ANZIAM J. **63** (2021), 359–375
doi:10.1017/S1446181121000316

ALGORITHM TO CONSTRUCT INTEGRO SPLINES

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(Received 16 September, 2020; accepted 17 May, 2021; first published online 13 September, 2021)

Abstract

We study some properties of integro splines. Using these properties, we design an algorithm to construct splines $S_{m+1}(x)$ of neighbouring degrees to the given spline $S_m(x)$ with degree m. A local integro-sextic spline is constructed with the proposed algorithm. The local integro splines work efficiently, that is, they have low computational complexity, and they are effective for use in real time. The construction of nonlocal integro splines usually leads to solving a system of linear equations with band matrices, which yields high computational costs.

2020 *Mathematics subject classification*: 65D05.

Keywords and phrases: super-convergence, numerical integration, differentiation and integration operations, polynomial and nonpolynomial, histospline.

1. Introduction

Interpolation of mean values or area-matching properties is often referred to as histospline or histopolation, which has been studied in the literature (see [3, 13] and the references therein). The convergence properties of the histospline density estimate were studied by Wahba [14], where the reader can also find several references to earlier results on this topic. Besides, polynomial integro-differential splines were first used by Kireev and Biryukova [8]. Recently, Burova and Doronina [5] discussed the construction of polynomial and nonpolynomial splines which use three integrals over subintervals in addition to the values of the function in the nodes. The differentiation and integration of $S(x)$, a cubic or quintic spline, produce splines of the corresponding degree and make the transfer of the spline connections between splines of neighbouring degrees [9, 12].

Theoretical studies [7] and real-life applications [22] require the approximation of unknown smooth functions using given successive integral values. In this case,

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the spline is called an *integro spline*. Recently, some authors have devoted much attention to the construction of integro splines [2, 17] and they studied the convergence properties of the constructed integro splines. Of course, in constructing integro splines, B-spline representation or piecewise polynomial representation is often used, and the construction leads to solving a system of linear equations. Solving the system needs certain additional end conditions. Fortunately, if exact conditions are not given, we can approximate the end conditions with artificial ones. However, these constructions are not related to each other in terms of integro splines of neighbouring degrees. Wu and Zhang [16] proposed the conjecture that splines with even degrees have a super-convergence order in an approximating function value at the knots. This conjecture has been confirmed in the quadratic [17], quartic [10] and sextic cases [16]. Zhanlav and Mijiddorj [20] attempted to explain the reason for the super-convergence by the local integro spline. The super-convergence phenomenon for the spline quasi-interpolant is interesting not only from theoretical, but also from practical, point of view. Therefore, several authors have considered and illustrated methods [1] to construct a super-convergent quintic quasi-interpolant applied to numerical integration.

The remainder of this paper is organized as follows. In Section 2, we formulate our task and give an integro spline with degree m based on Taylor expansion and consider its properties. Section 3 is devoted to the general construction of local integro splines. Then numerical computation and figure illustrations are presented in Section 4 to illustrate the constructed local integro-sextic spline. Section 5 concludes the paper.

2. Materials and methods

Let $y(x)$ be an unknown univariate real-valued function over $[a, b]$ and let $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, k$, be a uniform partition of the interval with mesh length

$$h = \frac{b-a}{k}.$$

Assume that the function values

$$y_i = y(x_i)$$

are not given, but the integral values I_i of $y(x)$ are known on the k subintervals $[x_{i-1}, x_i]$. Our task is to determine an integro spline $S(x)$ such that

$$\int_{x_{i-1}}^{x_i} S(x)\,dx = \int_{x_{i-1}}^{x_i} y(x)\,dx = I_i, \quad i = 1, 2, \ldots, k. \tag{2.1}$$

We assume that at least one of I_i is nonzero. Hence, the integro spline $S(x)$ is not identically equal to zero on $[a, b]$. This type of problem has many practical applications in statistics, numerical analysis, the fields of mechanics and so on. Particularly, Schoenberg [12, p. 115] showed the existence of a cubic/quintic spline $S(x)$ that

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satisfies the cumulative sequence $S(0) = 0$,

$$S(i) = \sum_{j=1}^{i} I_j,$$

with suitable end conditions and, letting

$$\sigma(x) = S'(x),$$

proved that $\sigma(x)$ is a quadratic/quartic histospline, which is a solution of the problem (2.1). In general, any solution to problem (2.1) is in fact the derivative of a one-degree-higher spline interpolant defined by the cumulative sequence. We now consider a method to find the solution of (2.1) from a given one-degree-higher or -lower solution of (2.1).

We will use the following notation: denote by $S_m(x)$ an integro spline with degree m satisfying (2.1), by $S_m^{(r)}(x)$ the rth derivative of $S_m(x)$, by $S_m^{(r)}(x_i)$ values of $S_m^{(r)}(x)$ at the knots and by $\tilde{S}_m^{(r)}(x_i)$ specific approximate values for the rth derivative integro spline. Also, $\hat{S}_m(x)$ is called a local integro spline whose coefficients are explicitly determined by integral values of I_i in a neighbourhood of x_i, without solving any system of equations. Lang and Xu [10, 18] studied the approximation properties of integro splines under the strong restriction $y \in C^{\infty}[a, b]$, but, in this study, we assume that an approximation function y belongs to $C^{m+2}[a, b]$ for the integro spline with degree m. For $y \in C^{m+2}[a, b], m = 2, 4, 6, \ldots$, by Taylor expansion of y,

$$I_i + I_{i+1} = 2 \sum_{j=0}^{m/2} \frac{h^{2j+1}}{(2j+1)!} y_i^{(2j)} + O(h^{m+3}), \tag{2.2}$$

where

$$y_i^{(r)} = y^{(r)}(x_i).$$

Then the following variant of the Euler–Maclaurin integration formula [11] holds.

LEMMA 2.1. *Let $y \in C^{m+2}[a, b]$ and $m = 2, 4, \ldots$. Then*

$$\begin{aligned} y_i = {} & l_1 \frac{1}{h}(I_i + I_{i+1}) + l_2 h(y'_{i+1} - y'_{i-1}) + l_3 h^3 (y'''_{i+1} - y'''_{i-1}) + \cdots \\ & + l_{m/2+1} h^{m-1} (y_{i+1}^{(m-1)} - y_{i-1}^{(m-1)}) + O(h^{m+2}), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} l_1 &= \frac{1}{2}, \\ l_2 &= -l_1 \frac{1}{3!} = -\frac{1}{12}, \\ l_3 &= -\left(l_2 \frac{1}{3!} + l_1 \frac{1}{5!} \right) = \frac{7}{720}, \end{aligned}$$

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$$l_4 = -\left(l_3\frac{1}{3!} + l_2\frac{1}{5!} + l_1\frac{1}{7!}\right) = -\frac{31}{30\,240},$$
$$\cdots$$
$$l_{m/2+1} = -\left(l_{m/2}\frac{1}{3!} + l_{m/2-1}\frac{1}{5!} + \cdots + l_2\frac{1}{(m-1)!} + l_1\frac{1}{(m+1)!}\right). \tag{2.4}$$

PROOF. The proof is by induction on m. When $m = 2$, substituting

$$y'_{i+1} - y'_{i-1} = 2hy''_i + O(h^3)$$

into (2.2),

$$y_i = \frac{1}{2h}(I_i + I_{i+1}) - \frac{h}{12}(y'_{i+1} - y'_{i-1}) + O(h^4),$$

so (2.3) and (2.4) are true for $m = 2$. Suppose that (2.3) and (2.4) are true for $m - 2$. Then, for m, using the Taylor expansion,

$$\begin{aligned}
y'_{i+1} - y'_{i-1} &= 2hy''_i + 2\frac{h^3}{3!}y^{(4)}_i + \cdots + 2\frac{h^{m-1}}{(m-1)!}y^{(m)}_i + O(h^{m+1}),\\
y'''_{i+1} - y'''_{i-1} &= 2hy^{(4)}_i + 2\frac{h^3}{3!}y^{(6)}_i + \cdots + 2\frac{h^{m-3}}{(m-3)!}y^{(m)}_i + O(h^{m-1}),\\
&\cdots\\
y^{(m-1)}_{i+1} - y^{(m-1)}_{i-1} &= 2hy^{(m)}_i + O(h^3).
\end{aligned} \tag{2.5}$$

Applying the induction hypothesis to (2.5),

$$\begin{aligned}
y''_i &= l_1\frac{1}{h}(y'_{i+1} - y'_{i-1}) + l_2h(y'''_{i+1} - y'''_{i-1}) + \cdots\\
&\quad + l_{m/2}h^{m-3}(y^{(m-1)}_{i+1} - y^{(m-1)}_{i-1}) + O(h^m),\\
y^{(4)}_i &= l_1\frac{1}{h}(y'''_{i+1} - y'''_{i-1}) + l_2h(y^{(5)}_{i+1} - y^{(5)}_{i-1}) + \cdots\\
&\quad + l_{m/2-1}h^{m-5}(y^{(m-1)}_{i+1} - y^{(m-1)}_{i-1}) + O(h^{m-2}),\\
&\cdots\\
y^{(m)}_i &= l_1\frac{1}{h}(y^{(m-1)}_{i+1} - y^{(m-1)}_{i-1}) + O(h^2).
\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.2), we obtain (2.3) with the coefficients (2.4). Hence, (2.3) and (2.4) hold for m, which completes the proof. □

The purpose of the equations (2.7) and (2.8) is to determine specific approximate values for $S^{(m-1)}_m(x_i)$ in the cases m even and odd. The Taylor expansion of $S_m(x)$ at the

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point $x = x_i$ for $m = 2, 3, 4, \ldots$ is

$$S_m(x) = S_m(x_i) + (x - x_i)S'_m(x_i) + \frac{(x - x_i)^2}{2}S''_m(x_i) + \frac{(x - x_i)^3}{6}S'''_m(x_i) + \cdots + \frac{(x - x_i)^m}{m!}S^{(m)}_m(x_i^-),$$

where we define the left derivative at x_i

$$S^{(m)}_m(x_i^-) = \frac{1}{h}(S^{(m-1)}_m(x_i) - S^{(m-1)}_m(x_{i-1})), \quad x \in [x_{i-1}, x_i].$$

Using the last expansion and (2.1) for $i = 1, 2, \ldots, k - 1$,

$$I_i + I_{i+1} = 2hS_m(x_i) + 2\frac{h^3}{3!}S''_m(x_i) + 2\frac{h^5}{5!}S^{(4)}_m(x_i) + \cdots + \frac{h^{m+1}}{(m+1)!}(S^{(m)}_m(x_i^-) + S^{(m)}_m(x_i^+)) \tag{2.7}$$

and

$$I_{i+1} - I_i = 2\frac{h^2}{2!}S'_m(x_i) + 2\frac{h^4}{4!}S'''_m(x_i) + \cdots + \frac{h^{m+1}}{m!}(S^{(m)}_m(x_i^+) - S^{(m)}_m(x_i^-)). \tag{2.8}$$

Dropping the last small terms in (2.7) and (2.8), we can find $S^{(m-1)}_m(x_i)$, which will be used later. Performing a similar computation as in the proof of Lemma 2.1, and using

$$S^{(m)}_m(x_i^-) + S^{(m)}_m(x_i^+) = \frac{1}{h}(S^{(m-1)}_m(x_{i+1}) - S^{(m-1)}_m(x_{i-1}))$$

in (2.7),

$$S_m(x_i) = l_1\frac{1}{h}(I_i + I_{i+1}) + l_2h(S'_m(x_{i+1}) - S'_m(x_{i-1})) + l_3h^3(S'''_m(x_{i+1}) - S'''_m(x_{i-1})) + \cdots + l_{m/2+1}h^{m-1}(S^{(m-1)}_m(x_{i+1}) - S^{(m-1)}_m(x_{i-1})), \quad m = 2, 4, \ldots, \tag{2.9}$$

and

$$S'_m(x_i) = l_1\frac{1}{h}(S_m(x_{i+1}) - S_m(x_{i-1})) + l_2h(S''_m(x_{i+1}) - S''_m(x_{i-1})) + \cdots + l_{(m+1)/2}h^{m-2}(S^{(m-1)}_m(x_{i+1}) - S^{(m-1)}_m(x_{i-1})), \quad m = 3, 5, \ldots. \tag{2.10}$$

Let $y \in C^{m+2}[a, b]$; the Taylor expansions at the point x_i yield

$$h^5y^{(4)}_i - \frac{h^6}{2}y^{(5)}_i = E(i) + O(h^7), \quad i = 3, 4, \ldots, k - 2,$$

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where

$$E(i) = I_{i-2} - 4I_{i-1} + 6I_i - 4I_{i+1} + I_{i+2}.$$

Hence,

$$E(i-1) - 2E(i) + E(i+1) = I_{i-3} - 6I_{i-2} + 15I_{i-1} - 20I_i + 15I_{i+1} - 6I_{i+2} + I_{i+3}$$
$$= h^7 y_i^{(6)} - \frac{h^8}{2} y_i^{(7)} + O(h^9), \quad i = 4, 5, \ldots, k-3. \tag{2.11}$$

Performing a similar computation as (2.11),

$$I_{i-m/2-1} - (m+2)I_{i-m/2} + \frac{(m+1)(m+2)}{2} I_{i-m/2+1} + \cdots$$
$$+ \frac{(m+1)(m+2)}{2} I_{i+m/2-1} - (m+2)I_{i+m/2} + I_{i+m/2+1} = O(h^{m+3}),$$
$$i = \frac{m}{2} + 2, \ldots, k - \frac{m}{2} - 1,$$
$$m = 2, 4, \ldots, \tag{2.12}$$

where the coefficients can be defined as the coefficients of x^i in the expansion of $(1-x)^{m+2}$.

3. General algorithm for local integro splines

The properties of the super-convergence of $S_m^{(r)}(x_i)$ proved in the literature [10, 16, 17] suggest the following conjectures.

CONJECTURE 3.1. If we use a similar technique as in [16, Remark 3.1] to construct integro interpolation by B-splines with degrees $m = 2, 4, \ldots$, then we conjecture that it possesses the super-convergence property

$$\max_{0 \le i \le k} |S_m^{(r)}(x_i) - y^{(r)}(x_i)| = O(h^{m+2-r}), \quad r = 0, 2, \ldots, m-2.$$

The super-convergence properties of $S_5'(x)$ and $S_5''(x)$ of the integro quintic spline are given by Theorem 3.4 in [23] as well as those $S_3'(x)$ of the integro cubic spline [21].

CONJECTURE 3.2. If we use a convenient technique to construct an integro spline with degrees $m = 3, 5, \ldots$, then we conjecture that it possesses the super-convergence property

$$\max_{0 \le i \le k} |S_m^{(r)}(x_i) - y^{(r)}(x_i)| = O(h^{m+2-r}), \quad r = 1, 3, \ldots, m-2.$$

Using the notation

$$S_m^{(r)}(x_i) = y_i^{(r)} + O(h^{d_r}), \ i = 0, 1, \ldots, k, \quad r = 0, 1, 2, \ldots,$$

and combining the two above-mentioned conjectures, we list the values of d_r of an integro spline with degree m in Table 1.

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TABLE 1. Conjectured values of d_r for $S_m^{(r)}(x_i)$, $r = 0, 1, 2, \ldots,\ i = 0, 1, \ldots, k$.

m	d_0	d_1	d_2	d_3	d_4	d_5	$\cdots$
2	4	2					
3	4	4	2				
4	6	4	4	2			
5	6	6	4	4	2		
6	8	6	6	4	4	2	
$\cdots$							

Algorithm 1: IntegroSpline $(m, \hat{S}_m(x))$

1. If m is even then
$$\tilde{S}_{m+1}^{(2k)}(x_i) = \tilde{S}_m^{(2k)}(x_i)$$
for $k = 0, 1, \ldots, m/2 - 1$, and $i = 0, 1, \ldots, k$, else
$$\tilde{S}_{m+1}^{(2k+1)}(x_i) = \tilde{S}_m^{(2k+1)}(x_i)$$
for $k = 0, 1, \ldots, (m-3)/2$, and $i = 0, 1, \ldots, k$.
2. If m is even then using (2.7), find $\tilde{S}_{m+1}^{(m)}(x_i)$,
else using (2.8), find $\tilde{S}_{m+1}^{(m)}(x_i)$.
3. If m is even then using (2.10), compute $\tilde{S}'_{m+1}(x_i)$, $\tilde{S}'''_{m+1}(x_i), \ldots, \tilde{S}_{m+1}^{(m-1)}(x_i)$,
else using (2.9), compute $\tilde{S}_{m+1}(x_i)$, $\tilde{S}''_{m+1}(x_i), \ldots, \tilde{S}_{m+1}^{(m-1)}(x_i)$. Simplify the explicit formulae by (2.12).
4. Using the same technique as in [5, 10], which we shall call an *artificial end conditions technique*, find the bordering values of $\tilde{S}_{m+1}^{(r)}(x_i)$.
5. Compute the coefficients of the local integro spline $\hat{S}_{m+1}(x)$ using $\tilde{S}_{m+1}^{(r)}(x_i)$ and properties of the B-spline.

Now, using given specific approximate formulae for the integro spline with degree m, we present a general algorithm to construct a local integro spline with degree $m - 1$ or $m + 1$, which provides the values in Table 1. For the general case, let us outline the general algorithm for constructing local integro splines. Suppose that $\hat{S}_m(x)$ is given. Then $\hat{S}_{m+1}(x)$, $m \geq 2$, is constructed by Algorithm 1.

REMARK 3.3. Suppose that $\hat{S}_m(x)$, $m \geq 3$, is given. Then, constructing an algorithm for $\hat{S}_{m-1}(x)$ is easily derived from Algorithm 1.

Let us explain this algorithm when a local integro quartic spline is given by constructing a local integro-sextic spline.

Now, we proceed to construct a local integro-sextic spline.

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Algorithm 2: IntegroSpline$(4, \hat{S}_4(x))$

Suppose that the local integro quartic spline $\hat{S}_4(x)$ is given by [22].

1. Set

$$\tilde{S}_5(x_i) = \tilde{S}_4(x_i), \quad \tilde{S}_5''(x_i) = \tilde{S}_4''(x_i), \quad i = 0, 1, \ldots, k,$$

where $\tilde{S}_4(x_i)$ are given by (3.18) and (3.32) in [22], $\tilde{S}_4''(x_i)$ are given by (3.17) and (3.33) in [22].

2. Using (2.7), compute $\tilde{S}_5^{(4)}(x_i)$, that is,

$$\tilde{S}_5^{(4)}(x_i) = \frac{120}{h^5}\left(\frac{1}{2}(I_{i+1} + I_i) - h\tilde{S}_5(x_i) - \frac{h^3}{6}\tilde{S}_5''(x_i)\right), \quad i = 1, 2, \ldots, k-1.$$

Find at the end points values of $\tilde{S}_5^{(4)}(x_0)$ and $\tilde{S}_5^{(4)}(x_k)$ by the artificial end conditions technique.

3. From (2.10) for $i = 1, 2, \ldots, k-1$, we have

$$\tilde{S}_5'''(x_i) = \frac{1}{2h}(\tilde{S}_5''(x_{i+1}) - \tilde{S}_5''(x_{i-1})) - \frac{h}{12}(\tilde{S}_5^{(4)}(x_{i+1}) - \tilde{S}_5^{(4)}(x_{i-1})), \tag{3.1}$$

and

$$\begin{aligned}\tilde{S}_5'(x_i) = {} & \frac{1}{2h}(\tilde{S}_5(x_{i+1}) - \tilde{S}_5(x_{i-1})) - \frac{h}{12}(\tilde{S}_5''(x_{i+1}) - \tilde{S}_5''(x_{i-1})) \\ & + \frac{7h^3}{720}(\tilde{S}_5^{(4)}(x_{i+1}) - \tilde{S}_5^{(4)}(x_{i-1})).\end{aligned} \tag{3.2}$$

When $\tilde{S}_5(x_i)$, $\tilde{S}_5''(x_i)$, and $\tilde{S}_5^{(4)}(x_i)$ are known, find the rest of unknowns $\tilde{S}_5^{(r)}(x_i)$, $r = 1, 3$, by (3.1) and (3.2). Simplifying (3.1) and (3.2) by (2.11), we obtain the following explicit formulae

$$\begin{aligned}\hat{S}_5'(x_i) = {} & \frac{1}{180h^2}(-2I_{i-2} + 25I_{i-1} - 245I_i \\ & + 245I_{i+1} - 25I_{i+2} + 2I_{i+3}), \quad i = 3, 4, \ldots, k-3,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\hat{S}_5'''(x_i) = {} & \frac{1}{6h^4}(I_{i-2} - 11I_{i-1} + 28I_i - 28I_{i+1} + 11I_{i+2} - I_{i+3}), \\ & i = 3, 4, \ldots, k-3.\end{aligned} \tag{3.4}$$

4. Find the bordering values of $\tilde{S}_5'(x_i)$ and $\tilde{S}_5'''(x_i)$ by the artificial end conditions technique.

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Algorithm 3: IntegroSpline$(5, \hat{S}_5(x))$

Suppose that the $\tilde{S}_5^{(r)}(x_i)$, $r = 0, 1, \ldots, 4$, are given by IntegroSpline$(4, \hat{S}_4(x))$ in Algorithm 2.

1. Set
$$\tilde{S}_6'(x_i) = \tilde{S}_5'(x_i), \quad \tilde{S}_6'''(x_i) = \tilde{S}_5'''(x_i), \quad i = 0, 1, \ldots, k.$$
2. Compute $\tilde{S}_6^{(5)}(x_i)$, that is, from (2.8) when $m = 6$ and get
$$I_{i+1} - I_i = h^2\tilde{S}_6'(x_i) + \frac{h^4}{12}\tilde{S}_6'''(x_i) + \frac{h^6}{360}\tilde{S}_6^{(5)}(x_i) + \frac{h^7}{6!}(S_6^{(6)}(x_i^+) - S_6^{(6)}(x_i^-)), \quad i = 1, 2, \ldots, k-1. \tag{3.5}$$
By (3.3) and (3.4), dropping the small term in (3.5), find $\tilde{S}_6^{(5)}(x_i)$. Find the bordering values of $\tilde{S}_6^{(5)}(x_i)$ by the artificial end conditions technique. In this case, we get
$$\begin{aligned}
\tilde{S}_6^{(5)}(x_0) &= \frac{1}{h^6}(-4I_1 + 23I_2 - 55I_3 + 70I_4 - 50I_5 + 19I_6 - 3I_7),\\
\tilde{S}_6^{(5)}(x_i) &= \frac{360}{h^6}\left(I_{i+1} - I_i - h^2\tilde{S}_i' - \frac{h^4}{12}\tilde{S}_i'''\right), \quad i = 1, 2, \ldots, k-1,\\
\tilde{S}_6^{(5)}(x_k) &= \frac{1}{h^6}(4I_k - 23I_{k-1} + 55I_{k-2} - 70I_{k-3} + 50I_{k-4}\\
&\quad - 19I_{k-5} + 3I_{k-5}),
\end{aligned} \tag{3.6}$$
or
$$\tilde{S}_6^{(5)}(x_i) = \frac{1}{h^6}(-I_{i-2} + 5I_{i-1} - 10I_i + 10I_{i+1} - 5I_{i+2} + I_{i+3}), \quad i = 3, 4, \ldots, k-3. \tag{3.7}$$
We can easily check that $\tilde{S}_6^{(5)}(x_i) = y_i^{(5)} + O(h^2)$ using (3.6) and (3.7).
3. Compute $\tilde{S}_6^{(r)}(x_i)$ for $r = 0, 2, 4$ and $i = 4, 5, \ldots, k-4$ using (2.9).
4. Find the remainder values by the artificial end conditions technique.

Thus, we have the Appendix, where we give the errors of $\tilde{S}_6(x)$ at knots.

THEOREM 3.4 *Let $y \in C^8[a, b]$ and $\tilde{S}_6(x_i)$ be defined by the above IntegroSpline $(5, \hat{S}_5(x))$. Then the convergence orders of*
$$\tilde{S}_6^{(j)}(x_i), \quad i = 0, 1, \ldots, k, \quad j = 0, 1, \ldots, 5, \tag{3.8}$$
are 8, 6, 6, 4, 4, 2, respectively.

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PROOF. For $1 \le i+j \le k$, using the Taylor expansion of $y \in C^8[a,b]$ at the point x_i,

$$I_{i+j} = \int_{x_{i+j-1}}^{x_{i+j}} y(x)dx = \int_{x_{i+j-1}}^{x_{i+j}} \left(\sum_{n=0}^{7} \frac{(x-x_i)^n}{n!} y_i^{(n)} + O(x-x_i)^8 \right) dx$$

$$= \sum_{n=1}^{8} y_i^{(n-1)} \frac{h^n}{n!} (j^n - (j-1)^n) + O(h^9). \tag{3.9}$$

Substituting (3.9) into the formulae (3.6)–(3.7) as well as the explicit formulae in the Appendix, we obtain (3.8). □

In order to construct a local integro-sextic spline using IntegroSpline$(5, \hat{S}_5(x))$, we use the B-spline representation of $S_m(x)$

$$S_m(x) = \sum_{i=1}^{k+m} \alpha_i B_{i,m}(x), \tag{3.10}$$

where $\{B_{i,m}, i \in \mathcal{J}\}$ with $\mathcal{J} = \{1, 2, \ldots, k+m\}$ is a basis of the space of splines of degree m and of class C^{m-1}, the support of $B_{i,m}$ being $supp(B_{i,m}) = [x_{i-m-1}, x_i]$ (see, for example, [6, 15, 19]). As usual, we add multiple knots at the end points $x_{-m} = \cdots = x_{-1} = x_0 = a$ and $b = x_k = x_{k+1} = \cdots = x_{k+m}$. Using the properties of sextic B-splines when $m = 6$ in (3.10),

$$\alpha_{i+4} = S_6(x_i) + \frac{h}{2} S_6'(x_i) - \frac{h^2}{6} S_6''(x_i) - \frac{h^3}{8} S_6'''(x_i) + \frac{h^4}{90} S_6^{(4)}(x_i)$$
$$+ \frac{h^5}{60} S_6^{(5)}(x_i), \quad i = 0, 1, \ldots, k. \tag{3.11}$$

From (2.11),

$$I_{i-3} - 7I_{i-2} + 21I_{i-1} - 35I_i + 35I_{i+1} - 21I_{i+2} + 7I_{i+3} - I_{i+4} = O(h^8),$$
$$i = 4, 5, \ldots, k-4. \tag{3.12}$$

Substituting the values of $\tilde{S}_6^{(r)}(x_i)$, $r = 0, 1, \ldots, 5$, at the knots into (3.11), and using (3.12), we can find the coefficients $\hat{\alpha}_i$, $i = 4, 5, \ldots, k+4$, in the local integro-sextic spline,

$$\hat{\alpha}_{i+4} = \frac{1}{15120h} (-311I_{i-2} + 3168I_{i-1} - 14913I_i + 39232I_{i+1}$$
$$- 14913I_{i+2} + 3168I_{i+3} - 311I_{i+4}), \quad i = 3, 4, \ldots, k-4. \tag{3.13}$$

Hence, the B-spline representation of the local integro-sextic spline is

$$\hat{S}_6(x) = \sum_{i=1}^{k+6} \hat{\alpha}_i B_{i,6}(x), \tag{3.14}$$

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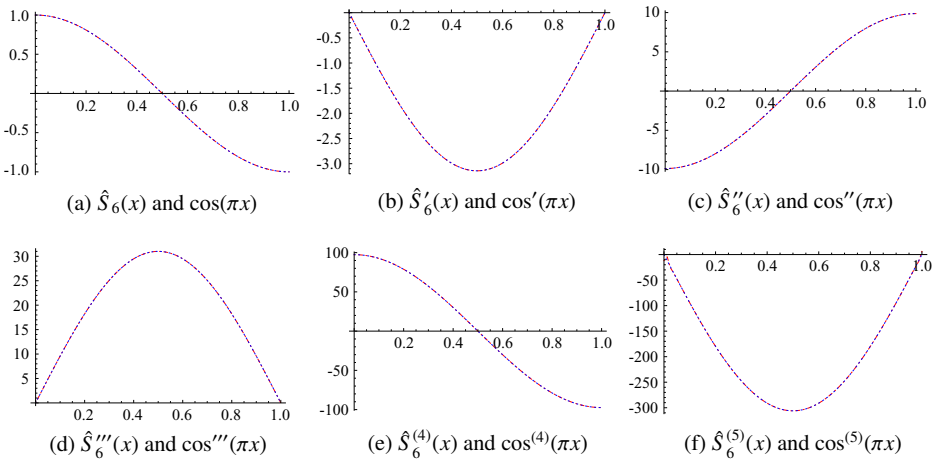


FIGURE 1. $\hat{S}_6(x)$ (red dots) and $\cos(\pi x)$ (blue dots) on $[0, 1]$, and their derivatives up to fifth order, where $k = 80$. (Colour available online.)

where the coefficients $\hat{\alpha}_i$ are given by (3.13). By virtue of the properties of the sextic B-spline, we have $(i = 0, 1, \ldots, k)$

$$\begin{aligned}
\hat{S}_6(x_i) &= \frac{\hat{\alpha}_{i+1} + 57\hat{\alpha}_{i+2} + 302\hat{\alpha}_{i+3} + 302\hat{\alpha}_{i+4} + 57\hat{\alpha}_{i+5} + \hat{\alpha}_{i+6}}{720},\\
\hat{S}_6'(x_i) &= \frac{-\hat{\alpha}_{i+1} - 25\hat{\alpha}_{i+2} - 40\hat{\alpha}_{i+3} + 40\hat{\alpha}_{i+4} + 25\hat{\alpha}_{i+5} + \hat{\alpha}_{i+6}}{120h},\\
\hat{S}_6''(x_i) &= \frac{\hat{\alpha}_{i+1} + 9\hat{\alpha}_{i+2} - 10\hat{\alpha}_{i+3} - 10\hat{\alpha}_{i+4} + 9\hat{\alpha}_{i+5} + \hat{\alpha}_{i+6}}{24h^2},\\
\hat{S}_6'''(x_i) &= \frac{-\hat{\alpha}_{i+1} - \hat{\alpha}_{i+2} + 8\hat{\alpha}_{i+3} - 8\hat{\alpha}_{i+4} + \hat{\alpha}_{i+5} + \hat{\alpha}_{i+6}}{6h^3},\\
\hat{S}_6^{(4)}(x_i) &= \frac{\hat{\alpha}_{i+1} - 3\hat{\alpha}_{i+2} + 2\hat{\alpha}_{i+3} + 2\hat{\alpha}_{i+4} - 3\hat{\alpha}_{i+5} + \hat{\alpha}_{i+6}}{2h^4},\\
\hat{S}_6^{(5)}(x_i) &= \frac{-\hat{\alpha}_{i+1} + 5\hat{\alpha}_{i+2} - 10\hat{\alpha}_{i+3} + 10\hat{\alpha}_{i+4} - 5\hat{\alpha}_{i+5} + \hat{\alpha}_{i+6}}{h^5}.
\end{aligned} \tag{3.15}$$

From (3.15),

$$\hat{\alpha}_{i+6} - \hat{\alpha}_{i+1} = 5h\tilde{S}_6'(x_i) + \frac{15h^3\tilde{S}_6'''(x_i)}{4} + \frac{h^5\tilde{S}_6^{(5)}(x_i)}{3}. \tag{3.16}$$

The remaining coefficients $\hat{\alpha}_i$ of (3.14) for $i = 1, 2, 3$ and $i = k + 5, k + 6$ are determined from (3.11) and (3.16). Thus, we find a local integro-sextic spline $\hat{S}_6(x) \in C^5[a, b]$ (cf. Figure 1) with (3.14).

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REMARK 3.5. If $y \in C^8[a,b]$, then

$$|\tilde{S}_6^{(r)}(x_i) - \hat{S}_6^{(r)}(x_i)| = O(h^{8-r}), \quad r = 0,2,4, \quad i = 0,1,\ldots,k,$$
$$|\tilde{S}_6^{(r)}(x_i) - \hat{S}_6^{(r)}(x_i)| = O(h^{7-r}), \quad r = 1,3,5.$$

4. Numerical examples

We present the approximation behaviour (see Table 1 and Theorem 3.4) of $\tilde{S}_6^{(j)}(x_i)$, $j = 0,2,4$, for given $y \in C^8[0,1]$ as

$$y_1(x) = e^x, \quad y_2(x) = \cos(\pi x), \quad y_3(x) = \frac{1}{x+2}.$$

Computational results are listed in Tables 2–4. In Tables 2–4, $R_{2r}(k)$ is computed by

$$R_{2r}(k) = \max_{0\le i\le k} |\hat{S}_6^{(2r)}(x_i) - y_j^{(2r)}(x_i)|, \quad r = 0,1,2, \quad j = 1,2,3, \quad \text{and}$$

$$\text{NCO} = \log_2 \left| \frac{R_{2r}(k)}{R_{2r}(2k)} \right|.$$

The numerical convergence order (NCO) values in Tables 2–4 provide empirical support for the conjectured super-convergence properties. All these computations are carried out using Mathematica, MachinePrecision in 64.

Finally, Figure 2 shows the proximity of the constructed spline and the given function histograms.

The process of integral approximation consisted of two stages: the measurement data were plotted on a grid of rectangular coordinates, the points were connected with a smooth curve and the appropriate formula describing the curve was selected. Then the parameters of the formulae that correspond to the accepted analytical expression

TABLE 2. Maximum errors of $\hat{S}_6(x)$ for $y_1(x) = e^x$.

k	R_0	NCO	R_2	NCO	R_4	NCO
160	6.876×10^{-19}	–	3.096×10^{-13}	–	3.016×10^{-08}	–
320	2.716×10^{-21}	7.98	4.891×10^{-15}	5.98	1.905×10^{-09}	3.98
640	1.067×10^{-23}	7.99	7.684×10^{-17}	5.99	1.197×10^{-10}	3.99

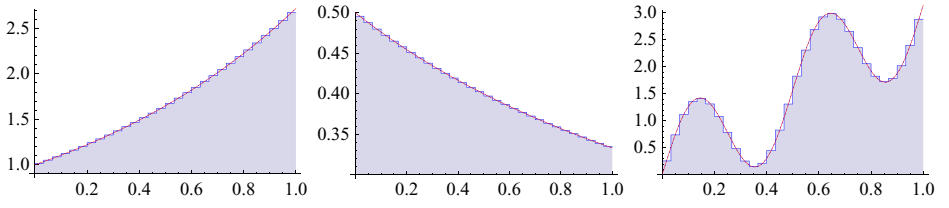
TABLE 3. Maximum errors of $\hat{S}_6(x)$ for $y_2(x) = \cos(\pi x)$.

k	R_0	NCO	R_2	NCO	R_4	NCO
160	2.448×10^{-15}	–	1.102×10^{-09}	–	1.072×10^{-04}	–
320	9.582×10^{-18}	7.99	1.725×10^{-11}	5.99	6.714×10^{-06}	3.99
640	3.745×10^{-20}	7.99	2.696×10^{-13}	5.99	4.198×10^{-07}	3.99

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TABLE 4. Maximum errors of $\hat{S}_6(x)$ for $y_3(x) = 1/(x+2)$.

k	R_0	NCO	R_2	NCO	R_4	NCO
160	1.843×10^{-17}	–	8.314×10^{-12}	–	8.137×10^{-07}	–
320	7.567×10^{-20}	7.92	1.364×10^{-13}	5.92	5.324×10^{-08}	3.93
640	3.031×10^{-22}	7.96	2.184×10^{-15}	5.96	3.405×10^{-09}	3.96
1280	1.199×10^{-24}	7.98	3.454×10^{-17}	5.98	2.153×10^{-10}	3.98
2560	4.713×10^{-27}	7.99	5.431×10^{-19}	5.99	1.353×10^{-11}	3.99



(a) Histogram of e^x, $x \in [0, 1]$

(b) Histogram of $1/(x+2)$, $x \in [0, 1]$

(c) Histogram of $\sin(4\pi x) + \pi x$, $x \in [0, 1]$

FIGURE 2. Functions (blue dots) of integral values in an histogram (blue line) and the plot of the constructed $\hat{S}_6(x)$ spline (red line), where $k = 30$. (Colour available online.)

were calculated. When constructing an integral approximation, the minimum deviation of one function from another was observed. Comparing the position of the constructed spline and the given function histograms, we can state that there is no oscillation. Also, the horizontal axis is intentionally moved to display the image more clearly.

5. Conclusions

We proposed a general algorithm satisfying the super-convergence properties to construct the local integro splines with degree $m+1$ when the local integro spline is given with degree m. Of course, this algorithm is converted into an algorithm to determine $\hat{S}_{m-1}(x)$ from $\hat{S}_m(x)$. As we know, construction of nonlocal integro splines leads to solving a system of linear equations with band matrices.

Therefore, they have high computational costs. In contrast, our algorithm is based on specific approximate values of an integro spline and its derivatives up to $m-1$ order. Using these approximate values, we can find a local integro B-spline representation. We constructed a local integro-sextic spline with the proposed algorithm. The local integro splines work efficiently; they have low computational complexity and they are suitable for real-time situations.

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Appendix

$$\tilde{S}_6(x_0) = \frac{1}{840h}(2283I_1 - 4437I_2 + 7323I_3 - 8357I_4 + 6343I_5 - 3065I_6 + 855I_7 - 105I_8),$$

$$\tilde{S}_6(x_1) = \frac{1}{840h}(105I_1 + 1443I_2 - 1497I_3 + 1443I_4 - 1007I_5 + 463I_6 - 125I_7 + 15I_8),$$

$$\tilde{S}_6(x_2) = \frac{1}{840h}(-15I_1 + 225I_2 + 1023I_3 - 657I_4 + 393I_5 - 167I_6 + 43I_7 - 5I_8),$$

$$\tilde{S}_6(x_3) = \frac{1}{840h}(5I_1 - 55I_2 + 365I_3 + 743I_4 - 307I_5 + 113I_6 - 27I_7 + 3I_8),$$

$$\tilde{S}_6(x_i) = \frac{1}{840h}(-3I_{i-3} + 29I_{i-2} - 139I_{i-1} + 533I_i + 533I_{i+1} - 139I_{i+2} + 29I_{i+3} - 3I_{i+4}),$$

$$i = 4(1)k - 4,$$

$$\tilde{S}_6(x_{k-3}) = \frac{1}{840h}(5I_k - 55I_{k-1} + 365I_{k-2} + 743I_{k-3} - 307I_{k-4} + 113I_{k-5} - 27I_{k-6} + 3I_{k-7}),$$

$$\tilde{S}_6(x_{k-2}) = \frac{1}{840h}(-15I_k + 225I_{k-1} + 1023I_{k-2} - 657I_{k-3} + 393I_{k-4} - 167I_{k-5} + 43I_{k-6} - 5I_{k-7}),$$

$$\tilde{S}_6(x_{k-1}) = \frac{1}{840h}(105I_k + 1443I_{k-1} - 1497I_{k-2} + 1443I_{k-3} - 1007I_{k-4} + 463I_{k-5} - 125I_{k-6} + 15I_{k-7}),$$

$$\tilde{S}_6(x_k) = \frac{1}{840h}(2283I_k - 4437I_{k-1} + 7323I_{k-2} - 8357I_{k-3} + 6343I_{k-4} - 3065I_{k-5} + 855I_{k-6} - 105I_{k-7}),$$

$$\tilde{S}'_6(x_0) = \frac{1}{180h^2}(-938I_1 + 3076I_2 - 4835I_3 + 4655I_4 - 2725I_5 + 893I_6 - 126I_7),$$

$$\tilde{S}'_6(x_1) = \frac{1}{180h^2}(-126I_1 - 56I_2 + 430I_3 - 425I_4 + 245I_5 - 79I_6 + 11I_7),$$

$$\tilde{S}'_6(x_2) = \frac{1}{180h^2}(11I_1 - 203I_2 + 175I_3 + 45I_4 - 40I_5 + 14I_6 - 2I_7),$$

$$\tilde{S}'_6(x_{k-2}) = \frac{1}{180h^2}(2I_{k-6} - 14I_{k-5} + 40I_{k-4} - 45I_{k-3} - 175I_{k-2} + 203I_{k-1} - 11I_k),$$

$$\tilde{S}'_6(x_{k-1}) = \frac{1}{180h^2}(-11I_{k-6} + 79I_{k-5} - 245I_{k-4} + 425I_{k-3} - 430I_{k-2} + 56I_{k-1} + 126I_k),$$

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$$\tilde{S}_6'(x_k) = \frac{1}{180h^2}(126I_{k-6} - 893I_{k-5} + 2725I_{k-4} - 4655I_{k-3} + 4835I_{k-2} - 3076I_{k-1} + 938I_k),$$

$$\tilde{S}_6''(x_0) = \frac{1}{240h^3}(2403I_1 - 11557I_2 + 25149I_3 - 32235I_4 + 26045I_5 - 13083I_6 + 3747I_7 - 469I_8),$$

$$\tilde{S}_6''(x_1) = \frac{1}{240h^3}(469I_1 - 1349I_2 + 1575I_3 - 1115I_4 + 595I_5 - 219I_6 + 49I_7 - 5I_8),$$

$$\tilde{S}_6''(x_2) = \frac{1}{240h^3}(5I_1 + 429I_2 - 1209I_3 + 1295I_4 - 765I_5 + 315I_6 - 79I_7 + 9I_8),$$

$$\tilde{S}_6''(x_3) = \frac{1}{240h^3}(-9I_1 + 77I_2 + 177I_3 - 705I_4 + 665I_5 - 261I_6 + 63I_7 - 7I_8),$$

$$\tilde{S}_6''(x_i) = \frac{1}{240h^3}(7I_{i-3} - 65I_{i-2} + 273I_{i-1} - 215I_i - 215I_{i+1} + 273I_{i+2} - 65I_{i+3} + 7I_{i+4}),$$

$$i = 4(1)k - 4,$$

$$\tilde{S}_6''(x_{k-3}) = \frac{1}{240h^3}(-9I_k + 77I_{k-1} + 177I_{k-2} - 705I_{k-3} + 665I_{k-4} - 261I_{k-5} + 63I_{k-6} - 7I_{k-7}),$$

$$\tilde{S}_6''(x_{k-2}) = \frac{1}{240h^3}(5I_k + 429I_{k-1} - 1209I_{k-2} + 1295I_{k-3} - 765I_{k-4} + 315I_{k-5} - 79I_{k-6} + 9I_{k-7}),$$

$$\tilde{S}_6''(x_{k-1}) = \frac{1}{240h^3}(469I_k - 1349I_{k-1} + 1575I_{k-2} - 1115I_{k-3} + 595I_{k-4} - 219I_{k-5} + 49I_{k-6} - 5I_{k-7}),$$

$$\tilde{S}_6''(x_k) = \frac{1}{240h^3}(2403I_k - 11557I_{k-1} + 25149I_{k-2} - 32235I_{k-3} + 26045I_{k-4} - 13083I_{k-5} + 3747I_{k-6} - 469I_{k-7}),$$

$$\tilde{S}_6'''(x_0) = \frac{1}{6h^4}(-56I_1 + 277I_2 - 575I_3 + 644I_4 - 412I_5 + 143I_6 - 21I_7),$$

$$\tilde{S}_6'''(x_1) = \frac{1}{6h^4}(-21I_1 + 91I_2 - 164I_3 + 160I_4 - 91I_5 + 29I_6 - 4I_7),$$

$$\tilde{S}_6'''(x_2) = \frac{1}{6h^4}(-4I_1 + 7I_2 + 7I_3 - 24I_4 + 20I_5 - 7I_6 + I_7),$$

$$\tilde{S}_6'''(x_{k-2}) = \frac{1}{6h^4}(-I_{k-6} + 7I_{k-5} - 20I_{k-4} + 24I_{k-3} - 7I_{k-2} - 7I_{k-1} + 4I_k),$$

$$\tilde{S}_6'''(x_{k-1}) = \frac{1}{6h^4}(4I_{k-6} - 29I_{k-5} + 91I_{k-4} - 160I_{k-3} + 164I_{k-2} - 91I_{k-1} + 21I_k),$$

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$$\tilde{S}_6'''(x_k) = \frac{1}{6h^4}(21I_{k-6} - 143I_{k-5} + 412I_{k-4} - 644I_{k-3} + 575I_{k-2} - 277I_{k-1} + 56I_k)$$

and

$$\begin{aligned}
\tilde{S}_6^{(4)}(x_0) &= \frac{1}{6h^5}(81I_1 - 494I_2 + 1296I_3 - 1899I_4 + 1681I_5 - 900I_6 \\
&\quad + 270I_7 - 35I_8), \\
\tilde{S}_6^{(4)}(x_1) &= \frac{1}{6h^5}(35I_1 - 199I_2 + 486I_3 - 664I_4 + 551I_5 - 279I_6 + 80I_7 - 10I_8), \\
\tilde{S}_6^{(4)}(x_2) &= \frac{1}{6h^5}(10I_1 - 45I_2 + 81I_3 - 74I_4 + 36I_5 - 9I_6 + I_7), \\
\tilde{S}_6^{(4)}(x_3) &= \frac{1}{6h^5}(10I_2 - 45I_3 + 81I_4 - 74I_5 + 36I_6 - 9I_7 + I_8), \\
\tilde{S}_6^{(4)}(x_i) &= \frac{1}{6h^5}(-I_{i-3} + 8I_{i-2} - 18I_{i-1} + 11I_i + 11I_{i+1} - 18I_{i+2} + 8I_{i+3} - I_{i+4}), \\
i &= 4(1)k - 4, \\
\tilde{S}_6^{(4)}(x_{k-3}) &= \frac{1}{6h^5}(10I_{k-1} - 45I_{k-2} + 81I_{k-3} - 74I_{k-4} + 36I_{k-5} - 9I_{k-6} + I_{k-7}), \\
\tilde{S}_6^{(4)}(x_{k-2}) &= \frac{1}{6h^5}(10I_k - 45I_{k-1} + 81I_{k-2} - 74I_{k-3} + 36I_{k-4} - 9I_{k-5} + I_{k-6}), \\
\tilde{S}_6^{(4)}(x_{k-1}) &= \frac{1}{6h^5}(35I_k - 199I_{k-1} + 486I_{k-2} - 664I_{k-3} + 551I_{k-4} - 279I_{k-5} \\
&\quad + 80I_{k-6} - 10I_{k-7}), \\
\tilde{S}_6^{(4)}(x_k) &= \frac{1}{6h^5}(81I_k - 494I_{k-1} + 1296I_{k-2} - 1899I_{k-3} + 1681I_{k-4} - 900I_{k-5} \\
&\quad + 270I_{k-6} - 35I_{k-7}).
\end{aligned}$$

Acknowledgements

The authors thank the reviewers for their thoughtful comments and suggestions to improve the manuscript. R. Mijiddorj acknowledges financial support from the Mongolian National University of Education (Grant No. 20-0247). This work was partially supported by the Foundation of Science and Technology of Mongolia (No. SST _18/2018).

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https://doi.org/10.1017/S1446181121000316 Published online by Cambridge University Press