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Control systems and differential inequalities

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1. *Introduction.* Let J denote the half line $0 \leqslant t < \infty$, R^n, the Euclidian n-space and $R^+ = [0, \infty)$. Let $\|x\|$ denote the Euclidian norm of $x \in R^n$, and S denote the set $[x: \|x\| < \rho]$.

We shall consider a control system of the form

$$x' = f(t, x) + \phi(t, u), \; [' \equiv d/dt], \tag{1}$$

where $x, f \in R^n$, $\phi \in R^m$, f is continuous on $J \times R^n$ and ϕ is continuous on $J \times R^m$. Let $y = y(t)$ be any continuously differentiable function for $t \in J$, which takes values from R^n.

One can pose the problem: Does there exist control function $u = u(t)$ under the influence of which the solution $x(t)$ of (1) can be approximated so as to be arbitrarily close to the target function $y(t)$ for all $t \geqslant t_0$, provided that the function $f(t, x)$ is sufficiently well-behaved. This question has been investigated by Bellman (**1**, **2**), Barbasin (**3**), and Liberman (**4**, **5**). Quite recently Kayande and Muley (**6**) considered the problem from the point of eventual stability introduced by LaSalle and Rath (**7**).

We in the present paper employ the concept of Lyapunov's functions and the theory of differential inequalities to study the problem with a unified approach by characterizing the controls in different ways. It is possible to conclude from our results that there exist controls such that the target function $y(t)$ is, in fact, asymptotically stable.

2. *Preliminaries.* Let $y(t)$ be the given target function. Consider the transformation.

$$z = x - y(t).$$

It follows that

$$z' = x' - y'(t) = f(t, x) + \phi(t, u(t)) - y'(t),$$

where $u = u(t)$ is a control function defined for $t \in J$. Define

$$F(t, z) = f(t, z + y(t)) - f(t, y(t)),$$

$$\psi(t, y(t)) = y'(t) - f(t, y(t)),$$

and

$$\xi(t) = \phi(t, u(t)) - \psi(t, y(t)), \tag{2}$$

then the control system reduces to

$$z' = F(t, z) + \xi(t). \tag{3}$$

Observing that $F(t, o) \equiv 0$, these considerations imply that it is enough to study the stability properties of the reduced control system (3) with respect to the set $z = 0$. We shall therefore, in what follows, concentrate on the control system (3).

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Let $z(t)\phi = z(t, t_0, x_0)$ be any solution of (3).

DEFINITION (i). *The set $z = 0$ is said to be* stable *with respect to the control system* (3), *if, for any $\epsilon > 0$, $t_0 \in J$, there exists a $\delta > 0$ such that $\|z_0\| \leqslant \delta$ implies $\|z(t)\| < \epsilon, t \geqslant t_0$.*

DEFINITION (ii). *The set $z = 0$ is said to be* asymptotically stable *with respect to the control system* (3), *if it is stable and in addition $\|z(t)\| \to 0$ as $t \to \infty$.*

The control system (3) may also be treated as a perturbed system corresponding to the differential equation

$$z' = F(t, z). \tag{4}$$

Then we can adopt the notion of integral stability introduced by Vrkoč (9), if the control function is subjected to integrability conditions. However, our proofs are general. They stem from Halanay's treatment (10).

DEFINITION (iii). *The trivial solution $z = 0$ of* (4) is *said to be* integrally stable, *if for every $\alpha > 0$ and $t_0 \in J$ there exists a positive function $\beta = \beta(\alpha)$, which is increasing in α and $\beta(0) = 0$, such that for every solution $z(t)$ of the control system* (3), *the inequality*

$$\|z(t)\| < \beta, \quad t \geqslant t_0$$

holds, provided

$$\|z_0\| < \alpha$$

and for every

$$T > 0,$$

$$\int_{t_0}^{t_0+T} \|\xi(s)\|\, ds < \alpha.$$

DEFINITION (iv). *The trivial solution $z = 0$ of* (4) *is said to be* asymptotically integrally stable *if* (iii) *holds and for every $\epsilon > 0$, $\alpha > 0$, and $t_0 \in J$, there exist two positive numbers $T = T(\alpha, \epsilon)$ and $\gamma = \gamma(\alpha, \epsilon)$ such that for every solution of the control system* (3), *the inequality*

$$\|z(t, t_0, z_0)\| < \epsilon \quad (t \geqslant t_0 + T)$$

holds, provided that, $\|z_0\| < \alpha$ and

$$\int_{t_0}^{\infty} \|\xi(s)\|\, ds < \gamma.$$

While discussing the latter two notions we shall need the scalar differential equations

$$W' = g(t, w), \tag{5}$$

and

$$W' = g(t, w) + \zeta(t), \tag{6}$$

where g is continuous on $J \times R^+$, $g(t, 0) \equiv 0$, and $\zeta(t)$ is continuous on J. Analogous to the definitions (iii) and (iv) we have the following definitions:

DEFINITION (v). *The null solution of* (5) *is said to be* integrally stable, *if for every $\alpha_1 > 0$, $t_0 \in J$, there exists a positive function $\beta_1 = \beta_1(\alpha_1)$, which is increasing in α_1, and $\beta_1(0) = 0$, such that whichever be the function ξ with*

$$\int_{t_0}^{t_0+T} \zeta(s)\, ds < \alpha_1,$$

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for every $T > 0$, *every solution* $w(t, t_0, \omega_0)$ *of the differential equation* (6) *satisfies the inequality*

$$w(t, t_0, \omega_0) < \beta_1 \quad (t \geqslant t_0),$$

provided

$$\omega_0 < \alpha_1.$$

DEFINITION (vi). *The null solution of* (6) *is said to be* asymptotically integrally stable *if* (v) *holds and for every* $\epsilon_1 > 0$, $\alpha_1 > 0$ *and* $t_0 \in J$, *there exist two positive numbers* $T_1 = T_1(\alpha_1, \epsilon_1)$ *and* $\gamma_1 = \gamma_1(\alpha_1, \epsilon_1)$ *such that, every solution of* (6) *satisfies*

$$w(t, t_0, \omega_0) < \epsilon_1 \quad (t \geqslant t_0 + T)$$

provided

$$w_0 < \alpha_1 \quad \text{and} \quad \int_{t_0}^{\infty} \zeta(s)\, ds < \gamma.$$

Finally, we require the following basic theorem on differential inequalities (8).

THEOREM 1: *Let* $m(t) \geqslant 0$ *be continuous on* J. *Suppose that* $G(t, u)$ *is continuous on* $J \times R^+$ *and*

$$D^+ m(t) \leqslant G(t, m(t)),$$

where

$$D^+ m(t) = \lim_{h \to 0^+} \sup (1/h)\,[m(t+h) - m(t)].$$

Let $\gamma(t)$ *be the maximal solution of* $u' = G(t, u)$, $u(t_0) = u_0 \geqslant 0$, *existing to the right of* t_0. *Then,* $m(t_0) = u_0$ *implies*

$$m(t) \leqslant \gamma(t)\, t \geqslant t_0.$$

3. *Main results.*

THEOREM 2. *Assume that* $F(t, z)$ *is continuous on* $J \times S_\rho$, $\partial F/\partial z$ *exists and is continuous for* $(t, z) \in J \times S$ *and* $\phi(t, u)$ *is continuous on* $J \times R^m$. *In addition, let there exist a function* $V(t, z)$ *such that* (i) $V(t, z)$ *is continuous on* $J \times S_\rho$,

$$\lambda \|z\| \leqslant V(t, z) \quad (\lambda > 0)$$

and satisfies Lipschitz's condition in z *for a constant* $M > 0$;

(ii) $\lim\limits_{h \to 0^+} \sup (1/h)\, V(t+h, z + h\{F_z(t, o)\, z\}) - V(t, z) \leqslant -\sigma V(t, z), \sigma > 0.$

Also let $\xi(t)$ *defined in* (3) *satisfy*

(iii) $\lim\limits_{t \to \infty}$

$$\int_t^{t+1} \eta(s)\, ds \to 0,$$

where

$$\eta(t) = \|\xi(t)\|.$$

Then, there exists a $T_0 \geqslant 0$ *such that if* $t_0 \geqslant T_0$, *the set* $z = 0$ *is asymptotically stable with respect to control system* (3).

Proof. Let ϵ be given such that $0 < \epsilon < \min(\sigma\lambda/m, \rho)$. Choose $T_0 \geqslant 1$ so large that for $t \geqslant T_0$, we have

$$\lambda \int_1^t \exp -\left(\sigma - \frac{\epsilon M}{\lambda}\right)(t-s)\, \eta(s)\, ds < \frac{\delta(\epsilon)}{2} \equiv \delta_1, \tag{7}$$

where $\delta(\epsilon) \leqslant \epsilon$. This choice is possible because it can be shown as follows:

Let

$$p(t) = \int_t^{t+1} \eta(s)\, ds.$$

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Then
$$\int_{t_0-1}^{t} p(s)\,ds = \int_{t_0-1}^{t}\left[\int_{s}^{s+1}\eta(u)\,du\right]ds$$
$$\geqslant \int_{t_0}^{t}\left[\int_{u-1}^{u}\eta(u)\,ds\right]du = \int_{t_0}^{t}\left[\eta(u)\int_{u-1}^{u}ds\right]du$$
$$= \int_{t_0}^{t}\eta(u)\,du \quad (t\geqslant t_0\geqslant 1).$$

For any $\beta > 0$, we can write

$$\lambda\int_{t_0}^{t} e^{\beta S}\eta(s)\,ds \leqslant \int_{t_0-1}^{t}\lambda\left[\int_{s}^{s+1} e^{\beta u}\eta(u)\,du\right]ds = \lambda\int_{t_0}^{t} e^{\beta(S+1)}p(s)\,ds. \tag{8}$$

whence, inequality (8), for $t_0 = 1$ can be expressed

$$\lambda e^{-\beta t}\int_{1}^{t} e^{\beta(s)}\eta(s)\,ds \leqslant \lambda\, e^{-\beta t}\int_{t_0}^{t} e^{\beta(s+1)}p(s)\,ds. \tag{9}$$

Applying L'Hospital's rule on the right-hand side of (9) it results in

$$\lim_{t\to\infty}\frac{\lambda}{e^{\beta t}}\int_{1}^{t} e^{\beta(s)}\eta(s)\,ds = 0. \tag{10}$$

For $\beta = (\sigma - [\epsilon M/\lambda])$, expression (10) can be written as

$$\lambda\int_{1}^{t}\exp\left\{-\left(\sigma-\frac{\epsilon M}{\lambda}\right)\right\}\eta(s)\,ds \leqslant \frac{\delta(\epsilon)}{z} \equiv \delta_1 \quad (\delta(\epsilon)\leqslant\epsilon). \tag{11}$$

We now proceed to show that whenever $\|z_0\| < \delta_1$, $\|z(t)\| < \delta(\epsilon)$, for $t \geqslant t_0$. For, otherwise, there exists a $t_1 > t_0 \geqslant T_0$ such that

$$\|z(t_1)\| = \delta(\epsilon) \tag{12}$$

and
$$\|z(t)\| \leqslant \delta(\epsilon), \quad t\epsilon[t_0, t_1].$$

Define for $t\epsilon[t_0, t_1]$, $m(t) = V(t, z(t))$. Then

$$m(t+h) - m(t) = V(t+h\,z(t) + h[F(t,z(t)) + \xi(t)]) - V(t,z(t)).$$

Since $F(t,0)\equiv 0$, we have $F(t,z) = F_z(t,0)\,z + H(t,z)$ where, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|H(t,z)\| \leqslant \epsilon\|z\|,$$

provided $\|z\| \leqslant \delta$ and $t \geqslant 0$. This implies that

$$D^+m(t) \leqslant -\sigma(m(t)) + M[\|H(t,z(t))\| + \|\xi(t)\|] \leqslant -(\sigma - [M\epsilon/\lambda])\,m(t) + M\,\gamma(s)$$

using the fact that $\lambda\|z\| \leqslant V(t,z)$.

Applying Theorem 1 (Comparison Principle Theorem), we obtain, for $t\epsilon[t_0, t_1]$,

$$M(t) \leqslant M(t_0)\exp\left\{-\left(\sigma-\frac{M\epsilon}{\lambda}\right)(t-t_0)\right\} + \int_{t_0}^{t}\exp-\left\{\left(\sigma-\frac{M\epsilon}{\lambda}\right)^{(t-s)}\right\}\eta(s)\,ds \tag{13}$$
$$\leqslant m(t_0) + \int_{1}^{t}\exp\left\{-\left(\sigma-\frac{M\epsilon}{\lambda}\right)^{(t-s)}\right\}\eta(s)\,ds.$$

Thus, for $t = t_1$ we get from relations (12), (13) and (11) the following absurdity

$$\delta(\epsilon) < \delta_1 + \delta_1 = \delta(\epsilon),$$

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which proves that if $\|z(t_0)\| < \delta(\epsilon)$, $\|z(t)\| < \delta(\epsilon)$, $t \geqslant t_0$. Therefore, (13) is valid for all $t \geqslant t_0$. Hence, there exist controls such that the target function $y(t)$ is asymptotically stable with respect to the control system (1).

With respect to Theorem 2 we state the following corollary. For this purpose, we need the following definition (11). For any matrix A the limit

$$\mu(A) = \lim_{h \to 0} \frac{\|I + hA\| - 1}{h},$$

exists, where $\|A\|$ is the square root of the largest characteristic root of A^*A, A^* being the transpose of A. (See (11), p. 41.)

COROLLARY 1. *Assume that $F(t,z)$ is continuous on $J \times S_\rho$, $F_z(t,z)$ exists and is continuous for $(t,z) \in J \times S_\rho$ and*

$$\mu[F_z(t,o)] \leqslant -\sigma \quad (\sigma > 0).$$

If, $r(t) = \|\xi(t)\|$ which satisfies

$$\int_t^{t+1} \eta(s)\, ds \to 0 \quad \text{as} \quad t \to \infty,$$

then, the conclusion of Theorem 2, remains valid.

Proof. Let $V(t,z) = \|z\|$. Then

$$\begin{aligned}
&\lim_{h \to 0^+} \sup (1/h) [V(t+h, z+h\{F_z(t,o)\, z\} - V(t,z)] \\
&\qquad = \lim_{h \to 0^+} \sup (1/h) [\|z + h\, F_z(t,o)\, z\| - \|z\|] \\
&\qquad \leqslant \lim_{h \to 0^+} \sup (1/h) [\|I + hF_z(t,o)\| - 1] \|z\| \\
&\qquad = \mu[F_z(t,o)]\, V(t,z) \\
&\qquad \leqslant -\sigma V(t,z),
\end{aligned}$$

which verifies the condition (ii) of Theorem 2. It is easy to see that with this choice of $V(t,z)$ all the assumptions of Theorem 2 are satisfied, hence the conclusions of the theorem remain valid.

THEOREM 3. *Assume that there exist functions $V(t,z)$ and $g(t,\omega)$ satisfying the properties:*

(i) *g is a continuous function defined on $J \times R^+$ and $g(t,o) \equiv 0$.*

(ii) *$V(t,z)$ is continuous on $J \times R^n$, is Lipschitzian in z for a constant $M > 0$ and*

$$b(\|z(t)\|) \leqslant V(t,z), \ (t,z) \in J \times R^n,$$

where $b(u) \geqslant 0$ is continuous for $u \geqslant 0$, $b(0) = 0$, $b(u)$ is strictly monotonic increasing in u and $b(u) \to \infty$ as $u \to \infty$.

(iii)

$$\begin{aligned}
D^+ V(t,z) &= \limsup_{h \to 0^+} (1/h) [V(t+h, hF(t,z)) - V(t,z)] \\
&\leqslant g(t, V(t,z)).
\end{aligned}$$

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Then, the integral stability of the null solution $\omega = 0$ of (5) implies the integral stability of the trivial solution of (4).

Proof. Let $\alpha > 0$ and $t_0 \in J$ be given and let $\|z(t_0)\| < \alpha$. Since $V(t,z)$ is Lipschitzian in z for a constant $M > 0$, we have

$$\|V(t,z) - V(t,\tilde{z})\| \leqslant M\|z - \tilde{z}\| \tag{22}$$

from which it follows that

$$V(t_0, z_0) \leqslant M\|z_0\| \leqslant M\alpha \equiv \alpha_1.$$

For sufficiently small $h \to 0$, we have,

$$V(t+h, z+h[F(t,z)+\xi(t)]) \leqslant Mh\|\xi(t)\| + V(t+h, z+hF(t,z)).$$

due to the fact that $V(t,z)$ is Lipschitzian in z for a constant $M > 0$. It then follows that

$$D^+V(t,z)_{(3)} \leqslant D^+V(t,z) + M\|\xi(t)\| \leqslant g(t, V(t,z)) + M\eta(t).$$

Define $\lambda(t) = M\|\xi(t)\|$ and choose $\omega_0 = V(t_0, z_0)$. An application of Theorem 1 shows that

$$V(t, z(t)) \leqslant r(t, t_0, \omega_0), \tag{23}$$

on the common interval of existence of $z(t)$ and $r(t, t_0, \omega_0)$, where $r(t, t_0, \omega_0)$ is the maximal solution of

$$\omega' = g(t,\omega) + \lambda(t), \quad \omega(t_0) = \omega_0.$$

Assume now that definition (v) holds. Then, given $\alpha_1 > 0$ and $t_0 \in J$, there exists a $\beta_1 = \beta_1(\alpha_1)$, β_1 non-decreasing function, for each t_0, such that, for every solution $\omega(t, t_0, \omega_0)$ of (6), the inequality

$$\omega(t, t_0, \omega_0) < \beta_1 \quad (t \geqslant t_0)$$

holds, whenever $\omega_0 \leqslant \alpha_1$ and every $T > 0$,

$$\int_{t_0}^{t_0+T} \zeta(s)\, ds < \alpha_1.$$

Since assumption (ii) holds, it is possible to choose a $\beta = \beta(\alpha)$ satisfying the relation

$$b(\beta) \geqslant \beta_1, \tag{24}$$

where β_1 is the function occurring in (v).

We claim that, with this β, the definition (iii) holds. If this is not true, there would exist $t_1 > t_0$ such that

$$\|z(t, t_0, z_0)\| = \beta \tag{25}$$

and

$$\|z(t, t_0, z_0)\| \leqslant \beta, \quad t \in [t_0, t_1].$$

For $t \in [t_0, t_1]$, take $\zeta(t)$

$$\zeta(t) = M\|\xi(t)\|.$$

Then, we have

$$\int_{t_0}^{t_1} \zeta(s)\, ds = \int_{t_0}^{t_1} M\|\zeta(s)\|\, ds \leqslant M\alpha = \alpha_1.$$

We extend $\zeta(t)$ continuously for all $t \geqslant t_1$, such that

$$\int_{t_0}^{\infty} \zeta(s)\, ds < \alpha_1.$$

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To do this, it is enough to take $t_2 > t_1$, satisfying the inequality

$$t_2 - t_1 < \frac{2\left(\alpha_1 - \int_{t_0}^{t_1} \zeta(s)\,ds\right)}{1+\zeta(t_1)},$$

to put $\zeta(t_2) = 0$ and to take $\zeta(t)$ linear on $[t_1, t_2]$ and $\zeta(t) = 0$ for $t \geqslant t_2$.

Let $r^*(t, t_0, \omega_0)$ be the maximal solution of the perturbed differential equation (6) with $\zeta(t)$ chosen as above. Because of definition (v) it would follow from $\omega_0 < \alpha_1$ and

$$\int_{t_0}^{t_0+T} \zeta(s)\,ds < \alpha_1,$$

for every $T > 0$, that $\quad r^*(t, t_0, \omega_0) < \beta_1 \quad (t \geqslant t_0)$.

But, on $[t_0, t_1]$, we have $\quad r^*(t, t_0, \omega_0) \equiv r(t, t_0, \omega_0)$,

because $\lambda(t)$ and $\zeta(t)$ are identical on this internal. Hence, $r(t_1, t_0, \omega_0) < \beta_1$. Thus, we get, from the relations (23), (24), (25) and assumption (ii) the following absurdity

$$b(\beta) \leqslant V(t_1, z(t_1)) \leqslant r(t_1, t_0, \omega_0) < \beta_1 \leqslant b(\beta).$$

This proves the integral stability of the trivial solution of (4).

THEOREM 4. *Let the assumption of Theorem* 3 *hold. Assume that the null solution of* (5) *is asymptotically integrally stable. Then, the null solution of* (4) *is also asymptotically integrally stable.*

Proof. From Theorem 3, the trivial solution of (4) is integrally stable. Let $\epsilon > 0$, $\alpha > 0$ and $t_0 \in J$ be given and also let $\|z_0\| < \alpha$. Similarly, as in Theorem 3, we define $\alpha_1 = M\alpha$. Let $\beta = \beta(\alpha)$ be the same function obtained by (24), for which definition (iii) holds.

From (vi) it follows that given $b(\epsilon) > 0$, $\alpha_1 > 0$ and $t_0 \in J$, there exists a pair of numbers $\gamma_1 = \gamma_1(\alpha_1, \epsilon)$ and $T = T(\alpha, \epsilon)$ such that, whichever be the function ζ continuous on $J \times R^+$ with

$$\int_{t_0}^{\infty} \zeta(s)\,ds < \gamma_1, \tag{26}$$

every solution $\omega(t, t_0, \omega_0)$ of the perturbed scalar differential equation (6) satisfies

$$\omega(t, t_0, \omega_0) < b(\epsilon) \quad (t \geqslant t_0 + T), \tag{27}$$

whenever $\omega_0 < \alpha_1$. Now we choose a positive number $\gamma = \gamma(\alpha, \epsilon)$ so that

$$M\gamma = \gamma_1, \tag{28}$$

and maintain that, with the positive number T and γ so defined, (iv) is satisfied. For, otherwise, let $\{t_l\}$ be a sequence such that $t_l \geqslant t_0 + T$, $t_l \to \infty$ as $l \to \infty$. Suppose that there is a solution $z(t) = z(t, t_0, z_0)$ of the control system (3) such that

$$\|z_0\| < \alpha \quad \text{and} \quad \|z(t_l, t_0, z_0)\| \geqslant \epsilon.$$

As in the proof of Theorem 3, the condition (iii), in view of the fact that $V(t, z)$ is Lipschitzian, gives

$$D^+V(t, z(t)) \leqslant g(t, V(t, z(t))) + M \quad \|\xi(t)\|. \tag{29}$$

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If we now define $\zeta(t) = M\|\xi(t)\|$, we have

$$\int_{t_0}^{\infty} \zeta(s)\,ds = M\int_{t_0}^{\infty} \|\xi(s)\|\,ds \leqslant M\gamma = \gamma_1,$$

using (28) and the fact that

$$\int_{t_0}^{\infty} \|\xi(s)\|\,ds < \gamma.$$

This implies that for solution $\omega(t, t_0, \omega_0)$ of (6), inequality (27) is true, because of (26). Moreover, by (29) and the definition of $\zeta(t)$, it follows from Theorem 1 that

$$V(t, z(t)) \leqslant r(t, t_0, \omega_0) \quad (t \geqslant t_0), \tag{30}$$

where $r(t, t_0, \omega_0)$ is the maximal solution of (6). Hence, the relations (30), (27) and the assumption (ii) of Theorem 3, lead us to the contradiction

$$b(\epsilon) \leqslant V(t_1, z(t_1)) \leqslant r(r_1, t_0, \omega_0) < b(\epsilon),$$

which proves the asymptotic integral stability of the solution $z = 0$ of (4).

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