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A characterization of pie limits

BY JOHN POWER

Laboratory for the Foundations of Computer Science, University of Edinburgh, Edinburgh EH9 3JZ, Scotland

AND EDMUND ROBINSON

School of Cognitive and Computing Sciences, University of Sussex, Falmer, Brighton BN1 6AA

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Introduction

It is well-known that limits in 2-categories are more complex than limits in ordinary categories. Most readers will at least be familiar with terms such as 'lax limit' and 'pseudo-limit'. In the more modern treatments, these become special cases of a more general class of 'weighted' or 'indexed' limits (see Kelly [7] and Section 1 of this paper).

The aim of this paper is to introduce and give a simple characterization of a closed class of weighted limits, the pie limits of the title. Pie is an acronym – pie limits are those constructible from products, inserters and equifiers. (There is a long tradition in category theory of using the worst nomenclature available. Nevertheless, the authors could not face epi-limits.) Our interest in this case stems primarily from the study of the categories of algebras for monads.

Now, given an arbitrary 2-monad T on an arbitrary 2-category $\mathscr{K}$, the forgetful functor from $\mathsf{T}\text{-}\mathbf{Alg}_s$, the 2-category of (strict) T-algebras and strict homomorphisms, to $\mathscr{K}$ creates all limits that exist in $\mathscr{K}$. It follows that if $\mathscr{K}$ is complete, then so is $\mathsf{T}\text{-}\mathbf{Alg}_s$.

However, we are more likely to be interested in the 2-category $\mathsf{T}\text{-}\mathbf{Alg}$, in which we have strict algebras, but non-strict homomorphisms. Unfortunately, $\mathsf{T}\text{-}\mathbf{Alg}$ is not necessarily complete, even when the base 2-category $\mathscr{K}$ is. However, $\mathsf{T}\text{-}\mathbf{Alg}$ has, and the forgetful functor to $\mathscr{K}$ preserves, products when $\mathscr{K}$ has products, inserters when $\mathscr{K}$ has inserters, and equifiers when $\mathscr{K}$ has equifiers (cf. [4]). It will follow that if $\mathscr{K}$ has pie limits, then so does $\mathsf{T}\text{-}\mathbf{Alg}$, for any monad T on $\mathscr{K}$.

As we indicated above, the main purpose of this paper is to characterize all those limits possessed by a 2-category by virtue of its having products, inserters, and equifiers. The characterization given is surprisingly simple and concrete, and the class is large enough to be useful. If a 2-category has all products, inserters, and equifiers, then it also has all pseudo-limits, all lax and op-lax limits, as well as various other common 2-categorical limits, such as cotensors and Eilenberg–Moore objects. In each case, the characterization given in this paper can be used to give a direct proof of this. Moreover, even if we are not directly interested in 2-limits, but in bilimits, then the fact that $\mathsf{T}\text{-}\mathbf{Alg}$ has pie limits is still of value. As observed in [7], if a category has all pseudo-limits, then it has all bilimits.

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Pie limits are closely related to another well-known class of weighted limits – the flexible limits. Flexible limits (or more precisely flexible weights) arise as part of a general theory of flexible algebra introduced in Bird [2] and Bird, Kelly, Power and Street [3]. Like pie limits, the flexible limits form a closed class, but a slightly larger one. The flexible limits can in fact be regarded as the closure of the class formed by taking all pie limits and adding the splittings of idempotents. They fit in well with the general theory of flexible algebra, in that if T is a flexible monad, then T-**Alg** has all flexible limits. However, if T is an arbitrary monad (or even an arbitrary finitary monad) then T-**Alg** need not have splittings of idempotents, although it will have all pie limits. Indeed, motivated by this, the authors of [3] specifically request an explicit description of the pie weights.

The current paper is divided into three sections. The first section contains a review of the general theory of weighted (or indexed) limits in 2-categories, together with examples of the limits with which we are most concerned. The treatment is more concrete than in most other versions, and this will turn out to be useful in subsequent sections. For more information on weighted limits, the reader is referred to Kelly's excellent survey [7].

The second section contains the main results of the paper. A notion of pie weight is introduced, and we show that all limits with pie weights can be constructed from products, inserters, and equifiers. An immediate corollary is that a 2-category has (resp. a functor preserves) these limits if and only if it has (preserves) products, inserters, and equifiers. In the third and final section, we show that the pie weights give a closed class of limits (and hence that if a limit is constructible from products, inserters, and equifiers, then it has a pie weight).

1. *Weighted limits*

In this section we recall the definition of weighted (or indexed) limits from, say [7] and also sketch what the definition means in concrete terms.

Let $\mathscr{D}$ be an arbitrary 2-category. We shall denote by $[\mathscr{D}, \mathbf{CAT}]$ the 2-category whose objects are 2-functors from $\mathscr{D}$ to **CAT**, whose 1-cells are 2-natural transformations, and whose 2-cells are modifications.

Definition. A weight on $\mathscr{D}$ is a 2-functor $\mathscr{D} \to \mathbf{CAT}$. (In the terminology of [7], this is an indexing type.)

Definition (cf. [7]). Let $\mathscr{K}$ be an arbitrary 2-category. Then the F-weighted limit of $G : \mathscr{D} \to \mathscr{K}$ ($\mathrm{Lim}\,(F, G)$) is a representing object in $\mathscr{K}$ for

$$[\mathscr{D}, \mathbf{CAT}](F, \mathscr{K}(-, G)).$$

This of course has to be interpreted in the sense appropriate to 2-categories. In other words we require an isomorphism from the category $\mathscr{K}(A, \mathrm{Lim}\,(F, G))$ to the category $[\mathscr{D}, \mathbf{CAT}](F, \mathscr{K}(A, G(-)))$, which is natural in A. We stress than this is an isomorphism of categories, and not an equivalence.

Now, for a given A in $\mathscr{K}$, an object of $[\mathscr{D}, \mathbf{CAT}](F, \mathscr{K}(A, G(-)))$, is a natural transformation from F to $\mathscr{K}(A, G(-))$. In [5], Kelly calls this an (F, A)-cylinder over G. We shall adopt terminology more in line with the classical terminology for limits, and call it an F-weighted cone over G with vertex A.

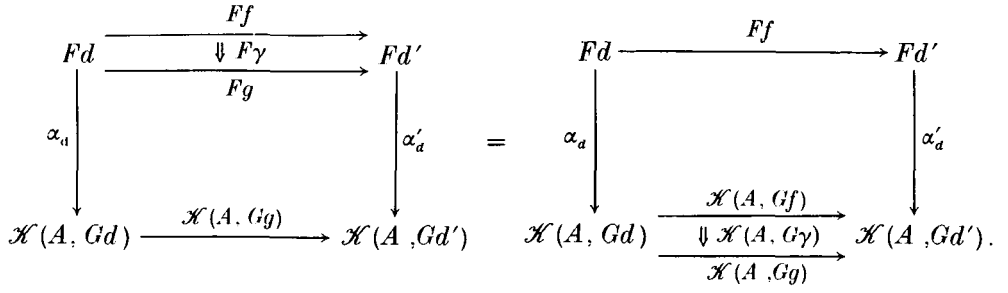
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In the classical case where we consider a 'conical' limit, a cone over a functor G with vertex A is given by providing for each vertex d of the diagram a single morphism $A \to Gd$. This choice is required to be preserved by the natural action of the morphisms (and where appropriate the 2-cells) of $\mathscr{D}$. The difference for weighted limits is that we require instead for each vertex d, a category isomorphic to the weight Fd of d of morphisms $A \to Gd$. The action of the 1-cells and 2-cells of $\mathscr{D}$ should then induce functors and natural transformations between these categories.

An F-weighted cone over G with vertex A is given by a family of functors

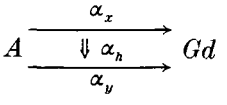
$$\alpha_d : Fd \to \mathscr{K}(A, Gd): \quad d \in \mathrm{ob}\,\mathscr{D}$$

satisfying the coherence requirement expressed by the pasting



Later on we shall find it useful to have a more concrete picture than this, and so we shall spell out the data and requirements for an F-weighted cone over a functor G with vertex A in some detail.

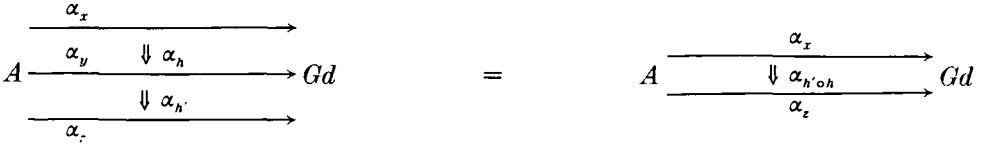
First, for each object d of $\mathscr{D}$ we require a functor $\alpha_d : Fd \to \mathscr{K}(A, Gd)$. In elementary terms this amounts to giving for each x an object of Fd a morphism in $\mathscr{K}$, $\alpha_x : A \to Gd$, and for each $h : x \to y$ in Fd, a 2-cell in $\mathscr{K}$,



such that

(0) (i) α preserves identities: $\alpha_{1_x} = 1_{\alpha_x}$

(ii) α preserves composition: $\alpha_{h'} \circ \alpha_h = \alpha_{h' \circ h}$

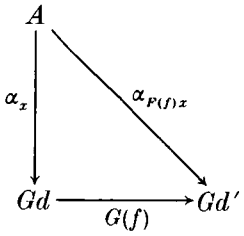


In order for these functors to form a natural transformation, we require them to satisfy the pasting condition given above. This is equivalent to the following elementary naturality conditions.

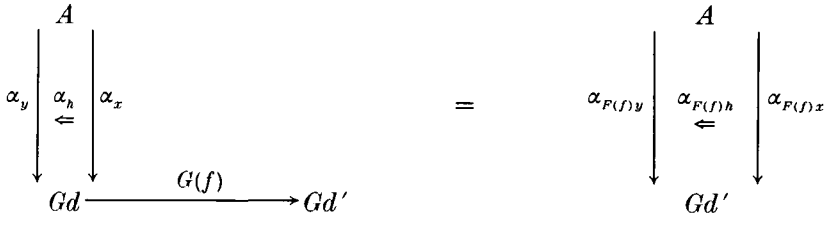
(1) The α_x are natural with respect to 1-cells $d \to d'$ (i.e. the α_x form a one-dimensional cone).

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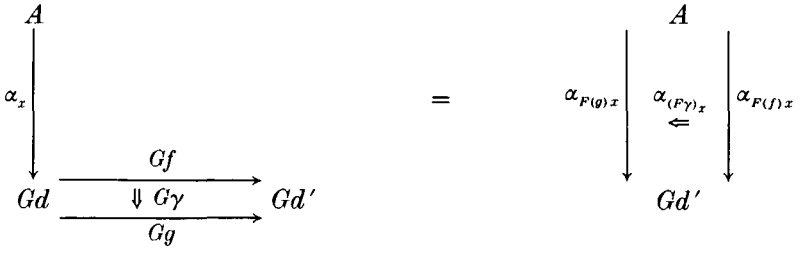
For all $f:d \to d'$, and for all x in Fd, $\alpha_{F(f)x} = G(f) \circ \alpha_x$.



(2) The α_h are natural with respect to 1-cells $d \to d'$. For all $f:d \to d'$, and for all $h:x \to y$ in Fd, $\alpha_{F(f)h} = (Gf)(\alpha_h)$.



(3) The α_x are natural with respect to 2-cells $d \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} d'$. For all $d \overset{f}{\underset{g}{\overset{\longrightarrow}{\Downarrow \gamma}{\longrightarrow}}} d'$, and for all x in Fd, $(G\gamma)\alpha_x = \alpha_{(F\gamma)_x}$.



Note that we only have to check naturality with respect to 2-cells for the object part of the functors. This is because we are comparing two natural transformations. We already know that they have the same domain and codomain, and so only need to check that they have the same components.

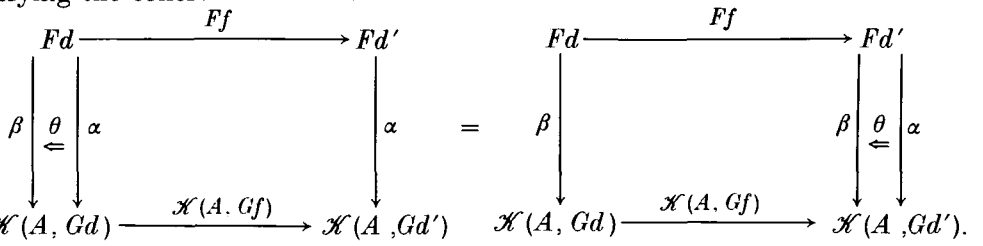
This completes our exploration of the one-dimensional aspect of weighted limits, which, however, is only half the picture. We recall that we require that the *category* of maps $\mathscr{K}(A, \mathrm{Lim}(F, G))$ be isomorphic to the category $[\mathscr{D}, \mathbf{CAT}](F, \mathscr{K}(A, G(-)))$, and not just that we have an isomorphism at the level of sets of objects. The morphisms of this category are modifications.

If α and β are two natural transformations from F to $\mathscr{K}(A, G(-))$, then a *modification* $\theta:\alpha \Rightarrow \beta$ is given by a family of natural transformations

$$\theta_d : \alpha_d \Rightarrow \beta_d : d \in \mathrm{ob}\,\mathscr{D}$$

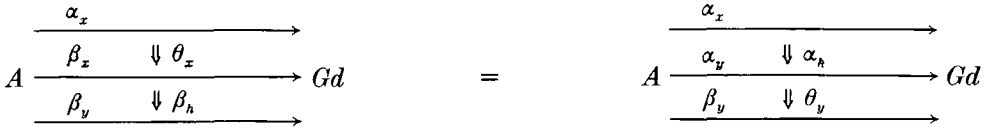
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satisfying the coherence condition

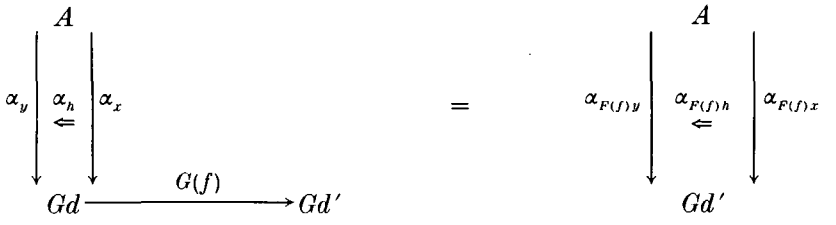


Concretely, the data required are for each (x, d) where x is an object of Fd, a 2-cell $\theta_x : \alpha_x \Rightarrow \beta_x$. The coherence conditions on these are as follows.

(*A*) The θ_x are natural with respect to x. For all $h : x \to y$, $\beta_h \circ \theta_x = \theta_y \circ \alpha_h$.



(*B*) The θ_x are natural with respect to d. For all $f : d \to d'$, and for all x in Fd, $(Gf)\theta_x = \theta_{F(f)x}$.

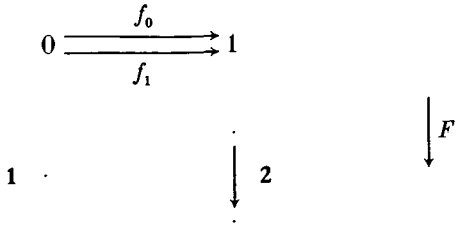


We conclude the section with some examples of weighted limits.

(1) Ordinary limits: Ordinary (or conical) limits are obtained by taking the weight to be the constant functor with value **1**, the category with only one object, and only the identity morphism. For example if **D** is a category, viewed as a 2-category with only identity 2-cells, and G is a functor $\mathbf{D} \to \mathcal{K}$, then an F-weighted cone over G is just a cone over G in the ordinary classical sense (requirements (0), (2), and (3) are trivial). Hence the one-dimensional aspect of the universal property is precisely the classical universal property of the limit. Of course there is also the two-dimensional aspect.

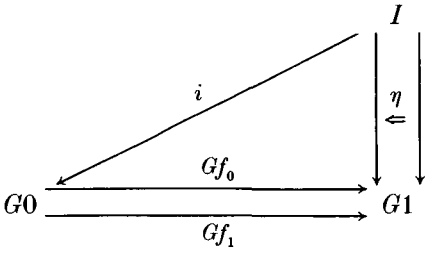
(2) Products: These are particular instances of conical limits. As in the classical case, the indexing category **D** is discrete. The one-dimensional property is the classical universal property of the product (as for all conical limits). The two-dimensional property tells us that if f and g are two morphisms from A to $\prod_{d \in \mathbf{D}} Gd$, then the 2-cells from f to g are in one-to-one correspondence with **D**-tuples of 2-cells from $\pi_d \circ f$ to $\pi_d \circ g$.

(3) Inserters: The diagram for an inserter is a parallel pair of arrows



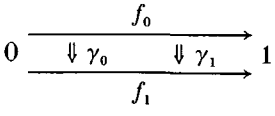
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and the weight F sends 0 to the category **1**, and 1 to the category **2**, the category with two objects and one non-identity arrow between them. However, $F(f_0): \mathbf{1} \to \mathbf{2}$ is the insertion of the domain of the non-identity arrow, and $F(f_1): \mathbf{1} \to \mathbf{2}$ is the insertion of the codomain. If now $G: \mathscr{D} \to \mathscr{K}$ is a functor, then giving a cone over G amounts to giving a map $\alpha_0: A \to G0$, together with a 2-cell between the composites $G(f_0) \circ \alpha_0$ and $G(f_1) \circ \alpha_0$. This limit therefore universally inserts a 2-cell, in the following sense. Let now



be the inserter. The one-dimensional side of the universal property of the limit says that factorizations of α_0 through i correspond to 2-cells $Gf_0 \circ \alpha_0 \Rightarrow Gf_1 \circ \alpha_0$. The two-dimensional side says that if $a, b: A \to I$, then 2-cells $a \Rightarrow b$ correspond to modifications of cones. It turns out that condition (B) implies that the data for such a modification can be reduced to a single 2-cell $\theta: i \circ a \Rightarrow i \circ b$, and that condition (A) is then equivalent to the equation $(\eta b)((Gf_0)\theta) = ((Gf_1)\theta)(\eta a)$.

(4) Equifiers: The diagram $\mathscr{D}$ for an equifier actually has non-trivial 2-cells, and so is not simply a category. It is given by a parallel pair of arrows, together with a pair of 2-cells between them as below.



The weight F for an equifier is similar to the weight for an inserter. $F0 = \mathbf{1}$, and $F1 = \mathbf{2}$, while $F(f_0)$ and $F(g_0)$ are the insertions of the domain and codomain as before. There is a unique natural transformation between these functors (its single component is the non-identity arrow of **2**), and both the 2-cells get sent to that. If $G: \mathscr{D} \to \mathscr{K}$ is a functor, then a cone over G is given by a map $a: A \to G0$ such that $\gamma_0 a = \gamma_1 a$. Moreover, if $a, b: A \to G0$ are two maps giving cones, then a modification between these cones is given by an arbitrary 2-cell $a \Rightarrow b$. It follows that if $E \to G0$ is the equifier, then the induced functor $\mathscr{K}(A, E) \to \mathscr{K}(A, G0)$ is full and faithful.

(5) Cotensors: These are in a sense the antithesis of conical limits – all the information for the cotensor is in the weight, while the shape $\mathscr{D}$ of the diagram is trivial. To form the cotensor $\mathbf{X} \pitchfork A$, let $\mathscr{D}$ be the 2-category **1**, the weight F be the functor which takes **1**'s unique object $*$ to the category $\mathbf{X}$, and G be the functor taking $*$ to A.

(6) Eilenberg–Moore objects: Street has observed that the construction of Eilenberg–Moore algebras for a monad can be regarded as a lax, and hence as a weighted limit. An explicit construction is given in [8]. Briefly, take $\mathscr{D}$ to be the 2-category with one object $*$, and with $\mathscr{D}(*, *) = \underline{\Delta}$, the simplicial category of finite ordinals and order-preserving maps. Composition in $\mathscr{D}$ is given by ordinal additions. Street observes that a monad in $\mathscr{K}$ amounts precisely to a 2-functor $G: \mathscr{D} \to \mathscr{K}$. The

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Eilenberg–Moore object is then constructed as the lax limit of this functor. We shall not reproduce in full Street's construction of the weight F appropriate for calculating a conical lax limit; we shall simply give the object part for this particular weight, which is all that we shall need in order to verify that the weight is pie. In this case, since $\mathscr{D}$ has only one object $*$, the construction becomes quite simple. The objects of $F(*)$ are the morphisms $\mathscr{D}(*, *)$, and so are in correspondence with the finite ordinals. The action of the morphisms of $\mathscr{D}$ is, essentially, by composition. If $n : * \to *$ in $\mathscr{D}$, then $F(n)$ acts on $F(*)$ as $n + (-)$.

2. *Pie limits*

We remarked in the introduction that **T-Alg** does not in general have all conical limits. In particular it often has neither equalizers nor pullbacks. As an example, take **T** to be the monad for categories with a terminal object, and **C** to be the codiscrete category on two objects (recall that in such a category there is a unique map from any given object to any other, and hence all objects are isomorphic and all are terminal). The two endo-functors `id` and `twist` on **C** have no equalizer in **T-Alg** (the empty category does not have a terminal object).

More illuminating from our point of view is the fact that for the same monad **T**, **T-Alg** does not have limits of infinite chains either. Here, $\mathscr{D}$ is the category

$$\ldots \to n \to \ldots \to 2 \to 1 \to 0.$$

To furnish an example, consider the functor G whose value on each object n is the codiscrete category on N. The value of G on an arrow $(m + n) \to m$ is to be the functor $(-) + n$. A moment's reflection shows that there is only one possible cone over this diagram, and that it has as vertex the empty category. It follows that the diagram has no limit in **T-Alg**.

The usual approach to these problems is to claim that they are caused by demanding the actual equality of two maps, as opposed to the existence of 2-cells between them. This suggests throwing away the 2-cells from our weight F, and looking instead at the composite $\mathscr{D} \xrightarrow{F} \mathbf{CAT} \xrightarrow{\mathrm{ob}} \mathit{Set}$.

Definition. If F is a functor $\mathscr{D} \to \mathbf{CAT}$, then we write *Gr-ob*($F$) for the Grothendieck category associated with the functor $\mathrm{ob} \circ F : \mathscr{D} \to \mathit{Set}$. The objects of this category are pairs (x, d) such that x is an object of Fd, and the morphisms $(x, d) \to (x', d')$ are maps $f : d \to d'$ in $\mathscr{D}$ such that $F(f)\, x = x'$.

Our intuition is that the existence of F-weighted limits in **T-Alg** corresponds to some sort of co-filteredness condition on the connected components of *Gr-ob*(F). The example of the infinite chain, in which *Gr-ob*(F) is actually co-filtered, prompts us to take this in its strongest possible form.

Definition. $F : \mathscr{D} \to \mathbf{CAT}$ is a pie weight if each connected component of *Gr-ob*(F) has an initial object.

Examples

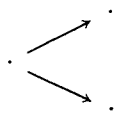
(1) If F is the conical weight of shape $\mathscr{D}$, then *Gr-ob*(F) is isomorphic to $\mathscr{D}$ (or more exactly to the category obtained from $\mathscr{D}$ by forgetting 2-cells). It follows that the

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weights for products are pie, and that the weights for pullbacks and for equalizers are not.

(2) The splitting of an idempotent can also be expressed as a conical limit. Here $\mathscr{D}$ has one object and a single idempotent non-identity arrow. This construction is not a pie limit. It is important to note the weakness of this observation. It does not imply that there are examples of 2-categories which have all pie-weighted limits but in which idempotents do not split, or even that there could not be some other construction of the splitting of idempotents as a pie-weighted limit. These will follow later from the closure properties for pie weights.

(3) If F is the weight for either an inserter or an equifier, then $Gr\text{-}ob(F)$ is a fork



It follows that both inserters and equifiers are pie limits.

(4) If F is the weight for a cotensor $\mathsf{X} \pitchfork -$, then $Gr\text{-}ob(F)$ is isomorphic to the discrete category on the objects of X. Thus cotensors are pie.

(5) When F is the weight for Eilenberg–Moore objects, $Gr\text{-}ob(F)$ is the category

$$1 \to 2 \to 3 \to 4 \to \dots$$

and hence F is pie.

Our first aim will be to show that if F is a pie weight, then F-weighted limits can be constructed out of products, inserters, and equifiers. There is an obvious analogy between this result and Freyd's classical construction of limits out of products and equalizers.

It is worth taking a little trouble to make this precise. In order to give limits in some 2-category $\mathscr{K}$ weighted by a functor F, we must be able to furnish three operations. The first takes an arbitrary diagram and produces the limiting cone, the second takes an ordinary cone, again over an arbitrary diagram, and produces its factorization through the limit, and the third takes a modification of cones with a common vertex and returns the 2-cell between the appropriate factorizations. In each case we may regard this as an operation that takes as input a diagram of one form and returns as result a diagram of another.

Definition. Given a collection $S = \{F_i : \mathscr{K}^{\mathscr{D}_i} \to \mathscr{K}^{\mathscr{D}'_i}\}$, we shall say that an operation $F : \mathscr{K}^{\mathscr{D}} \to \mathscr{K}^{\mathscr{D}'}$ is constructible from the elements of S if one of the following conditions holds:

1. F is an element of S;
2. $F = \mathscr{K}^{\theta} : \mathscr{K}^{\mathscr{D}} \to \mathscr{K}^{\mathscr{D}'}$, for some functor $\theta : \mathscr{D}' \to \mathscr{D}$;
3. $F = F_1 \circ F_2$, where F_1 and F_2 are constructible from S;
4. $\mathscr{D}'$ is of the form $\varinjlim \mathscr{D}_\alpha$, with inclusions $I_\alpha : \mathscr{D}_\alpha \to \mathscr{D}$, for each α we find that $F_\alpha : \mathscr{K}^{\mathscr{D}} \to \mathscr{K}^{\mathscr{D}_\alpha}$, and F is the unique map $\mathscr{K}^{\mathscr{D}} \to \mathscr{K}^{\mathscr{D}'}$ such that for each α, $F_\alpha = \mathscr{K}^{I_\alpha} \circ F$.

Definition. Given a weight $F : \mathscr{D} \to \mathsf{CAT}$, we shall say that F-weighted limits are constructible from products, inserters and equifiers (or more briefly that F is PIE) if the three operations we need for limits of weight F are constructible from the operations for products, inserters and equifiers, and, moreover, the uniqueness

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requirements on the operations that ensure an F-weighted limit rather than a weak limit follow from the uniqueness requirements on the operations for products, inserters and equifiers.

Thus our first major result is that PIE includes all pie weights.

PROPOSITION 2·1. *If $F:\mathscr{D}\to\mathbf{CAT}$ is a pie weight, and $G:\mathscr{D}\to\mathscr{K}$ is a 2-functor, then* Lim (F,G) *can be constructed from products, inserters, and equifiers.*

Given that, as we have seen, products, inserters and equifiers themselves have pie weights, this has an immediate corollary:

THEOREM 2·2. *A 2-category $\mathscr{K}$ has all pie-weighted limits if and only if it has all products, inserters, and equifiers. A 2-functor $\mathscr{K}\to\mathscr{K}'$ between 2-categories with pie-weighted limits preserves these pie-weighted limits if and only if it preserves products, inserters, and equifiers.*

It is now natural to ask whether all limits constructible from products, inserters and equifiers have pie weights. To show that this is indeed the case we take a slight diversion whose general structure is all too familiar from elementary algebra.

There is an evident Galois connection between classes of weights on the one hand, and subcategories of **CAT** on the other. In one direction this connection sends a class Φ of weights to the subcategory of **CAT** whose objects are those categories which have all Φ-weighted limits, and whose morphisms are those functors which preserve them. In the other direction it sends a subcategory **C** to the largest class Φ of weights such that all categories in **C** have all Φ-weighted limits, and all functors in **C** preserve them. As usual, composing these operations induces a closure operation on classes of weights, which we shall write $\Phi\mapsto\Phi^*$. We shall say that a class Φ of weights is *closed* if $\Phi=\Phi^*$.

We shall show

PROPOSITION 3·2. *The class of pie weights is closed.*

The reader will note that this result implies that all limits constructible from products, inserters and equifiers have pie weights.

Unfortunately, our proof of this proposition requires the use of Proposition 2·1, and we must therefore postpone it to the next section. We shall spend the rest of this section on Proposition 2·1.

Throughout the rest of this section, let $F:\mathscr{D}\to\mathbf{CAT}$ be a pie weight. Let U be the set of connected components of $Gr\text{-}ob(F)$, and let $\{(\chi_u,\delta_u)\mid u\in U\}$ be a choice of an initial object for each component. If (x,d) is an object of $Gr\text{-}ob(F)$, then we shall write (χ_x,δ_x) for the chosen initial object of the corresponding component, and $!_x$ for the unique map $\delta_x\to d$ sending χ_x to x.

We fix a map $G:\mathscr{D}\to\mathscr{K}$, and an object A of $\mathscr{K}$.

LEMMA 2·3. (i) *If α^Π is a map $A\to\prod_{u\in U}G(\delta_u)$, we can define a family of maps $\{\alpha_x:A\to Gd\mid(x,d)\in Gr\text{-}ob(F)\}$, by taking α_x to be the composite*

$$A\xrightarrow{\alpha^\Pi}\prod_{u\in U}G(\delta_u)\xrightarrow{\pi}G(\delta_x)\xrightarrow{G(!_x)}G(d).$$

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Then the α_x so defined are natural with respect to maps $f:d \to d'$ in $\mathscr{D}$ (condition (1) of Section 1). Moreover, any such natural family of maps α_x arises from some (unique) α^Π in this way.

(ii) *If α_x and β_x are the families of maps arising from α^Π, $\beta^\Pi : A \to \prod_{u \in U} G(\delta_u)$, as above, then there is a similar bijection between 2-cells $\theta^\Pi : \alpha^\Pi \Rightarrow \beta^\Pi$ and those families of 2-cells $\{\theta_x : \alpha_x \Rightarrow \beta_x \mid (x,d) \in Gr\text{-}ob(F)\}$, which are natural with respect to maps $f:d \to d'$ (condition (B) of the 2-dimensional property of the limit).*

Proof. Suppose that α_x arise from a map α^Π as in the statement, and let $f:d \to d'$. Then if (x,d) is an object of $Gr\text{-}ob(F)$, both (x,d) and $(F(f)x, d')$ lie in the same connected component. It follows that $(\chi_x, \delta_x) = (\chi_{F(f)x}, \delta_{F(f)x})$, and that $!_{F(f)x} = f \circ !_x$. Hence $\alpha_{F(f)x} = G(!_{F(f)x}) \circ \pi \circ \alpha^\Pi = G(f) \circ G(!_x) \circ \pi \circ \alpha^\Pi = G(f) \circ \alpha_x$ as required.

Conversely, suppose the α_x are natural with respect to maps $f:d \to d'$; then $\alpha_x = G(!_x) \circ \alpha_{\chi_x}$, and so the α_x arise from the map

$$\langle \alpha_{\chi_u} \rangle_{u \in U} : A \to \prod_{u \in U} G(\delta_u)$$

as required. The uniqueness of this representation is clear.

For part (ii) of the lemma we of course need the 2-dimensional rather than the 1-dimensional property of the product. If $\theta_x : \alpha_x \Rightarrow \beta_x$ is a family of 2-cells as in the statement, then the 2-dimensional property of the product tells us that there is a unique $\theta^\Pi : \alpha^\Pi \Rightarrow \beta^\Pi$ such that for each u in U the composite $\pi_u \theta^\Pi$ is equal to θ_{δ_u}. Note that it follows from the naturality property of the θ_x that in fact $\theta_x = G(!_x)\pi\theta^\Pi$.

Conversely, suppose $\theta^\Pi : \alpha^\Pi \Rightarrow \beta^\Pi$ is a 2-cell; then the composite $\theta_x = G(!_x)\pi\theta^\Pi$ is a 2-cell $\alpha_x \Rightarrow \beta_x$, by part (i). Moreover, the θ_x are natural with respect to maps $f:d \to d'$. The proof here is exactly as for part (i), using the fact that both (x,d) and $(F(f)x, d')$ lie in the same connected component of $Gr\text{-}ob(F)$. ∎

Lemma 2·3 is in fact the crucial part of the proof of Proposition 2·1. It tells us that we can use a product on its own, and not a product together with an equalizer, to classify cones at the level of 1-cells. It now only remains to deal with the level of 2-cells, and this is straightforward. However, before proceeding, we note that Lemma 2·3 already essentially characterizes the conical pie limits with shape an ordinary category.

COROLLARY 2·4. *All conical pie limits with shape an ordinary category can be constructed from products.*

Let $S = \{(h : x \to y, d) \mid h \in \mathrm{mor}(Fd)\}$. There are two obvious maps

$$\prod_{u \in U} G(\delta_u) \underset{\phi_1}{\overset{\phi_0}{\rightrightarrows}} \prod_{(h,d) \in S} Gd$$

whose projections onto the h component are respectively

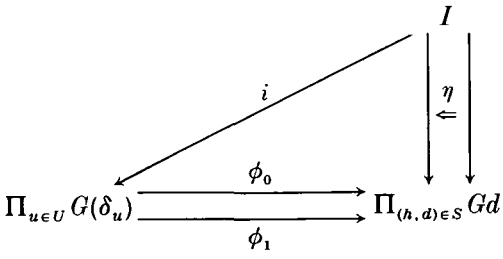
$$\prod_{u \in U} G(\delta_u) \xrightarrow{\pi} G\delta_x \xrightarrow{G(!_x)} Gd$$

and

$$\prod_{u \in U} G(\delta_u) \xrightarrow{\pi} G\delta_y \xrightarrow{G(!_y)} Gd.$$

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Let



be the inserter for these two maps.

LEMMA 2·5. *Composition with i induces a bijection between maps $\alpha^I : A \to I$ and families of maps $\{\alpha_x : A \to Gd \mid (x,d) \in Gr\text{-}ob(F)\}$, together with families of 2-cells $\{\alpha_h : \alpha_x \Rightarrow \alpha_y \mid h : x \to y \text{ in } Fd\}$ such that the α_x are natural with respect to d. Moreover, if $a^I, b^I : A \to I$, the composition with i also induces a bijection between 2-cells $\theta^I : a^I \Rightarrow b^I$ and families of 2-cells $\{\theta_x : \alpha_x \Rightarrow \beta_x \mid (x,d) \in Gr\text{-}ob(F)\}$ such that θ_x is natural with respect to both x and d.*

Note that the α_x and α_h form the data for an F-weighted cone over G with vertex A, and the θ_x form the data for a modification. Furthermore, the α_x satisfy requirement (1) of the requirements for being a cone, and the θ_x satisfy both requirements ((*A*) and (*B*)) for being a modification.

Proof. Lemma 2·3 shows that there is a bijection between maps $a^\Pi : A \to \prod_{u \in U} G(\delta_u)$, and families α_x as in the statement of the current lemma. The one-dimensional property of the inserter now tells us that composition with i induces a bijection between factorizations $A \xrightarrow{a^I} I \xrightarrow{i} \prod_{u \in U} G(\delta_u)$, and 2-cells from $\phi_0 \circ i \circ a^I$ to $\phi_1 \circ i \circ a^I$. The two-dimensional property of the product gives us a bijection with families of $\alpha_h : \alpha_x \Rightarrow \alpha_y$, as required.

The proof of the statement for the two-dimensional property is similar. Referring back to the characterization of the two-dimensional property of the inserter in the previous section, we see that a modification of cones is given by a 2-cell $\theta^I : i \circ a^I \Rightarrow i \circ b^I$, such that $(\eta b^I)(\phi_0 \theta^I) = (\phi_1 \theta^I)(\eta a^I)$. Using the two-dimensional side of Lemma 2·3, we find that such a θ^I corresponds to a family of 2-cells $\theta_x : \alpha_x \Rightarrow \beta_x$, natural with respect to d, and decoding the equation above, we find that it corresponds to their additional naturality with respect to x, as required. ∎

As we remarked above we have succeeded in classifying the data needed for cones and modifications. We now have to ensure that these data satisfy the relevant properties. We have already dealt with the requirements for modifications, and with requirement (1) for cones. We thus have to deal with requirements (0), (2), and (3) for cones. These are all in the form of equations between 2-cells, and so we shall use an equifier. Let

$$S_0 = \{(x,d) \mid (x,d) \in Gr\text{-}ob(F)\},$$
$$S_1 = \{(h : x \to y, h' : y \to z, d) \mid h, h' \in \mathrm{mor}(Fd)\},$$
$$S_2 = \{(h : x \to y, f : d \to d') \mid h \in \mathrm{mor}(\mathrm{Fd})\},$$
$$S_3 = \{(x, \gamma : f \Rightarrow g) \mid f, g : d \to d', x \in \mathrm{ob}\,(Fd\},$$

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corresponding to the forms of the various equations we have to satisfy (S_0 to (0)(i), S_1 to (0)(ii), S_2 to (2), and S_3 to (3)).

Then we have an equifier diagram

$$I \xrightarrow[\psi_1]{\overset{\psi_0}{\eta_0 \Downarrow \Downarrow \eta_1}} \prod_{(x,d)\in S_0} Gd \times \prod_{(h,h',d)\in S_1} Gd \times \prod_{(h,f)\in S_2} Gd' \times \prod_{(x,\gamma)\in S_3} Gd'$$

in which the 2-cells correspond to the equations we have to satisfy. For example, if we project onto the third component of this product, what we get corresponds to the equation $\alpha_{F(f)h} = (Gf)\alpha_h$ (requirement (2) for cones). In order to produce this, the projection of η_0 onto the (h,f) component of this factor is given by

$$I \xrightarrow[\Downarrow \eta]{} \Pi_{(h,d)\in S} Gd \xrightarrow{\pi_{(F(f)h,d')}} Gd'$$

and the projection of η_1 by

$$I \xrightarrow[\Downarrow \eta]{} \Pi_{(h,d)\in S} Gd \xrightarrow{\pi_{(h,d)}} Gd \xrightarrow{Gf} Gd'$$

Take the equifier $e : E \to I$ of this diagram.

LEMMA 2·6. *Composition with $e : E \to I$ induces a bijection between maps $a : A \to E$, and cones over G with vertex A. Moreover, if $a, b : A \to E$, then composition with e induces a bijection between 2-cells from a to b, and modifications between the corresponding cones.*

Proof. We already know from the lemma above that there is a correspondence between maps from A to I, and families α_x and α_h giving the data for cones. The one-dimensional property of the equifier tells us that a map from A to I factors through E if and only if the corresponding α_x and α_h actually form a cone. The two-dimensional property comes from the result already established for I, and the fact that the map $e : E \to I$ induces a functor which is fully faithful on 2-cells (cf. Section 1). ∎

We now note that since the isomorphism conducted in Lemma 2·6 is given by composition with specified morphisms, it is natural in A, and that $\mathrm{Lim}(F, G)$ is therefore given by E. This therefore completes the proof of Proposition 2·1.

Furthermore, since, as we remarked at the beginning of this section, products, inserters, and equifiers are themselves pie-weighted limits, we obtain Theorem 2·2 as an immediate corollary. We have thus completed the proofs of our first two major results, and are now finally in a position to prove Proposition 3·2.

There is a sense in which Proposition 2·1 is stronger and more useful than Theorem 2·2. In order to prove that some form of limit is constructible from products, inserters, and equifiers, we simply have to check that it has a pie weight. Thus, we see from the remarks at the beginning of this section that cotensors and Eilenberg–Moore objects are so constructible. In order further to illustrate the point, we sketch how similar techniques can be used to show that lax and pseudo limits are also pie.

We first observe, with Kelly [7], that if $(-)^\dagger$ is the left adjoint to the inclusion $[\mathbf{C}, \mathbf{CAT}] \to Lax[\mathbf{C}, \mathbf{CAT}]$, then the F-weighted *lax* limit of a functor G is the same as

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the $F^\dagger$-weighted limit of G. Thus, to show that lax limits are constructible as pie limits, it suffices to show that each $F^\dagger$ is a pie weight. A similar analysis holds for pseudo-limits, with respect to the adjoint $(-)'$ to the inclusion $[\mathsf{C}, \mathsf{CAT}] \to Ps[\mathsf{C}, \mathsf{CAT}]$. Unfortunately, an explicit description of these adjoints is not simple (there is one, however, in [**2**]). But, as we have noted before in the context of the Eilenberg–Moore object, we do not need a complete description of the weight F itself in order to prove that F is pie; we only need a description of $\mathrm{ob} \circ F$. In the case at hand, as in that, this observation greatly decreases the number of details we need to present.

In fact, since $\mathrm{ob} \circ F'$ is isomorphic to $\mathrm{ob} \circ F^\dagger$, we can handle both lax and pseudo-limits at once. Specifically, if $F : \mathsf{C} \to \mathsf{CAT}$, and A is an object of C, then $(\mathrm{ob} \circ F^\dagger)(A) = \{(x, f) \mid f : B \to A,\ x \in \mathrm{ob}(FB)\}$. If $g : A \to A'$, then g acts on (x, f) by taking it to $(x, g \circ f)$. It is clear from this description that for any functor F, $Gr\text{-}ob(F^\dagger) \cong Gr\text{-}ob(F')$ is pie, since $((x, \mathrm{id}_B), B)$ is an initial object for the connected component $((x, f), A)$.

It is natural to ask whether the results of this section can be extended to cover the finitary case. In fact they can. This does not involve any new ideas. However, a slightly more careful construction of the limit is needed.

Definition. A weight $F : \mathscr{D} \to \mathsf{CAT}$ is called finite (Kelly [**6**]) if

(i) $\mathrm{ob}\,\mathscr{D}$ is finite,

(ii) each $\mathscr{D}(d, d')$ is a finitely-presentable category,

(iii) for each $d \in \mathrm{ob}\,\mathscr{D}$, the category Fd is finitely presentable.

We now have, in analogy to Proposition 2·1,

Proposition 2·7. *If $F : \mathscr{D} \to \mathsf{CAT}$ is a finite pie weight, then F-weighted limits can be constructed from finite products, inserters, and equifiers.*

This of course has a corollary corresponding to Theorem 2·2.

Theorem 2·8. *A 2-category $\mathscr{K}$ has all finite pie-weighted limits if and only if it has all finite products, inserters, and equifiers. A 2-functor $\mathscr{K} \to \mathscr{K}'$ between 2-categories with these limits preserves them if and only if it preserves products, inserters, and equifiers.*

3. *Pie weights are closed*

In this section we shall give our proof that the class of pie weights is closed. Our essential tool is the following result of Albert and Kelly [**1**]$^{\mathrm{op}}$, which we use in conjunction with Proposition 2·1.

Theorem 3·1. *A weight $\phi : \mathscr{D} \to \mathsf{CAT}$ is in the closure Φ^* of Φ if and only if it is contained in the closure of the representables in $[\mathscr{D}, \mathsf{CAT}]$ under Φ-weighted colimits. Hence, a class Φ of weights is closed if and only if for each $\mathscr{D}$ it includes the representables in $[\mathscr{D}, \mathsf{CAT}]$, and is closed under Φ-weighted colimits.*

As might be expected, a Φ-weighted colimit in $\mathscr{K}$ is defined to be a Φ-weighted limit in $\mathscr{K}^{\mathrm{op}}$.

Proposition 3·2. *The class of pie weights is closed.*

Proof. Combining Proposition 2·1 with the theorem above, we see that we have to show that the class of pie weights includes the representables, and is closed under coproducts, coinserters, and coequifiers.

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(1) If $\mathscr{D}(d, -)$ is a representable, then $(1_d, d)$ is an initial object of $Gr\text{-}ob(\mathscr{D}(d, -))$, which thus has only one component, and hence is pie.

(2) Coproducts in $[\mathscr{D}, \mathbf{CAT}]$ are calculated pointwise. We thus have an isomorphism $Gr\text{-}ob(\oplus_i F_i) \cong \oplus_i Gr\text{-}ob(F_i)$. It follows that the connected components for the coproducts are simply the connected components for the individual F_i, and hence that $(\oplus_i F_i)$ is pie if and only if each F_i is.

(3) Coinserters are again calculated pointwise. Suppose

$$A \underset{G}{\overset{F}{\rightrightarrows}} B$$

is a diagram in **CAT**. We calculate its coinserter by formally adjoining for each object a of A an arrow $\alpha_a : Fa \to Ga$ in B, and requiring that the α_a form a natural transformation $F \Rightarrow G$. Note that the objects of B are left unchanged by the passage to the coinserter. It follows that for

$$F' \rightrightarrows F$$

the passage to the coinserter $F \to I$ gives rise to a natural isomorphism between the functors $\mathrm{ob} \circ F$ and $\mathrm{ob} \circ I : \mathscr{D} \to Set$. Hence $Gr\text{-}ob(F) \cong Gr\text{-}ob(I)$: and so the coinserter is pie if and only if F is.

(4) The argument for coequifiers is essentially the same as the argument for coinserters. Once again, coequifiers are calculated pointwise. The coequifier $B \to E$ of

$$A \underset{\Downarrow \alpha \Downarrow \beta}{\rightrightarrows} B$$

is obtained by formally imposing the equations $\alpha_a = \beta_a$ on B. Thus the passage to the coequifier again gives rise to an isomorphism at the level of objects, just as the passage to the coinserter. ∎

COROLLARY 3·3. *The limits constructible from products, inserters and equifiers are precisely the pie-weighted limits.*

Proof. We have already seen in Proposition 2·1 that the pie-weighted limits are constructible from products, inserters and equifiers. On the other hand, the limits constructible from products, inserters and equifiers must plainly be contained in the closure of the class, and so are pie-weighted. ∎

We can now see that the splittings of idempotents cannot be given by a pie-weighted limit. We recall (from Section 2) that the splitting of an idempotent is the conical limit of a diagram with one vertex and a single (idempotent) non-identity arrow. Since the pie weights form a closed class, if the splitting of an idempotent could be expressed as a pie limit, then this conical limit would yield a pie weight, which clearly it does not. This fact tells us that the pie weights form a strict subclass of the flexible weights (for which see [3]).

It is, however, interesting to have an example of a 2-category that has pie limits, but not all flexible limits. Perhaps the obvious example of a category of algebras that does not have splittings of idempotents is the category of strict monoidal categories and monoidal functors. It is not however easy to give a specific example of a non-

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splitting idempotent in this category, and to prove that it has no splitting. A technically simpler example is to be found instead in $Ps[\mathbf{C}, \mathbf{CAT}]$ (objects are functors and the morphisms pseudo-natural transformations) in the case that $\mathbf{C}$ is the free living retraction ($\mathbf{C}$ has two objects 0 and 1, 0 being a retract of 1), for a discussion of which see [3].

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