Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 March 2020.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

103.I (Martin Lukarevski)

Let x, y, z be positive real numbers such that $x + y + z = xyz$. Prove that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{8}{\sqrt{(1 + x^2)(1 + y^2)(1 + z^2)}} \geqslant 2.$$

When does equality hold?

103.J (Joseph Tonien)

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that:

- $f(2019) = 2018$;
- $f\big(n - f(n)\big) = 0$ for all $n \in \mathbb{Z}$;
- there is an integer t such that, for any fixed $n \in \mathbb{Z}$, the equation $f(x) = n$ has exactly t solutions for x.

103.K (Yagub N. Aliyev)

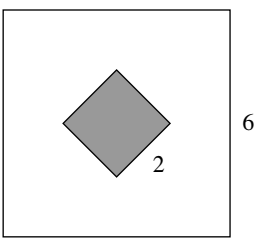
Given triangles ABC and $A_1B_1C_1$, denote by K, L and M the intersection points of BC_1 and B_1C, CA_1 and C_1A, AB_1 and A_1B respectively. Prove that the lines AA_1, BB_1 and CC_1 are concurrent if, and only if, the intersection points of BC and LM, CA and MK, AB and KL are collinear.

103.L (Tony Crilly and Stan Dolan)

Let a, b and c be positive integers such that an $a \times a$ square can strictly contain a $b \times c$ rectangular hole.

Find, with proof, all possibilities for which the area of this *Emmental square* is numerically equal to its perimeter.

For example, the Emmental square in the figure with a square hole has area $36 - 4 = 32$ and perimeter $4 \times 6 + 4 \times 2 = 32$.



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Solutions and comments on **103.A**, **103.B**, **103.C**, **103.D** (March 2019).

103.A (Michael Fox)

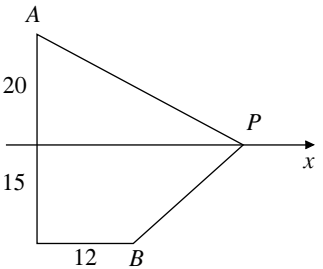
The point A is $(6,\ 16,\ 12)$, B is $(18,\ -9,\ 12)$ and P is a variable point on the x-axis.

Find the ranges of possible values of (i) $AP + BP$, (ii) $AP - BP$, and (iii) AP/BP.

Answer: The respective ranges are (i) $37 \leqslant AP + BP < \infty$, (ii) $-12 < AP - BP \leqslant 13$, (iii) $\frac{4}{5} \leqslant \frac{AP}{BP} \leqslant \frac{5}{3}$.

Most solvers used differentiation and curve sketching to resolve this problem but, in fact, we do not need to find explicitly the extreme positions of P or use calculus. The solution below is based on those given by Stan Dolan and the proposer, Michael Fox.

(i) The problem is not affected by rotating B around the x-axis so we can suppose the set-up is planar, as shown in the figure.

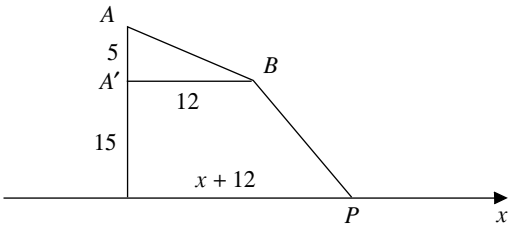


By the triangle inequality, $AP + PB \geqslant AB$ with equality when P is on the line AB. Therefore

$$AP + PB \geqslant \sqrt{35^2 + 12^2} = 37$$

so that $37 \leqslant AP + PB < \infty$.

(ii) Now use the configuration of A and B shown in the figure below.



By the triangle inequality, $AB + BP \geqslant AP$ with equality when P is on the line AB. Also, $BA' + A'P > BP$ with equality approached as $x \to -\infty$.

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Therefore

$$-BA' < A'P - BP < AP - BP \leqslant AB$$

$$\text{so that } -12 < AP - BP \leqslant 13.$$

(iii) With the notation as in part (ii),

$$\frac{AP^2}{BP^2} = \frac{(x + 12)^2 + 400}{x^2 + 225} = y, \text{ say}.$$

Then $(1 - y)x^2 + 24x + (544 - 225y) = 0$ with real solutions for x if, and only if, $576 - 4(1 - y)(544 - 225y) \geqslant 0$ which rearranges and factorises to give $(9y - 25)(25y - 16) \leqslant 0$ so that

$$\frac{16}{25} \leqslant y = \frac{AP^2}{BP^2} \leqslant \frac{25}{9} \quad \text{and} \quad \frac{4}{5} \leqslant \frac{AP}{BP} \leqslant \frac{5}{3}.$$

A number of respondents commented admiringly on how carefully the numbers had been chosen to give pleasant answers for each part and Chris Starr investigated the conditions needed for this to occur.

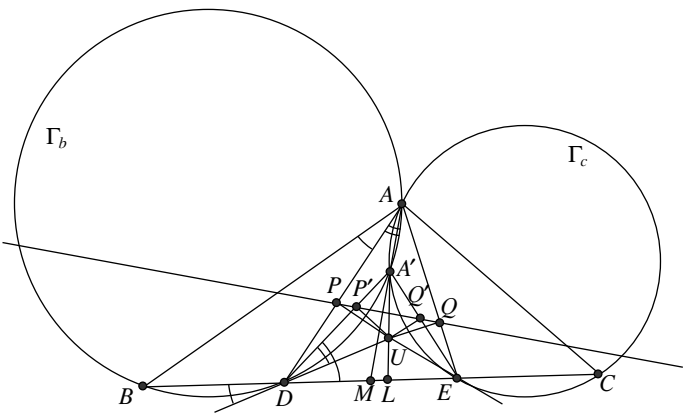
Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, A. P. Harrison, G. Howlett, P. F. Johnson, J. A. Mundie, I. D. Sfikas, C. Starr and the proposer Michael Fox.

103.B (Michel Bataille)

Let ABC be a triangle and D, E on the side BC be such that $BD = DE = EC$. Let the circumcircles Γ_b and Γ_c of $\triangle ABD$ and $\triangle ACE$, respectively, intersect at A and A' and let the tangents to Γ_b at D and to Γ_c at E intersect in U.

Prove that the orthogonal projections P, Q, P', Q' of U onto AD, AE, $A'D$, $A'E$, respectively, are collinear and that $PP' = QQ'$.

This intriguing geometrical problem attracted clear and carefully argued solutions. Stan Dolan, Graham Howlett and Li Zhou, whose proof follows, all observed that we just need to assume $BD = EC$ (rather than $BD = DE = EC$) to obtain the required result.



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Let AA' intersect BC in M. Then

$$MD(MD + DB) = MD \cdot MB = MA' \cdot MA$$
$$= ME \cdot MC = ME(ME + EC),$$

thus $MD = ME$ by virtue of $BD = EC$. Let R_b and R_c be the circumradii of Γ_b and Γ_c respectively. Then

$$2R_b \sin\angle EDU = 2R_b \sin\angle BAD = BD = EC = 2R_c \sin\angle CAE = 2R_c \sin\angle DEU,$$

so $\dfrac{DU}{EU} = \dfrac{\sin\angle DEU}{\sin\angle EDU} = \dfrac{R_b}{R_c}$ (*).

Let $A'U$ intersect BC in L. Then

$$A'U \sin\angle DUL = A'D \sin\angle A'AD = \frac{(A'D)^2}{2R_b}.$$

Likewise, $A'U \sin\angle EUL = \dfrac{(A'E)^2}{2R_c}$. Therefore,

$$\frac{DL}{LE} = \frac{DU \sin\angle DUL}{EU \sin\angle EUL} = \frac{A'D^2}{A'E^2},$$

which is well known to imply that $A'U$ is the A'-symmedian in $\Delta A'DE$.

Observing that $A'P'UQ'$, $PDUP'$ and $QEUQ'$ are cyclic quadrilaterals, we conclude that $\angle Q'P'U = \angle Q'A'U = \angle DA'M = \angle PDU$ and $\angle P'Q'U = \angle P'A'U = \angle EA'M = \angle QEU$, from which it follows that P, P', Q, Q' are collinear. Finally,

$$PP' = DU \sin\angle PDP' = \frac{DU}{2R_b} \cdot 2R_b \sin\angle ADA' = \frac{DU \cdot AA'}{2R_b},$$

and similarly, $QQ' = \dfrac{EU \cdot AA'}{2R_c}$, so we conclude from (*) that $PP' = QQ'$.

Other features of the configuration were noted by solvers; for example, the line $PP'Q'Q$ is parallel to the line joining the centres of Γ_b and Γ_c, and hence perpendicular to AA'. Graham Howlett proved a generalisation of **103.B**: Let ABC be a triangle and m the median through A. Draw any circle Γ_b through A and B which is not tangent to m. Let Γ_b meet m again at A' and let Γ_c be the circumcircle of $\Delta AA'C$. Let the tangents to Γ_b at B and to Γ_c at C meet at U. Then the orthogonal projections P, Q, P', Q' of U onto AB, AC, $A'B$, $A'C$ respectively are collinear with $PP' = QQ'$.

Correct solutions were received from: S. Dolan, M. G. Elliott, G. Howlett, J. A. Mundie, L. Zhou and the proposer Michel Bataille.

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103.C (K. S. Bhanu and M. N. Deshpande)

Consider the following method of generating a permutation of the integers $(1, 2, \ldots, n)$. We have equal numbers of balls of three colours in a bag. Balls are drawn randomly one by one n times with replacement. For $1 \leqslant k \leqslant n$, the colour of the kth ball drawn is noted and we use this to generate at most three cycles $(a_1, \ldots, a_{k_1})$, $(b_1, \ldots, b_{k_2})$ and $(c_1, \ldots, c_{k_3})$ where the a_i, b_i and c_i denote the draw numbers of balls of the same colour and $k_1 + k_2 + k_3 = n$. From the product of these cycles, we generate a random permutation $(\pi_1, \ldots, \pi_n)$.

Show that, for large n, $\mathrm{E}(\pi_1) \approx 4$ and $\mathrm{E}(\pi_n) \approx 3$.

[For example, if $n = 7$ and the draw number of balls of each colour are (1, 4), (2, 3, 6) and (5, 7), then the generated permutation is (4, 3, 6, 1, 7, 2, 5).]

Almost all respondents gave a solution along the following lines: several commented on the surprising nature of the results.

Consider the values taken by π_1.

- $\pi_1 = 1$ if, and only if, the first cycle is (1), i.e. all balls chosen after the first have a different colour from the first ball. Thus $\mathrm{P}(\pi_1 = 1) = (\frac{2}{3})^{n-1}$.
- $\pi_1 = k > 1$ if, and only if, the first and kth balls have the same colour and the 2nd, 3rd, … $(k-1)$th balls have a different colour from them. Thus $\mathrm{P}(\pi_1 = k) = \frac{1}{3}(\frac{2}{3})^{k-2}$.

It follows that

$$\begin{aligned}
\mathrm{E}(\pi_1) &= \sum_{k=1}^{n} k\,\mathrm{P}(\pi_1 = k) = \left(\frac{2}{3}\right)^{n-1} + \sum_{k=2}^{n} k\frac{1}{3}\left(\frac{2}{3}\right)^{k-2} \\
&= \left(\frac{2}{3}\right)^{n-1} + \frac{1}{2}\sum_{k=2}^{n} k\left(\frac{2}{3}\right)^{k-1} \\
&= \left(\frac{2}{3}\right)^{n-1} + \frac{1}{2}\left[\frac{1 - (n+1)(\frac{2}{3})^n + n(\frac{2}{3})^{n+1}}{(1-\frac{2}{3})^n} - 1\right] \qquad (*) \\
&= 4 - (n+2)\left(\frac{2}{3}\right)^{n-1} \to 4 \text{ as } n \to \infty
\end{aligned}$$

where, at (*), we have used the standard summation

$$\sum_{k=1}^{n} kx^{k-1} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

In a similar way, consider the values taken by π_n.

- $\pi_n = n$ if, and only if, the cycle containing n is (n), i.e. all the previous balls chosen differ in colour from the last ball. Thus $\mathrm{P}(\pi_n = n) = (\frac{2}{3})^{n-1}$.

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- $\pi_n = k < n$ if, and only if, the kth and the last ball chosen have the same colour and the first $k-1$ balls chosen have a different colour from them. Thus $\mathrm{P}(\pi_n = k) = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$.

It follows that

$$\mathrm{E}(\pi_n) = \sum_{k=1}^{n-1} k\frac{1}{3}\left(\frac{2}{3}\right)^{k-1} + n\left(\frac{2}{3}\right)^{n-1}$$

$$= \frac{1}{3}\left[\frac{1 - n\left(\frac{2}{3}\right)^{n-1} + (n-1)\left(\frac{2}{3}\right)^{n}}{\left(1 - \frac{2}{3}\right)^{n}}\right] + n\left(\frac{2}{3}\right)^{n-1},$$

using the summation at (*)

$$= 3 - 2\left(\frac{2}{3}\right)^{n-1} \to 3 \text{ as } n \to \infty.$$

The proposers, K. S. Bhanu and M. N. Deshpande, remarked that by similar methods we can derive

$$\mathrm{E}(\pi_1^2) = 22 - (n^2 + 6n + 14)\left(\frac{2}{3}\right)^{n-1}$$

$$\text{and } \mathrm{E}(\pi_n^2) = 15 - 3(2n+5)\left(\frac{2}{3}\right)^{n}$$

from which for large n, we have $\mathrm{var}(\pi_1) \approx \mathrm{var}(\pi_n) \approx 6$.

Correct solutions were received from: S. Dolan, M. G. Elliott, A. P. Harrison, G. Howlett and the proposers K. S. Bhanu and M. N. Deshpande.

103.D (Ovidiu Furdui)

Let $b, c, d \in \mathbb{R}$ with $bc > 0$. Calculate

$$\lim_{n\to\infty}\begin{pmatrix} 1 & \sin\frac{b}{n} \\ \sin\frac{c}{n} & \cos\frac{d}{n}\end{pmatrix}^n.$$

Answer: The limit is $\begin{pmatrix} \cosh\sqrt{bc} & \sqrt{\frac{b}{c}}\sinh\sqrt{bc} \\ \sqrt{\frac{c}{b}}\sinh\sqrt{bc} & \cosh\sqrt{bc}\end{pmatrix}$, independent of d.

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The following solution collates arguments used by several solvers.

Just as for limits of sequences of real numbers (but with respect to a matrix norm), if $A_n \to A$ then $\lim_{n\to\infty}\left(I + \frac{A_n}{n}\right)^n = \exp(A)$ where $\exp(A) = \sum_{n=0}^{\infty}\frac{1}{n!}A^n$. Writing $\begin{pmatrix} 1 & \sin\frac{b}{n} \\ \sin\frac{c}{n} & \cos\frac{d}{n}\end{pmatrix} = I + \frac{A_n}{n}$ where

$$A_n = \begin{pmatrix} 0 & n\sin\frac{b}{n} \\ n\sin\frac{c}{n} & n\left(\cos\frac{d}{n} - 1\right)\end{pmatrix} \to A = \begin{pmatrix} 0 & b \\ c & 0\end{pmatrix} \text{ as } n \to \infty,$$

we see that

$$\lim_{n\to\infty}\begin{pmatrix} 1 & \sin\frac{b}{n} \\ \sin\frac{c}{n} & \cos\frac{d}{n}\end{pmatrix}^n = \exp\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}.$$

To calculate $\exp\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}$, solvers followed two paths.

- Direct calculation gives $\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}^{2n} = \begin{pmatrix} b^n c^n & 0 \\ 0 & b^n c^n\end{pmatrix}$ and $\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & b^{n+1}c^n \\ b^n c^{n+1} & 0\end{pmatrix}$ so that

$$\exp\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\begin{pmatrix} b^n c^n & 0 \\ 0 & b^n c^n\end{pmatrix} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\begin{pmatrix} 0 & b^{n+1}c^n \\ b^n c^{n+1} & 0\end{pmatrix}$$

$$= \begin{pmatrix} \cosh\sqrt{bc} & 0 \\ 0 & \cosh\sqrt{bc}\end{pmatrix} + \begin{pmatrix} 0 & \sqrt{\frac{b}{c}}\sinh\sqrt{bc} \\ \sqrt{\frac{c}{b}}\sinh\sqrt{bc} & 0\end{pmatrix},$$

as required.

- Diagonalisation gives $\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix} = P\begin{pmatrix} \sqrt{bc} & 0 \\ 0 & -\sqrt{bc}\end{pmatrix}P^{-1}$ where $P = \begin{pmatrix} \sqrt{b} & -\sqrt{b} \\ \sqrt{c} & \sqrt{c}\end{pmatrix}$ so that $\exp\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix} = P\begin{pmatrix} e^{\sqrt{bc}} & 0 \\ 0 & e^{-\sqrt{bc}}\end{pmatrix}P^{-1}$

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which evaluates as $\begin{pmatrix} \cosh\sqrt{bc} & \sqrt{\frac{b}{c}}\sinh\sqrt{bc} \\ \sqrt{\frac{c}{b}}\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{pmatrix}$.

An alternative route used by some solvers, including the proposer Ovidiu Furdui, was to calculate $\begin{pmatrix} 1 & \sin\frac{b}{n} \\ \sin\frac{c}{n} & \cos\frac{d}{n} \end{pmatrix}^n$ by diagonalsing $\begin{pmatrix} 1 & \sin\frac{b}{n} \\ \sin\frac{c}{n} & \cos\frac{d}{n} \end{pmatrix}$ and then taking the limit as $n \to \infty$. Peter Johnson observed that the limit for $bc < 0$ corresponds to replacing $\sqrt{bc}$ by $i\sqrt{|bc|}$ in the analysis above.

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, A. P. Harrison, G. Howlett, A. Izmailov, P. F. Johnson, P. Kitchenside (3 solutions), M. Lukarevski, J. A. Mundie, I. D. Sfikas (2 solutions) and the proposer Ovidiu Furdui.

10.1017/mag.2019.129 N. J. L.

From the Editor

As you can see from the preamble to Student Problem Corner on the next page, this is Stan Dolan's last issue in charge. Stan has been the editor since taking over from Tim Cross in 2010, and he has maintained the high quality of problem setting and marking of his predecessor. He has also dealt with innovations – some more welcome than others – such as the publication of the problems on the MA website and the requirements of GDPR legislation. I am indebted to him for the commitment he has shown, but, of course, I hope he will continue to write and referee articles for the *Gazette*. I am also very grateful that Lewis Roberts has agreed to take over the role of editor of SPC.

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