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On decoupling and restriction estimates

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Abstract

In this short paper, we prove that the restriction conjecture for the (hyperbolic) paraboloid in $\mathbb{R}^d$ implies the l^p-decoupling theorem for the (hyperbolic) paraboloid in $\mathbb{R}^{2d-1}$. In particular, this gives a simple proof of the l^p decoupling theorem for the (hyperbolic) paraboloid in $\mathbb{R}^3$.

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1. *Introduction*

Let $d \geq 2$ and $\vec{v} = (v_1, \dots, v_{d-1}) \in \{-1, 1\}^{d-1}$. Consider

$$\mathcal{H}^{d-1}_{\vec{v}} := \left\{ (\xi_1, \dots, \xi_{d-1}, \sum_{i=1}^{d-1} v_i \xi_i^2) : \xi_i \in [0,1]^{d-1} \right\}. \tag{1·1}$$

Define the extension operator for $\mathcal{H}^{d-1}_{\vec{v}}$ by

$$\widehat{fd\sigma}(x) := \int_{[0,1]^{d-1}} f(\xi) \left((x \cdot (\xi_1, \dots, \xi_{d-1}, \sum_{i=1}^{d-1} v_i \xi_i^2) \right) d\xi, \tag{1·2}$$

where $e(t) = e^{2\pi i t}$. Let us state the well-known conjecture in restriction theory.

CONJECTURE 1·1 (Restriction conjecture, [**Ste79**]). *For $p > 2 + 2/(d-1)$, we have*

$$\|\widehat{fd\sigma}\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p([0,1]^{d-1})}. \tag{1·3}$$

The restriction conjecture is introduced by E. Stein. The conjecture is solved when $n = 2$ by Fefferman [**Fef70**], but it is still open for higher dimensions. For the case $v_1 = \dots = v_{d-1}$, we refer to [**WW24**] for the best known result. For the general case $\vec{v} \in \{-1, 1\}^{d-1}$, we refer to [**HI22**] and references therein.

The purpose of this paper is to relate a restriction conjecture and a decoupling theorem. A decoupling inequality is first introduced by [**Wol00**]. The l^2 decoupling inequality for the paraboloid is proved by [**BD15**] and the l^p decoupling inequality for the hyperbolic paraboloid is proved by [**BD17**]. Decoupling inequalities have a number of applications. We refer to [**Gut23**] for references.

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Let us introduce some notations to state a decoupling inequality. Denote by $\mathcal{N}_{\mathcal{H}^{d-1}_{\vec{v}}}(R^{-1})$ the R^{-1}-neighbourhood of the hyperbolic paraboloid $\mathcal{H}^{d-1}_{\vec{v}}$. Given a parameter $\vec{\alpha} = (\alpha_1, \dots, \alpha_{d-1}) \in [0,1]^{d-1}$, let $\Gamma_{\vec{\alpha}}(R^{-1})$ be the collection defined by

$$\Big\{(B \times \mathbb{R}) \cap \mathcal{N}_{\mathcal{H}^{d-1}_{\vec{v}}}(R^{-1}): \\ B = \big((c_1, \dots, c_{d-1}) + [0, R^{-\alpha_1}] \times \cdots \times [0, R^{-\alpha_{n-1}}]\big),\ c_i \in R^{-\alpha_i}\mathbb{Z}\Big\}.$$

For each $B \subset \mathbb{R}^d$ and each $F{:}\mathbb{R}^d \to \mathbb{C}$ we define

$$\mathcal{P}_B F(x) := \int_B \widehat{F}(\xi) e(\xi \cdot x)\, d\xi. \tag{1·4}$$

Let us state the l^p decoupling theorem for the hyperbolic paraboloid $\mathcal{H}^{d-1}_{\vec{v}}$ in $\mathbb{R}^d$.

THEOREM 1·2 ([**BD17**]). *Let* $\vec{\alpha} = (1/2, \dots, 1/2)$. *For* $2 \le p \le 2 + 4/(d-1)$ *and* $\epsilon > 0$,

$$\|F\|_{L^p(\mathbb{R}^d)} \le C_{p,\epsilon} R^{\frac{1}{2}(d-1)(\frac{1}{2}-\frac{1}{p})+\epsilon} \left(\sum_{\theta \in \Gamma_{\vec{\alpha}}(R^{-1})} \|F_\theta\|^p_{L^p} \right)^{\frac{1}{p}} \tag{1·5}$$

for all functions $F{:}\mathbb{R}^d \to \mathbb{C}$ *whose Fourier transforms are supported on* $\mathcal{N}_{\mathcal{H}^{d-1}_{\vec{\alpha}}}(R^{-1})$.

Our theorem is as follows.

THEOREM 1·3. *Let* $d \ge 2$. *Conjecture* 1·1 *for the (hyperbolic) paraboloid in* $\mathbb{R}^d$ *implies Theorem* 1·2 *for the (hyperbolic) paraboloid in* $\mathbb{R}^{2d-1}$.

Note that the critical exponents of p of the restriction conjecture for the hyperbolic paraboloid in $\mathbb{R}^d$ and the decoupling theorem for the hyperbolic paraboloid in $\mathbb{R}^{2d-1}$ are equal. As mentioned before, Conjecture 1·1 is proved by Fefferman for the case $d = 2$. Hence, our theorem gives an alternative proof of the l^p decoupling theorem for the (hyperbolic) paraboloid in $\mathbb{R}^3$.

2. *Proof of Theorem* 1·3

Let us begin with the equivalent formulation of the restriction conjecture.

PROPOSITION 2·1 ([**GMO24**, Proposition 1·5]). *Let* $d \ge 2$ *and* $\vec{\alpha} = (1, \dots, 1)$. *If Conjecture* 1·1 *is true for some d, then for* $2 \le p \le 2 + 2/(d-1)$ *and* $\epsilon > 0$, *we have*

$$\|F\|_{L^p(\mathbb{R}^d)} \le C_{p,\epsilon} R^{(d-1)(\frac{1}{2}-\frac{1}{p})+\epsilon} \left(\sum_{\gamma \in \Gamma_{\vec{\alpha}}(R^{-1})} \|\mathcal{P}_\gamma F\|^p_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{p}} \tag{2·1}$$

for all functions $F{:}\mathbb{R}^d \to \mathbb{C}$ *whose Fourier transforms are supported on* $\mathcal{N}_{\mathcal{H}^{d-1}_{\vec{\alpha}}}(R^{-1})$.

The proof of Theorem 1·3 relies on an iterative argument. This argument is inspired by [**FSWW18**]. Consider the hyperbolic paraboloid $\mathcal{H}^{2d-2}_{\vec{v}}$ in $\mathbb{R}^{2d-1}$ associated with

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$\vec{v} = (v_1, \ldots, v_{2d-2}) \in \{-1, 1\}^{2d-2}$. Our aim is to prove that for $2 \le p \le 2 + 4/(2d-2)$, $\vec{\alpha} = (1/2, \ldots, 1/2)$, and $\epsilon > 0$,

$$\|F\|_{L^p(\mathbb{R}^{2d-1})} \le C_{p,\epsilon} R^{\frac{1}{2}(2d-2)(\frac{1}{2}-\frac{1}{p})+\epsilon} \left(\sum_{\theta \in \Gamma_{\vec{\alpha}}(R^{-1})} \|F_\theta\|^p_{L^p(\mathbb{R}^{2d-1})} \right)^{\frac{1}{p}} \tag{2·2}$$

for all functions $F: \mathbb{R}^{2d-1} \to \mathbb{C}$ whose Fourier supports are in $\mathcal{N}_{\mathcal{H}^{2d-2}_{\vec{\alpha}}}(R^{-1})$. For the rest of the proof, assume that $2 \le p \le 2 + 4/(2d-2)$.

LEMMA 2·2. *Suppose that* $0 \le a_2 \le a_1 \le 1/2$. *Define two vectors*

$$\begin{aligned} \vec{\alpha} &:= (a_1, \ldots, a_1, a_2, \ldots, a_2) \in [0,1]^{d-1} \times [0,1]^{d-1} \\ \vec{\beta} &:= (a_1, \ldots, a_1, 2a_1 - a_2, \ldots, 2a_1 - a_2) \in [0,1]^{d-1} \times [0,1]^{d-1}. \end{aligned} \tag{2·3}$$

Under the assumption of Conjecture 1·1 *for the hyperbolic paraboloid in* $\mathbb{R}^d$ *for some d, for given* $\tau \in \Gamma_{\vec{\alpha}}(R^{-1})$ *and* $\epsilon' > 0$, *we have*

$$\|F\|_{L^p(\mathbb{R}^{2d-1})} \le C_{p,\epsilon'} R^{(2a_1-2a_2)(d-1)(\frac{1}{2}-\frac{1}{p})+\epsilon'} \left(\sum_{\theta \in \Gamma_{\vec{\beta}}(R^{-1})} \|\mathcal{P}_\theta F\|^p_{L^p(\mathbb{R}^{2d-1})} \right)^{\frac{1}{p}} \tag{2·4}$$

for all functions F whose Fourier supports are contained in τ. *The constant* $C_{p,\epsilon'}$ *is independent of the choice of* τ, a_1, a_2.

Let us assume the lemma and continue the proof. Fix $\epsilon > 0$. By the assumption on the Fourier support of F, we may write

$$F = \sum_{\theta_1 \in \Gamma_{\vec{\alpha_1}}(R^{-1})} \mathcal{P}_{\theta_1} F, \tag{2·5}$$

where $\vec{\alpha_1} := (\epsilon, \ldots, \epsilon, 0, \ldots, 0) \in [0,1]^{d-1} \times [0,1]^{d-1}$. Since the cardinality of the collection $\Gamma_{\vec{\alpha_1}}(R^{-1})$ is comparable to $R^{(d-1)\epsilon}$, by the triangle inequality and Hölder's inequality, we have

$$\|F\|_{L^p(\mathbb{R}^{2d-1})} \le C_p R^{(d-1)\epsilon(1-\frac{1}{p})} \left(\sum_{\theta_1 \in \Gamma_{\vec{\alpha_1}}(R^{-1})} \|\mathcal{P}_{\theta_1} F\|^p_{L^p(\mathbb{R}^{2d-1})} \right)^{\frac{1}{p}}. \tag{2·6}$$

To simplify the notation, let us introduce $C := (d-1)(1-1/p)$.

Define

$$\vec{\alpha}_n := \begin{cases} \epsilon(n-1, \ldots, n-1, n, \ldots, n) & \text{if } n \text{ is even} \\ \epsilon(n, \ldots, n, n-1, \ldots, n-1) & \text{if } n \text{ is odd.} \end{cases} \tag{2·7}$$

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By applying Lemma 2·2 to the right-hand side of (2·6) with $\epsilon' = \epsilon^2$, $a_1 = \epsilon$, and $a_2 = 0$, we have

$$\|F\|_{L^p(\mathbb{R}^{2d-1})} \le C_{p,\epsilon^2} C_p R^{C\epsilon} R^{2\epsilon(d-1)(\frac{1}{2}-\frac{1}{p})+\epsilon^2} \left(\sum_{\theta_2 \in \Gamma_{\vec{\alpha_2}}(R^{-1})} \|\mathcal{P}_{\theta_2} F\|^p_{L^p(\mathbb{R}^{2d-1})} \right)^{\frac{1}{p}}.$$

Note that we can apply Lemma 2·2 again to the right-hand side of the above inequality with $\epsilon' = \epsilon^2$, $a_1 = 2\epsilon$, and $a_2 = \epsilon$ (when we apply the lemma, we switch $\xi_d, \dots, \xi_{2d-2}$ variables and $\xi_1, \dots, \xi_{d-1}$ variables). By repeating this process $[\epsilon^{-1}/2]$-times, we have

$$\|F\|_{L^p(\mathbb{R}^{2d-1})} \le (C_{p,\epsilon^2})^{[\epsilon^{-1}/2]} C_p R^{C\epsilon} R^{2\epsilon[\epsilon^{-1}/2](d-1)(\frac{1}{2}-\frac{1}{p})+\frac{\epsilon}{2}} \times \left(\sum_{\theta \in \Gamma_{\vec{\alpha}_{[\epsilon^{-1}/2]}}(R^{-1})} \|\mathcal{P}_{\theta} F\|^p_{L^p(\mathbb{R}^{2d-1})} \right)^{\frac{1}{p}}.$$

By the triangle inequality, we finally obtain

$$\|F\|_{L^p(\mathbb{R}^{2d-1})} \le \widetilde{C}_{p,\epsilon} R^{2C\epsilon+\frac{\epsilon}{2}}(R)^{(d-1)(\frac{1}{2}-\frac{1}{p})} \left(\sum_{\theta \in \Gamma_{\vec{\alpha}}(R^{-1})} \|\mathcal{P}_{\theta} F\|^p_{L^p(\mathbb{R}^{2d-1})} \right)^{\frac{1}{p}}, \tag{2·8}$$

where $\vec{\alpha} = (1/2, \dots, 1/2)$ and $\widetilde{C}_{p,\epsilon} = (C_{p,\epsilon^2})^{[\epsilon^{-1}/2]} C_p$. This is the desired inequality. It remains to prove the lemma.

Proof of Lemma 2·2. Let us fix $\tau \in \Gamma_{\vec{\alpha}}(R^{-1})$ and suppose that the Fourier support of F is contained in τ. We need to prove (2·4). We write

$$\tau = \Big(\big((c_1, \dots, c_{2d-2}) + ([0, R^{-a_1}]^{n-1} \times [0, R^{-a_2}]^{n-1}) \big) \times \mathbb{R} \Big) \cap \mathcal{N}_{\mathcal{H}^{2d-2}_{\vec{v}}}(R^{-1}). \tag{2·9}$$

To simplify the notation, we introduce

$$\begin{aligned} &c := (c_1, \dots, c_{2d-2}), \quad \xi = (\xi', \xi_{2d-1}) \in \mathbb{R}^{2d-2} \times \mathbb{R}, \\ &P(\xi') := v_1 \xi_1^2 + \cdots + v_{2d-2} \xi_{2d-2}^2. \end{aligned} \tag{2·10}$$

Define the affine transformation

$$L(\xi) := (\xi' + c, \xi_{2d-1} + P(c) + \xi' \cdot \nabla P(c)), \quad \xi \in \mathbb{R}^{2d-1} \tag{2·11}$$

and the function G_0 satisfying $\hat{G}_0(\xi) := \hat{F}(L\xi)$. Then the Fourier support of G_0 is contained in the following set

$$\big([0, R^{-a_1}]^{d-1} \times [0, R^{-a_2}]^{d-1} \times \mathbb{R} \big) \cap \mathcal{N}_{\mathcal{H}^{2d-2}_{\vec{v}}}(16R^{-1}). \tag{2·12}$$

Recall that $a_2 \le a_1$. Define the linear mapping L_1 and the function G so that

$$L_1(\xi) := (R^{-a_2}\xi', R^{-2a_2}\xi_{2d-1}), \quad \hat{G}(\xi) := \hat{G}_0(L_1(\xi)) = \hat{F}(L_1 L(\xi)). \tag{2·13}$$

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Then the Fourier support of G is contained in

$$\big([0,R^{a_2-a_1}]^{d-1}\times[0,1]^{d-1}\times\mathbb{R}\big)\cap\mathcal{N}_{\mathcal{H}^{2d-2}_{\vec{v}}}(16R^{2a_2-1}). \tag{2·14}$$

Note that $R^{2a_2-1}\le R^{-2a_1+2a_2}$. By performing the linear change of variables (2·13), the desired inequality (2·4) follows from

$$\|G\|_{L^p(\mathbb{R}^{2d-1})}\le C_{p,\epsilon'}R^{(2a_1-2a_2)(d-1)(\frac12-\frac1p)+\epsilon'}\left(\sum_{\theta\in\Gamma_{\vec{\beta}_0}(\widetilde{R}^{-1})}\|\mathcal{P}_\theta G\|^p_{L^p(\mathbb{R}^{2d-1})}\right)^{\frac1p},$$

where $\widetilde{R}^{-1}:=16R^{-2a_1+2a_2}$ and $\vec{\beta}_0:=(1/2,\dots,1/2,1,\dots,1)\in\mathbb{R}^{d-1}\times\mathbb{R}^{d-1}$. To prove the above inequality, let us first observe that the set (2·14) is contained in

$$[0,R^{a_2-a_1}]^{d-1}\times\mathcal{N}_{\mathcal{H}^{d-1}_{\vec{v}_0}}(d\widetilde{R}^{-1}), \tag{2·15}$$

where $\vec{v}_0=(v_d,\dots,v_{2d-2})$. To prove this, suppose that $\xi=(\xi_1,\dots,\xi_{2d-1})$ is an element of (2·14). We need to verify that

$$\left|\xi_{2d-1}-\sum_{i=d}^{2d-2}v_i\xi_i^2\right|\le d\widetilde{R}^{-1}. \tag{2·16}$$

This is true because

$$\begin{aligned}&\left|\sum_{i=1}^{2d-2}v_i\xi_i^2-\sum_{i=d}^{2d-2}v_i\xi_i^2\right|\le\sum_{i=1}^{d-1}|\xi_i|^2\le(d-1)R^{-2a_1+2a_2},\\&\left|\xi_{2d-1}-\sum_{i=1}^{2d-2}v_i\xi_i^2\right|\le 10R^{2a_2-1}\le 10R^{-2a_1+2a_2}.\end{aligned} \tag{2·17}$$

Since the Fourier support of G is contained in (2·15), to get the desired inequality, it suffices to decompose frequencies for $\xi_d,\dots,\xi_{2d-2}$-variables. We apply Proposition 2·1 for the last d-variables with R^{-1} replaced by $d\widetilde{R}^{-1}$, and obtain

$$\|G\|_{L^p(\mathbb{R}^{2d-1})}\le C_{p,\epsilon'}R^{(2a_1-2a_2)(d-1)(\frac12-\frac1p)+\epsilon'}\left(\sum_{\theta\in\Gamma_{\vec{\beta}_0}(d\widetilde{R}^{-1})}\|\mathcal{P}_\theta G\|^p_{L^p(\mathbb{R}^{2d-1})}\right)^{\frac1p}.$$

It suffices to apply the triangle inequality and obtain

$$\|G\|_{L^p(\mathbb{R}^{2d-1})}\le C'_{p,\epsilon'}R^{(2a_1-2a_2)(d-1)(\frac12-\frac1p)+\epsilon'}\left(\sum_{\theta\in\Gamma_{\vec{\beta}_0}(\widetilde{R}^{-1})}\|\mathcal{P}_\theta G\|^p_{L^p(\mathbb{R}^{2d-1})}\right)^{\frac1p}.$$

This completes the proof.

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