Math. Proc. Camb. Phil. Soc. (1992), **111**, 219
Printed in Great Britain

Strongly torsion generated groups

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(*Received* 11 *March* 1991)

Introduction

It has long been known that the integral homology of a non-trivial finite group must be non-zero in infinitely many dimensions [**17**]. Recent work on the Sullivan conjecture in homotopy theory has made it possible to extend this result to locally finite groups. For more general groups with torsion it becomes more difficult to make such a strong statement. Nevertheless we prove that when a non-perfect group is generated by torsion elements its integral homology must also be non-zero in infinitely many dimensions. Remarkably, this result is best possible, in that for perfect torsion generated groups all (finite or infinite) sequences of abelian groups are shown below to be attainable as higher homology groups.

Surprisingly, it is often preferable to work with a special subclass of torsion generated groups, here called strongly torsion generated groups, which is of interest in its own right.

Statement of results

A group is said to be torsion generated if it is generated by its elements of finite order. A group G is strongly n-torsion generated ($n \geqslant 2$) if there is an element $g \in G$ of order n such that the conjugates of g generate G, that is, the normal closure of g is G. A group G is strongly torsion generated if it is strongly n-torsion generated for every $n \geqslant 2$.

For example, any finite group is torsion generated; so is any locally finite group, that is, a group whose finitely generated subgroups are all finite. Any non-trivial finite simple group G is strongly p-torsion generated for every prime p dividing the order of G. The group A_∞ of even finitary permutations of a countable set is strongly torsion generated; for A_∞ is the direct limit of the alternating groups A_n. In fact, any simple group which has elements of each finite order is strongly torsion generated. In particular, existentially closed groups (that is, non-trivial algebraically closed groups [**12**]) are strongly torsion generated.

Other examples of strongly torsion generated groups include the subgroup $E(R)$ of the stable general linear group $\mathrm{GL}(R) = \operatorname{dir\,lim} \mathrm{GL}_n(R)$ generated by all elementary matrices and the Steinberg groups $\mathrm{St}(R)$, where R is an associative ring with 1 (see [**8**] and Corollary 12 below).

https://doi.org/10.1017/S0305004100075319 Published online by Cambridge University Press

Evidently, a strongly torsion generated group must be infinite. Moreover, as we shall show below (Lemma 7), strongly torsion generated groups are necessarily perfect and have no proper subgroups of finite index. In this paper we investigate the integral homology groups of (strongly) torsion generated groups using constructions from combinatorial group theory and substantial results from homotopy theory. In particular, we apply Haynes Miller's solution [**15**] to the Sullivan conjecture to obtain some of our results. The general picture that emerges is that there is a huge gulf between the conditions forced upon the homology of groups of a locally finite group and those on the homology of a torsion generated group.

Strongly torsion generated groups were considered by Berrick in [**8**] where the following two theorems were established via algebraic K-theory:

THEOREM A. If A is any abelian group, there is a strongly torsion generated group H such that A is isomorphic to the centre $Z(H)$ of H.

THEOREM B. If A is any abelian group and $m \geqslant 2$, then there is a strongly torsion generated group G_m whose integral homology groups satisfy the conditions $A \cong H_m(G_m, \mathbb{Z})$ and $H_i(G_m, \mathbb{Z}) = 0$ $(1 \leqslant i < m)$.

Recall that a group G is acyclic if all of its integral homology groups $H_n(G, \mathbb{Z})$ vanish for $n \geqslant 1$. Using suitable acyclic groups and the techniques of Baumslag, Dyer and Miller [**3**] we extend theorem B (and negatively answer question 1) of [**8**] by showing the following:

THEOREM 1. *Let $A_2, A_3, \ldots$ be a sequence of abelian groups. Then there exists a strongly torsion generated group G such that $H_n(G, \mathbb{Z}) \cong A_n$ for all $n \geqslant 2$. Moreover, if λ is an infinite cardinal and if each A_k has cardinality $\leqslant \lambda$, then G can be chosen to be of cardinality λ and to have trivial centre.*

Remark. We do not know whether if each A_k is countable, G can be chosen finitely generated.

In contrast, the homology of a locally finite group is heavily constrained as is shown by the following result of Henn [**11**]:

HENN'S THEOREM: If G is a locally finite group having only finitely many non-zero integral homology groups $H_n(G, \mathbb{Z})$, then G is acyclic.

Note that because strongly torsion generated groups are perfect, the condition $n \geqslant 2$ is necessary in Theorem 1. More generally, although a torsion generated group G need not be perfect, constraints are imposed on $H_1(G, \mathbb{Z})$ by its higher homology sequence $H_k(G, \mathbb{Z})$ for $k \geqslant 2$, as illustrated by the following:

THEOREM 2. *Let G be a torsion generated group having only finitely many non-zero integral homology groups $H_n(G, \mathbb{Z})$. Then G is perfect.*

This result is a consequence of the following theorem whose proof involves the use of homotopy theory as mentioned previously:

THEOREM 3. *Let G be a group having only finitely many non-zero integral homology groups $H_n(G, \mathbb{Z})$. Then any complex linear representation $\phi: G \to \mathrm{GL}_k(\mathbb{C})$ is trivial on any finite subgroup of G.*

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This generalizes the main theorem of [7]: the finite-dimensional complex representations of an acyclic group restrict trivially to any finite subgroup. For an application of Theorem 3 to the homology of soluble minimax groups, see [10].

A similar application of homotopy theory gives the following:

THEOREM 4. *Let $N \hookrightarrow U \twoheadrightarrow Q$ be a central extension where Q is a group having only finitely many non-zero integral homology groups $H_k(Q, \mathbb{Z})$. If Q is strongly n-torsion generated (respectively, strongly torsion generated, torsion generated), then so is $[U, U]$.*

Combining Theorem 4 with Theorem 1 we give an alternative proof of theorem A of [8] which was quoted above.

Our proof of Theorem 1 has a certain unnatural aspect. One would like to start with any perfect group P and construct an embedding of P into a strongly torsion group G so that the induced homology maps $H_k(P, \mathbb{Z}) \to H_k(G, \mathbb{Z})$ are all isomorphisms. Our methods do not quite accomplish this. The best we have been able to do along these lines is the following:

THEOREM 5. *Let P be a perfect group which is the normal closure of λ of its elements (where λ may be an infinite cardinal). Then P can be embedded in a strongly torsion generated group G so that the induced homology map*

$$H_k(P, \mathbb{Z}) \to H_k(G, \mathbb{Z})$$

is an isomorphism for $k \neq 2$ and for $k = 2$ is the monomorphism in the split exact sequence

$$0 \to H_2(P, \mathbb{Z}) \to H_2(G, \mathbb{Z}) \to \mathbb{Z}^{(\lambda)} \to 0$$

where $\mathbb{Z}^{(\lambda)}$ denotes the free abelian group of rank λ. Moreover, if P is finitely generated (respectively, finitely presented), then G can be chosen finitely generated (respectively, finitely presented).

In the course of proving this result, the following interesting group is constructed:

THEOREM 6. *There exists a universal finitely presented acyclic group which is strongly torsion generated.*

We recall that the term 'universal' here means that the group G of the theorem contains an isomorphic copy of each finitely presented group. Theorem 6 is applied in [9] to show that every possible localization of the classifying space BG is contractible.

In the next section we collect the proofs of our results which employ homotopy theory and defer our group-theoretic constructions until the subsequent section.

Torsion generated groups and the Sullivan conjecture

We begin by recording the following observations about strongly torsion groups. For additional properties of torsion generated groups having no proper subgroups of finite index see [7].

LEMMA 7. *Suppose that G is a strongly torsion generated group. Then*

(1) *Any non-trivial quotient of G is strongly p-torsion generated for every prime p.*
(2) *G is perfect.*

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(3) *G has no proper subgroups of finite index.*

(4) *Every non-central element of G has infinitely many distinct conjugates.*

(5) *If N is a normal subgroup of G which is finitely generated and residually finite, then N is central.*

If in addition G is finitely generated, then

(6) *G has no non-trivial finite-dimensional linear representations.*

Proof. Suppose that G is a strongly torsion generated group. If $g \in G$ is an element of prime order p whose normal closure is G, then the image of g in any non-trivial quotient group again has these properties. Thus any non-trivial quotient of G is strongly p-torsion generated for every prime p. This proves the first assertion.

Remark. We do not know whether a non-trivial quotient of a strongly torsion generated group must again be strongly torsion generated, although this is clearly the case when the kernel is torsion-free.

Now an abelian group is strongly n-torsion generated if and only if it is a cyclic group of order n. So an abelian group cannot be strongly p-torsion generated for more than a single prime p. Hence the second assertion follows from the first.

In any group a subgroup of finite index contains a normal subgroup of finite index. So if G has a proper subgroup of finite index, then G has a non-trivial finite quotient group. But a finite group cannot be strongly p-torsion generated for infinitely many primes p, so the third assertion follows from the first.

If x is a non-central element of G, then the conjugacy class $C(x)$ of x has more than one element and G acts on $C(x)$ by conjugation. If $C(x)$ were finite, the stabilizer of x in G would be a proper subgroup of finite index, contrary to the third assertion.

If N is a normal subgroup which is finitely generated and residually finite, then its automorphism group $\mathrm{Aut}(N)$ is also residually finite. Now G acts on N by conjugation and so if N were not central, then G would have a non-trivial representation in $\mathrm{Aut}(N)$, contradicting the third assertion.

If G is finitely generated and has a non-trivial finite-dimensional representation $\phi : G \to \mathrm{GL}_k(F)$ where F is a field, then its image $\phi(G)$ is residually finite by a theorem of Malcev [13]. But then $\phi(G)$ has a non-trivial finite quotient group which is also a quotient of G, contradicting the third assertion. This completes the proof.

The proofs of several of the results that follow rely on Haynes Miller's solution of the Sullivan conjecture on maps from classifying spaces. We state this for reference as follows:

MILLER'S THEOREM. Let G be a locally finite group with classifying space BG, and let X be a connected finite-dimensional CW-complex. Then for $i \geqslant 0$, any map on the i-fold suspension $\Sigma^i BG \to X$ is null-homotopic, that is, $[\Sigma^i BG, X] = 0$ for all i.

Since Henn's Theorem quoted in the introduction is unpublished and related to the present work, we provide its proof which is an application of Miller's Theorem.

Proof of Henn's Theorem. Let G be a locally finite group having only finitely many non-zero homology groups. Then ΣBG is homotopy equivalent to a finite-dimensional CW-complex. By Miller's Theorem $[\Sigma BG, \Sigma BG] = 0$ and so $\Sigma BG \simeq *$ and G is acyclic. This proves the result.

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Proof of Theorem 3. Suppose on the contrary that the representation $\phi: G \to \mathrm{GL}_k(\mathbb{C})$ is non-trivial on some finite subgroup of G. Then G contains a non-trivial cyclic subgroup C of order p^r for some prime p such that the composition $C \hookrightarrow G \to \mathrm{GL}_k(\mathbb{C})$ is non-trivial. As the irreducible representations of C in $\mathrm{GL}_k(\mathbb{C})$ are one-dimensional, by a change of basis we can arrange for the images of elements of C to be diagonal matrices whose diagonal elements are p^rth roots of unity. Thus we can assume the image of C is contained in the unitary group U_k.

Passing to classifying spaces and using Bott periodicity, we obtain a commutative diagram

$$\begin{array}{ccccccc} BC & \longrightarrow & BU_k & \longrightarrow & BU & \xrightarrow{\simeq} & \Omega^2 BU. \\ \downarrow & & \downarrow \simeq & & & & \\ BG & \longrightarrow & B\mathrm{GL}_k(\mathbb{C}) & & & & \end{array}$$

Taking adjunctions of the composition $BC \to BG \to \Omega^2 BU$ we have maps

$$\Sigma^2 BC \to \Sigma^2 BG \to BU.$$

Since G has only finitely many non-zero integral homology groups, $\Sigma^2 BG$ is homotopy equivalent to a finite-dimensional CW-complex. Hence, by Miller's Theorem $[\Sigma^2 BC, \Sigma^2 BG] = 0$. Therefore the maps $C \to U_k \hookrightarrow U$ induce a null-homotopic map $BC \xrightarrow{*} BU$. Now since C is a finite p-group, results of Atiyah (see (6·11) and (7·2) of [1]) show that the complex representation ring $R(C)$ of C embeds naturally in $\mathbb{Z} \times [BC, BU]$. Hence our given representation $C \to U_k$ is trivial. This contradiction completes the proof of the theorem.

Corollary 8. *Suppose that G is a group and denote by $T(G)$ the subgroup generated by its elements of finite order. If G has only finitely many non-zero integral homology groups $H_n(G, \mathbb{Z})$, then $T(G) \subseteq [G, G]$.*

Proof. Let $x \in T(G) \backslash [G, G]$. Let $G_{\mathrm{ab}} = G/[G, G]$ denote the abelianization of G and denote by $A \neq 0$ the (finite) cyclic subgroup generated by the image of x in G_{ab}. Since $\mathbb{Q}/\mathbb{Z}$ is injective as an abelian group, any embedding of A in $\mathbb{Q}/\mathbb{Z}$ extends to a homomorphism $\psi: G_{\mathrm{ab}} \to \mathbb{Q}/\mathbb{Z}$ such that $\psi(A) \neq 0$. Since $\mathbb{Q}/\mathbb{Z}$ embeds in $\mathrm{GL}_1(\mathbb{C})$, it follows that there is a representation $\phi: G \to \mathrm{GL}_1(\mathbb{C})$ such that $x \notin \ker \phi$, contrary to Theorem 3. This completes the proof.

Remark. One can complete a more elementary proof of Corollary 8 without appealing to Theorem 3, by showing that $x \in \ker \phi$ as follows. Let C denote the cyclic group generated by x. Since G has only finitely many non-zero homology groups, the restriction of ϕ to C induces a map $H_*(C, \mathbb{Z}) \to H_*(\mathrm{GL}_1(\mathbb{C}), \mathbb{Z})$ which is non-zero in only finitely many dimensions. Now $\mathrm{GL}_1(\mathbb{C})$ is divisible and so is a direct sum of copies of $\mathbb{Q}$ and C_{p^∞}s. Using the fact that homology commutes with direct limits and the homology of finite cyclic groups is non-zero and periodic, one can conclude $x \in \ker \phi$ as required.

Proof of Theorem 2. Using the notation and result of Corollary 8, since G is torsion generated we have $G = T(G) \subseteq [G, G]$. Thus G is perfect as desired.

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For our proof of Theorem 4 we require the following lemma whose proof uses Miller's Theorem.

LEMMA 9. *Let $N \hookrightarrow U \twoheadrightarrow Q$ be a group extension with trivial coupling $Q \to \mathrm{Out}(N)$, where Q is a group having only finitely many non-zero integral homology groups $H_k(Q, \mathbb{Z})$. Then any homomorphism $\phi : H \to Q$ where H is a locally finite group lifts to a homomorphism $\psi : H \to U$.*

Proof. Because the coupling is trivial, it follows from [**6**] that the extension is classified by a map $f : BQ \to K(Z(N), 2)$ where $Z(N)$ denotes the centre of N. So the obstruction to the lifting is

$$f \circ B\phi : BH \to BQ \to K(Z(N), 2) \simeq \Omega K(Z(N), 3).$$

The adjoint composition $\Sigma BH \to \Sigma BQ \to K(Z(N), 3)$ is null-homotopic because $[\Sigma BH, \Sigma BQ] = 0$ by Miller's Theorem. Thus $f \circ B\phi$ is null-homotopic as required.

In the proof of Theorem 4, we apply this lemma to the case of a central extension where couplings are always trivial. However, for Q as in the lemma, the coupling is also trivial in the following situations. When $\mathrm{Out}(N)$ is known to be residually finite (or, more generally, residually linear) then by Theorem 3 the coupling is trivial. If further Q is torsion generated, then the coupling is trivial whenever $\mathrm{Out}(N)$ admits a descending series each of whose factors is soluble (using Theorem 2), or residually finite (using Theorem 3) or torsion free (see [**8**], (4·7)).

LEMMA 10. *Let $N \hookrightarrow U \overset{\phi}{\twoheadrightarrow} P$ be a central extension where P is a perfect group. Let M be a subgroup of $[U, U]$ whose image $\phi(M)$ normally generates P. Then M normally generates $[U, U]$.*

Proof. Since P is perfect, it is generated by commutators and so $[U, U]$ maps onto P and $U = N[U, U]$. Let K denote the normal closure of M in $[U, U]$. Since $\phi(M)$ normally generates P, we have $U = NK$. Now since N is central, on forming commutators one has

$$[U, U] = [NK, NK] = [K, K] \subseteq K.$$

Hence M normally generates $[U, U]$ as desired.

Proof of Theorem 4. Under each of the alternative hypotheses Q is torsion generated and so Q is perfect by Theorem 2. First consider the case when Q is strongly n-torsion generated. Then there is an element $\bar{x} \in Q$ of order n such that $\bar{x}$ normally generates Q. By Lemma 9, $\bar{x}$ is the image of some element $x \in [U, U]$ of order n. Let M denote the cyclic subgroup generated by x. By Lemma 10, M normally generates $[U, U]$. Thus $[U, U]$ is strongly n-torsion generated.

In the case when Q is only torsion generated, the argument is similar. Each element of finite order lifts to an element of the same order in $[U, U]$. Taking M to be the subgroup of $[U, U]$ generated by all of these lifts, the result follows as before. This completes the proof.

We now describe a procedure for constructing strongly torsion generated groups from given ones of a certain type.

Recall that a group L is superperfect if $H_1(L, \mathbb{Z}) = H_2(L, \mathbb{Z}) = 0$. If L is superperfect then L is perfect and every central extension by L splits. The universal central extension of a perfect group is always superperfect (see [**5**, **16**]).

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LEMMA 11. *Let H be a simple group which for each $n \geqslant 2$ has a superperfect subgroup L_n containing an element of order n. Suppose that G is a group containing H such that the normal closure of H in G is G. Then both G and the universal central extension U of G are strongly torsion generated.*

Proof. That G is strongly torsion generated is immediate, and so G is perfect. Consider the universal central extension U of G, say $N \hookrightarrow U \overset{\phi}{\twoheadrightarrow} G$ where $N \cong H_2(G, \mathbb{Z})$. Now U is (super)perfect so $U = [U, U]$. Let L be a superperfect subgroup of G. The central extension U restricts to a central extension $N \hookrightarrow V \twoheadrightarrow L$ where $V = \phi^{-1}(L) \subseteq U$. Since L is superperfect this extension splits as $V = NL'$ where ϕ maps L' isomorphically onto L. By Lemma 10, if L normally generates G then L' normally generates $U = [U, U]$. So from the hypothesis on H and G it follows that U is strongly torsion generated. This completes the proof.

Now A_∞ is an infinite simple group which satisfies the conditions on H in Lemma 11. This is implied by the following familiar or easily verified facts. (1) Any finite group can be embedded in A_k for sufficiently large k ($\geqslant 5$). (2) The groups A_k are non-abelian finite simple groups for $k \geqslant 5$. (3) The universal central extension of a finite perfect group is a finite superperfect group. (4) If a quotient group Q of a finite group G has an element of order n, then G has an element of order n. Now for a fixed $n \geqslant 2$ the cyclic group of order n can be embedded in some A_k where $k \geqslant 5$. Then the universal central extension U_k of A_k is superperfect and finite and contains an element of order n. But U_k is embedded in A_m for sufficiently large m and hence in A_∞. Thus for each $n \geqslant 2$, A_∞ has a superperfect subgroup containing an element of order n.

COROLLARY 12. *The Steinberg group* $\mathrm{St}(R)$ *where R is an associative ring with* 1 *is strongly torsion generated. Also the universal central extension of A_∞ is strongly torsion generated.*

Proof. The subgroup $E(R)$ of the stable general linear group $\mathrm{GL}(R)$ generated by the elementary matrices is the normal closure of a copy of A_∞ (see [**8**]). In particular, $E(R)$ is strongly torsion generated. The Steinberg group $\mathrm{St}(R)$ is the universal central extension of $E(R)$ having centre $K_2(R)$ (see [**16**]). Hence by Lemma 11 and the properties of A_∞ it follows that $\mathrm{St}(R)$ is strongly torsion generated. The second assertion is immediate from the lemma and the properties of A_∞.

Constructions of groups

The following well known fact is just a combination of the Higman embedding theorem and the usual constructions embedding a countable group in a two-generator group.

LEMMA 13. *If the group K can be presented by a recursive set of generators subject to a recursively enumerable set of defining relations, then K can be embedded in a two-generator, finitely presented group L. Under this embedding the given generators of K are represented by a recursive set of words in the generators of L.*

Using this and the results of Baumslag, Dyer, Miller [**3**] we prove the following which provides a method for ensuring strong torsion generation:

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LEMMA 14. *There is a finitely presented acyclic group K which contains an element w of infinite order with the following property: for every $n \geqslant 2$, there are elements y_n and z_n in K such that $w = [y_n, z_n]$ and z_n has order n. Moreover, K can be chosen to be a universal finitely presented group, that is, to contain an isomorphic copy of every finitely presented group.*

Proof. For each $n \geqslant 2$ put $B_n = \langle y_n, z_n \,|\, z_n^n = 1 \rangle \cong C_n * \mathbb{Z}$. In each of these groups the element $[y_n, z_n]$ has infinite order and so we can form the amalgamated free product of all of them amalgamating the infinite cycles that these elements generate. The resulting group can be presented as say

$$T = \langle w, y_2, y_3, \ldots, z_2, z_3, \ldots \,|\, w = [y_n, z_n], z_n^n = 1 \quad \text{for} \quad n = 2, 3, \ldots \rangle.$$

By the previous lemma, T can be embedded in a finitely presented group which in turn can be embedded in a universal finitely presented acyclic group K by [3]. This proves the lemma.

The following lemma is a variation on a similar one in Miller [14].

LEMMA 15. *Let K be a universal finitely presented acyclic group of the previous lemma and let $w, y_n, z_n (n \geqslant 2)$ be the elements described there. Suppose that K has presentation*

$$K = \langle x_1, x_2, \ldots, x_n \,|\, R_1 = 1, R_2 = 1, \ldots, R_m = 1 \rangle.$$

Define L to be the finitely presented group obtained from K by adding three new generators a, b, c together with defining relations

$$a^{-1}ba = c^{-1}b^{-1}cbc, \tag{1}$$

$$a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2, \tag{2}$$

$$a^{-3}[w, b]\,a^3 = c^{-3}bc^3, \tag{3}$$

$$a^{-(3+i)}x_i\,ba^{(3+i)} = c^{-(3+i)}bc^{(3+i)} \quad (i = 1, 2, \ldots, n), \tag{4}$$

where $[w, b]$ is the commutator of w and b. Then

(1) *K is embedded in L_w by the inclusion map on generators.*

(2) *L is normally generated by w.*

(3) *For each $n \geqslant 2$, L is normally generated by z_n which is an element of order n (in particular, L is strongly torsion generated).*

Note also that L is perfect.

Proof. By hypothesis $w \neq_K 1$. In the free group $\langle b, c \,|\, \rangle$ on generators b and c consider the subgroup C generated by b together with the right-hand sides of equations (1)–(4). It is easy to check that the indicated elements are a set of free generators for C, since in forming the product of two powers of these elements or their inverses some of the conjugating symbols will remain uncancelled and the middle portions will be unaffected.

Similarly, in the ordinary free product $K * \langle a, b \,|\, \rangle$ of K with the free group on generators a and b consider the subgroup A generated by b together with the left-hand sides of equations (1)–(4). Using the fact that $w \neq_K 1$ it is again easy to check that the indicated elements are a set of free generators for A.

Thus the indicated presentation for L together with the equation identifying the

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symbol b in each of the two factors is the natural presentation for the free product with amalgamation

$$(K * \langle a, b \,|\, \rangle) \underset{A=C}{*} \langle b, c \,|\, \rangle.$$

So K is embedded in L, establishing the first claim.

Now $[w, b]$ lies in the normal closure of w. So by equation (3), b lies in the normal closure of w. But equations (1) and (2) ensure that a, b, c are all conjugate and so lie in the normal closure of w. Finally, since each of the system of equations (4) can be solved to express x_i in terms of a, b, c, it follows that x_i lies in the normal closure of w for $i = 1, 2, \ldots, n$. Thus each of the generators of L lies in the normal closure of w. This verifies the second assertion.

The third assertion follows from the properties of K listed in the previous lemma. This completes the proof.

Proof of Theorem 6. The proof is a variation of one in Baumslag and Miller[**4**] concerning embeddings into groups generated by divisible elements.

Let K, L and w be as in Lemma 15. Since K is a universal finitely presented group there is an embedding $\phi : L \to K \subset L$. Form the corresponding HNN-extension $M = \langle L, t \,|\, t^{-1}ut = \phi(u), u \in L \rangle$. Since L is finitely presented, M can also be finitely presented. The normal closure L^M of L in M is the ascending union of conjugates of the acyclic group K and hence is acyclic (see [**3**]). But M can also be viewed as an HNN-extension of L^M by the stable letter t and so by the Mayer–Vietoris sequence for HNN-extensions $H_n(M) = 0$ for $n \geqslant 2$. Also $H_1(M) \cong \langle t \,|\, \rangle \cong \mathbb{Z}$.

Now form the HNN-extension $P = \langle M, s \,|\, s^{-1}ts = t^2 \rangle$. One can easily check that in P the elements w and s freely generate a free subgroup of rank 2. Also by the Mayer–Vietoris sequence for HNN-extensions $H_n(P) = 0$ for $n \geqslant 2$ and one has $H_1(P) \cong \langle s \,|\, \rangle \cong \mathbb{Z}$.

Now consider the group $Q = \langle s, u, v \,|\, u^{-1}su = s^2, v^{-1}uv = u^2 \rangle$. This group can be viewed as the group obtained from the infinite cyclic group generated by s by successively forming two HNN-extensions. One can check that the elements s and v freely generate a free subgroup of rank two in Q. By the Mayer–Vietoris sequence for HNN-extensions $H_n(Q) = 0$ for $n \geqslant 2$ and $H_1(Q) \cong \langle v \,|\, \rangle \cong \mathbb{Z}$.

Finally form the amalgamated free product $G = \langle P * Q \,|\, s = s, v = w \rangle$. Since w lies in an acyclic subgroup of P, by the Mayer–Vietoris sequence for amalgamated free products it follows that G is acyclic. Relative to a presentation for M this group G can be presented as

$$G = \langle M, s, u \,|\, s^{-1}ts = t^2, u^{-1}su = s^2, w^{-1}uw = u^2 \rangle$$

where w and t are the specified elements of M. Now G is normally generated by w. For by the equation $w^{-1}uw = u^2$ the element u lies in the normal closure of w. So by the equations $u^{-1}su = s^2$ and then $s^{-1}ts = t^2$ both s and t lie in the normal closure of w. But L is normally generated by w and M is generated by t and L. Thus G is normally generated by w as claimed.

By the previous lemmas, it follows that for each $n \geqslant 2$, G is normally generated by z_n which is an element of order n. In particular, G is strongly torsion generated as desired. This completes the proof.

Proof of Theorem 1. Let $A_2, A_3, \ldots$ be a sequence of abelian groups. Then by a result of Baumslag, Dyer and Miller[**3**] there is a group M such that $H_{n-1}(M, \mathbb{Z}) \cong A_n$ for

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$n \geqslant 2$. Moreover, if λ is an infinite cardinal and if each A_n is of cardinality $\leqslant \lambda$, then M is of cardinality λ.

Let E be an existentially closed group which contains an isomorphic copy of M and has the same cardinality as M. According to a result of Baumslag, Dyer and Heller [2], E is acyclic. Define $G = E *_M E$, the free product of two copies of E with amalgamation of the corresponding subgroups isomorphic to M. Since E is acyclic, the Mayer–Vietoris sequence for amalgamated free products shows that $H_n(G, \mathbb{Z}) \cong H_{n-1}(M, \mathbb{Z}) \cong A_n$ for $n \geqslant 2$. Evidently since E is strongly torsion generated and simple and $M \neq 1$, G is also strongly torsion generated. Clearly G has trivial centre and cardinality λ. This proves the theorem.

COROLLARY 16 (theorem A of [8]). *If A is any abelian group, there is a strongly torsion generated group H such that A is isomorphic to the centre $Z(H)$ of H.*

Proof. By Theorem 1 there is a strongly torsion generated group G with $H_2(G, \mathbb{Z}) \cong A$ and $H_k(G, \mathbb{Z}) = 0$ for $k > 2$. We can assume the centre of G is trivial. Now G is perfect, so let H be the universal central extension of G. Then $Z(H) \cong H_2(G, \mathbb{Z}) \cong A$ and H is perfect (even superperfect – see chapter 8 of [5], [16]). Hence by Theorem 4, H is strongly torsion generated. This completes the proof.

Proof of Theorem 5. By hypothesis, P is a perfect group which is the normal closure of λ of its elements, say $\{x_i \mid i \in \Lambda\}$ where Λ is an index set of cardinality λ.

If λ is countable, let E be the universal finitely presented acyclic group of Theorem 6; otherwise, let E be an existentially closed group of cardinality λ which contains a free subgroup of rank λ. In either case, E contains two sets of elements $\{e_i \mid i \in \Lambda\}$ and $\{f_j \mid j \in \Lambda\}$ which together freely generate a free subgroup of E.

For each $i \in \Lambda$, let $Q_i = \langle a_i, b_i, c_i \mid b_i^{-1} a_i b_i = a_i^2, c_i^{-1} b_i c_i = b_i^2 \rangle$. As in the proof of Theorem 6, $H_n(Q_i, \mathbb{Z}) = 0$ for $n \geqslant 2$ and $H_1(Q_i, \mathbb{Z}) \cong \langle c_i \mid \rangle \cong \mathbb{Z}$. Also the elements a_i and c_i freely generate a free subgroup of rank two in Q_i. Let Q_Λ be the ordinary free product of all of the Q_i ($i \in \Lambda$). Finally form the amalgamated free product G of the groups $(P * E)$ and Q_Λ where the amalgamated subgroup is the free subgroup on the indicated generators as follows:

$$\begin{aligned} G &= (P * E) * Q_\Lambda \\ x_i e_i &= a_i \quad (i \in \Lambda) \\ f_i &= c_i \quad (i \in \Lambda). \end{aligned}$$

When used with the properties of Q and the fact that E is acyclic, the Mayer–Vietoris sequence for amalgamated free products shows that the embedding $P \to G$ induces isomorphisms $H_k(P, \mathbb{Z}) \cong H_k(G, \mathbb{Z})$ for $k \geqslant 3$. Since P and E are perfect, it follows easily from the defining relations that G is perfect. So in low dimensions we have the exact sequence:

$$0 \to H_2(P, \mathbb{Z}) \to H_2(G, \mathbb{Z}) \to \mathbb{Z}^{(\lambda)} \oplus \mathbb{Z}^{(\lambda)} \to 0 \oplus \mathbb{Z}^{(\lambda)} \to 0$$

where $\mathbb{Z}^{(\lambda)}$ denotes the free abelian group of rank λ. Note that the images of the c_is are a homology basis for the second summand of both of the terms $\mathbb{Z}^{(\lambda)} \oplus \mathbb{Z}^{(\lambda)}$ and $0 \oplus \mathbb{Z}^{(\lambda)}$. The images of the a_i, f_i and $x_i e_i$ all map trivially into H_1 of the factors. Thus we have the desired exact sequence

$$0 \to H_2(P, \mathbb{Z}) \to H_2(G, \mathbb{Z}) \to \mathbb{Z}^{(\lambda)} \to 0.$$

https://doi.org/10.1017/S0305004100075319 Published online by Cambridge University Press

Now E is strongly torsion generated, and the normal closure of E contains each c_i by the second amalgamating relations. Then by the relations of the Q_i, the normal closure of c_i contains b_i and a_i. So by the first amalgamating relations, the normal closure of E contains P and hence all of G. Thus G is strongly torsion generated.

Finally, the assertions concerning finite generation and finite presentation follow easily from the construction.

Remark. In the proof of the previous theorem, the presence of the unwanted summand $\mathbb{Z}^{(\Lambda)}$ is caused by our use of the equations $x_i e_i = a_i$ where a_i is homologous to 0 in the second factor $H_1(Q_\Lambda, \mathbb{Z})$. The other type of equation $f_i = c_i$ is harmless since the c_is are a homology basis for $H_1(Q_\Lambda, \mathbb{Z})$. To avoid increasing the size of H_2 in the construction one would like to add to $P * E$ the same number of new generators as defining relations. One must ensure the result is strongly torsion generated and identify its homology as being induced from the subgroup P. A more complicated construction might fulfil these requirements, but we have not yet found one which works.

REFERENCES

[1] M. F. Atiyah. Characters and cohomology of finite groups. *Inst. Hautes Études Sci. Publ. Math.* **9** (1961), 23–64.

[2] G. Baumslag, E. Dyer and A. Heller. The topology of discrete groups. *J. Pure Appl. Algebra* **16** (1980), 1–47.

[3] G. Baumslag, E. Dyer and C. F. Miller III. On the integral homology of finitely presented groups. *Topology* **22** (1983), 27–46.

[4] G. Baumslag and C. F. Miller III. Some odd finitely presented groups. *Bull. London Math. Soc.* **20** (1988), 239–244.

[5] A. J. Berrick. *An Approach to Algebraic K-Theory*. Research Notes in Math. no. 56 (Pitman, 1982).

[6] A. J. Berrick. Group extensions and their trivialisation. *Enseign. Math.* (2) **31** (1985), 151–172.

[7] A. J. Berrick. Remarks on the structure of acyclic groups. *Bull. London Math. Soc.* **22** (1990), 227–232.

[8] A. J. Berrick. Torsion generators for all abelian groups. *J. Algebra* **139** (1991), 190–194.

[9] A. J. Berrick and C. Casacuberta. Groups and spaces with all localizations trivial. In *Groups and Classifying Spaces*, Lecture Notes in Math. (Springer-Verlag), to appear.

[10] A. J. Berrick and P. H. Kropholler. In preparation.

[11] H.-W. Henn. A note on the homology of locally finite groups. Unpublished manuscript.

[12] R. C. Lyndon and P. E. Schupp. *Combinatorial Group Theory* (Springer-Verlag, 1977).

[13] A. I. Malcev. On isomorphic matrix representations of infinite groups. *Mat. Sb.* **8** (1940), 405–422.

[14] C. F. Miller III. Decision problems for groups: survey and reflections. In *Proceedings of the Workshop on Algorithms, Word Problems and Classification in Combinatorial Group Theory* (Springer-Verlag, 1990).

[15] H. Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math.* (2) **120** (1984), 39–87.

[16] J. Milnor. *Introduction to Algebraic K-Theory. Ann. of Math. Studies*, no. 72 (Princeton University Press, 1971).

[17] R. G. Swan. The nontriviality of the restriction map in the cohomology of groups. *Proc. Amer. Math. Soc.* **11** (1960), 885–887.

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