEOREM 4. *If the operator L is defined by equations* (6), (7) *and* (8), *then the integral equation*

$$J(\eta) = L_\eta\{J(\xi)\} \tag{24}$$

has a non-zero solution (unique apart from multiplicative constant) in the class $C(\ln \eta^{-1}, \eta^\alpha)$ provided $\alpha \geqslant 1$. Further, if we denote by $J(\eta)$ the solution such that $\lim_{\eta\to 0+}\{J(\eta)\} = 1$, then

$$J(\eta) = \sqrt{2}\,\eta + 2h_2 - \frac{1}{\pi}\int_0^1 \frac{\sqrt{(1-x^2)}}{H(x)}\exp(-\eta/x)\,dx, \tag{25}$$

where h_2 is defined by equation (16) *and $H(x)$ is given by equation* (15).

Proof. For the proof of the existence of $J(\eta)$ see ((1), Theorem 22·1). An expression for it has been found by Dr Busbridge (see (1), equations 23·15 (with $k = 0$) and 23·8) by the Weiner–Hopf method. It is

$$J(\eta) = h_1^{-1}(\eta + h_2/h_1) - \frac{1}{2\pi i}\int_{\lambda \cup \lambda'} \frac{\exp(-\eta/z)}{zH(z)\,T(z)}dz,$$

where the z-plane is cut along $[0, +1]$ on the real axis and the integral is taken along λ (the lower edge of the cut) from 0 to 1 and along λ' (the upper edge of the cut) from 1 to 0. Substituting the expression for $T(z)$ from Theorem 1, and for h_1 from equation (18), we obtain the required result.

Moreover, from Theorem 23·1 of (1), we have

THEOREM 5. *The Laplace transform of $J(\eta)$ satisfies the relation*

$$\mathscr{L}_{1/\mu}\{J(\eta)\} \equiv \frac{1}{\mu}\int_0^\infty J(\eta)\exp(-\eta/\mu)\,d\eta = H(\mu). \tag{26}$$

7. *The inhomogeneous Milne equation.* Again summarizing results (see (1), sections 27 and 28) we have

THEOREM 6. *For any given $\sigma > 0$, the function $J(\eta, \sigma)$ given by*

$$J(\eta, \sigma) = J(0, \sigma)\left\{J(\eta) - \sigma\int_0^\eta J(\xi)\exp[-\sigma(\eta-\xi)]\,d\xi\right\} \tag{27}$$

is bounded for all $\eta \geqslant 0$, and satisfies the equation

$$J(\eta, \sigma) - L_\eta\{J(\xi, \sigma)\} = \exp(-\eta\sigma). \tag{28}$$

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THEOREM 7. *For $\mu \geqslant 0$, $\sigma > 0$, $s > 0$, we have*

$$J(0, \mu^{-1}) = H(\mu), \tag{29}$$

$$\int_0^\infty J(\eta, \sigma) \exp(-s\eta)\, d\eta = \frac{H(s^{-1})\, H(\sigma^{-1})}{s+\sigma}. \tag{30}$$

8. *The solution of the equations of the problem.* If we replace σ by $\operatorname{cosec}\theta$ in Theorem 6, we have the auxiliary equation

$$J(\eta, \operatorname{cosec}\theta) - L_\eta\{J(\xi, \operatorname{cosec}\theta)\} = \exp(-\eta \operatorname{cosec}\theta).$$

Hence, for any given value of β, we have

$$\frac{1}{2\pi}\int_0^\pi J(\eta, \operatorname{cosec}\theta)\{\epsilon(\theta,\beta)+\phi(0)\}\, d\theta - L_\eta\left\{\frac{1}{2\pi}\int_0^\pi J(\xi, \operatorname{cosec}\theta)\{\epsilon(\theta,\beta)+\phi(0)\}\, d\theta\right\}$$
$$= \frac{1}{2\pi}\int_0^\pi \exp(-\eta \operatorname{cosec}\theta)\{\epsilon(\theta,\beta)+\phi(0)\}\, d\theta, \tag{31}$$

where the known properties of $\epsilon(\theta,\beta)$ and of the functions K_n are used together with Fubini's theorem to justify the change of order of integration. On subtracting (31) from (9) we see that

$$\phi(\eta) - \frac{1}{2\pi}\int_0^\pi J(\eta, \operatorname{cosec}\theta)\{\epsilon(\theta,\beta)+\phi(0)\}\, d\theta$$
$$= L_\eta\left\{\phi(\xi) - \frac{1}{2\pi}\int_0^\pi J(\xi, \operatorname{cosec}\theta)[\epsilon(\theta,\beta)+\phi(0)]\, d\theta\right\}.$$

From the physics of the problem, we know that $\phi(\eta) \in C(\ln \eta^{-1}, \eta)$ and therefore, from Theorem 4, we have

$$\phi(\eta) = \frac{1}{2\pi}\int_0^\pi J(\eta, \operatorname{cosec}\theta)\{\epsilon(\theta,\beta)+\phi(0)\}\, d\theta + A(\beta)\, J(\eta), \tag{32}$$

where $A(\beta)$ depends only on the value of β.

Using Theorems 5–7 and the known properties of the H-function, it can be proved that, for large η,

$$J(\eta, \operatorname{cosec}\theta) = \sqrt{2}\sin\theta\, H(\sin\theta) + O\{\exp(-\tfrac{1}{2}\eta)\};$$

and hence, for large η,

$$\phi(\eta) = \frac{\sqrt{2}}{\pi}\int_0^{\frac{1}{2}\pi} \{\epsilon(\theta,\beta)+\phi(0)\}\, H(\sin\theta)\sin\theta\, d\theta + A(\beta)\{\sqrt{2}\,\eta + 2h_2\} + O\{\exp(-\tfrac{1}{2}\eta)\},$$

i.e. for large η,
$$\phi(\eta) = \sqrt{2}\, A(\beta)\{\eta + \eta_0\} + O\{\exp(-\tfrac{1}{2}\eta)\}, \tag{33}$$

where
$$\eta_0 = \sqrt{2}\, h_2 + \frac{1}{\pi A(\beta)}\int_0^{\frac{1}{2}\pi} \sin\theta\, H(\sin\theta)\{\epsilon(\theta,\beta)+\phi(0)\}\, d\theta, \tag{34}$$

and it is the constant η_0 which we wish to evaluate.

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If we now put $\mu = \sin\theta$ and

$$\phi(0)f(\mu,\beta) = \epsilon(\theta,\beta) + \phi(0) \tag{35}$$

we have

$$\eta_0 = \sqrt{2}h_2 + \frac{\phi(0)}{A(\beta)}\int_0^1 \mu H(\mu)\Psi(\mu)f(\mu,\beta)\,d\mu, \tag{36}$$

where from equations (32) and (10)

$$\phi(\eta) = \phi(0)\int_0^1 \Psi(\mu)J(\eta, 1/\mu)f(\mu,\beta)\,d\mu + A(\beta)J(\eta), \tag{37}$$

$$\phi(0)f(\mu,\beta)\exp(i\beta\mu) = \frac{1}{\mu}\int_0^\infty \phi(\xi)\exp(-\xi/\mu)\,d\xi. \tag{38}$$

On putting $\eta = 0$ in equation (37), we obtain

$$A(\beta) = \phi(0)\left\{1 - \int_0^1 \Psi(\mu)H(\mu)f(\mu,\beta)\,d\mu\right\}. \tag{39}$$

From equations (37), (38) and (39) we obtain the equation

$$f(\mu,\beta)\exp(i\beta\mu) = H(\mu) - \mu H(\mu)\int_0^1 \Psi(x)f(x,\beta)H(x)\frac{dx}{\mu+x}, \tag{40}$$

on using Theorems 5 and 7 (the change of order of integration being justified by Fubini's theorem).

Now put

$$F(\mu,\beta) = f(\mu,\beta)H(\mu) \tag{41}$$

and

$$F_n(\beta) = \int_0^1 F(\mu,\beta)\Psi(\mu)\mu^n\,d\mu. \tag{42}$$

Then

$$\eta_0 = \sqrt{2}h_2 + F_1(\beta)/\{1 - F_0(\beta)\}, \tag{43}$$

where

$$F(\mu,\beta) = \{H(\mu)\}^2\exp(-i\beta\mu)\left\{1 - \mu\int_0^1 \Psi(x)F(x,\beta)\frac{dx}{\mu+x}\right\}. \tag{44}$$

The best method of computing $F(\mu,\beta)$ and its moments has yet to be decided.

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APPENDIX

PROPOSED EXPERIMENT AND PHYSICAL BACKGROUND

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Proposed experiment. Take a very pure metal single crystal, so that the mean free path l is long at low temperatures. Evaporate onto a plane surface a row of strips of a superconductor (e.g. niobium), of which alternate members are joined so as to form two intersecting combs. Pass current from one comb to the other and measure the

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resistance. Apply a magnetic field parallel to the teeth of the combs (see Figure 1*a*). The resistance is expected to oscillate as the magnetic field is changed.

Theory. Consider a free electron metal and let l be infinite. We have to be careful about what happens at the interface between the metal and the niobium potential leads, and we make a specific simplifying assumption; the surface is supposed to be covered with a layer of the same metal in which the mean free path is very short, so that the layer can be many free paths thick and yet thin compared to the comb spacing (Figure 1*b*). We now expand the current input at the level of the comb into its Fourier components and treat a typical component $\exp(iqx)$ separately. By keeping the thickness of the resistive layer short compared to any q^{-1} of interest we allow ourselves the simplification of assuming only J_y to exist in the resistive layer and to be constant through the thickness of the layer. Moreover, we may ignore the curvature of the electron paths in the layer.

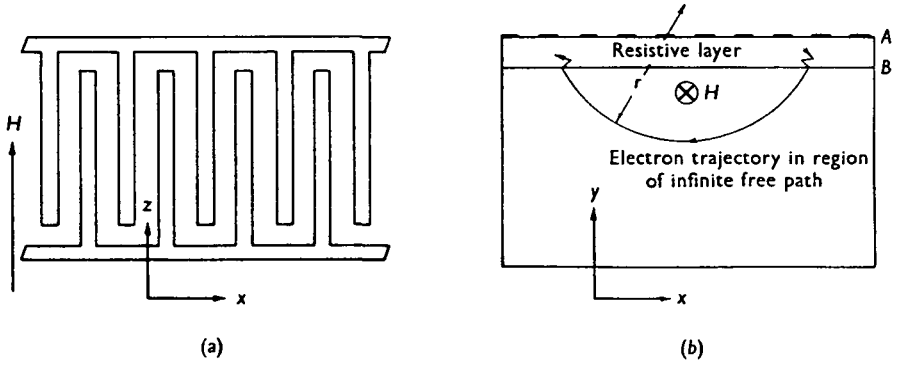


Fig. 1

Finally, it is sufficient in the first instance to consider a two-dimensional metal. It can be extended to three dimensions without any new principles being involved.

We adopt the 'hard Fermi surface' method, characterizing the electron distribution at any point in the metal by the distorted form of the Fermi surface, as defined by the displacement $\epsilon(\theta)$ of the Fermi energy for electrons moving in a direction θ.

Let us now consider the relation between the disturbed distributions at different points along the surface B. Let the Fermi surface of electrons leaving B be distorted by an amount $\epsilon_0'(\theta)\exp(iqx)$ and let the potential along the surface be $V(0)\exp(iqx)$ (V is understood as the potential energy of an electron). Then the electron leaves with a total energy of $\epsilon_0' + V(0)$, and since there are no collisions it arrives back again at the surface with the same energy, but $2r\sin\theta$ further along, if its orbit radius is r (see figure 1*b*). Its angle of arrival is the same as its angle of departure. If we write $\epsilon_0(\theta)\exp(iqx)$ for the Fermi energy of arriving electrons, we have

$$\{\epsilon_0 + V(0)\}\exp(2iqr\sin\theta) = \epsilon_0' + V(0). \tag{I}$$

Now V adjusts itself by space-charge so that the total area enclosed by the Fermi surface is the same at all points, i.e.

$$\int_0^{\pi} (\epsilon_0' + \epsilon_0)\, d\theta = 0,$$

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or, from (I),
$$\int_0^\pi \{[\epsilon_0 + V(0)] + [\epsilon_0 + V(0)] \exp(i\beta \sin\theta)\} d\theta = 0, \tag{II}$$

where
$$\beta = 2qr. \tag{III}$$

We must now discuss the relation between the disturbed distributions at different levels in the resistive layer AB. Measure y from B, and let $V = V(y)\, e^{iqx}$. Consider a Fermi electron moving upwards at an angle θ to the horizontal and calculate the average excess energy weighted over all past histories. The probability that the last collision was between $y = t$ and $t + dt\,(t \geqslant 0)$ is $dt \exp[-(y-t)/l \sin\theta]/l \sin\theta$, and if the electron reverted to the local Fermi surface at the collision it arrives at y with excess kinetic energy $V(t) - V(y)$. In writing this we assume $q(y-t) \ll 1$ so that the obliquity of the path is of no consequence. All electrons colliding last between 0 and y make a contribution to the average excess kinetic energy,

$$\int_0^y [V(t) - V(y)] \exp\{-(y-t)/l \sin\theta\}\, l^{-1} \operatorname{cosec}\theta\, dt.$$

Now the electrons which reach y without colliding in the resistive layer started at $y = 0$ with total excess energy $\epsilon_0 + V(0)$ and hence have excess kinetic energy $\epsilon_0 + V(0) - V(y)$ on arrival. The probability of arrival is $\exp\{-y/l \sin\theta\}$, and we may write for the average excess kinetic energy of electrons moving upwards

$$\epsilon(y, \theta) = \{\epsilon_0 + V(0) - V(y)\} \exp(-y/l \sin\theta) + \int_0^y \{V(t) - V(y)\} \exp\{-(y-t)/l \sin\theta\}\, l^{-1} \operatorname{cosec}\theta\, dt.$$

For electrons coming down to y, we assume A is effectively at infinity and by the same argument derive the result

$$\epsilon'(y, \theta) = -V(y) + l^{-1} \operatorname{cosec}\theta \int_y^\infty V(t) \exp[-(y-t)/l \sin\theta]\, dt. \tag{IV}$$

The condition that the electron density is constant is that

$$\int_0^\pi [\epsilon + \epsilon']\, d\theta = 0,$$

i.e.
$$\frac{1}{2\pi} \int_0^\pi [\epsilon_0 + \phi(0)] \exp[-\eta \operatorname{cosec}\theta]\, d\theta + \int_0^\infty \phi(\xi)\, K_1(|\xi - \eta|)\, d\xi = \phi(\eta), \tag{V}$$

where $\eta = y/l$, $\xi = t/l$, $\phi(\eta) = V(y)$ and

$$K_1(u) = \frac{1}{\pi} \int_1^\infty \frac{\exp(-ux)}{(x^2 - 1)^{\frac{1}{2}}}\, dx. \tag{VI}$$

On writing $\beta = qr$, we have from (I) and (IV) with $y = 0$

$$\epsilon_0 + \phi(0) = \operatorname{cosec}\theta \exp(-i\beta \sin\theta) \int_0^\infty \phi(\xi) \exp(-\xi \operatorname{cosec}\theta)\, d\xi. \tag{VII}$$

The function ϵ_0 depends on both θ and β and if we now write $\epsilon(\theta, \beta)$ instead of ϵ_0 in equations (V), (VI) and (VII) we obtain equations (1), (3), (2) of section 1.

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Now in practice we are not concerned with the detailed form of $\phi(\eta)$, but only with the relation between potential and current at the plane A. If the resistive layer is thick, $\phi(\eta)$ at A may be expanded as a series

$$\phi(\eta) = \text{const.}\,(\eta + \eta_0 + \ldots),$$

the higher terms being negligible as $\eta \to \infty$. The current density at A is determined, when B is many free paths distant from A, by the gradient of ϕ, and we may think of $\eta + \eta_0$ as defining the effective thickness of the resistive layer. Since η_0 is determined by the boundary conditions at B, i.e. by the electron behaviour in the collision-free region, we may expect η_0 to contain all the interesting effects resulting from focusing of the electrons, and it is the variation of η_0 with magnetic field strength which needs to be evaluated.

In the actual experiment, of course, the potential variation across the plane A is not sinusoidal but involves a complete set of Fourier components. The relation between the spectrum $\eta_0(q)$ and the behaviour of the comb of electrodes is not a trivial problem, being one of mixed boundary conditions, but this aspect of the experiment may be treated as a separate issue and will not be discussed further here.

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