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Polynomial shape adic systems are inherently expansive

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Abstract. To study any dynamical system, it is useful to find a partition that allows essentially faithful encoding (injective, up to a small exceptional set) into a subshift. Most topological and measure-theoretic systems can be represented by Bratteli–Vershik (or adic, or BV) systems. So, it is natural to ask when can a BV system be encoded essentially faithfully. Extending what was known previously for special diagrams and particular orders, we show here that for BV diagrams defined by homogeneous positive integer multivariable polynomials, and a wide family of their generalizations, which we call polynomial shape diagrams, for every choice of the edge ordering, the coding according to initial path segments of a fixed finite length is injective off of a negligible exceptional set.

Key words: Bratteli–Vershik system, adic system, expansiveness, symbolic dynamics
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1. *Introduction*

Measure-preserving and topological dynamical systems have representations as successor (Vershik) maps on path spaces of Bratteli diagrams [17, 24, 25]. Whether they can be coded as subshifts is the question of expansiveness. The combinatorial nature of these symbolic dynamical representations provides a viewpoint that can suggest new questions and new methods to deal with them. Previously, Méla [20, 21] showed that in the famous Pascal adic system, with the left–right ordering, the orbits of infinite paths from the root that do not eventually follow only minimal or only maximal edges can be faithfully coded by the partition determined by the first edge. Frick [12, Theorem 4.3] extended this kind of expansiveness to *limited scope* systems, including those defined by polynomials in one variable (in our current setting, two variables) as well as higher-dimensional Pascal systems. The proofs relied on the particular structure and ordering of the diagrams and

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explicit descriptions of the maps. We showed [10] that for *every* ordering of the Pascal diagram, the set of paths not in the orbit of a minimal or maximal path can be faithfully coded by the first three edges, so that this system is *inherently expansive*. These results raise the natural question of what is it about the diagram and order (and maybe measure) that guarantees this kind of essential expansiveness, and then, what is it about the diagram that guarantees expansiveness for *every* order? (There are easy examples of diagrams that are essentially expansive for some orders, but not others.)

Our aim here is to extend this result to a fairly wide family of diagrams, which we call *polynomial shape*, that includes the diagrams defined by homogeneous multivariable polynomials, the Pascal, Euler, and reverse Euler diagrams in higher dimensions, and more, generalizing systems previously studied in, for example, [2, 3, 13–15, 18–21, 23, 25]. Recent papers related to expansiveness of Bratteli–Vershik systems include [1, 5, 7, 8, 10, 11].

As mentioned below (Remark 6.4), it may be possible to formulate general conditions on a diagram that guarantee inherent expansiveness. The class of diagrams considered in this paper is wide enough to include many interesting examples and their variants. The multidimensional polynomial structure enables the use of coordinate descriptions for precision and visualization of important aspects of the argument. Nevertheless, the main argument is necessarily indirect, so the problem of finding a satisfactory characterization of inherently expansive diagrams (as well as essentially expansive systems) remains open.

The next section begins by recalling basic definitions and terminology about Bratteli diagrams and Bratteli–Vershik systems. When coding orbits, it is necessary to discard those of maximal and minimal paths, and our proofs require paths with dense orbits. Proposition 2.3 establishes that, under mild conditions on the diagram, for each ordering the *exceptional set* (one-sided orbits of maximal and minimal paths together with the non-dense orbits) is meager (first category). The set of invariant measures depends only on the tail relation and so is independent of the choice of ordering. For the diagrams dealt with in this paper, the exceptional set also has measure zero for every fully supported ergodic invariant probability measure—see Remark 2.4 and Proposition 3.7.

In §3, we define the *polynomial diagrams* and their generalization *polynomial shape diagrams* that are the subject of this paper, establish terminology and notation for keeping track of connections between vertices, and identify *distinguished source vertices* which are useful as pivots for moving around in the diagram. (When working in higher dimensions, the coordinate representation is essential to make sure that the motion continues in a consistent direction.) In the proof of the main theorem, it is essential to observe the intersections of *source sets* of certain vertices (v is in the source set of w if there is an edge from v to w, see Definition 3.3), so in §4 we determine exactly which vertices at a given level have their source sets inside those of other vertices at that level. To prove that orbits of paths not in the exceptional set can be distinguished from one another by observing their initial finite segments of a fixed length, one necessarily proceeds by contradiction. In §5, based on the hypothesis that there exist pairs of paths that cannot be so distinguished, we build machinery (chains and links) within the diagram that connect the vertices through which these paths pass at two adjacent levels. Although all of this is fictional, since the ultimate aim is to show that none of it is possible, in the final §6 the constructions do

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produce a contradiction to how the pair of paths would have to move through the diagram, thereby proving the main theorem (Theorem 6.3): Every polynomial shape diagram is inherently expansive.

Now, we summarize with more detail the main steps of the argument.

The idea is to show that if a *depth i pair* (Definition 2.10) did indeed exist, then we could construct a long linearly ordered *chain* (Definition 5.1) consisting of *uncovered* (Definition 4.1) *shared* and *splitting vertices* (Definition 5.1). We then argue that this leads to a contradiction, as follows.

Assuming that the orbits of both paths are dense, eventually one of them, say x', must be the first to change vertices at level $i + 2$. For i sufficiently large, the constructed chain will be long enough so that (1) the first vertex at level $i + 1$ that x' passes through after switching vertices twice at level $i + 2$ is still part of the chain; and (2) if z is the second vertex at level $i + 2$ through which the forward orbit of x' passes while in the chain, then there is a vertex in its source set $S(z)$ that is not one of the splitting vertices met by the orbit of x' during the time that it moves through the chain. (This is argued in the proof of the main theorem using Lemma 6.2.) Hence, the orbit of x' does not meet every vertex in $S(z)$ before changing vertices again at level $i + 2$. However, according to the definition of the Vershik map, it must. Therefore, there can be no depth i pair.

To produce the straight chain used in this proof, we first show that there exist a large region of uncovered vertices at level $i + 1$ (Lemma 5.4, Proposition 4.2) and a time when the orbits of x and x' meet distinct vertices in this region, w_0 and w_1, respectively (Lemma 6.1). The vertices w_0 and w_1 are then chosen to be the first and second splitting vertices of our chain.

Since w_1 is uncovered, there exists a distinguished direction and *distinguished vertex* (Definition 3.8) u in $S(w_1) \setminus S(w_0)$. To find the next splitting vertex of the chain, we consider the first time (forward or backward) that the orbit of x' meets u, while still at w_1. Since x and x' have the same i-coding, the orbit of x must switch vertices at level $i + 1$ by this time, while the orbit of x' does not leave w_1 until afterwards. The fact that w_0 is in our desired region of uncovered vertices (in particular, $S(w_0)$ is not contained in $S(w_1)$) is then sufficient (and necessary) for us to show that when the orbit of x leaves w_0, it cannot switch to w_1 (Lemma 5.4).

Hence, the next vertex w_2 at level $i + 1$ met by the orbit of x is distinct from both w_0 and w_1. We let w_2 be the third vertex in our chain. Since the region of uncovered vertices containing w_0 and w_1 is large enough to contain w_2, we can repeat the process, using w_1 and w_2, with the roles of x and x' reversed. We continue in this manner until we produce a sufficiently long chain whose splitting vertices are met, in an alternating fashion, by the orbits of x and x', respectively. The properties of the distinguished source vertices and our region ensure that each new splitting vertex is in our region and distinct from all previous splitting vertices in the chain. Thus, the chain has the properties necessary to complete the proof as sketched above (Theorem 6.3).

To highlight the essentials of the argument even more, we now very briefly describe how it applies to the two-dimensional Pascal system, which is defined by the polynomial in two variables $p(x_1, x_2) = x_1 + x_2$. The proof of the main theorem, when applied to this system, would require $i \geq 20$, but in fact we need only $i \geq 3$.

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If there were a depth i pair x, x', then at some time their orbits would follow different paths into level $i+1$, and since there is at most one edge between any two vertices, they would meet different vertices w_0 and w_1 at level $i+1$ at this time, which we can assume is time 0.

Note that each row of the two-dimensional Pascal diagram has only two covered vertices and they are at the ends, so every vertex after the root is in the source set of at most one covered vertex. Since x and x' meet the same vertex at level i, at least one of w_0 or w_1 is uncovered.

Consider the case where one of these vertices, say w_0, is covered. Replacing T with T^{-1} if necessary, assume that the orbit of x' meets the (automatically distinguished) vertex $u \in S(w_1) \setminus S(w_0)$ at some time $m > 0$, while still at w_1. Since $m > 0$, it is then necessarily the case that the edge the path x' follows into w_1 is minimal according to the edge ordering, while the edge that $T^m x'$ follows from u to w_1 is maximal. Then, $T^m x$ also meets u. Since $u \notin S(w_0)$, the vertex w_2 at level $i+1$ met by $T^m x$ is clearly distinct from w_0. It also follows that $w_2 \neq w_1$, since $T^m x$ follows a minimal edge into w_2 and the minimal edge into w_1 has source in $S(w_0)$.

Note that w_1 shares one of its source vertices with the covered vertex w_0 and the other with w_2. So since $i+1 \geq 3$, w_2 is clearly uncovered. Hence, we can now add to the current chain (whose splitting vertices are w_0, w_1, and w_2) by using the (distinguished) vertex in $S(w_2) \setminus S(w_1)$ to extend the subchain (link, w_1 to w_2) to a new vertex, w_3. We can continue to extend our chain, moving in a fixed direction away from w_0, at step $k+1$ adding a splitting vertex w_{k+1} whose source set intersects that of w_k and which is distinct from $w_0, \ldots, w_k$, until its last splitting vertex is covered (that is, until we reach the other edge of the diagram). The existence of the chain forces a consistent ordering (left to right or right to left) on the edges entering the vertices at level $i+1$. Each of the uncovered vertices at level $i+1 \geq 4$ (of which there are at least three) is a splitting vertex in our chain, and our construction ensures that collectively the orbits of x and x' meet them all, in an alternating manner, before one of them meets the last (covered) splitting vertex in the chain. We claim that this does not happen until after the orbit of x has changed vertices twice at level $i+2$.

Looking at the diagram structure, it is easy to see that two splitting vertices can both be in the source set of a vertex z at level $i+2$ only if they are adjacent in the chain (equivalently, next to each other on row $i+1$ of the diagram). Because the orbits of x and x' are skipping every other splitting vertex in the chain, every time one of them changes splitting vertices it changes vertices at level $i+2$. Furthermore, the chain has length at least 5 (counting the two covered vertices at its beginning and end), so the orbit of x, beginning at w_0, will meet at least 3 splitting vertices. Hence, the orbit of x changes vertices at level $i+2$ at least twice before reaching the end of the chain.

If z is the second vertex at level $i+2$ through which the forward orbit of x passes while in the constructed chain, $|S(z)| = 2$. The orbit of x meets z and a splitting vertex (at level $i+1$) that is adjacent to two other splitting vertices in the chain. It does not meet either of these adjacent vertices while moving through the chain, since it skips alternate splitting vertices in the chain, However, one of these splitting vertices must be in $S(z)$,

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which contradicts the fact that the orbit of x must meet every vertex in $S(z)$ before changing vertices again at level $i+2$. So there can be no depth i pair.

If we start near the middle of the diagram, with w_0 and w_1 both uncovered, we build the chain in both directions, using forward time for one direction and backward for the other (starting with (distinguished) vertices in $S(w_1) \setminus S(w_0)$ and $S(w_0) \setminus S(w_1)$, respectively). Replacing x and x' with paths in their orbits that simultaneously meet two splitting vertices of the chain, the first of which is covered, we will be in the previously considered case.

2. *Background*

We now recall (in fact, partly quote) from [22] the standard basic definitions about Bratteli diagrams, ordered Bratteli diagrams, and Bratteli–Vershik systems. For further terminology and background, see [6, 9, 10, 16, 17] and their references.

A Bratteli diagram (here also just called a *diagram*) is a countably infinite, directed, graded graph. For each $n = 0, 1, 2, \ldots$, there is a finite non-empty set of vertices $\mathcal{V}_n$. Here, $\mathcal{V}_0$ consists of a single vertex, called the 'root'. The set of edges is the disjoint union of finite non-empty sets $\mathcal{E}_n$, $n \geq 0$, with $\mathcal{E}_n$ denoting the set of edges with source in $\mathcal{V}_n$ and target in $\mathcal{V}_{n+1}$. The *source map* $s : \mathcal{E}_n \to \mathcal{V}_n$ and *target map* $t : \mathcal{E}_n \to \mathcal{V}_{n+1}$ are defined as usual. The *source set* of a vertex $w \in \mathcal{V}_{n+1}$ is $S(w) = s(t^{-1}\{w\})$. In words, v is in the source set of w if there is an edge from v to w. (To simplify notation, we write $t^{-1}\{w\} = t^{-1}w$.) We define $\mathcal{V} = \bigcup_n \mathcal{V}_n$, $\mathcal{E} = \bigcup_n \mathcal{E}_n$ and denote the diagram by $\mathcal{B} = (\mathcal{V}, \mathcal{E})$.

Every vertex other than the root has at least one incoming edge and at least one outgoing edge, and there can be multiple edges between pairs of vertices.

The space X is the set of infinite paths (sequences $x = x_0x_1 \cdots$, each x_i in $\mathcal{E}_i$) starting at the root at level $i = 0$. For a path x, we denote by $v_i(x)$ the vertex of the path at level i. In other words, $v_i(x) = s(x_i) = t(x_{i-1})$. The set X is a compact metric space when we specify that two paths have distance $1/2^n$ if they agree from levels 0 to n and disagree leaving level n. We avoid degenerate situations and only consider diagrams for which X is homeomorphic to the Cantor set. The cylinder sets $x_{[0,n]} = \{y \in X : y_i = x_i, i = 0, \ldots, n\}$, for $n \geq 0$ and $x \in X$, are clopen sets that generate the topology.

The edges entering each vertex can be totally ordered by specifying a map $\xi : \mathcal{E} \to \mathbb{N}$ such that for $n > 0$ and $w \in \mathcal{V}_n$, ξ restricted to $t^{-1}w$ is a bijective map with range $\{1, 2, \ldots, |t^{-1}w|\}$. The diagram together with such an order is called an *ordered Bratteli diagram*.

Two paths x and y are *comparable*, or *tail equivalent*, if they are cofinal: there is a smallest $N > 0$ such that $x_N = y_N$ for all $n \geq N$. In this case, $x_{N-1} \neq y_{N-1}$; we agree that $\xi(x) < \xi(y)$ if $\xi(x_{N-1}) < \xi(y_{N-1})$, and $x > y$ if not. The set of *minimal paths*, meaning all of those whose edges are minimal into all of their vertices, is denoted by $X_{\min}$ and, similarly, the set of maximal paths is denoted by $X_{\max}$. The Vershik, or adic, map $T = T_\xi$ is defined from the set of non-maximal paths to the set of non-minimal paths by mapping each path x to its *successor*, the smallest $y > x$. The pair (X, T) is called a *Bratteli–Vershik system*.

We do not always need an ordering or transformation to use dynamical concepts and terminology. Given a diagram, we think of the tail relation as the 'orbit' relation: two points are in the same 'orbit' if they agree from some level downward. If x and x' are tail

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related, then for each ordering ξ there is $n \in \mathbb{Z}$ such that $T_\xi^n x = x'$; thus, every tail-relation orbit is a T_ξ-orbit.

An *invariant set* is one that is a union of orbits, that is, it is saturated with respect to the tail relation. Note that each orbit is at most countably infinite. An *invariant measure* μ is one that for each vertex w assigns equal measure to all the cylinder sets determined by paths from the root to w: if $v_n(x) = v_n(y) = w$, then $\mu(x_{[0,n]}) = \mu(y_{[0,n]})$.

We denote by X'_ξ the set consisting of the backward T_ξ-orbits of maximal paths and the forward orbits of minimal paths. Thus, X'_ξ consists of the paths whose edges are all eventually minimal or all eventually maximal. (We make no assumptions that our diagrams are proper.)

Denote by X_0 the set of paths whose orbits are not dense. Then X_0 has measure 0 for every fully supported ergodic invariant measure on X. Unlike the situation for the two-dimensional Pascal system with left–right order [**20**, Proposition 2.3], even for a polynomial shape system the set X_0 need not coincide with X'_ξ for any order. See Proposition 3.6 for a precise description of X_0 in the case of polynomial shape systems.

Definition 2.1. Given an ordering ξ of a Bratteli diagram as above, we denote by $X(\xi)$ the set of paths in the path space X that have a dense orbit and are not in X'_ξ. The *exceptional set* is $X'_\xi \cup X_0 = X \setminus X(\xi)$.

We say that two vertices v, w are *connected* if there is a finite downward directed path of consecutive edges in the diagram that begins at one of the vertices and ends at the other. If for every n and every pair of vertices $v_1, v_2 \in \mathcal{V}_n$ there are $m > n$ and a vertex $w \in \mathcal{V}_m$ that is connected to both v_1 and v_2, we say the diagram is *connected*.

We say that a diagram is *topologically transitive* if given non-empty open sets U, V, there is a point (infinite path) in U which is tail equivalent to some point in V. Just as for homeomorphisms on compact metric spaces, there are several conditions equivalent to topological transitivity. Recall that a set is called *meager*, or *first category*, if it is a countable union of nowhere dense sets, and the complement of a meager set is called *comeager* or *residual*.

PROPOSITION 2.2. *The following properties of a Bratteli diagram are equivalent:*

(1) *the diagram is connected, in the sense just defined;*
(2) *the diagram is topologically transitive;*
(3) *every proper closed invariant set is nowhere dense;*
(4) *there is a point with dense orbit;*
(5) *the set of points with dense orbit is comeager.*

Proof. The equivalence of properties (1), (2), and (3) is immediate. Since X is a Baire space, (5) implies (4), and clearly (4) implies (2). We show now that (1) implies (5).

Given a finite path $\alpha = \alpha_0 \cdots \alpha_n$ starting at the root, denote by $[\alpha]$ the cylinder set consisting of all paths that begin with α:

$$[\alpha] = \{x \in X : x_{[0,n]} = \alpha\}. \tag{2.1}$$

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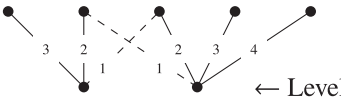


FIGURE 1. The edge ordering is given by the numbers on the edges. Any infinite paths that contain a dashed edge are in U_k^c and the rest are in U_k.

If x is a path whose orbit $\mathcal{O}(x)$ is not dense, then there is such a finite path α for which $\mathcal{O}(x) \cap [\alpha] = \emptyset$. For any finite path α starting at the root, let

$$A_\alpha = \{x \in X : \mathcal{O}(x) \cap [\alpha] = \emptyset\}. \tag{2.2}$$

Then each A_α is invariant (saturated) for the tail equivalence relation. Moreover, we show that it is closed and nowhere dense and thus the set $\bigcup_\alpha A_\alpha$ of points without a dense orbit is meager.

Suppose that $y \in A_\alpha^c$ and find $z \in \mathcal{O}(y)$ such that $z \in [\alpha]$. By connectedness of the diagram, there is a vertex w at some level $m > n$ such that $v_n(y)$ and $v_n(z)$ are connected to w. Now, any $y' \in X$ close enough to y that $y'_{[0,m]} = y_{[0,m]}$ also satisfies $\mathcal{O}(y') \cap [\alpha] \neq \emptyset$, so $y' \in A_\alpha^c$. Therefore, A_α^c is open.

Now, we show that A_α is nowhere dense by showing that its complement A_α^c is dense. Let β be any finite path starting at the root. Extending either α or β if necessary, we may assume that they end at vertices v_1 and v_2 at the same level $n > 0$. By connectedness of the diagram, α and β have extensions α' and β' that end at the same vertex w at some level $m > n$. Extend β' to an infinite path y'. Then, $\mathcal{O}(y') \cap [\alpha] \neq \emptyset$ and $\mathcal{O}(y') \cap [\beta] \neq \emptyset$, so that $y' \in A_\alpha^c \cap [\beta]$. □

PROPOSITION 2.3. *Suppose that a diagram is connected and every vertex has at least two outgoing edges. Then for every edge ordering the set of orbits of minimal paths is meager and, similarly, the set of orbits of maximal paths is meager; therefore, for every ordering ξ the set $X(\xi)$ as defined above (Definition 2.1) is comeager.*

Proof. Fix an arbitrary edge ordering. For each $k \geq 1$, let

$$U_k = \{x \in X : \text{ the edge entering } v_k(x) \text{ is not minimal}\}. \tag{2.3}$$

Then the set of orbits of minimal paths (the set of paths that eventually follow only minimal edges) is

$$E = \bigcup_n \bigcap_{k \geq n} U_k^c. \tag{2.4}$$

See Figure 1.

To show that E is meager, it suffices to show that each of the closed sets $\bigcap_{k \geq n} U_k^c$ is nowhere dense, equivalently that each of the open sets

$$\mathcal{E}_n = \bigcup_{k \geq n} U_k \tag{2.5}$$

is dense.

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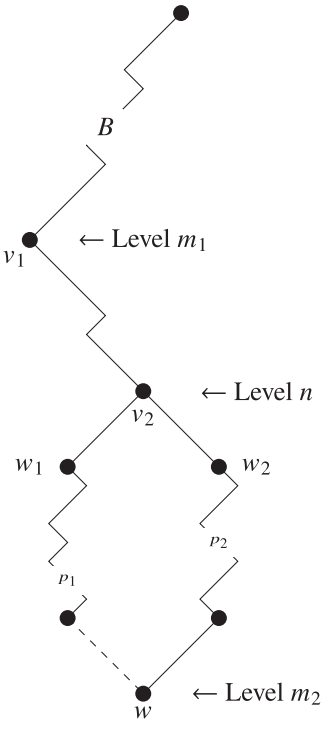


FIGURE 2. The dashed edge into w is non-minimal. The path $x \in \mathcal{E}_n \cap B$ follows the path p_1 into w through the non-minimal edge.

Given $n \geq 0$ and a basic open set B defined by an initial segment y from the root to a vertex v_1 at a level m_1, we find a point $x \in \mathcal{E}_n \cap B$ as follows. Since the sets $\mathcal{E}_n$ are decreasing, we may assume that $n > m_1$. Extend y arbitrarily to arrive at a vertex v_2 at level n. The vertex v_2 has at least two outgoing edges. Let us suppose first that all edges leaving v_2 arrive at the same terminal vertex, w. Then, one of these edges, call it e_1, is not minimal. Extend the finite path downward from v_2 along e_1 and then arbitrarily downward from w, producing a path $x \in \mathcal{E}_{n+1} \cap B \subseteq \mathcal{E}_n \cap B$.

If there are two edges leaving v_2 that arrive at different vertices w_1, w_2, since the diagram is connected there is a first level $m_2 > n + 1$ on which there exists a vertex w that connects to both w_1 and w_2, along paths p_1, p_2 down from v_2. One of these two paths, call it p_1, enters w along a non-minimal edge. In this case, we extend the path downward from v_2 along p_1 and then arbitrarily downward from w to produce the path $x \in \mathcal{E}_{m_2} \cap B \subseteq \mathcal{E}_n \cap B$. See Figure 2.

This proves the set X'_ξ is meager; then, by Proposition 2.2, $X(\xi)$ is comeager. □

Remark 2.4. The preceding argument shows that for every k, the set of minimal paths $\bigcap_{k\geq 1} U_k^c$ is nowhere dense (but it can be uncountable—see [**10**, Example 7.2]), similarly for the maximal paths. Using the argument in [**5**, Lemma 6.2], we can show (in Proposition 3.7) that in any polynomial shape diagram (see Definition 3.1), the sets of maximal and minimal paths (along with their orbits) are also negligible in the measure-theoretic

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sense and, therefore, $X_0 \cup X'_\xi$ is meager and has measure 0 for every fully supported ergodic invariant measure on X. Alternatively, in many cases, the ergodic measures can be identified explicitly (see Méla [19, 21] and the extension by Frick [2, 12], showing that the measure of every cylinder set is given by the product of weights that come from a finite set in (0, 1)), and then one can apply an argument such as in [10, Proposition 2.1]. We defer the details to Proposition 3.7 in §3, after the necessary definitions, notation, and properties have been presented.

We turn now to establish terminology and notation for the main question addressed in this paper, the effectiveness of coding of orbits by finite paths, edges, or vertices. For each $k \geq 1$, denote by A_k the finite alphabet whose elements are the finite paths (segments, strings of edges) from the root to level k. For each $a \in A_k$, the set $[a] = \{x \in X : x_0 \cdots x_{k-1} = a\}$ is a clopen cylinder set and $\mathcal{P}_k = \{[a] : a \in A_k\}$ is a partition of X into clopen sets. The map $\pi_k : X \to A_k^{\mathbb{Z}}$ is defined by $(\pi_k x)_n = a$ if and only if $T^n x \in [a]$. The sequence $\pi_k x$ is called the *k-coding of x*. We denote by Σ_k the closure of $\pi_k X$. Denote by σ the shift transformation on $A_k^{\mathbb{Z}}$. Then, $\pi_k : (X, T) \to (\Sigma_k, \sigma)$ is a Borel measurable map that commutes with the transformations and (Σ_k, σ) is a symbolic dynamical system.

Definition 2.5. The *coding by vertices at level $j < n$ of a vertex w at level n*, denoted by $C_j(w)$, is defined as follows. List, in the order determined by their lexicographical ordering in the diagram, the paths entering w from vertices at level j as $\{p_1, \ldots, p_r\}$ and denote the source of (the first edge of) p_i by u_i, $i = 1, \ldots, r$. Then, $C_j(w) = u_1 \cdots u_r$. (For $j = n - 1$, this is the 'morphism read on $\mathcal{V}_n$' in [9, p. 328]).

The *coding by edges from level $n - 1$ to level n of the orbit of a path x* is the sequence $Edge_n(x) = ((T^m x)_{n-1}, m \in \mathbb{Z})$.

The *coding by vertices at level n (or by $\mathcal{V}_n$) of the orbit of a path x* is the sequence $Vert_n(x) = (v_n(T^m x), m \in \mathbb{Z}) = ((Vert_n(x))_m, m \in \mathbb{Z})$.

Remark 2.6.

(1) The mapping that takes a path x to its coding by vertices at level n intertwines the action of T on X with the shift on $\mathcal{V}_n^{\mathbb{Z}}$.

(2) It might happen that there are $w_1, w_2 \in \mathcal{V}_n$ with $w_1 \neq w_2$ but, for example, $C_{n-1}(w_1) = C_{n-1}(w_2)$.

(3) Note that whenever m is such that $(T^m x)_{n-1}$ is the minimal edge entering $w = v_n(T^m x)$ from level $n - 1$, then the coding of the orbit of x by $\mathcal{V}_n$ begins a string of repeats of w of length equal to the number of paths from the root to w, which we denote by $\dim w$. This is because the coding by vertices $C_{n-1}(v_n(T^m x)) = u_1 u_2 \cdots$ of $v_n(x)$ by vertices at level $n - 1$ expands into the codings of the u_i by vertices at level $n - 2$, and each of those vertices has its coding by vertices of the previous level expand to codings by vertices of its previous level, and so on. Stated slightly differently, if we list in their lexicographical order in the diagram the paths from the root entering w as $\{q_1, \ldots, q_{\dim w}\}$, whenever $(T^m x)_{n-1}$ is the minimal edge to w from level $n - 1$, as we apply T the orbit of x uses each q_i in turn, and so in the coding by $\mathcal{V}_n$ of the orbit of x we see a string of length $\dim w$

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of repeated w terms with the 'dot' to its immediate left (the path being coded passes through the vertex just to the right of the dot).

In [1], a (measure-theoretic) rank-one system was defined to be *essentially k-expansive* if the partition $\mathcal{P}_k$ generates the full sigma-algebra under the transformation T. Since we are dealing here with more generality than rank one, in particular not assuming a fixed invariant measure, we need a slightly different definition. Also, we restrict attention to points that have dense full two-sided orbits.

Definition 2.7. We say that a topologically transitive BV system (with ordering ξ) is *(bilaterally) expansive* if there is an $i \geq 1$ such that the map $\pi_i : (X(\xi), T) \to (\Sigma_i, \sigma)$ is injective, in which case the system is called *(bilaterally) i-expansive*.

Definition 2.8. We say that a topologically transitive diagram is *inherently expansive* if for every ordering the resulting system is (bilaterally) expansive.

Remark 2.9. The concept of expansiveness for Bratteli–Vershik systems is tantamount to recognizability for substitutions, morphisms, or sequences of morphisms: see [4, 5, 7] and their references. These papers focus on systems of bounded width ('finite alphabet rank', possibly after telescoping) and recognizability for aperiodic points, while most of the examples we are interested in have unbounded width. The sequence of morphisms associated with a Bratteli–Vershik system is *recognizable* (at all levels, for full orbits) in the sense of [5, Definitions 2.1 and 4.1] if (briefly) for each $n \geq 1$ the coding of an orbit by vertices at level n uniquely determines (by 'desubstitution') its coding by vertices at level $n + 1$. We claim that a system is recognizable at all levels for full orbits if and only if it is 1-expansive for full orbits.

Let us note first that for an ordered Bratteli diagram with at least two vertices per level after the root, for each $i \geq 0$ codings of a path by edges between levels i and $i + 1$, and codings by vertices at levels i and $i + 1$, determine one another.

In each case, we assume that a coding also specifies the location of the central coordinate ('dot') in the given sequence of edges or symbols. A sequence of edges, $(T^m x)_i$, $m \in \mathbb{Z}$, clearly determines the sequences of vertices $v_i(T^m x) = u_m(x)$ and $v_{i+1}(T^m x) = w_m(x)$. Conversely, given the sequences (on m) $v_i(T^m x) = u_m(x)$ and $v_{i+1}(T^m x) = w_m(x)$, and the position of the dot in each sequence, in a string of repeats of a symbol $w_m(x)$ the path is entering that vertex along the edge whose label is given by the position of the dot. Thus, the codings by vertices determine for each $m \in \mathbb{Z}$ a unique edge $e \in \mathcal{E}_i$ such that $s(e) = u_m(x)$ and $t(e) = w_m(x)$.

Suppose that $X \setminus X_\xi$ is 1-expansive and we are given the coding $(v_n(T^m x), m \in \mathbb{Z})$ of a path x by vertices at some level $n \geq 1$. By Remark 2.6(3), each coding of x by $\mathcal{V}_n$ determines its coding by $\mathcal{V}_{n-1}$, using the 'substitution read on $\mathcal{V}_n$', including the position of the 'dot'. Thus, the coding by $\mathcal{V}_n$ determines the codings by vertices on all levels $j \leq n$, and hence, by the paragraph above, the coding by edges for all levels before level n, in particular, the 1-coding, hence (by 1-expansiveness) the path x itself, and hence the coding by vertices at level $n + 1$.

Conversely, if $X \setminus X'_\xi$ is recognizable at all levels, then for each path $x \in X \setminus X'_\xi$ the coding by vertices at each level $n \geq 1$ of the orbit of x determines its coding by vertices

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at level $n+1$, so we may just reverse the process in the previous paragraph: if $X \setminus X'_\xi$ is recognizable at all levels, then coding by edges to level 1, which determines coding by vertices at level 1, by recognizability determines the coding by vertices at all levels, hence the coding by edges at all levels, hence the edge traversed by each path at every level, and hence each path itself, so that the system is 1-expansive.

If a system is not i-expansive, there exist pairs of points in $X \setminus X'_\xi$ such that at all times in their orbits they agree on their first i edges. The following definition is from [8].

Definition 2.10. Let $i \geq 0$. We say two paths $x, x' \in X \setminus X'_\xi$ form a pair of *depth i* if they have the same i-coding but not the same $(i+1)$-coding.

PROPOSITION 2.11. *Consider an ordered Bratteli diagram for which there are at least two vertices at every level after the root. Then for each $i \geq 1$ and every depth i pair, x and x' in $X(\xi)$, there is $m \in \mathbb{Z}$ such that $v_{i+1}(T^m x) \neq v_{i+1}(T^m x')$.*

Proof. Since the pair x, x' is depth i, there is $j \in \mathbb{Z}$ such that $(T^j x)_i \neq (T^j x')_i$. If it happens that $v_{i+1}(T^j x) = w = v_{i+1}(T^j x')$, one of x, x' must reach the maximal path to w and change vertices at level $i+1$ before the other, since there are at least two vertices at every level and both x and x' have dense orbits. Thus, there is a smallest $k \geq 0$ such that $v_{i+1}(T^{j+k} x) \neq v_{i+1}(T^{j+k} x')$. Letting $m = k + j$, we have $v_{i+1}(T^m x) \neq v_{i+1}(T^m x')$. □

3. *Multivariable polynomial Bratteli diagrams*

NOTE: in this section we use the notation x_i for variables (the arguments in multivariable polynomials) and *not* for the edges of paths in the diagram.

Multivariable polynomial adic systems associated to multivariable polynomials of degree d can be represented by a two-dimensional diagram where the vertices are totally ordered on each level. However, to fully exploit the structure given by the polynomials it is convenient to think of the diagrams in higher dimensions.

Let $p = p(x)$ be a homogeneous positive integer multivariable polynomial of degree d in q variables,

$$p(x) = p(x_1, \ldots, x_q) = \sum_{m_1+m_2+\cdots+m_q=d} a_{(m_1,m_2,\ldots,m_q)} x_1^{m_1} x_2^{m_2} \cdots x_q^{m_q},$$

where $m_1, \ldots, m_q$ are non-negative integers and $a_{(m_1,m_2,\ldots,m_q)} \in \mathbb{N}$. The polynomial p spawns Bratteli diagrams in the following manner. Each vertex $w \in \mathcal{V}_n$ is a monomial $x_1^{m_1} \cdots x_q^{m_q}$ in $(p(x))^n$, with $\sum_{i=1}^q m_i = nd$. We identify the monomial $x_1^{m_1} \cdots x_q^{m_q}$ with the vector $(m_1, \ldots, m_q) = \sum_{i=1}^q m_i e_i$, where e_i is the ith standard basis vector in $\mathbb{R}^q$. We write $w = (m_1, \ldots, m_q) = (w(1), \ldots, w(q))$. By $w \geq 0$, we mean $w(\ell) \geq 0$ for all $\ell \in \{1, \ldots, q\}$, and $w \geq w'$, $w - w' \geq 0$, and $|w| = w_1 + w_2 + \cdots + w_q$.

Definition 3.1. For each $n \geq 0$, let there be specified a set $\mathcal{E}_n$ of edges connecting the set $\mathcal{V}_n$ of vertices at level n to the set $\mathcal{V}_{n+1}$ of vertices at level $n+1$. If for each $n \geq 0$ the set $\mathcal{E}_n$ has the property that there is at least one edge from $u \in \mathcal{V}_n$ to $w \in \mathcal{V}_{n+1}$ if and only if there is $s \in S$ such that $u + s = w$, we say that the diagram is (or has) *polynomial shape* and *is*

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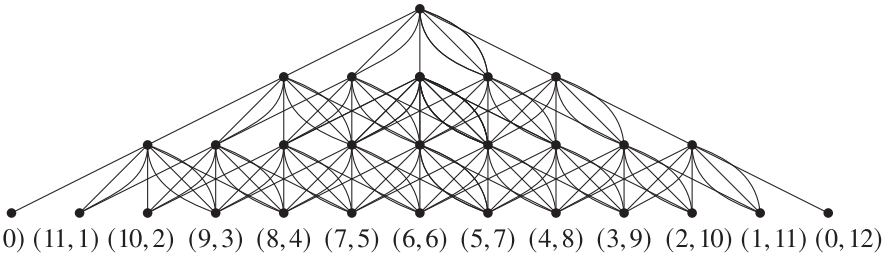


FIGURE 3. The polynomial adic system associated with $p(x, y) = x^4 + 2x^3y + x^2y^2 + 3xy^3 + y^4$, has $q = 2$ variables and is of degree $d = 4$. The vertices on level 3 corresponding to terms in $(p(x, y))^3$ are labeled.

associated to p. If in addition the number of edges between u and w is the coefficient of the monomial $x_1^{s_1}x_2^{s_2}\cdots x_q^{s_q}$ in p that corresponds to the vector $(s_1, s_2, \ldots s_q) = w - u$, we say that the diagram is *a polynomial diagram* and *is defined by p*.

For each $n \in \mathbb{N}$, the number of vertices in $\mathcal{V}_n$ equals the number of terms in $(p(x))^n$, equivalently, the number of ways nd can be written as a sum of q non-negative integers. In other words,

$$|\mathcal{V}_n| = \binom{nd + q - 1}{q - 1}.$$

Definition 3.2. The set of *source vectors* is $S = \{s = (s_1, s_2, \ldots, s_q) : s \geq 0 \text{ and } |s| = d\}$.

Thinking of $\mathcal{V}_1$ as a set of vectors, we have that $\mathcal{V}_1 = S$ and hence each $s \in S$ also corresponds to a monomial in $p(x)$.

In a polynomial diagram, the number of paths from the root vertex to a vertex $u = (m_1, m_2, \ldots, m_q)$ is the coefficient of the monomial $x_1^{m_1}x_2^{m_2}\cdots x_q^{m_q}$ in $(p(x_1, \ldots, x_q))^n$. See Figure 3.

Definition 3.3. For $n \geq 1$ and $w \in \mathcal{V}_{n+1}$, we define the *source set* of w to be

$$S(w) = \{u \in \mathcal{V}_n : w - u \in S\}. \tag{3.1}$$

This agrees with the definition in §2 in terms of source and target maps, $S(w) = s(t^{-1}\{w\})$. Simply stated, $u \in S(w)$ if and only if there is an edge from u to w. Note that $S(w)$ can also be described as the set of non-negative vectors in $w - S = \{w - s : s \in S\}$. Similarly, for any $k \leq n$ the set of vectors in $\mathcal{V}_k$ that connect to w via a path of $j = n + 1 - k$ edges is the set of non-negative vectors in $w - (S + S + \cdots + S)$, where S is repeated j times. (Here $S + \cdots + S$ (j times) means $\{s_{i_1} + \cdots + s_{i_j} : s_{i_\ell} \in S, \ell = 1, \ldots, j\}$.) Since vector addition corresponds to multiplication of monomials, $u \in S(w)$ if and only if the monomial corresponding to u can be multiplied by a monomial of $p(x)$ to obtain the monomial corresponding to w.

Example 3.4. For the diagram in Figure 3 associated with the polynomial $p(x, y) = x^4 + 2x^3y + x^2y^2 + 3xy^3 + y^4$, $V_1 = S = \{(4, 0), (3, 1), (2, 2), (1, 3), (0, 4)\}$. The source sets for each of the eight $w \in V_2$ are given in the following table. As a reminder, the vertices are associated to monomials. For instance, $(8, 0) \sim x^8$ and $(7, 1) \sim x^7y$.

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w	$S(w)$
(8, 0)	{(4, 0)}
(7, 1)	{(4, 0), (3, 1)}
(6, 2)	{(4, 0), (3, 1), (2, 2)}
(5, 3)	{(4, 0), (3, 1), (2, 2), (1, 3)}
(4, 4)	{(4, 0), (3, 1), (2, 2), (1, 3), (0, 4)}
(3, 5)	{(3, 1), (2, 2), (1, 3)(0, 4)}
(2, 6)	{(2, 2), (1, 3), (0, 4)}
(1, 7)	{(1, 3), (0, 4)}
(0, 8)	{(0, 4)}

Every polynomial diagram has polynomial shape, but a polynomial shape diagram associated to a polynomial p, while having connections between the same pairs of vertices as in the polynomial diagram defined by p, can have any positive number of edges between connected vertices. When $q = 1$, a polynomial shape diagram is that of either an odometer or a finite set of points.

The incidence matrices for polynomial shape diagrams change at each level but have a predictable structure. They have size $|\mathcal{V}_n| \times |\mathcal{V}_{n+1}|$, where the entry for (v, w) is equal to the number of edges from $v \in \mathcal{V}_n$ to $w \in \mathcal{V}_{n+1}$. The number of non-zero entries on each row is exactly $|\mathcal{V}_1|$. For polynomial diagrams those non-zero entries correspond exactly to the coefficients of $p(x)$. Further, for polynomial diagrams the column corresponding to a vertex $w \in \mathcal{V}_{n+1}$, with $|S(w)| = |\mathcal{V}_1|$, will have exactly $|\mathcal{V}_1|$ non-zero entries that again correspond exactly to the coefficients of $p(x)$. This is most easily seen in polynomial diagrams of degree d associated to two variables, x and y. In this case, the associated diagrams can be drawn in such a manner that the incidence matrix representing the connections between levels n and $n + 1$ for each row i has non-zero entries in each position $(i, i + k)$, where $0 \leq k \leq d$. Further, the $(i, i + k)$ entry in the matrix is the coefficient of $x^{d-k}y^k$ in $p(x, y)$. For example, the connections between level 2 and 3 of the polynomial diagram associated to $p(x, y) = x^4 + 2x^3y + x^2y^2 + 3xy^3 + y^4$ can be described by the 9 by 13 incidence matrix

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 1 \end{bmatrix}.$$

See Figure 3. Multivariable polynomial shape systems have an analogous representation using higher-dimensional arrays.

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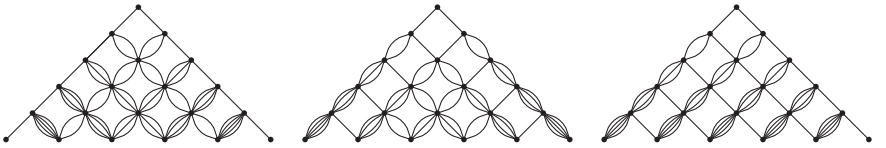


FIGURE 4. The Euler, reverse Euler, and Stirling Bratteli diagrams.

For the rest of this work we assume that X refers to the path space of a polynomial shape diagram for which $q \geq 2$. This class of diagrams satisfies all the conditions mentioned in §2. Some polynomial shape diagrams that have previously been studied include the Euler and reverse Euler diagrams in any dimension, as well as the Stirling diagrams. See Figure 4.

Remark 3.5. If $u \in S(w)$, then for all $j \in \{1, 2, \ldots, q\}$, $u(j) \leq w(j) \leq u(j) + d$. This implies that if $w, w' \in \mathcal{V}_{n+1}$ and $S(w) \cap S(w') \neq \emptyset$, then for each $j \in \{1, \ldots q\}$, $|w(j) - w'(j)| \leq d$. Likewise, if $u, u' \in S(w)$, then for each $j \in \{1, \ldots q\}$, $|u(j) - u'(j)| \leq d$. Further, the coordinates of each vertex in $\mathcal{V}_{n+1}$ must sum to $(n+1)d$, and since there are exactly q coordinates, at least one coordinate is at least $(n+1)d/q$.

The following proposition gives a complete description of which paths in a polynomial shape diagram have dense orbits.

PROPOSITION 3.6. *In a polynomial shape diagram with path space X, a path* $x \in X$ *has dense orbit if and only if*

$$\lim_{n\to\infty} \min\{v_n(x)(j) : j = 1, \ldots, q\} = \infty. \tag{3.2}$$

Proof. Recall (see the comments after Definition 3.3) that the set of vertices at level $k \leq n$ that connect to a vertex $v_n(x)$ at level n via a path of $j = n - k$ edges is the set of non-negative vectors u in $v_n(x) - (S + \cdots + S)$ (j times). Each such vertex u is on a path in the orbit of x. So for any k, given a vertex $u \in \mathcal{V}_k$, if for each $i = 1, \ldots, q$, $v_n(x)(i)$ grows without bound as n goes to infinity, for large enough n we will have $u \in v_n(x) - (S + \cdots + S)$ (with S repeated $n - k$ times), and hence the orbit of x will meet all cylinder sets determined by paths from the root to u. Conversely, if for some $j = 1, \ldots, q$ the entry $v_n(x)(j) \leq M$ for all n, no vector u with $u(j) > M$ can be on any path in the orbit of x. For example, one could take $u = (M+1)e_j \sim x_j^{M+1}$. □

PROPOSITION 3.7. *For polynomial shape diagrams, for every ordering the sets of maximal and minimal paths have measure zero for every fully supported ergodic invariant (Borel probability) measure.*

Proof. As mentioned above in Remark 2.4, we can adapt an argument from [5]. In [5], a diagram was defined to be *everywhere growing* if $\min\{\dim v : v \in \mathcal{V}_n\} \to \infty$ as $n \to \infty$. Polynomial shape diagrams need not have this property, but we show that they almost do, in the sense that for each such diagram there is a finite (possibly empty) set of paths $x^{(i)}$, $i = 1, \ldots, q$, for which $\dim v_n(x^{(i)})$ is bounded from above, while $\min\{\dim v : v \in \mathcal{V}_n \setminus \bigcup_{1\leq i\leq q} v_n(x^{(i)})\} \to \infty$ as $n \to \infty$.

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Let ξ be an ordering of a polynomial shape diagram $\mathcal{B}$ that is associated to a polynomial p, and let μ be a fully supported ergodic measure on the path space X. Since larger coefficients lead to larger dimensions for the vertices, we may assume that all coefficients of p are equal to 1, so that

$$p(x_1, \ldots, x_q) = \sum_{n_1+\cdots+n_q=d} x_1^{n_1} \cdots x_q^{n_q}. \tag{3.3}$$

For each $i = 1, \ldots, q$, consider the path $x^{(i)}$ such that $v_n(x^{(i)}) = x_i^{nd} \sim nde_i$ for all $n \geq 1$. Then $\dim v_n(x^{(i)}) = 1$ for each i and n. Since μ is a fully supported ergodic measure on X, $\mu\{x^{(i)}\} = 0$ for each $i = 1, \ldots, q$. So to show that for any ordering the sets of minimal and maximal paths have measure 0, we focus on paths not in this finite set.

As just mentioned above, the dimension of vertex $v = (m_1, \ldots, m_q)$ (with $\sum_i m_i = nd$) at level n is the coefficient of $x_1^{m_1} \cdots x_q^{m_q}$ in $p(x_1, \ldots, x_q)^n$. If $v = (m_1, \ldots, m_q)$ is not one of the vertices x_i^{nd} (that is, at least two of the m_i are not equal to 0), then $\dim v \geq n$.

(This may be seen as follows. The statement is obviously true for $n = 1$. Any vertex $v = x_1^{m_1} \cdots x_q^{m_q}$ at level $n > 1$ is not one of the special x_i^{nd} if and only if it has at least two of the exponents m_i positive; equivalently, if and only if it has at least two different source vertices. Thus, the statement is also true for $n = 2$. Assume now that $n > 2$, that $\dim w \geq n - 1$ for all $w \in \mathcal{V}_{n-1}$ for which w is not one of the special x_i^{nd}, and that $v \in \mathcal{V}_n$ has at least two source vertices, v_{n-1} and v'_{n-1}. Then at least one of these, call it v_{n-1}, again has two source vertices, since otherwise both would equal some $x_i^{(n-1)d}$, and this cannot occur for $n > 2$. Since $\dim v \geq \dim v_{n-1} + \dim v'_{n-1}$ and $\dim v_{n-1} \geq n - 1$, we have $\dim v \geq n$.)

Let $v \in \mathcal{V}_{n+1}$ be a vertex at level $n + 1$. Each path $\alpha = \alpha_0 \cdots \alpha_n$ from the root to v defines a cylinder set $[\alpha] = \{x \in X : x_{[0,n]} = \alpha\}$, and all these cylinder sets have equal measure

$$\mu[\alpha] = \frac{\mu[v]}{\dim v}, \tag{3.4}$$

where $[v] = \{x \in X : v_{n+1}(x) = v\}$. Denote by $\mathcal{V}'_{n+1}$ the set of vertices in $\mathcal{V}_{n+1}$ that are not any of the $(n + 1)de_i$ and by M_n the set of finite minimal paths with terminal vertex in $\mathcal{V}'_{n+1}$. If x is a minimal path that is not one of the $x^{(i)}$, then $x_{[0,n]} \in M_n$ for all large enough n. For each $v \in \mathcal{V}_{n+1}$, there is exactly one finite minimal path from the root to v, so, as in [5, Lemma 6.2],

$$\mu \bigcup \{[\alpha] : \alpha \in M_n\} \leq \sum_{v \in \mathcal{V}'_{n+1}} \frac{\mu[v]}{\dim v} \leq \frac{1}{n+1} \sum_{v \in \mathcal{V}'_{n+1}} \mu[v] \leq \frac{1}{n+1} \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Similarly, the set of maximal paths has measure 0. □

Definition 3.8. For each $j \in \{1, \ldots, q\}$, let e_j denote the standard jth basis vector in q-space. Given $w \in \mathcal{V}_{n+1}$, the *distinguished source vertex of w in direction j*, denoted $DSV(w, j)$, is the vertex $w - de_j \in \mathcal{V}_n$.

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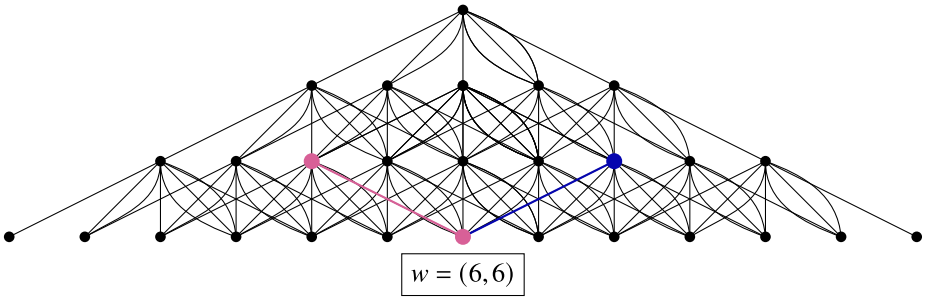


FIGURE 5. $DSV(w, 1) = (2, 6)$ is shown in blue and $DSV(w, 2) = (6, 2)$ is shown in red. Another vertex with $DSV(w, 1)$ in its source set has first coordinate less than $w(1) = 6$. Likewise, a vertex with $DSV(w, 2)$ in its source set has second coordinate less than $w(2) = 6$. (Color online.)

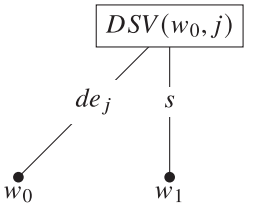


FIGURE 6. The $DSV(w_0, j)$ is in the source set of w_1 and, hence, $s(j) < d$, which implies $w_1(j) < w_0(j)$.

Remark 3.9. A vertex $w \in \mathcal{V}_{n+1}$ has a distinguished source vertex in direction j if and only if $w(j) \geq d$. The $DSV(w, j)$ is the vertex in the source set of w obtained by division of the monomial $x_1^{w(1)} \cdots x_q^{w(q)}$ by x_j^d, reducing the exponent of x_j by the maximal possible amount while the other exponents are unchanged. See Figure 5.

LEMMA 3.10. *Let $w_0, w_1 \in \mathcal{V}_n$ with $w_0 \neq w_1$. If $DSV(w_0, j) \in S(w_1)$, then*

$$1 \leq w_0(j) - w_1(j).$$

Proof. Assume $DSV(w_0, j) \in S(w_1)$. Then there exists $s \in S$ such that $DSV(w_0, j) + s = w_1$. This gives

$$w_0 - de_j + s = w_1,$$
$$\text{so that } s = w_1 - w_0 + de_j.$$

Since the coordinates of s are non-negative and sum to d, $0 \leq s(j) = w_1(j) - w_0(j) + d \leq d$, implying that

$$0 \leq w_0(j) - w_1(j).$$

If $w_0(j) - w_1(j) = 0$, this implies that $s = de_j$. However, then $w_1 = DSV(w_0, j) + de_j = w_0$. Since we assumed that $w_0 \neq w_1$, we must have $w_0(j) - w_1(j) \neq 0$. Hence,

$$1 \leq w_0(j) - w_1(j).$$

See Figure 6. □

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LEMMA 3.11. *Let* $w_0, w_1 \in \mathcal{V}_n$. *If* $DSV(w_0, j_1) \in S(w_1)$ *and* $DSV(w_0, j_2) \in S(w_1)$, *where* $j_1 \neq j_2$, *then* $w_1 = w_0$.

Proof. Recall from the definition of $DSV(w, j)$ that

$$w(i) = \begin{cases} DSV(w, j)(i), & i \neq j, \\ d + DSV(w, j)(i), & i = j. \end{cases}$$

Therefore, $w_0(j_1) = DSV(w_0, j_2)(j_1)$. Since $DSV(w_0, j_2) \in S(w_1)$, by Remark 3.5, $DSV(w_0, j_2)(j_1) \leq w_1(j_1)$. So we have $DSV(w_0, j_1) \in S(w_1)$ and $w_0(j_1) \leq w_1(j_1)$. Then, by Lemma 3.10, it must be the case that $w_1 = w_0$. □

4. *Covered and uncovered vertices*

We are trying to prove that for polynomial shape diagrams, for every ordering and all large enough i there are no depth i pairs (outside of a negligible exceptional set), that is, no paths x, x' that have the same i coding but different $i+1$ codings. If x and x' were a depth i pair, then for some $m \in \mathbb{Z}$, $w = v_{i+1}(T^m x) \neq v_{i+1}(T^m x') = w'$, and the source sets $S(w)$ and $S(w')$ intersect. Thus, we need to keep track of a key aspect of the interconnections among vertices in a diagram, namely which vertices at a given level have their source sets contained in those of other vertices at that level. This will allow precise tracking of vertices through which the depth i pair passes.

Definition 4.1. We say that a vertex w is *covered* if there is a vertex $w' \neq w$ (at the same level as w) such that $S(w) \subseteq S(w')$, in which case we say that w' *covers* w. Otherwise w is said to be *uncovered*.

We show that for any depth i pair x, x', the time m can be chosen so that w and w' are uncovered. For i sufficiently large, this will ensure that after time m the orbits of x and x' continue to pass through distinct uncovered vertices at level $i+1$ for quite a while, essentially chasing each other at this level, while staying together at level i.

Lemma 3.11 implies that if a vector w has at least two distinct coordinates, j_1, j_2, for which $DSV(w, j_1)$ and $DSV(w, j_2)$ are defined, then w must be uncovered. While this is sufficient, it is not necessary. For example, for $q \geq 3$, the vertex on level 2 given by $w = (d, d-1, 1, 0, \ldots, 0)$ is uncovered (see Proposition 4.2), but the only distinguished source vertex for w that exists is $DSV(w, 1)$ (we need to subtract exactly d from a single entry of w to get a DSV). The following discussion gives necessary and sufficient conditions for a vertex to be covered, and in the case that it is covered, explicitly describes the vertices by which it is covered.

PROPOSITION 4.2. *A vertex* w *at a level* $n > q \geq 2$ *is covered if and only if there is a* $j \in \{1, \ldots, q\}$ *such that* $w(j) > (n-1)d$.

Proof. We prove the first direction by proving the contrapositive. Suppose that $w(i) \leq (n-1)d$ for all $i \in \{1, \ldots, q\}$. Because $n > q$, by Remark 3.5 there is at least one j with

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$w(j) \geq d$. Since $\sum_{i=1}^{q} w(i) = nd$ and $n > q \geq 2$,

$$\sum_{i \neq j} w(i) \geq d. \tag{4.1}$$

We make two vertices u and u' in the source set of w that cannot both be in the source set of any $w' \neq w$, as follows. Define

$$u(i) = \begin{cases} w(j) - d & \text{if } i = j, \\ w(i) & \text{if } i \neq j \end{cases} \tag{4.2}$$

(so that $u = DSV(w, j)$). Note that $\sum_i u(i) = (n-1)d$.

For each $i \neq j$, choose $n(i) \in [0, w(i)]$ so that $\sum_{i \neq j} n(i) = \sum_{i \neq j} w(i) - d$, and let

$$u'(i) = \begin{cases} w(j) & \text{if } i = j, \\ n(i) & \text{if } i \neq j. \end{cases} \tag{4.3}$$

The idea is that to get u' from w, we subtract, in any possible way, a total of d from coordinates not equal to j. Hence, $\sum_i u'(i) = (n-1)d$.

By Lemma 3.10, for every w' such that $u = DSV(w, j) \in S(w')$ and $w' \neq w$, we have $w'(j) < w(j)$. If $s' = w' - u'$ is a non-negative vector with $|s'| = d$, and $s'(j) \geq 0$, then $w'(j) \geq w(j)$. So it is not possible that there is a w' with u and u' both in $S(w')$. Therefore, such a vertex w is uncovered.

To prove the converse, suppose that there is a $j \in \{1, \ldots, q\}$ such that $w(j) > (n-1)d$; we will prove that then w is covered. First, we have $\sum_{i \neq j} w(i) < d$, so that

$$c(w) = d - \sum_{i \neq j} w(i) \in [1, d]. \tag{4.4}$$

Choose $b \in [1, c(w)]$, let σ be an integer vector with q components such that $\sigma(j) = -b$, choose $\sigma(i) \geq 0$ for $i \neq j$ such that $\sum_{i \neq j} \sigma(i) = b$, and let $w' = w + \sigma$. We claim that w' covers w.

Suppose that $u \in S(w)$. Then, $u = w - s$ for some s; and $s(j) \geq c(w)$, since $s(j) < c(w)$ implies $|s| < c(w) + \sum_{i \neq j} s(i) \leq c(w) + \sum_{i \neq j} w(i) = d$. Therefore, such a triple w, s, j satisfies the following useful condition. □

CONDITION 4.3. *The vertex $w \in \mathcal{V}_n$, non-negative integer vector s with $|s| = d$, and $j \in \{1, \ldots, q\}$ satisfy $w(j) > (n-1)d$, $s(i) \leq w(i)$ for all i, and $s(j) \in [c(w), d]$.*

(Conversely, for any w, j, and s that satisfy Condition 4.3, $u = w - s \in S(w)$. In fact, the converse holds even if we no longer assume $s(j) \in [c(w), d]$.)

We know that $1 \leq b \leq c(w) \leq s(j)$, so $s(j) \geq b$. Therefore, $s' = \sigma + s \geq 0$ and $u + s' = w'$, showing that $u \in S(w')$. Hence, $S(w) \subseteq S(w')$, and w is covered.

Remark 4.4. The previous proposition is stated in terms of the definition of covered. We frequently use it to determine that a vertex is uncovered. In particular, Proposition 4.2 could be stated as follows: for $n > q \geq 2$, a vertex $w \in \mathcal{V}_n$ is uncovered if and only if for

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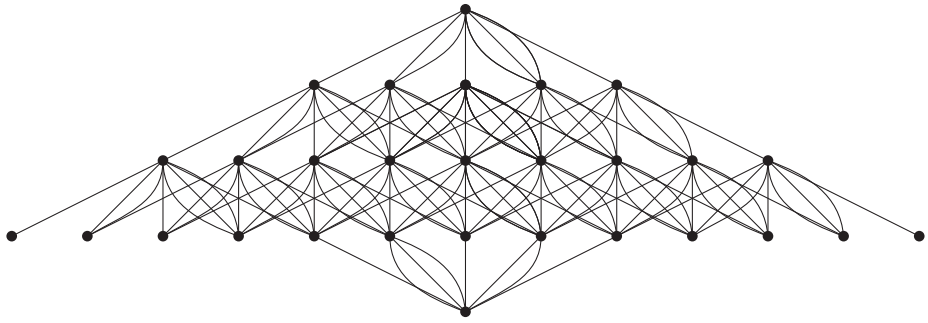


FIGURE 7. Level 4 is the first level to have a vertex whose entire source set is uncovered. For a vertex source set to be entirely uncovered, all coordinates should be less than or equal to $(n-2)d$ which, in this case, is $2(4) = 8$.

every $l \in \{1, \ldots, q\}$ we have $w(\ell) \leq (n-1)d$ or, equivalently, if there exists j for which $d \leq w(j) \leq (n-1)d$.

Continue to assume through the end of Remark 4.6 that $n > q \geq 2$. Now we will specify, given a covered vertex w, exactly which vertices $w' \neq w$ cover w. Since this information is not used in our main argument, the proof is omitted.

PROPOSITION 4.5. *Suppose that $n > q \geq 2$. Let $w, w' \in \mathcal{V}_n$ with $w \neq w'$. Assume that $j \in \{1, \ldots, q\}$ satisfies $w(j) > (n-1)d$, so that w is covered. Let $\sigma = w' - w$ (so that $\Sigma\sigma(i) = 0$). Then w' covers w if and only if σ satisfies the following equivalent conditions with respect to w.*

(1) *For each choice of s satisfying Condition 4.3 with respect to w and j, we have $s \neq \sigma + s \geq 0$.*
(2) *Since $w \in \mathcal{V}_n$ and $j \in \{1, \ldots, q\}$ satisfy $w(j) > (n-1)d$, defining $b = -\sigma(j) \in [1, c(w)]$, we have $\sigma(i) \geq 0$ for all $i \neq j$, and $\sum_{i \neq j} \sigma(i) = b$.*

Moreover, $w' = w + \sigma$ covers w and is itself uncovered if and only if σ satisfies condition (2) with $b = c(w)$.

Remark 4.6. The central part of Proposition 4.5 could be restated as follows. If there exists a $j \in \{1, 2, \ldots, q\}$ such that $w(j) > (n-1)d$, then w' covers w if and only if for $\sigma = w - w'$ we have $-d \leq \sigma(j) < 0$, and for $i \neq j$, $\sigma(j) \geq 0$.

We mention a few more useful observations about uncovered vertices. By Remark 3.5, every vertex in $\mathcal{V}_n$ has at least one coordinate greater than or equal to nd/q. Therefore, the hypothesis of the following lemma (Lemma 4.7) can be satisfied only if $nd/q \leq (n-2)d$, which implies that $2q/(q-1) \leq n$. See Figure 7.

LEMMA 4.7. *For $n > q \geq 2$, if $w \in \mathcal{V}_n$ is such that for every $l \in \{1, \ldots, q\}$, $w(\ell) \leq (n-2)d$, then every vertex in $S(w)$ is uncovered.*

Proof. Let $u \in S(w)$. By Remark 3.5, for all $l \in \{1, \ldots, q\}$, $u(\ell) \leq w(\ell)$. Since $u \in \mathcal{V}_{n-1}$, by Remark 4.4 u is uncovered, since for every $\ell \in \{i, \ldots, q\}$, $u(\ell) \leq (n-2)d$. □

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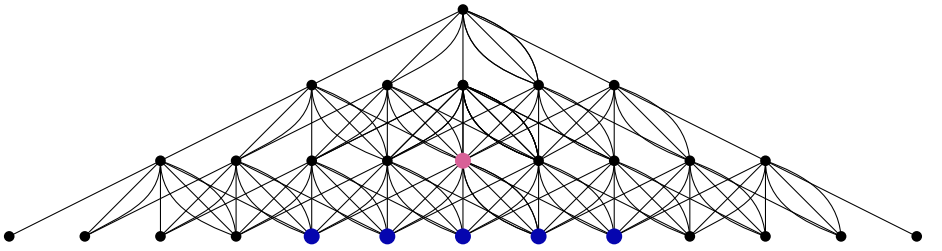


FIGURE 8. Vertex (4, 4), pictured in red, in level 2 is uncovered, and every vertex in level 3 with (4, 4) its source set, pictured in blue, is also uncovered. (Color online.)

COROLLARY 4.8. *Given $w \in \mathcal{V}_n$, if there exists $j \in \{1, \dots, q\}$ such that $2d \le w(j) \le (n-2)d$, then every vertex in $S(w)$ is uncovered.*

Proof. Since $2d \le w(j)$, we have that

$$2d + \sum_{l \neq j} w(\ell) \le w(j) + \sum_{l \neq j} w(\ell) = nd.$$

Hence, $\sum_{l \neq j} w(\ell) \le (n-2)d$. Therefore, for each $l \neq j$ we have that $w(\ell) \le (n-2)d$, and, by assumption, $w(j) \le (n-2)d$. Hence, for all $\ell \in \{1, \dots, q\}$, $w(\ell) \le (n-2)d$. Then, by Lemma 4.7, every vertex in $S(w)$ is uncovered. □

LEMMA 4.9. *For $n - 1 > q \ge 2$, if $u \in \mathcal{V}_{n-1}$ is uncovered and $u \in S(w)$, then w is uncovered.*

Proof. Assume $u \in \mathcal{V}_{n-1}$ is uncovered. By Remark 4.4, for all $l \in \{1, \dots, q\}$, $u(\ell) \le (n-2)d$. Further, for all $\ell \in \{1, \dots, q\}$, $w(\ell) \le u(\ell) + d$ (by Remark 3.5). So, $w(\ell) \le u(\ell) + d \le (n-2)d + d = (n-1)d$. Then, by Remark 4.4, w is uncovered. See Figure 8. □

5. *Chains and links*

In this section we develop some of the machinery that is used in the main theorem, continuing to work in a fixed polynomial shape diagram with path space X.

As mentioned at the beginning of §4, we show that any depth i pair will, at some time, meet distinct uncovered vertices at level $i + 1$, and these vertices are linked together via edges connecting them to the vertex at level i met by both orbits at this time. The orbits of x and x' will then continue to meet distinct uncovered vertices (and for i large, this will continue for quite a while), essentially chasing each other at level $i + 1$, while staying together at level i. Tracking these orbits will produce a sequence of vertices at level $i + 1$ that are met in an alternating manner by the orbits of x and x', respectively. Any two consecutive vertices in the sequence will be linked by a vertex at level i that is met simultaneously by both orbits, producing what we refer to as an x, x' *chain*.

Definition 5.1. Given $k, n \ge 1$, a *chain of length k at level n* is a double sequence

$$w_0 \overset{u_0}{\longleftrightarrow} w_1 \overset{u_1}{\longleftrightarrow} w_2 \cdots w_{k-1} \overset{u_{k-1}}{\longleftrightarrow} w_k \tag{5.1}$$

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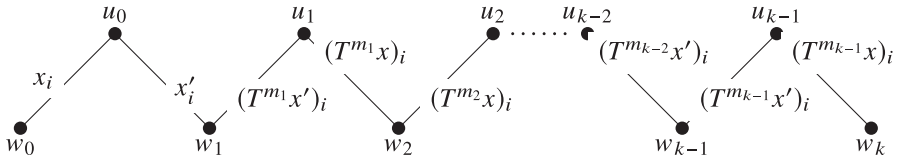


FIGURE 9. An example of an x, x' chain of length k, where k is even. Other vertices are allowed, but those pictured are those that make up the chain.

of vertices $w_0, \ldots, w_k$ at level n, called *splitting vertices*, and $u_0, \ldots, u_{k-1}$ at level $n-1$, called *shared vertices*, such that for every $0 \le \ell < k$, $u_\ell \in S(w_\ell) \cap S(w_{\ell+1})$. We say that the chain *links the two vertices* w_0 *and* w_k. The chain is *straight* if all vertices are distinct. A chain of length 1 is called a *link*.

Definition 5.2. We say that a straight chain

$$w_0 \overset{u_0}{\longleftrightarrow} w_1 \overset{u_1}{\longleftrightarrow} w_2 \overset{u_2}{\longleftrightarrow} w_3 \overset{u_3}{\longleftrightarrow} w_4 \overset{u_4}{\longleftrightarrow} \ldots \overset{u_{k-1}}{\longleftrightarrow} w_k$$

is a *distinguished chain in direction* j if for each $1 \le \ell < k$, u_ℓ is $DSV(w_\ell, j)$.

We do not want to impose in the above definition any restrictions on u_0 because our ultimate goal is to extend any arbitrary link to a distinguished chain. The coordinate description of the distinguished source vertex allows us to specify a direction (direction j, for some j) in which to keep moving, to ensure that our chain is straight. The following definition will be applied to a depth i pair to find sequences of vertices at levels i and $i+1$ that eventually cause a contradiction.

Definition 5.3. For $i \ge 1$ and a pair of paths x, x', a chain

$$w_0 \overset{u_0}{\longleftrightarrow} w_1 \overset{u_1}{\longleftrightarrow} w_2 \cdots w_{k-1} \overset{u_{k-1}}{\longleftrightarrow} w_k$$

of length k at level $i+1$ is called an x, x' *chain* if there is a strictly monotonic sequence of integers $m_0 = 0, m_1, \ldots, m_{k-1}$, called *the switching times*, such that

for each $1 \le \ell \le k$, $u_\ell = v_i(T^{m_\ell}x) = v_i(T^{m_\ell}x')$; $w_0 = v_{i+1}(x)$, $w_1 = v_{i+1}(x')$;
for odd $\ell \in [1, k)$, $w_\ell = v_{i+1}(T^{m_{\ell-1}}x') = v_{i+1}(T^{m_\ell}x')$;
for even $\ell \in [2, k)$, $w_\ell = v_{i+1}(T^{m_{\ell-1}}x) = v_{i+1}(T^{m_\ell}x)$;
and $w_k = v_{i+1}(T^{m_{k-1}}x)$ if k is even, $w_k = v_{i+1}(T^{m_{k-1}}x')$ if k is odd.
An x, x' chain of length 1 is called an x, x' *link*. See Figure 9.

LEMMA 5.4. [Link Lemma] *For $i > q \ge 2$, if x, x' is a depth i pair of paths, then every straight x, x' link $w_0 \overset{u_0}{\longleftrightarrow} w_1$ with $v_{i+1}(x) = w_0$, $v_{i+1}(x') = w_1$, such that there exists a $j \in \{1, \ldots, q\}$ for which $2d \le w_1(j) \le w_0(j) \le (i-1)d$, extends to a straight x, x' chain of length 2*

$$w_0 \overset{u_0}{\longleftrightarrow} w_1 \overset{DSV(w_1,j)}{\longleftrightarrow} w_2,$$

with m_1, as in the definition of an x, x' chain, satisfying $v_{i+1}(T^{m_1}x') = w_1$, $v_{i+1}(T^{m_1}x) = w_2$, and moreover for all $m \in [0, m_1]$, $v_{i+1}(T^m x') = w_1$ and $v_{i+1}(T^m x) \in \{w_0, w_2\}$.

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Proof. By assumption, there exists a j such that $2d \leq w_1(j) \leq w_0(j) \leq (i-1)d$, which by Remark 4.4 is more than sufficient to ensure that w_0 and w_1 are both uncovered. Since $w_1(j) \leq w_0(j)$, by Lemma 3.10 $DSV(w_1, j) \notin S(w_0)$. In particular, $DSV(w_1, j) \neq u_0 = v_i(x')$. Choose $m_1 \neq 0$ with $|m_1|$ minimal such that $v_i(T^{m_1}x') = DSV(w_1, j)$, and for all $m \in [0, m_1]$ (or $[m_1, 0]$ if $m_1 < 0$), $v_{i+1}(T^m x') = w_1$. Then define $w_2 = v_{i+1}(T^{m_1}x)$. Replacing T by T^{-1} if necessary, assume $m_1 > 0$. Since x, x' are depth i, $v_i(T^{m_1}x) = DSV(w_1, j)$ and m_1 is also the first time that the forward orbit of x hits $DSV(w_1, j)$.

Since $DSV(w_1, j) \notin S(w_0)$, $w_2 \neq w_0$. We claim that also $w_2 \neq w_1$. Suppose, to the contrary, that $w_2 = w_1$. By the way we chose m_1, x' is still at w_1 at time m_1. Look at the edge numbers based at w_1. The orbit of x has entered w_1 before or at time m_1, but after x' (which is at w_1 at time 0), so $\xi((T^{m_1}x)_i) < \xi((T^{m_1}x')_i)$. In other words, $(T^{m_1}x)_i$ connects w_1 and $DSV(w_1, j)$, as does $(T^{m_1}x')_i$, and $(T^{m_1}x)_i$ has smaller edge label than $(T^{m_1}x')_i$.

Going backwards in time, the orbit of x' has to hit $DSV(w_1, j)$ again. Hence, it passes through edge $(T^{m_1}x)_i$ at some time $m_2 < m_1$. Specifically, the backward orbit of x' goes through edges based at w_1, starting at edge number $\xi((T^{m_1}x')_i)$ and working down to edge number $\xi((T^{m_1}x)_i)$. It stays at w_1 all this time, at least through time m_2. By the minimality of m_1, there is no time m in $[0, m_1)$ such that $v_i(T^m x') = DSV(w_1, j)$. Therefore, it is necessarily the case that $m_2 < 0$. For all $m \in [m_2, m_1]$, since $v_{i+1}(T^m x') = w_1$, we have $v_i(T^m x') = v_i(T^m x) \in S(w_1)$. Further, since $DSV(w_1, j) \notin S(w_0)$, we know that $v_i(T^{m_2}x) \neq w_0$. Therefore, we have that $v_i(T^{m_2}x) \neq w_0$, $v_i(x) = w_0$, and $v_i(T^{m_1}x) \neq w_0$. Then, by the definition of the Vershik map, the orbit of x must pass through every vertex in $S(w_0)$ between time m_2 and m_1. Hence, it is necessarily the case that $S(w_0) \subseteq S(w_1)$ and w_1 covers w_0. This contradicts our assumption that w_0 is uncovered, proving that $w_2 \neq w_1$ and

$$w_0 \overset{u_1}{\longleftrightarrow} w_1 \overset{DSV(w_1,j)}{\longleftrightarrow} w_2 \tag{5.2}$$

is a straight x, x' chain of length 2.

We claim that between times 0 and m_1, the orbit of x cannot pass through any vertices at level $i+1$ other than w_0 and w_2. For all m in $[0, m_1]$, $v_{i+1}(T^m x') = w_1$ and hence $v_i(T^m x') \in S(w_1)$; further, since x and x' are a depth i pair, $v_i(T^m x) = v_i(T^m x') \in S(w_1)$. Therefore, if the orbit of x leaves w_0 and passes through some vertex $w \neq w_2$ between time 0 and m_1, then w must be covered by w_1.

However, since $2d \leq w_1(j) \leq (i-1)d$, by Corollary 4.8 every vertex in $S(w_1)$ is uncovered. In particular, for each $m \in [0, m_1]$, $v_i(T^m x)$ is uncovered. Then, by Lemma 4.9, $v_{i+1}(T^m x)$ is also uncovered. In particular, $v_{i+1}(T^m x)$ cannot be covered by w_1. Therefore, between time 0 and m_1 the orbit of x can only pass through the vertices w_0 and w_2. □

6. *Inherent expansiveness of polynomial shape diagrams*

Here, X continues to denote the path space of a fixed polynomial shape diagram (Definition 3.1). We intend to prove that outside of the negligible exceptional set defined above (Definition 2.1), there are no pairs of paths of any large enough depth. The idea is to

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show that for any sufficiently large i and any possible depth i pair x, x' in $X(\xi)$ (the set of paths which are not in the orbit of any maximal or minimal path and have dense biinfinite orbits), one can construct a distinguished x, x' chain in which the splitting vertices are uncovered and the orbits of x and x' meet vertices of the chain in an alternating manner. The chain will have sufficient length $(2d + 3)$ to ensure that when we look at level $i + 2$, we find that the alternation contradicts the definition of the Vershik map. To construct this chain, it will be necessary to find two uncovered vertices forming a link (Lemma 6.1), each with jth component of size at least $2d^2 + 4d$ for some $j \in \{1, \ldots, q\}$, with which to start the chain. The following lemma allows us to find such a link if we look low enough in the diagram.

LEMMA 6.1. (Chain starting) *Let an ordering ξ be given, $N = 4dq + 6q$, and $i > N$. For any depth i pair $x, x' \in X(\xi)$, there exists a time in their orbits when, at level $i + 1$, they pass through distinct uncovered vertices v and v' for which there is a $j \in \{1, 2, \ldots, q\}$ such that*

$$2d^2 + 4d \leq v'(j) < v(j) \leq (i - 1)d < id. \tag{6.1}$$

Proof. Let ξ be given, $i > N$, and suppose that $x, x' \in X(\xi)$ is a depth i pair. Then, by Proposition 2.11, there is a time in their orbits such that x and x' pass through different vertices at level $i + 1$, say w_0 and w_0', respectively. Without loss of generality, assume this happens at time 0.

Since $w_0, w_0' \in \mathcal{V}_{i+1}$ and $i + 1 \geq 4dq + 6q$, Remark 3.5 tells us that they each have a component (not necessarily the same one) of size at least

$$(i + 1)d/q \geq (4dq + 6q)d/q = 4d^2 + 6d.$$

Let $j \in \{1, \ldots, q\}$ such that $w_0'(j) \geq 4d^2 + 6d$. Since $v_i(x) = v_i(x')$, w_0 and w_0' share a common source vertex and hence each of their components differ by at most d. (See Remark 3.5.) Therefore, the jth components of *both* vertices are of size at least $4d^2 + 5d$. Switching names if necessary, assume $4d^2 + 5d \leq w_0(j) \leq w_0'(j)$.

We do not know if our initial w_0 and w_0' meet the conditions of (6.1) or not. However, the following procedure produces two new vertices that meet the conditions of (6.1), regardless of whether w_0 and w_0' do. We produce an x, x' chain of length $2d + 2$ starting with the link between w_0' and w_0 for which the jth components of the splitting vertices are decreasing by at least one and by at most d each time. Since we know that $4d^2 + 5d \leq w_0(j) \leq (i + 1)d$, the splitting vertex w_{2d+1} at the end of the chain must have jth component at most $(i + 1)d - (2d + 1) = (i - 1)d - 1 < (i - 1)d$, but at least $4d^2 + 5d - (2d + 1)d = 2d^2 + 4d > 2d$, and hence be uncovered and have all vertices in its source set uncovered, by Corollary 4.8. (The ultimate goal is to have the end of this chain start (in the next theorem) a new chain of adequate length for which every splitting vertex has an uncovered source set. The above-mentioned vertex w_{2d+1} having jth component $2d^2 + 4d$ accomplishes this.)

Switching to T^{-1} if necessary, let $m_1 > 0$ be the first time in the orbit of x such that $v_i(T^{m_1}x) = DSV(w_0, j)$ and for all m in $[0, m_1]$, $v_{i+1}(T^m x) = w_0$. First, consider the case in which $v_{i+1}(T^{m_1}x') = w_0$. Then the orbit of x' enters w_0 after x, so $\xi((T^{m_1}x')_i) < \xi((T^{m_1}x)_i)$. By density, the orbits of x and x' must eventually leave w_0, but $\xi((T^{m_1}x')_i) <$

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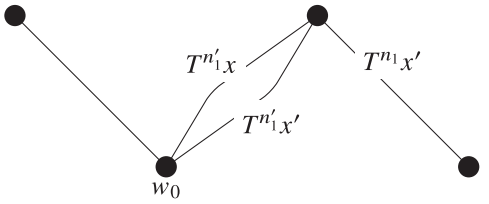


FIGURE 10. $\xi((T^{n_1'}x)_i) < \xi((T^{n_1'}x')_i)$ and so the orbit of x must pass through $(T^{n_1'}x')_i$ at some time greater than or equal to time n_1, and hence, is still at vertex w_0 at time n_1.

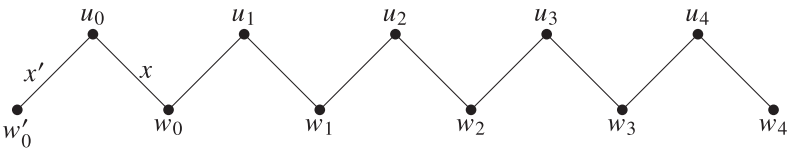


FIGURE 11. We build the chain where for $l \geq 1$, $u_l = v_i(T^{n_l}x) = v_i(T^{n_l}x')$ and w_l is $v_{i+1}(T^{n_l}x)$ or $v_{i+1}(T^{n_l}x')$, and w_{l-1} is the other vertex.

$\xi((T^{m_1}x)_i)$ implies this must occur at different times. Let $n_1 > m_1$ be the first time that $v_i(T^{n_1}x) = v_i(T^{n_1}x') = DSV(w_0, j)$ and $v_{i+1}(T^{n_1}x) \neq v_{i+1}(T^{n_1}x')$. In other words, n_1 is the first time after one of these orbits (of x and x') leave w_0 that both orbits meet the $DSV(w_0, j)$.

Without loss of generality, assume the orbit of x' leaves w_0 first. We claim that $v_{i+1}(T^{n_1}x) = w_0$. Consider the last time $n_1' < n_1$ that $v_i(T^{n_1'}x) = v_i(T^{n_1'}x') = DSV(w_0, j)$ and $v_{i+1}(T^{n_1'}x) = v_{i+1}(T^{n_1'}x') = w_0$. Then $\xi((T^{n_1'}x)_i) < \xi((T^{n_1'}x')_i)$ (since x' leaves w_0 first) implies the forward orbit of $T^{n_1'}x$ must still pass through $(T^{n_1'}x')_i$ before it can leave w_0. In particular, this happens after time n_1'. Since n_1 is the first time after n_1' that the orbits of x and x' visit $DSV(w_0, j)$, it must be the case that the orbit of x is still at w_0 at time n_1. So we have that $v_{i+1}(T^{n_1}x) = w_0$ and $v_{i+1}(T^{n_1}x') \neq w_0$. See Figure 10.

For the second case in which $v_{i+1}(T^{m_1}x') \neq w_0$, let $n_1 = m_1$. Then, in either case, we have $v_i(T^{n_1}x) = v_i(T^{n_1}x') = DSV(w_0, j)$, $v_{i+1}(T^{n_1}x) = w_0$, and $v_{i+1}(T^{n_1}x') \neq w_0$. Call $v_{i+1}(T^{n_1}x') = w_1$. Then $w_1 \neq w_0$, but $DSV(w_0, j) \in S(w_1)$. So by Lemma 3.10 and Remark 3.5, $1 \leq w_0(j) - w_1(j) \leq d$. Furthermore, since $w_0'(j) \geq w_0(j)$, Lemma 3.10 also implies that $DSV(w_0, j) \notin S(w_0')$, indicating that $w_1 \neq w_0'$.

Since $S(w_1) \cap S(w_0) \neq \emptyset$, by Remark 3.5 $w_1(j) \geq 4d^2 + 4d > d$. Hence, by Remark 3.9, $DSV(w_1, j)$ exists. Arguing as above, there exists a smallest $n_2 > n_1$ such that $v_i(T^{n_2}x) = v_i(T^{n_2}x') = DSV(w_1, j)$ and $v_{i+1}(T^{n_2}x) \neq v_{i+1}(T^{n_2}x')$. Therefore, one of $v_{i+1}(T^{n_2}x)$ and $v_{i+1}(T^{n_2}x')$ must be w_1. Let the other be w_2. Then we have $2 \leq w_0(j) - w_2(j) \leq 2d$. Continue in this manner to define $w_3, w_2, \ldots, w_{2d}, w_{2d+1}$. See Figure 11.

As ℓ increases from 0 to $2d$, the jth coordinate of w_ℓ decreases by at least one each time and by at most d each time. Therefore, by the time we produce w_{2d}, the jth coordinate has decreased by at least $2d$ from the jth coordinate of w_0, ensuring that $w_{2d}(j) \leq w_0(j) - 2d \leq (i+1)d - 2d = (i-1)d$. Further, the jth coordinate decreases by at most $2d^2$.

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Then we have

$$w_0(j) - w_{2d}(j) \leq 2d^2,$$

so that

$$w_{2d}(j) \geq w_0(j) - 2d^2 \geq 4d^2 + 5d - 2d^2 = 2d^2 + 5d.$$

Also, $w_{2d}(j) - w_{2d+1}(j) \leq d$ implies that

$$d \geq w_{2d}(j) - w_{2d+1}(j) \geq 2d^2 + 5d - w_{2d+1}(j),$$

and hence

$$w_{2d+1}(j) \geq 2d^2 + 5d - d = 2d^2 + 4d.$$

Further, since the jth component is decreasing each time, w_{2d} and w_{2d+1} are distinct and $2d^2 + 4d \leq w_{2d+1}(j) < w_{2d}(j) \leq (i-1)d < id$. Hence, by Remark 4.4, $v = w_{2d}$ and $v' = w_{2d+1}$ are distinct uncovered vertices. Therefore, we have shown that the (dense) orbits of the depth i pair x, x' pass through distinct uncovered vertices v and v' for which $2d^2 + 4d \leq v'(j) < v(j) \leq (i-1)d < id$. □

LEMMA 6.2. *Let $z \in \mathcal{V}_{n+1}$ and $j \in \{1, 2, \ldots, q\}$ be such that $d \leq z(j) \leq nd$. Then there exist $w_0, w_1, \ldots, w_d \in S(z)$ such that $w_\ell(j) = z(j) - \ell$ for all $\ell = 0, 1, \ldots, d$.*

Proof. Since $d \leq z(j) \leq nd$ and $\sum_{i=1}^{q} z(i) = (n+1)d$, we have $d \leq \sum_{i \neq j} z(i) \leq nd$. We can then find an $s_0 \in S$ such that $s_0(j) = 0$, $\sum_{i \neq j} s_0(i) = d$ and $z - s_0$ has all non-negative coordinates. Hence, $w_0 = z - s_0 \in \mathcal{V}_n$. This means $w_0 \in S(z)$ and $w_0(j) = z(j)$. We proceed in a recursive manner. Since $\sum_{i \neq j} s_0(i) = d$, there exists a $j_1 \neq j$ such that $s_0(j_1) \geq 1$. Define $s_1 = s_0 - e_{j_1} + e_j$ (recalling that e_{j_1} and e_j are standard basis vectors in q-space) and let $w_1 = z - s_1$. Then w_1 has all coordinates non-negative, and hence is in $\mathcal{V}_n$ and $S(z)$. Further, $\sum_{i \neq j} s_1(i) = d - 1$, $s_1(j) = 1$, and $w_1(j) = z(j) - s_1(j) = z(j) - 1$. We continue recursively, producing $s_\ell \in S(z)$ from $s_{\ell-1}$ for each $\ell \leq d$. Specifically, given that $\sum_{i \neq j} s_{\ell-1}(i) = d - (\ell - 1)$ and $s_{\ell-1}(j) = \ell - 1$ for some $\ell \leq d$, choose $j_\ell \neq j$ such that $s_{\ell-1}(j_\ell) \geq 1$ and define $s_\ell = s_{\ell-1} - e_{j_\ell} + e_j$. Then define $w_\ell = z - s_\ell$. So, $s_\ell(j) = \ell$, $\sum_{i \neq j} s_\ell(i) = d - \ell$, and $w_\ell \geq 0$. Hence, $w_\ell \in S(z)$ and $w_\ell(j) = z(j) - \ell$. □

Recall that, given an ordering ξ, $X(\xi)$ denotes the set of paths which are not in the orbit of any maximal or minimal path and have a dense (biinfinite) orbit. By Propositions 2.3 and 3.7 (and Remark 2.4), $X(\xi)$ is comeager and has full measure with respect to every fully supported ergodic invariant measure on X.

THEOREM 6.3. *Suppose we have an unordered polynomial shape Bratteli diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, with path space X. Then, for any ordering ξ of the diagram, there exists N such that for each $i \geq N$ there is no depth i pair of paths in $X(\xi)$. That is, the Bratteli diagram defining X is inherently expansive: for every ordering ξ of X the Vershik map is expansive on $X(\xi)$.*

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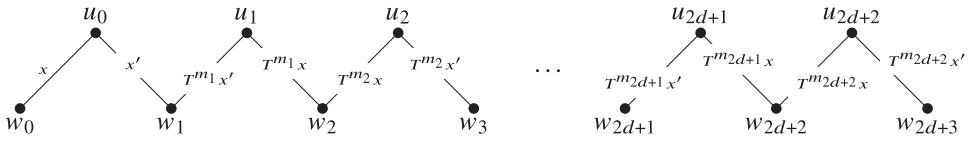


FIGURE 12. For $l \geq 1$, $u_l = DSV(w_l, j)$.

Proof. Let ξ be an ordering of $\mathcal{B}$ and choose $N = 4dq + 6q$. Suppose to the contrary that for some $i \geq N$ there exists a depth i pair x, x' in $X(\xi)$. Then, by Lemma 6.1, there exists a time when the orbits of x and x' pass through distinct uncovered vertices w_0 and w_1, respectively, at level $i+1$ such that for some $j \in \{1, 2, \ldots, q\}$, $2d^2 + 4d \leq w_1(j) < w_0(j) \leq (i-1)d$. Then, by Lemma 5.4, the link $w_0 \overset{u_1}{\longleftrightarrow} w_1$ extends to a distinguished x, x' chain of length 2,

$$w_0 \overset{u_1}{\longleftrightarrow} w_1 \overset{DSV(w_1,j)}{\longleftrightarrow} w_2$$

for which there exists an m_1 such that $v_{i+1}(T^{m_1}x') = w_1$, $v_{i+1}(T^{m_1}x) = w_2$, and for all $m \in [0, m_1]$ we have $v_{i+1}(T^m x') = w_1$ and $v_{i+1}(T^m x) \in \{w_0, w_2\}$.

Since $DSV(w_1, j) \in S(w_2)$, by Lemma 3.10 and Remark 3.5 $1 \leq w_1(j) - w_2(j) \leq d$. Together with the fact that $2d^2 + 4d \leq w_1(j) \leq (i-1)d$, we have that $2d < 2d^2 + 3d \leq w_2(j) \leq (i-1)d$. Hence, by Lemma 5.4, we can extend the chain again. Continue in this manner to produce a distinguished x, x' chain:

$$w_0 \overset{u_1}{\longleftrightarrow} w_1 \overset{DSV(w_1,j)}{\longleftrightarrow} w_2 \longleftrightarrow \cdots \longleftrightarrow w_{2d+3}.$$

Specifically, given $1 < \ell \leq 2d+2$ and a w_ℓ, we have $2d \leq 2d^2 + 4d - (\ell-1)d \leq w_\ell(j) \leq (i-1)d$. Then use the Link Lemma to extend the chain to $w_{\ell+1}$, where for $\ell > 1$, $u_{\ell+1} = DSV(w_\ell, j)$ and, by Lemma 3.10, $w_0(j) > w_1(j) > \cdots > w_{2d+3}(j)$. This means all the splitting vertices are distinct and, by Remark 4.4, uncovered.

By construction of the distinguished chain via the Link Lemma (Lemma 5.4), the orbit of x only passes through even numbered splitting vertices and the orbit of x' only passes through odd numbered splitting vertices, while moving through the chain. Further, since the jth component of the splitting vertices are strictly decreasing, for all $\ell \geq 2$, $w_\ell(j) - w_{\ell+1}(j) \geq 1$ for all $\ell \in \{1, \ldots, 2d+2\}$. So the jth components of odd vertices differ by at least 2. Since the orbit of x' passes through only odd numbered vertices, then for each $l \in \{0, 1, \ldots, d+1\}$, there is no $m \in [0, m_{2d+2}]$ such that $v_{i+1}(T^m x')(j) = w_{2l+1}(j) - 1$ or $w_{2l+1}(j) + 1$. See Figure 12.

To understand the switching of vertices at level $i+1$ of the two orbits, it is necessary to consider vertices at levels lower than $i+1$. In fact, we show that we do not have to look any lower than level $i+2$. We show that while the chain is being traversed, the orbit of x' hits a vertex z at level $i+2$ such that it cannot hit all vertices in $S(z)$.

Let $v_{i+2}(x') = z'$. Then $w_1 \in S(z')$. Recall that for $l \geq 2$, $w_1(j) - w_l(j) \geq l - 1$. Since the jth coordinates of vertices in the same source set differ by at most d (Remark 3.5), for $\ell \geq d+2$, $w_l \notin S(z')$. Therefore, there is an $m \in (0, m_{d+2}]$ such that $T^m x'$ is minimal into level $i+2$ and $v_{i+2}(T^m x') = z \neq z'$. Further, $v_{i+1}(T^m x')$ is some odd numbered

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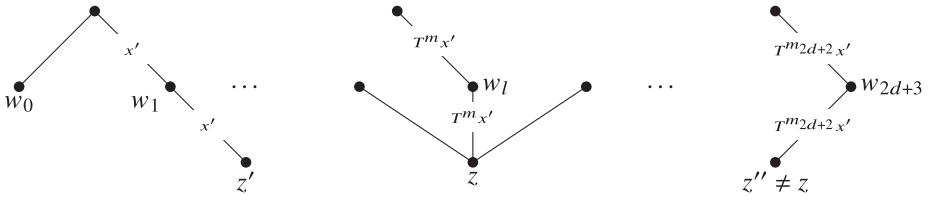


FIGURE 13. The orbit of x' must pass through at least 3 distinct vertices at level $i+2$ between time 0 and m_{2d+2}. So, it must pass through all the vertices in $S(z)$ between time 1 and m_{2d+2}.

splitting vertex w_t with $t \leq d+2$. By the way we have constructed our chain, we have that $2d \leq w_t(j) \leq (i-1)d$. Since $w_t \in S(z)$, it follows that $2d \leq z(j) \leq id$.

Let $r = w_t(j)$, where w_t is as above, so that $z(j) \geq r$. In the case $z(j) = r$, by Lemma 6.2 we would have a vertex $v \in S(z)$ such that $v(j) = z(j) - 1 = r - 1$. In the case that $r+1 \leq z(j) \leq r+d$, $z(j) - (r+1)$ is in the set $\{0, \ldots, d-1\}$. Then, by Lemma 6.2, there is a vertex $v \in S(z)$ such that $v(j) = z(j) - (z(j) - (r+1)) = r+1$. Combining these two cases, we have that there is a vertex $v \in S(z)$ such that either $v(j) = r-1$ or $v(j) = r+1$.

Note that since $t \leq d+2$, $w_t(j) - w_{2d+3}(j) \geq w_{d+2}(j) - w_{2d+3}(j) \geq 2d+3-(d+2) = d+1$. Hence (again by Remark 3.5), $v_{i+1}(T^{m_{2d+2}} x') = w_{2d+3} \notin S(z)$. See Figure 13.

By the definition of the Vershik map, between time m and time m_{2d+2} the orbit of x' must pass through all of the vertices in $S(z)$. However, this contradicts the fact that there is no $m' \in [0, m_{2d+2}]$ such that $v_{i+1}(T^{m'} x')(j) = r-1$ or $r+1$. Therefore, there can be no depth i pair of paths, each with a dense infinite orbit. □

Remark 6.4. The constructions and arguments presented above might extend to more general systems, beyond polynomial shape. As a first step, one could consider systems defined by different polynomials at different levels, adjusting the degree and the number of variables at each level. Even more generally, the properties of polynomial shape systems that allow for the construction of chains can be abstracted to define a class of diagrams, free of coordinate representations for vertices and all the accompanying structure, for which one might still be able to prove inherent expansiveness. Complete details have not yet been established.

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