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RESEARCH ARTICLE

The analytic moduli space of holomorphic submersions

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Abstract

We construct a moduli space that parametrises stable proper holomorphic submersions over a fixed compact Kähler base. Stability is described in terms of the existence of a canonical relatively Kähler metric on the submersion, called an optimal symplectic connection. The construction of the moduli space combines techniques from geometric invariant theory with the study of the geometric PDE defining an optimal symplectic connection. A special case of this moduli space is the moduli space of stable vector bundles over a compact Kähler manifold. We also show that the moduli space is a Hausdorff complex space equipped with a Weil-Petersson type Kähler metric.

Contents

1 Introduction **1**
2 Preliminaries **4**
2.1 Deformation theory of cscK manifolds . . . 5
2.2 Fibrations . . . 7
2.3 Optimal symplectic connections . . . 9
3 Construction of the moduli space **12**
3.1 Openness of the setting . . . 12
3.2 Openness of the space of optimal symplectic connections . . . 17
3.3 The moduli space of optimal symplectic connections . . . 21
4 A Weil-Petersson type Kähler metric **24**
4.1 The determinant line bundle for the Weil-Petersson metric . . . 27
References **30**

1. Introduction

A fundamental result in the study of holomorphic vector bundles is the construction of the moduli space of stable vector bundles. Seshadri [43] and Newstead [37, 38] gave the first construction of such a moduli space for vector bundles over a curve, while Mumford [35] introduced semistability in the sense of geometric invariant theory (GIT) to study this moduli space and established its structure as a global GIT quotient. These moduli spaces also have an analytic interpretation: the Hitchin-Kobayashi correspondence of Donaldson, Uhlenbeck and Yau [12, 46] states that a holomorphic vector bundle on a compact Kähler manifold is slope-stable if and only if it admits a Hermite-Einstein connection, thus establishing

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an equivalence between an algebro-geometric stability condition and the solvability of a geometric PDE. Fujiki–Schumacher [20] and Lübke–Okonek [31] later directly constructed the moduli space of Hermite-Einstein vector bundles over a fixed compact Kähler manifold using analytic techniques. These moduli spaces remain a central object of study to this day, and we refer to Greb-Sibley-Toma-Wentworth [23] for recent work containing a discussion of analytic and algebraic compactifications.

Motivated by the Hitchin-Kobayashi correspondence and the construction of the moduli space of stable vector bundles, we extend these results to the context of proper holomorphic submersions which are not necessarily isotrivial. We recall that a holomorphic vector bundle $F \to B$ induces a proper holomorphic submersion by considering its projectivisation $\mathbb{P}(F) \to B$.

More precisely, let $\pi_Y : (Y, H_Y) \to (B, L)$ be a proper holomorphic submersion, where B is a smooth projective manifold with an ample line bundle L and H_Y is a relatively ample line bundle over Y. The base (B, L) is considered fixed throughout. The stability condition we impose on such a submersion is given by a stability condition on the fibres, defined in terms of K-stability, and a global condition on the fibration, given in terms of a canonical choice of a relatively Kähler metric, called an *optimal symplectic connection*.

When all the fibres admit a constant scalar curvature Kähler metric (hence are K-polystable), Dervan and Sektnan [9] introduced optimal symplectic connections as solutions to a geometric PDE on the fibration, which produces a canonical relatively Kähler metric. In particular, when the fibration is obtained as the projectivisation of a stable holomorphic vector bundle, an optimal symplectic connection is precisely induced by and induces the Hermite-Einstein connection on the vector bundle [9, §3.5]. Similarly, on isotrivial fibrations, the optimal symplectic connection condition becomes the Hermite-Yang-Mills condition on an associated principal bundle [33].

In [40], the author generalised the notion of optimal symplectic connections to smooth fibrations whose fibres may not have a constant scalar curvature Kähler metric (cscK), but instead are *analytically K-semistable*, that is, they degenerate to cscK manifolds. Allowing K-semistable fibres is essential for the construction of the moduli space of fibrations with an optimal symplectic connection. Indeed, when deforming a fibration with cscK fibres, one cannot expect that the fibres remain cscK. Analytic K-semistability, on the other hand, is an open condition, and this allows us to study the local behaviour of families of fibrations with an optimal symplectic connection. Our main result is the following.

Theorem 1.1. *Let G be a fixed complex Lie group. There exists a moduli space $\mathcal{M}$ that parametrises holomorphic submersions over a fixed base B such that*

1. *the fibres are analytically K-semistable, and the degenerations vary holomorphically in B;*
2. *the connected components of the identity of the groups of automorphisms of the fibres of the relatively cscK degenerations are all isomorphic to G;*
3. *they have a discrete relative automorphism group;*
4. *they admit an optimal symplectic connection.*

The space $\mathcal{M}$ is a Hausdorff analytic complex space and it carries a Weil-Petersson type Kähler metric.

In the statement, the relative automorphism group of a fibration $Y \to B$ is the group of automorphisms of Y that preserve the projection onto B. Moreover, the first two assumptions are essential in the definition of an optimal symplectic connection, and thus the group of automorphisms of the fibres of the relatively cscK degeneration is considered to be a fixed datum in our construction of the moduli space.

We now explain briefly the optimal symplectic connection condition. The assumption that the fibres are analytically K-semistable means that they each admit a degeneration to a cscK manifold. We further assume that these degenerations vary holomorphically in B, so that we have a family $(\mathcal{X}, \mathcal{H}) \to B \times S$, parametrised by $S \subseteq \mathbb{C}$, such that $(\mathcal{X}_s, \mathcal{H}_s) \to B$ is isomorphic to $(Y, H_Y) \to B$ for $s \neq 0$ and at $s = 0$ we have a fibration $(X, H_X) \to B$ with cscK fibres. Applying a relative version of Ehresmann's fibration theorem, Y and X are isomorphic as smooth manifolds. The Chern classes $c_1(H_Y)$ and $c_1(H_X)$, viewed as cohomology classes on the underlying smooth manifold, are equal because they are integral classes. Thus we can take the perspective of fixing the underlying symplectic fibration and deforming

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the complex structure. We also fix the complex structure of the base and assume that the deformations keep the projection to B holomorphic.

A relatively symplectic form ω on $(Y, H_Y) \to (B, L)$ induces a splitting of the tangent bundle of Y into a vertical part, defined as the tangent bundle to the fibres, and a horizontal part, defined using orthogonality with respect to ω. In the language of symplectic geometry, ω is called a symplectic connection. A relatively Kähler form ω is an *optimal symplectic connection* on (Y, H_Y) if it satisfies the geometric PDE

$$p_E(\Delta_\nu(\Lambda_{\omega_B}(\gamma^* F_{\mathcal{H}})) + \Lambda_{\omega_B}\rho_{\mathcal{H}}) + \frac{\lambda}{2}\nu = 0. \tag{1.1}$$

In this expression $F_{\mathcal{H}}$ and $\rho_{\mathcal{H}}$ are curvature quantities which depend on ω, ν is a curvature quantity that depends on the infinitesimal change in the complex structure and $\lambda > 0$ is a constant. The map p_E is the projection onto the global sections of the real vector bundle $E \to B$ whose fibre over b is the vector space of holomorphy potentials with respect to the relatively cscK complex structure of X, defined in §2.2. Thus, the left-hand side is a smooth function on Y; in particular, the equation should be interpreted as a second-order elliptic PDE on the vector bundle E.

We construct the moduli space $\mathcal{M}$ by gluing local charts around fibrations that admit an optimal symplectic connection. If $(Y, H_Y) \to (B, L)$ is such a fibration, the local moduli space around Y is given as the quotient

$$W_Y / \mathrm{Aut}(\pi_Y),$$

where W_Y is the complex space of deformations of Y which also admit an optimal symplectic connection, and we quotient by the action of the discrete group $\mathrm{Aut}(\pi_Y)$ of relative automorphisms, which is finite.

The definition of W_Y essentially involves two steps. The first step, established in §3.1, consists of finding a locally closed analytic space which parametrises all small deformations of the complex structure of Y that admit a degeneration to a fibration with cscK fibres. To do so, we combine the theory of deformations of cscK manifolds and the theory of deformations of fibrations, developed by the author in [40]. More precisely, we consider a fibration $(Y, H_Y) \to (B, L)$ with an optimal symplectic connection and a degeneration to a fibration $(X, H_X) \to (B, L)$ with cscK fibres. A fibration version of Kuranishi's theorem allows us to parametrise the compatible vertical deformations of the complex structure of X with a complex space V_π of harmonic $(0,1)$-forms with values in the $(1,0)$-vertical tangent bundle. Working locally in B, we then establish that the deformations of Y which degenerate to a relatively cscK fibration form a locally closed analytic subvariety V_π^+ of V_π. We explicitly construct the relatively cscK degeneration using techniques from GIT; although we do not use it directly, our construction is related to the Byałinicki-Birula decomposition [3, 29]. This is the key new step in our construction, not present in other constructions of moduli spaces.

The second step, established in §3.2, consists of proving that, in a small open neighbourhood W_Y of the point associated to Y in V_π^+, all fibrations admit an optimal symplectic connection. The proof of this essentially relies on the implicit function theorem and employs the linearisation of the equation studied in [40] with respect to the complex structure. It is at this step that we use the assumption that the relative automorphism group, that is, the group of automorphisms which commute with the projection to the base, is discrete. Moreover, the optimal symplectic connection condition guarantees that the moduli space is Hausdorff, as the Hermite-Einstein condition does for the moduli space of simple vector bundles [20].

While our work concentrates on the analytic aspects of the moduli space of holomorphic submersions, it is natural to ask if it is possible to give an algebro-geometric construction based on stability. In fact, there exists a notion of stability of fibrations developed by Dervan, Sektnan [10], and Hallam [24], which comes from K-stability and is equivalent to slope stability when the fibration is the projectivisation of a vector bundle. An algebro-geometric construction would lead to the structure of a variety on the moduli space rather than the structure of a complex space, and would naturally allow singular fibres.

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Such a moduli space would parametrise certain stable fibrations which degenerate to a fibration whose general fibre is K-polystable. Moreover, the automorphism group of the general fibre of the degeneration should be fixed. When singular fibres are present, we expect that the stability of the fibration should be interpreted in terms of the existence of solutions to the PDE (1.1) on the smooth locus having a prescribed behaviour near the singularities.

Recently, Hashizume and Hattori [25] have constructed a moduli space of adiabatically K-stable fibrations over a curve where the generic fibre is Calabi-Yau, using algebro-geometric techniques. Adiabatic K-stability [26] is a condition defined on a fibration with K-polystable fibres over a twisted K-polystable base, and is related to Dervan-Sektnan's stability of fibrations and the existence of optimal symplectic connections. In the special case where the fibres are all smooth, our moduli space constitutes an alternative construction of Hashizume-Hattori's moduli space. However, our construction of the moduli space of optimal symplectic connections does not require any condition on the base of the submersion and allows the fibres to have automorphisms. For example, the projectivisation of a stable simple vector bundle over a K-unstable base is not adiabatically K-stable, yet it admits an optimal symplectic connection, thus it is represented by a point in our moduli space. This observation is based on a discussion with M. Hattori, whom we gratefully thank.

Outline

In Section 2 we give preliminary definitions and results on deformations of cscK manifolds and we describe the deformation theory of submersions with cscK fibres. We then explain briefly the theory of optimal symplectic connections. In Section 3 we construct the moduli space of holomorphic submersions admitting an optimal symplectic connection. Then in Section 4 we describe a Weil-Petersson type Kähler metric on the moduli space, along with a natural line bundle.

2. Preliminaries

This section recalls some basic definitions and properties of Kähler manifolds and introduces the theory of optimal symplectic connections. Let (M, L) be a compact Kähler manifold of complex dimension n with an ample line bundle L, and let $\omega \in c_1(L)$ be a Kähler form. The *scalar curvature* of ω is the function on M defined as the contraction of the Ricci curvature:

$$\mathrm{Scal}(\omega) = \Lambda_\omega \mathrm{Ric}(\omega),$$

where

$$\mathrm{Ric}(\omega) = -\frac{i}{2\pi}\partial\bar{\partial}\log\det(\omega).$$

We will be interested in Kähler metrics with constant scalar curvature, where the constant is a topological constant given by the intersection product

$$\widehat{S} = \frac{n\, c_1(M)\cdot c_1(L)^{n-1}}{c_1(L)^n}.$$

Let g_J be the Riemannian metric on M induced by J and ω. A smooth function h on M is called a *holomorphy potential* if the $(1,0)$-part of the Riemannian gradient of h, denoted $\nabla_g h$, is a holomorphic vector field. Consider the operator $\mathcal{D} : C^\infty(M, \mathbb{C}) \to \Omega^{0,1}(T^{1,0}M)$ defined by

$$\mathcal{D}(\varphi) = \bar{\partial}\nabla_g^{1,0}\varphi.$$

The *Lichnerowicz operator* is the operator $\mathcal{D}^*\mathcal{D}$, where the adjoint is defined with respect to the $L^2(g)$-inner product. The Lichnerowicz operator is a fourth-order elliptic operator, and its kernel, which by ellipticity is the kernel of $\mathcal{D}$, coincides with the space of holomorphy potentials on M.

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2.1. Deformation theory of cscK manifolds

In this section, we describe the deformations of the complex structure of a Kähler manifold with constant scalar curvature, following the notation of Székelyhidi [44, §3], [45, §6]. A similar description was also treated by Brönnle [5, Part 1], and the author [41].

Let (M, ω) be a symplectic manifold. Define the infinite-dimensional manifold

$$\mathcal{J} = \{J : TM \to TM \text{ almost complex structure compatible with } \omega\}.$$

We denote by $\mathcal{J}^{\text{int}}$ the complex subspace of *integrable* almost complex structures. The tangent space at a point $J \in \mathcal{J}$ can be identified with

$$T_J^{0,1}\mathcal{J} = \{\alpha \in \Omega^{0,1}(T^{1,0}M) | \omega(\alpha(u), v) + \omega(u, \alpha(x)) = 0\}.$$

Recall that a vector field ξ_h is Hamiltonian with respect to ω if there exists $h \in C^\infty(M, \mathbb{R})$ such that $\omega(\xi_h, \cdot) = -\mathrm{d}h$. On a Kähler manifold we can characterise a Hamiltonian vector field with Hamiltonian function h as $\xi_h = J\nabla^g(h)$, where $\nabla^g h$ is the Riemannian gradient of h.

Let $\mathcal{G}$ be the group of Hamiltonian symplectomorphisms of (M, ω). Then $\mathcal{G}$ acts on $\mathcal{J}$ by pull-back. The Lie algebra of $\mathcal{G}$ can be identified with the space of smooth functions on M with mean value zero, denoted $C_0^\infty(M)$. The following theorem is due to Fujiki [19] and Donaldson [13].

Theorem 2.1. *The action $\mathcal{G} \curvearrowright \mathcal{J}$ is Hamiltonian with moment map*

$$\begin{aligned} \mathcal{S} : \mathcal{J} &\longrightarrow \mathrm{Lie}(\mathcal{G})^* \\ J &\longmapsto \mathrm{Scal}(\omega, J) - \widehat{S}. \end{aligned} \tag{2.1}$$

Therefore, cscK metrics on M correspond to $J \in \mathcal{J}^{\text{int}}$ such that $\mu(J) = 0$. The function $\mathrm{Scal}(\omega, J) - \widehat{S}$ is the Hermitian scalar curvature if J is nonintegrable, which coincides with the Riemannian scalar curvature in the integrable case. We view it as an element in $C_0^\infty(M)^*$ by identifying $\mathrm{Lie}(\mathcal{G})$ with its dual.

Let J be an integrable compatible complex structure on (M, ω), and denote by $\mathcal{K}_J(\omega)$ the set of Kähler metrics in the same Kähler class of ω with respect to J, that is,

$$\mathcal{K}_J(\omega) = \{\omega' \in [\omega] \mid \omega' = \omega + i\partial_J\bar{\partial}_J\phi \text{ for some } \phi \in C^\infty(M, \mathbb{R})\}.$$

For fixed $J \in \mathcal{J}$, consider the infinitesimal action of $\mathcal{G}$ on $\mathcal{J}$, defined by the operator

$$\begin{aligned} P : C_0^\infty(M, \mathbb{R}) &\to T_J\mathcal{J} \\ h &\mapsto \mathcal{L}_{\xi_h} J, \end{aligned} \tag{2.2}$$

where ξ_h is the Hamiltonian vector field with function h; let $P^{\mathbb{C}}$ be its extension to $C_0^\infty(M, \mathbb{C})$. The following proposition (see also [45, §6.1]) justifies the fact that instead of moving the Kähler metric inside the Kähler class with respect to a fixed J, one can think of ω as fixed and move the complex structure.

Proposition 2.2 [14, p.17]. *For every $\omega_\phi \in \mathcal{K}_J(\omega)$ there exists an $f \in \mathrm{Diff}_0(M)$ such that $f^*\omega_\phi = \omega$ and (M, ω_ϕ, J) is isomorphic to (M, ω, f^*J). In particular, there is a map F that sends a Kähler potential ϕ to $F_\phi(J) = f^*J$. The differential at 0 of F is given by*

$$\mathrm{d}_0 F(\phi) = J\mathcal{L}_{\xi_\phi(\omega)} J.$$

Although a complexification of $\mathcal{G}$ does not genuinely exist, we can think of the image of the map F as an infinitesimal complexified orbit of the action of $\mathcal{G}$ on $\mathcal{J}$. The proposition says that a variation of

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the Kähler form in a given Kähler class for J fixed corresponds to a variation of the complex structure J in the same $\mathcal{G}^c$-orbit, for ω fixed.

The following result characterises P as the adjoint operator of the differential of the scalar curvature when varying the complex structure.

Lemma 2.3 [13, §4]. *Let Q be the differential of the scalar curvature map (2.1) at a point $J \in \mathcal{J}$. Then if $\alpha \in T_J\mathcal{J}$ and $\varphi \in C^\infty(M)$,*

$$\langle Q(\alpha), \varphi\rangle_{L^2} = \frac{1}{2}\langle \alpha, P(\varphi)\rangle_J$$

so $P^* = 2Q$, and

$$Q(P(\phi)) = \mathrm{Re}(\mathcal{D}^*\mathcal{D}\phi).$$

We next let (M, L) be a compact Kähler manifold with an ample line bundle L, and assume that $\omega \in c_1(L)$ has constant scalar curvature with respect to the complex structure J. The deformations of J compatible with ω can be considered as elements of the vector space

$$\widetilde{H}^1 = \ker\{\Box := PP^* + (\bar\partial^*\bar\partial)^2\} \tag{2.3}$$

on $T_J\mathcal{J}$. Let K be the stabilizer of J with respect to the action of $\mathcal{G}$ on $\mathcal{J}$. Then K is the group of Hamiltonian isometries of (ω, J). The complexification $K^{\mathbb{C}}$ is the connected component of the identity of the group of automorphisms of M which lift to L, denoted $\mathrm{Aut}_0(M, L)$. The group K (as well as its complexification) act on $\widetilde{H}^1$ by pull-back, and the origin is a fixed point of the action.

The following theorem is a symplectic version of Kuranishi's theorem [30] and an infinite-dimensional version of Luna's slice theorem [32]: it provides a parametrisation of compatible infinitesimal deformations of J. A proof that takes into account the compatibility with the symplectic structure is explained in [7, Lemma 6.1].

Theorem 2.4. *There exists a ball around the origin $V \subset \widetilde{H}^1$ and a K-equivariant embedding*

$$\Psi : V \to \mathcal{J} \tag{2.4}$$

such that $\Psi(0) = J$ and

1. *the $\mathcal{G}^c$-orbit of every integrable complex structure near J_0 intersects the image of Ψ;*
2. *If $x, x' \in V$ are in the same orbit for the complexified action of K, and $\Psi(x)$ is integrable, then $\Psi(x), \Psi(x')$ are in the same $\mathcal{G}^c$-orbit;*
3. *$\mathrm{Scal}(\omega, \Psi(x))$ is an element of the Lie algebra of K.*

In particular, a moment map for the action of K on V is given by

$$x \mapsto \mathrm{Scal}(\omega, \Psi(x)) - \widehat{S}. \tag{2.5}$$

The Kuranishi space is the base of a deformation family $\mathcal{M} \to V$ centred at J.

Definition 2.5. We say that the deformation family is *complete* if, for another deformation family $\mathcal{M}_1 \to V_1$, there exists a map

$$\tau : V_1 \to V$$

such that $\mathcal{M}_1 = \tau^*\mathcal{M}$. Moreover, if the differential of τ at the origin is unique we say that the deformation family is *versal*, and *universal* if τ itself is unique.

We remark that Kuranishi's theorem [30] also gives the existence of a versal deformation family centred at J which is complete for nearby complex structures.

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2.2. Fibrations

In this section, we collect some results on Kähler metrics on holomorphic submersions. Let $\pi_X : X \to B$ be a holomorphic submersion of compact complex manifolds. Let n be the dimension of B, m the dimension of the fibres, so $\dim(X) = n + m$. We also assume that the spaces $H^0(X_b, T^{1,0}X_b)$ of holomorphic vector fields on X_b are isomorphic as Lie algebras for all b.

Definition 2.6. A line bundle H_X on the total space X of the submersion π_X is said to be *relatively ample* if its restrictions to the fibres of π_X are ample. Similarly, a representative $\omega \in c_1(H_X)$ is called a *relatively Kähler metric* if its restrictions to the fibres of π_X are Kähler metrics.

Let ω be a relatively Kähler metric on X and ω_B a Kähler metric on B. We can define a Kähler metric on X by taking the relative Kähler metric ω and adding a large multiple of the pull-back of the base metric:

$$\omega_k := \omega + k\omega_B \qquad k \gg 0.$$

The relative symplectic form ω determines a splitting of the tangent space

$$TX = \mathcal{V} \oplus \mathcal{H}^\omega, \tag{2.6}$$

where $\mathcal{V}_x = T_x X_{\pi(x)}$ is the tangent space to the fibre X_x, and

$$\mathcal{H}^\omega_x = \{u \in T_x X | \omega(u, v) = 0 \ \forall v \in \mathcal{V}_x\}.$$

In the context of symplectic fibrations ω is called a *symplectic connection* [34, Chapter 6] and the splitting (2.6) induces a splitting on all tensor bundles.

In local computations, we make the following notational conventions:

$\{w^1, \dots, w^m\}$ denote vertical holomorphic coordinates; indices are denoted with the letters $a, b, c, \dots$;
$\{z^1, \dots, z^n\}$ denote holomorphic coordinates on the base; indices are denoted with the letters $i, j, k, \dots$;
$\{\zeta^1, \dots, \zeta^{n+m}\}$ denote holomorphic coordinates on the total space; indices are denoted with the letters $p, q, r, \dots$.

Lemma 2.7. *Let α be a covariant 2-tensor of type $(1, 1)$. Then*

$$\Lambda_{\omega_k}\alpha = \Lambda_{\mathcal{V}}\alpha + \frac{1}{k}\Lambda_{\omega_B}\alpha + O\left(k^{-2}\right),$$

where Λ_{ω_k} denotes the contraction with the Kähler metric ω_k, Λ_{ω_B} the contraction with the (pull-back of the) base metric ω_B and $\Lambda_{\mathcal{V}}$ the contraction with the vertical part of the relatively Kähler metric ω.

Proof. At each point $x \in X_b \subset X$, the matrix $[\omega_k]$ is block-diagonal:

$$\begin{pmatrix} [\omega_b] & 0 \\ 0 & [\omega_{k,\mathcal{H}}] \end{pmatrix}.$$

Therefore in local coordinates around the point x

$$(\omega_k)^{p\bar{q}}\alpha_{p\bar{q}} = (\omega_b)^{a\bar{b}}\alpha_{a\bar{b}} + (\omega_{k,\mathcal{H}})^{i\bar{j}}\alpha_{i\bar{j}}.$$

The horizontal part of ω_k, denoted $\omega_{k,\mathcal{H}}$ splits as $\omega_{\mathcal{H}} + k\omega_B$, where $\omega_{\mathcal{H}}$ is the horizontal part of ω. Let $[\omega_{k,\mathcal{H}}]$, $[\omega_{\mathcal{H}}]$ and $[\omega_B]$ be the matrices of coefficients of the two-forms $\omega_{k,\mathcal{H}}$, $\omega_{\mathcal{H}}$ and ω_B respectively.

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We can write

$$[\omega_{k,\mathcal{H}}] = k\left(\frac{[\omega_{\mathcal{H}}][\omega_B]^{-1}}{k} + \mathrm{Id}\right)[\omega_B],$$

where Id is the identity matrix and the base form ω_B induces a Riemannian metric on the horizontal tangent bundle, so its inverse is well defined. The inverse of the matrix $[\omega_{k,\mathcal{H}}]$ can be expanded in inverse powers of k as

$$\begin{aligned}[\omega_{k,\mathcal{H}}]^{-1} &= k^{-1}[\omega_B]^{-1}\left(\frac{[\omega_{\mathcal{H}}][\omega_B]^{-1}}{k} + \mathrm{Id}\right)^{-1} \\ &= k^{-1}[\omega_B]^{-1}\sum_{i=0}^{\infty}\left(-\frac{[\omega_{\mathcal{H}}][\omega_B]^{-1}}{k}\right)^{i} \\ &= \frac{1}{k}[\omega_B]^{-1} + O\left(k^{-2}\right).\end{aligned}$$

This implies the claim. □

Let us now restrict to the case when the fibres of X each admit a cscK metric. The following lemma explains that one can define a relatively Kähler metric on the total space which is relatively cscK.

Lemma 2.8 [9, Lemma 3.8]. *For any $b \in B$, let ω_b be a cscK metric on the fibre X_b in the class $c_1(H_X|_b)$. Then there exists a $\omega \in c_1(H_X)$ which is relatively cscK.*

Under the relatively cscK assumption, we now explain how the function space $C^\infty(X,\mathbb{R})$ splits in a way that takes into account the fibration structure. We begin by considering the vertical Lichnerowicz operator,

$$\mathcal{D}_{\mathcal{V}}^*\mathcal{D}_{\mathcal{V}} : C^\infty(X,\mathbb{R}) \to C^\infty(X,\mathbb{R}),$$

defined fibrewise as $\left(\mathcal{D}_{\mathcal{V}}^*\mathcal{D}_{\mathcal{V}}\varphi\right)|_{X_b} = \mathcal{D}_b^*\mathcal{D}_b\varphi|_{X_b}$. It is a real operator since the fibrewise metric is cscK. By integrating a function $\varphi \in C^\infty(X,\mathbb{R})$ over the fibres of π, we define a projection

$$\begin{aligned} C^\infty(X,\mathbb{R}) &\longrightarrow C^\infty(B,\mathbb{R}) \\ \varphi &\longmapsto \int_{X/B}\varphi\omega^m.\end{aligned}$$

Its kernel is given by the space $C_0^\infty(X,\mathbb{R})$ of functions that have fibrewise mean value zero. A key step in the study of optimal symplectic connections is that we can further split this space as follows.

Consider the real vector bundle $E \to B$ [9, §3.1], whose fibre over $b \in B$ is the real finite-dimensional vector space $\ker_0(\mathcal{D}_b^*\mathcal{D}_b)$ of holomorphy potentials on the fibre X_b with mean-value zero with respect to ω_b. It is well defined as a vector bundle since we assume that the complex dimension of the space $H^0(X_b, T^{1,0}X_b)$ of holomorphic vector fields on X_b is independent of b [24, Proposition 2.16]. The space of smooth global sections of E, denoted by $C^\infty(E)$, is given by the kernel over the fibrewise mean-value-zero functions of the vertical Lichnerowicz operator $\mathcal{D}_{\mathcal{V}}^*\mathcal{D}_{\mathcal{V}}$.

Remark 2.9. In [24, Lemma 2.7], Hallam used the Cartan decomposition for the space $\mathfrak{h}(X_b)$ of holomorphic vector fields of the fibre to show that E_b can be also viewed as the vector space of all Kähler potentials φ_b on X_b of mean-value zero for which $\omega_b + i\partial\bar\partial\varphi_b$ is still cscK.

We can split $C_0^\infty(X)$ as

$$C_0^\infty(X,\mathbb{R}) = C^\infty(E) \oplus C^\infty(R),$$

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where $C^\infty(R)$ is the fibrewise L^2-orthogonal complement with respect to the fibre metric ω_b, that is, for all $\varphi \in \ker_0 \mathcal{D}_b^* \mathcal{D}_b$, $\psi \in C^\infty(R)$

$$\langle \varphi, \psi \rangle_b := \int_{X_b} \varphi \psi \omega_b^m = 0.$$

So we obtain

$$C^\infty(X, \mathbb{R}) = C^\infty(B) \oplus C^\infty(E) \oplus C^\infty(R). \tag{2.7}$$

We denote by $p_E : C^\infty(X) \to C^\infty(E)$ the projection.

Definition 2.10. We denote by $\mathcal{K}_E$ the space of functions $\varphi \in C^\infty(X)$ such that $\omega + i\partial\bar{\partial}\varphi$ is still a *fibrewise cscK metric*.

In particular, if we change the relatively cscK metric ω to $\omega + i\partial\bar{\partial}\varphi$ with $\varphi \in \mathcal{K}_E$, the vector bundles $E(\omega)$ and $E(\omega + i\partial\bar{\partial}\varphi)$ are isomorphic. The following result [11, Lemma 4.20] relates the space $\mathcal{K}_E$ to the vector bundle $E \to B$.

Lemma 2.11. *Let $\varphi_t : [0,1] \to \mathcal{K}_E$ be a smooth path. Then for all t*

$$\dot{\varphi}_t \in C^\infty(B) \oplus C^\infty(E(\omega + i\partial\bar{\partial}\varphi_t))$$

that is to say that, up to a function on the base, $\dot{\varphi}_t$ is a fibrewise holomorphy potential with respect to $\omega + i\partial\bar{\partial}\varphi_t$.

2.3. Optimal symplectic connections

Let $(Y, H_Y) \to (B, L)$ be a holomorphic submersion with a relatively ample line bundle $H_Y \to Y$. In this section we recall the theory of optimal symplectic connections [40]. The following assumptions restrict the class of admissible fibrations to those whose fibres satisfy a stability property defined in terms of K-stability. More precisely we assume that:

(i) the fibres Y_b are analytically K-semistable, which means by definition that they each admit a degeneration to a cscK manifold X_b. We also assume that the deformation is compatible with the fibration structure in the following sense: there exists a holomorphic map $\varpi : (\mathcal{X}, \mathcal{H}) \to (B, L) \times S$, parametrised by a disk S, such that for $s \neq 0$, the family $(\mathcal{X}_s, \mathcal{H}_s) \to B$ is isomorphic to the original fibration $\pi_Y : (Y, H_Y) \to B$ and the central fibration at $s = 0$ is a family $\pi_X : (X, H_X) \to B$ whose fibres are cscK;
(ii) the automorphism groups $\mathrm{Aut}_0(X_b, H_b)$ of the fibres are all isomorphic.

The first hypothesis is a stability assumption. We will refer to the submersion $X \to B$ as the *relatively cscK degeneration* of $Y \to B$. The second hypothesis holds if and only if the spaces $H^0(X_b, T^{1,0}X_b)$ are isomorphic as Lie algebras, which we assumed in §2.2 to define the vector bundle $E \to B$. It is needed to define optimal symplectic connection and it is a key hypothesis for the construction of the moduli space.

A relative version of Ehresmann's theorem [40, Proposition 4.5] implies that X and Y are diffeomorphic. Let M denote the underlying smooth manifold. Since the Chern classes are integral classes we have that $c_1(H_X)$ coincides with $c_1(H_Y)$ as cohomology classes on M. Since Y is a small deformation of X, the cohomology class $c_1(H_X)$ is of type $(1,1)$ on Y, so ω is a $(1,1)$-form with respect to the complex structure of Y [28, §6.1]. By Moser's theorem [6, Theorem 7.2], we can modify the complex structure of Y by a small diffeomorphism so that ω restricted to the fibres of π_Y is compatible with the restriction of the complex structure. Thus we can assume that ω is relatively Kähler on Y.

Therefore we can view $Y \to B$ and $X \to B$ as the same relatively symplectic fibration $(M, \omega) \to B$ with two different integrable almost complex structures J and I where (ω, I) is relatively cscK and

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(ω, J) is just relatively Kähler. The family $\mathcal{X} \to B \times S$ corresponds to a family of complex structures $\{J_s\}$ on $(M, \omega) \to B$, such that for $s \neq 0$, J_s is isomorphic to J and J_0 is isomorphic to I.

The definition of optimal symplectic connections involves various curvature quantities and a description of the complex structure of Y as a deformation of the one on X. To describe the deformations of the complex structure I, we begin by considering the space $\mathcal{J}_\pi$ of almost complex structures compatible with ω and such that $\mathrm{d}\pi \circ J = J_B \circ \mathrm{d}\pi$. The tangent space at I to $\mathcal{J}_\pi$ can be identified with

$$T_I^{0,1}\mathcal{J}_\pi = \{\alpha \in \Omega^{0,1}(\mathcal{V}^{1,0}) | \omega_F(\alpha\cdot, \cdot) + \omega_F(\cdot, \alpha\cdot) = 0\},$$

where ω_F is the vertical part of ω. In local holomorphic coordinates an element $\alpha \in T_J^{0,1}\mathcal{J}_\pi$ is written as

$$\alpha = \alpha^a_{\bar{b}} \partial_{w^a} \otimes \mathrm{d}\bar{w}^b + \alpha^a_{\bar{j}} \partial_{w^a} \otimes \mathrm{d}\bar{z}^j,$$

where [40, Lemma 4.1]

$$\alpha^a_{\bar{j}} = \alpha^a_{\bar{c}} (\omega_F)^{d\bar{c}} (\omega)_{d\bar{j}}. \tag{2.8}$$

Consider the map

$$\begin{aligned} P_{\mathcal{V}} : C_0^\infty(X, \mathbb{R}) &\longrightarrow T_I^{0,1}\mathcal{J}_\pi \\ \varphi &\longmapsto \bar{\partial}(\mathrm{grad}^{\omega_F}\varphi)^{1,0}, \end{aligned}$$

which is the relative version of the map (2.2). Let $\widetilde{H}^1_{\mathcal{V}}$ be the kernel of the elliptic operator

$$\Box_{\mathcal{V}} = P_{\mathcal{V}} P^*_{\mathcal{V}} + (\bar{\partial}^* \bar{\partial})^2,$$

where the adjoint is computed with respect to any Kähler metric on X which restricts to ω_F vertically. The operator $\Box_{\mathcal{V}}$ is fibrewise elliptic because $P_{\mathcal{V}} P^*_{\mathcal{V}}$ is elliptic in the fibre direction, and globally elliptic because $(\bar{\partial}^* \bar{\partial})^2$ is. The space $\widetilde{H}^1_{\mathcal{V}}$ is the space of integrable first-order deformations of I. Let K_π be the group of biholomorphisms of I which restrict to an isometry on each fibre, with respect to the fibrewise metric defined by (ω, I). The group K_π acts on $\widetilde{H}^1_{\mathcal{V}}$ by pull-back. The following theorem is a fibrewise version of Theorem 2.4.

Theorem 2.12 [40, Theorem 4.7]. *There exists a neighbourhood of the origin $V_\pi \subset \widetilde{H}^1_{\mathcal{V}}$ and a K_π-equivariant map*

$$\Phi : V_\pi \to \mathcal{J}_\pi \tag{2.9}$$

such that:

1. $\Phi(0) = I$;
2. *If $v_1, v_2 \in V_\pi$ and $v_1 \in K_b^{\mathbb{C}} \cdot v_2$, and if $\Phi(v_1)$ is integrable, then $\Phi(v_1)$ and $\Phi(v_2)$ are isomorphic and for all $b \in B$ the complex structure $\Phi(v_1)|_{X_b}$ is in the same $\mathcal{G}_b^c$-orbit as $\Phi(v_2)|_{X_b}$;*
3. *For any $J \in \mathcal{J}_\pi$ integrable close to I, there exists J' in the image of Φ such that, for all b, J'_b is in the same $\mathcal{G}_b^c$-orbit as J_b;*
4. $\mathrm{Scal}_{\mathcal{V}}(\omega_X, \Phi(x)) \in C^\infty(E, I)$.

We now describe the curvature quantities involved in the definition of optimal symplectic connections. The symplectic connection ω determines the following curvature quantities:

(i) the *symplectic curvature* is a two-form on B with values in the fibrewise Hamiltonian vector fields defined for $v_1, v_2 \in \mathfrak{X}(B)$ as

$$F_{\mathcal{H}}(u_1, u_2) = [u_1^\sharp, u_2^\sharp]^{\mathrm{vert}},$$

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where $u_j^\sharp$ denotes the horizontal lift. Let γ be the map which associates to a fibrewise Hamiltonian vector field its fibrewise Hamiltonian function with fibrewise mean value zero. Thus we consider $\gamma^*(F_{\mathcal{H}})$, which is a two-form on B with values in $C_0^\infty(Y,\mathbb{R})$, and we pull it back to Y;

(ii) the curvature ρ of the Hermitian connection induced by (ω, I) on the top wedge power $\bigwedge^m \mathcal{V}$. We will primarily consider its purely horizontal part $\rho_{\mathcal{H}}$;

(iii) the curvature of the deformation family ν, which is defined as follows. On each fibre X_b of the relatively cscK degeneration $X \to B$, let $\widetilde{H}_b^1$ be the deformation space (2.3) and V_b the Kuranishi space described in Theorem 2.4. Let $v_b \in \widetilde{H}_b^1$ and denote by $\mathbf{v}_b$ a vector field on V_b such that $\mathbf{v}_b|_0 = v_b$. For any $f \in \mathfrak{k}_b$ let

$$A_f v_b := -(\mathcal{L}_{\mathrm{grad}^\omega f}\, \mathbf{v}_b)|_0.$$

Define ν_b as

$$\langle \nu_b(v_b), f\rangle = \frac{1}{2}\Omega_0(A_f v_b, v_b).$$

In particular, ν_b is the moment map for the action of the group K_b on the tangent space at the origin to V_b. Therefore we can define a global section ν of the vector bundle $E \to B$ of relatively holomorphy potentials of $X \to B$, as

$$\nu(b) = \nu_b(v_b). \tag{2.10}$$

Given $x \in V_\pi$, $v \in \widetilde{H}_{\mathcal{V}}^1$, we will write $\nu_x(v)$ when we want to underline the dependence of the map ν on the complex structure $\Phi(x)$ and on the deformation v.

Definition 2.13 [40, Definition 3.3]. A relatively Kähler metric ω on $Y \to B$ is called an *optimal symplectic connection* if

$$p_E\big(\Delta_{\mathcal{V}}(\Lambda_{\omega_B}\gamma^*(F_{\mathcal{H}})) + \Lambda_{\omega_B}\rho_{\mathcal{H}}\big) + \frac{\lambda}{2}\nu = 0, \tag{2.11}$$

for a positive number λ. In the following, we will use the notation $\Theta(\omega, J) = \Delta_{\mathcal{V}}(\Lambda_{\omega_B}\gamma^*(F_{\mathcal{H}})) + \Lambda_{\omega_B}\rho_{\mathcal{H}}$.

Equation (2.11) is a second-order elliptic equation on the vector bundle $E \to B$. It arises by expanding the scalar curvature of the Kähler metric (ω_k, J_s) in inverse powers of k and powers of s and then relating the two parameters to obtain a single expansion. The left-hand side of (2.11) is then the projection onto $C^\infty(E(\omega, I))$ of the sub-leading order term of such an expansion. The equation with $\nu = 0$ is the condition for an optimal symplectic connection in the sense of [9], where all the fibres are required to be cscK.

The linearisation of the equation at a solution is given by the operator [40, §5.3]

$$\widehat{\mathcal{L}} = \mathcal{R}^*\mathcal{R} + \mathcal{A}^*\mathcal{A}$$

on $C^\infty(E)$, where

$$\mathcal{R}(\varphi_E) = \bar\partial_B \nabla_{\mathcal{V}}^{1,0}\varphi_E \tag{2.12}$$

and

$$\mathcal{A}(\varphi_E) = \mathrm{d}_0\Phi\big(A_{\varphi_E} v\big). \tag{2.13}$$

The adjoint is computed with respect to $\omega_F + \omega_B$. Here $\nabla_{\mathcal{V}}^{1,0}\varphi_E$ is a section of the holomorphic tangent bundle; the vertical part of $\bar\partial\nabla_{\mathcal{V}}^{1,0}\varphi_E$ vanishes since $\varphi_E \in C^\infty(E)$ and the horizontal part is denoted by the expression (2.12). The operator (2.12) can be described as follows [9, §4.3]: let $\mathcal{D}_k^*\mathcal{D}_k$ be the

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Lichnerowicz operator with respect to the Kähler metric ω_k. It admits a power series expansion in negative powers of k:

$$\mathcal{D}_k^*\mathcal{D}_k = \mathcal{L}_0 + k^{-1}\mathcal{L}_1 + O\left(k^{-2}\right),$$

where $\mathcal{L}_0$ is the *vertical* Lichnerowicz operator $\mathcal{D}_{\mathcal{V}}^*\mathcal{D}_{\mathcal{V}}$. Then for φ, ψ fibrewise holomorphy potentials

$$\int_X \varphi\mathcal{L}_1(\psi)\omega^m \wedge \omega_B^n = \int_X \langle \mathcal{R}\varphi, \mathcal{R}\psi\rangle_{\omega_F+\omega_B}\omega^m \wedge \omega_B^n.$$

This means that the operator $\mathcal{R}^*\mathcal{R}$ can actually be seen as $p_E \circ \mathcal{L}_1$ restricted to $\mathcal{C}_E^\infty(X)$. The kernel of $\mathcal{R}$, thus of $\mathcal{R}^*\mathcal{R}$, consists of fibrewise holomorphy potentials which are global holomorphy potentials on X with respect to ω_k, and it is independent of k. The operator (2.13) is described as

$$\langle \mathrm{d}_v\nu(\mathcal{L}_{\nabla_{\mathcal{V}}\varphi}\nu), \psi\rangle = \int_X \langle \mathrm{d}_0\Phi\big(A_\varphi\nu\big), \mathrm{d}_0\Phi\big(A_\psi\nu\big)\rangle_{\omega_F}\omega_F^m \wedge \omega_B^n, \tag{2.14}$$

where $\varphi, \psi \in C^\infty(E)$. A function $\psi \in C^\infty(E)$ is in the kernel of A if and only if ψ is a fibrewise holomorphy potential with respect to all J_s, that is, $\psi \in C^\infty(E, J_s)$. Thus the kernel of $\widehat{\mathcal{L}}$ consists of those functions $\psi \in C^\infty(E, J_0)$ such that $\bar\partial_s(\nabla^{1,0}_{s,\mathcal{V}}\psi) = 0$ for all s.

Remark 2.14. Although we are considering only integral Kähler classes, this is not essential. Indeed, we expect that all our results should hold if $c_1(H_X)$ and $c_1(L)$ are replaced respectively by a relative Kähler class and a Kähler class that do not come from holomorphic line bundles. For details, see [39, Remark 3.2].

3. Construction of the moduli space

Consider a submersion $Y \to B$ that degenerates to a relatively cscK holomorphic submersion $X \to B$, as described in §2.2. We prove that the set of deformations of $Y \to B$ that still degenerate to a relatively cscK fibration forms a locally closed subvariety of the relative Kuranishi space. Throughout, by subvariety we mean complex analytic subvariety. The relatively cscK degeneration of a deformation might differ from $X \to B$, but the group of automorphisms of the fibre is fixed and should be considered a fixed datum in the construction of the moduli space. We then prove that, when the relative automorphism group is discrete, the solutions to the optimal symplectic connection equation (2.11) form an open set inside the locally closed subset of admissible deformations. This allows us to define a local moduli space of optimal symplectic connections. Finally, we glue the local moduli spaces and we prove that we obtain a global Hausdorff complex analytic space $\mathcal{M}$ which parametrises optimal symplectic connections.

3.1. Openness of the setting

Given a fibration $\pi_Y : (Y, H_Y) \to (B, L)$ with analytically K-semistable fibres, we have introduced in (i) the assumption that there exists a degeneration of π_Y to a fibration $\pi_X : (X, H_X) \to (B, L)$ such that the fibres of π_X are cscK. We have explained in §2.2 that we can consider $Y \to B$ and $X \to B$ as the same symplectic fibration $\pi : (M, \omega) \to B$, and the degeneration as a deformation of a complex structure J to I, where (ω, I) has fibrewise constant scalar curvature. The goal of this section is to understand for which deformations J' of J we can still find a relatively cscK degeneration X', and to construct such a degeneration.

We begin by working locally in B. Let $\mathcal{U} \subseteq B$ be a coordinate open subset of B. The fibre over the origin of $\mathcal{U}$, denoted M_0, has a constant scalar curvature metric (ω_0, I_0). Let K be the group of Hamiltonian isometries of (ω_0, I_0) and let $\widetilde{H}^1_0$ be the vector space (2.3) parametrising first-order deformations of

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(M_0, ω_0, I_0). Since (ω_0, I_0) has constant scalar curvature, the complexification of K is the group

$$G = \mathrm{Aut}_0(M_0, H_0).$$

We will employ the following definition of GIT-stability for the action of a reductive group on an affine space [38, §3] [27, §4.5, §4.6].

Definition 3.1. Let $\mathbb{A}^d$ be an affine space of dimension d and G a reductive affine group acting on it. Let $(z_1, \ldots, z_d)$ be a system of coordinates on $\mathbb{A}^d$. The space $\mathbb{A}^d$ can be embedded in the projective space $\mathbb{P}^d$ as a coordinate chart with the map

$$(z_1, \ldots, z_d) \mapsto [1 : z_1 : \ldots : z_d].$$

We extend the action of G to an action on $\mathbb{P}^d$ by acting trivially on the first coordinate. Let $x \in \mathbb{A}^d$ and let $\widehat{x}$ be its image in $\mathbb{P}^d$. We say that x is

(i) *semistable* if there exists a nonconstant homogeneous polynomial in $[z_0 : \ldots : z_d]$ on $\mathbb{P}^d$ which is G-invariant and does not vanish at $\widehat{x}$;
(ii) *polystable* if it is semistable and its orbit is closed;
(iii) *stable* if it is semistable, its orbit is closed and its stabiliser is finite.

In particular, the polynomial $P(\mathbf{z}) = z_0$ is a G-invariant homogeneous polynomial that does not vanish at any point of $\mathbb{A}^d$. So every point of $\mathbb{A}^d$ is semistable.

Remark 3.2. The fact that every point is semistable holds because we defined the action on $\mathbb{P}^d$ to be trivial on the first homogenous coordinate. In principle, one can choose to extend the action in a nontrivial way and can still define stability as above, but unstable points might appear. We do not treat this case here, so we have included the trivial extension of the action in the definition of stability.

We apply the definition of stability to the vector space $\widetilde{H}^1_0$. The complex structure I_0 corresponds to the origin in the Kuranishi space V_0, which is fixed by the action of G. Therefore its orbit is closed and its stabiliser is the group G itself, so it is a polystable point. A key result for our construction is the following.

Lemma 3.3 [27, Corollary 4.31]. *The closure of the orbit of every point in $\widetilde{H}^1_0$ contains a unique polystable orbit.*

Let V_0 be the subspace of the Kuranishi space which parametrises integrable almost complex structures; it is a locally closed analytic subspace of $\widetilde{H}^1_0$ because it is defined by the vanishing of the Nijenhuis tensor. Thus the family $X \to B$ can be described locally over $\mathcal{U}$ as a family $\{x_b\}$ in V_0. The family $Y \to B$ can be described locally over $\mathcal{U}$ as a family $\{y_b \mid b \in \mathcal{U}\}$ of points such that for each b the closure of the G-orbit of y_b contains the polystable point x_b [7, Theorem 1.3].

The key step in the construction is the following: by our hypothesis ii, the automorphism group of (M_b, H_b) is isomorphic to G for all $b \in B$. Therefore the points $\{x_b \mid b \in \mathcal{U}\}$ are all fixed by the action of G and are hence polystable. By Lemma 3.3, x_b is the only polystable orbit in the closure of the orbit of y_b. Let V_0^+ be the set of all semistable points that have a fixed point in the closure of their orbit, it follows that the map

$$F : V_0^+ \to V_0 \tag{3.1}$$

that maps a semistable point to the corresponding fixed point is well-defined. Note that the action of G on V_0^+ is effective, since by definition if $x \in V_0^+$, then the closure of the orbit of x is in V_0^+.

Lemma 3.4. *The set V_0^+ is an analytic subvariety of V_0 and the map (3.1) is holomorphic.*

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Proof. The space V_0 is an open subset of the vector space $\widetilde{H}^1$. Let d be the dimension of $\widetilde{H}^1$, so each point $z \in V_0$ has coordinates

$$(z_1, \dots, z_d).$$

Let us begin with the case when G is isomorphic to a complex torus $(\mathbb{C}^*)^r$. The fixed points of the action can be described by the vanishing of the coordinates

$$z_{i_1} = \cdots = z_{i_h} = 0 \quad \text{for} \quad i_1, \dots, i_h \in \{1, \dots, d\},$$

thus they form an analytic subspace of V_0.

By the Hilbert-Mumford criterion [36, Theorem 2.1], a point y_j is semistable if it is semistable for the action of every 1-parameter subgroup of the group G. Let $\rho(t) : \mathbb{C}^* \hookrightarrow G$ be a 1-parameter subgroup. The action of $\rho(t)$ can be written as $(t^{a_1}, \dots, t^{a_d})$, where the numbers a_j are the *weights* of the action. Then V_0 splits into a sum of weight spaces $V_0^{\text{pos}} \oplus V_0^{\text{fix}} \oplus V_0^{\text{neg}}$, where $\rho(t)$ acts on V_0^{pos} with positive weights, on V_0^{neg} with negative weights and fixes the subspace V_0^{fix}. A semistable point that has a fixed point in the closure of its orbit is described by the following condition: if a coordinate in V_0^{pos} is nonzero, then all coordinates in V_0^{neg} vanish. Applying this condition to all 1-parameter subgroups yields a set of polynomial equations that define the semistable points in V_0^+. Thus, the semistable points correspond to an analytic subset of V_0. The map F is the projection onto the set described by $\{z_{i_1} = \dots = z_{i_h} = 0\}$, thus it is holomorphic.

Let now G be any reductive group, and let y be a semistable point and x be a polystable and fixed point in the closure of its orbit. The fixed points of G form a vector subspace V_0^{fix} also in this case. Moreover there exists a 1-parameter subgroup $\lambda_x : \mathbb{C}^* \hookrightarrow G$ such that

$$\lim_{t \to 0} \lambda_x(t) \cdot y = x.$$

It follows that the map $y \mapsto x$ is the projection onto the vector subspace of fixed points for λ_x, hence it is holomorphic. Consider the composite map

$$V_0 \to V_0 \xrightarrow{pr} V_0^{\text{fix}} \tag{3.2}$$

where the first map is the projection onto the vector subspace of fixed points for λ_x and the second map is the projection onto the subspace of fixed points for the whole group G. The map is holomorphic because it is a composition of holomorphic projections. We prove that it coincides with the map (3.1). Let $\widetilde{x}' = \lim_{t \to 0} \lambda_x(t) \cdot y'$. Then

$$\overline{G \cdot \widetilde{x}'} \subseteq \overline{G \cdot y'}.$$

The unique polystable orbit contained in $\overline{G \cdot \widetilde{x}'}$ is also a polystable orbit in $\overline{G \cdot y'}$, so it must coincide with the fixed-point orbit $\{x'\}$. Thus flowing along the orbit of $\widetilde{x}'$ amounts to projecting onto the subspace of fixed points, and so the map (3.2) maps any point y' to the fixed point x' in the closure of its orbit. □

Remark 3.5. The map (3.1) is analogous to the one given by the Byałinicki-Birula decomposition [3, 29].

Let V_π be the Kuranishi space of the fibration $X \to B$ defined in Theorem 2.12. Consider the subspace

$$V_\pi^+ := \{y \in V_\pi \mid y|_{X_b} \in V_b^+\}.$$

We remark that V_π^+ depends on the complex structure of the reference fibration $X \to B$ and its deformation $Y \to B$. We denote by $\mathcal{J}_\pi^+$ the image of V_π^+ via the relative Kuranishi map (2.9).

Lemma 3.6. *V_π^+ is a locally closed complex analytic subvariety of V_π.*

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Proof. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two open coordinate subsets of B with nonempty intersection and let 0_1 be the origin of $\mathcal{U}_1$ and 0_2 be the origin of $\mathcal{U}_2$. Denote by I_1 and I_2 the complex structures of the fibres X_{0_1} and X_{0_2}, and G_1, G_2 their groups of automorphisms. By assumption, G_1 and G_2 are isomorphic, and we will denote them by G. Recall that the Kuranishi space is *versal* and more specifically that it is a complete deformation space for the nearby fibres. We use the versality of the Kuranishi map to glue the spaces V_1^+ and V_2^+ constructed in Lemma 3.4 to a variety V^+ and prove that V_π^+ is obtained as the intersection of said variety with the relative Kuranishi space V_π. More precisely, versality of the Kuranishi space means that there is a map

$$\tau_{21} : V_2 \to V_1,$$

not necessarily unique.

The map τ_{21} can be taken to be G-equivariant. In fact, the G-equivariance can be traced back to the proof of Kuranishi's Theorem 2.4. The map τ_{21} is defined using the implicit function theorem, which can be applied to a K-equivariant map to provide an implicit inverse function which is K-equivariant. Since G is the complexification of K, we obtain that τ_{21} is G-equivariant. The equivariance also implies that the image of V_2^+ is V_1^+, so we can restrict τ_{21} to

$$\widetilde{\tau}_{21} : V_2^+ \to V_1^+.$$

Moreover, the map $\widetilde{\tau}_{21}$ has an inverse that is constructed by reversing the roles of V_1 and V_2, so it is an isomorphism. In fact, although the map τ_{21} is not canonical, the restriction to $\widetilde{\tau}_{21}$ is fixed by the reference K-semistable fibration $Y \to B$. Each Kuranishi space V_b is a complex subspace of the vector space $\widetilde{H}_b^1$, described as the kernel of the elliptic operator $P_b P_b^* + (\bar{\partial}_b^* \bar{\partial}_b)^2$ (2.3). So we can use the isomorphism $\widetilde{\tau}_{21}$ to glue the spaces V_b^+ to a subvariety V^+ of the kernel of the fibrewise elliptic operator

$$P_{\mathcal{V}} P_{\mathcal{V}}^* + (\bar{\partial}_{\mathcal{V}}^* \bar{\partial}_{\mathcal{V}})^2.$$

Therefore the intersection

$$V_\pi^+ = V^+ \cap V_\pi$$

is a locally closed subvariety of V_π. □

The following lemma shows that we can glue the local fibration constructed in Lemma 3.4 to a global fibration over B.

Lemma 3.7. *Let $Y' = (M, \omega, J') \to B$ be a fibration with complex structure J' represented by $y' \in V_\pi^+$. Then Y' degenerates to $X' = (M, \omega, I') \to B$ such that*

1. *(ω, I') is relatively cscK;*
2. *the groups $\mathrm{Aut}(X_b', H_b')$ are isomorphic for all $b \in B$.*

Proof. Let $\mathcal{U} \subseteq B$ be an open coordinate subset. By the relative Kuranishi Theorem 2.4, for each $b \in \mathcal{U}$ there exists a point $y_b' \in V_0$ such that $\Phi_0(y_b')$ is in the same $\mathcal{G}^c$-orbit of J_b'. The fact that the map (3.1) is holomorphic implies that we can find polystable points $\{x_b'\}$ such that $\{\Phi_0(x_b')\}$ are a holomorphic family of cscK complex structures over $\mathcal{U}$ that are deformations of $\{J_b'\}$. Then we can construct a local relatively cscK fibration from the pullback diagram

$$\begin{array}{ccc} X_{\mathcal{U}}' & \longrightarrow & \mathcal{M}_{\mathcal{U}} \\ \downarrow & & \downarrow p_{\mathcal{U}} \\ \mathcal{U} & \xrightarrow[i]{} & V_0 \end{array} \tag{3.3}$$

where $p_{\mathcal{U}} : \mathcal{M}_{\mathcal{U}} \to V_0$ is Kuranishi's versal family and $i(b) = x_b'$.

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Now we glue the local fibrations to a fibration $X' \to B$. Let $\mathcal{U}_1, \mathcal{U}_2$ and G be as in Lemma 3.6. The associated Kuranishi maps are denoted respectively by $\Phi_1 : V_1 \to \mathcal{J}$ and $\Phi_2 : V_2 \to \mathcal{J}$.

For $b \in \mathcal{U}_1 \cap \mathcal{U}_2$, consider the complex structure J'_b. Since J'_b can be regarded as a deformation of both I_1 and I_2, there exist points $y'_{b,1} \in V_1$ and $y'_{b,2} \in V_2$ such that

$$\Phi_1(y'_{b,1}) = J'_b = \Phi_2(y'_{b,2}).$$

The diagram (3.3) produces two fibrations $X'_1 \to \mathcal{U}_1$ and $X'_2 \to \mathcal{U}_2$. In order to glue the local fibrations, we need to prove that there is an isomorphism

$$\phi_{12} : X'_1|_{\mathcal{U}_1 \cap \mathcal{U}_2} \xrightarrow{\sim} X'_2|_{\mathcal{U}_1 \cap \mathcal{U}_2}$$

and that it satisfies the cocycle condition

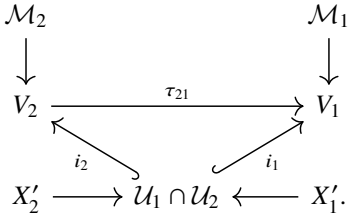
$$\phi_{23} \circ \phi_{12} = \phi_{13}$$

on a triple intersection $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ of open subsets of B. In particular, the first condition produces a global compact complex manifold X', while the cocycle condition implies that X' admits a submersion onto B.

To prove the existence of the isomorphism ϕ_{12} we use again the fact that the Kuranishi space induces complete deformations on the nearby fibres. The map

$$\tau_{21} : V_2 \to V_1$$

is such that $\mathcal{M}_2 = \tau_{21}^* \mathcal{M}_1$. In particular, we have the following diagram

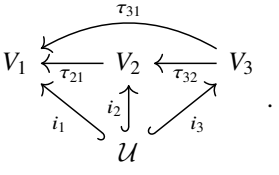


If we show that the diagram is commutative, that is, $\tau_{21} \circ i_2 = i_1$, then

$$X'_2|_{\mathcal{U}_1 \cap \mathcal{U}_2} = i_2^* \mathcal{M}_2 \simeq (\tau_{21} \circ i_2)^* \mathcal{M}_1 = i_1^* \mathcal{M}_1 = X'_1|_{\mathcal{U}_1 \cap \mathcal{U}_2}.$$

The commutativity follows from the fact that the map τ_{21} is G-equivariant. Indeed the points $i_1(b)$ and $i_2(b)$ are defined as the fixed-point limits of semistable orbits. Moreover, an equivariant map between two spaces on which there is an action of the same group G sends fixed points to fixed points and the closure of the orbit of $i_1(b)$ to the closure of the orbit of $i_2(b)$.

We now show the cocycle condition. Let $\mathcal{U}$ be a triple intersection $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ and consider the following diagram



On V_3 we have two families pulled-back from V_1, namely $\tau_{31}^* \mathcal{M}_1$ and $(\tau_{21} \circ \tau_{32})^* \mathcal{M}_1$. They induce two distinct families on $\mathcal{U}$, pulled-back using i_3. Although in general it is *not* true that τ_{31} is equal to the

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composition $\tau_{21} \circ \tau_{32}$, the commutativity of the arrows proved above implies that

$$\tau_{21} \circ \tau_{32} \circ i_3 = \tau_{31} \circ i_3.$$

Therefore the two families $\tau_{31}^* \mathcal{M}_1$ and $(\tau_{21} \circ \tau_{32})^* \mathcal{M}_1$ coincide. □

3.2. Openness of the space of optimal symplectic connections

Let $Y \to B$ be a fibration with K-semistable fibres, and assume that it degenerates to a fibration $X \to B$ with cscK fibres, in the sense of §2.2. Let $Y' \to B$ be a deformation of $Y \to B$ in $\mathcal{J}_\pi^+$. Then $Y' \to B$ admits a degeneration to $X' \to B$, whose fibres are cscK, as explained in §3.1. The goal of this section is to show that if Y admits an optimal symplectic connection, then Y' also does.

We denote by I the complex structure of X, by J the complex structure of Y, and we assume that $Y \to B$ is represented by $v_0 \in V_\pi$. We also assume that (ω, J) is an optimal symplectic connection. As before, let V_π^+ be the subvariety of V_π which describes the family of complex structures $\mathcal{J}_\pi^+$.

Proposition 3.8. *For every $\varphi \in \mathcal{K}_E(I)$ there exists $f \in \mathrm{Diff}_0(M)$ such that $f^*\omega_\varphi = \omega$ and $(M, \omega_\varphi, I) \to B$ is isomorphic to $(M, \omega, f^*I) \to B$.*

Proof. Let us consider a potential $\varphi \in \mathcal{K}_E(I)$ and a path $\{\varphi_t\}$ in $\mathcal{K}_E(I)$ from 0 to φ. [24, Theorem 3.3] guarantees that this path exists and that it is smooth. We define the relatively cscK metrics

$$\omega_t = \omega + 2i\partial\bar{\partial}\varphi_t$$

and the Kähler metrics

$$\omega_{k,t} = \omega_t + k\omega_B,$$

where the $\partial, \bar{\partial}$ operators are taken with respect to the relatively cscK complex structure I. From Proposition 2.11, we have that $\dot{\varphi}_t \in C^\infty(E(\omega_t, I)) \oplus C^\infty(B)$, for all t. Thus the fibrewise Hamiltonian vector fields

$$\eta_t := \mathrm{grad}^{\omega_t} \dot{\varphi}_t$$

are well-defined. Consider the vertical vector fields

$$\xi_t := (\nabla^{g_t} \dot{\varphi}_t)_{\mathcal{V}} = (I\mathrm{grad}^{\omega_t} \dot{\varphi}_t)_{\mathcal{V}}.$$

Then fibrewise

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = -\mathcal{L}_{I\eta_t}\omega_t = \mathcal{L}_{\xi_t}\omega_t. \tag{3.4}$$

Let $\{f_t, t \in [0,1]\}$ be the *isotopy* of the time-dependent vector field ξ_t, that is, the collection of diffeomorphisms of M such that

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = \xi_t(f_t), \quad f_0 = \mathrm{id}.$$

Since ξ_t is vertical, $f_t \in \mathrm{Diffeo}(M, \pi)$. Recall the following property [6, §6.4], for a smooth family σ_t of tensors:

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t^* \sigma_t = f_t^* \left(\mathcal{L}_{\xi_t} \sigma_t + \frac{\mathrm{d}\sigma_t}{\mathrm{d}t} \right),$$

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where ξ_t and f_t are a time-dependent vector field and its isotopy, respectively. Applying it to ω_t gives the fibrewise relation

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t^* \omega_t = 0,$$

which implies

$$f_t^* \omega_t = f_0^* \omega = \omega.$$

Define $J_t = f_t^* I$. Then, for $t = 1$, the two metrics $g(\omega_X, f_1^* I)$ and $g(\omega_X + 2i\partial\bar{\partial}\varphi, I)$ are fibrewise isometric, that is,

$$(M, \omega, f_1^* I) \simeq (M, \omega + 2i\partial\bar{\partial}\varphi, I)$$

as relative Kähler manifolds with fibrewise constant scalar curvature. □

Thus, given an integrable relatively cscK complex structure I in $\mathscr{J}_\pi$, we have a map

$$\begin{aligned} F_\pi : \mathcal{K}_E(I) &\longrightarrow \mathscr{J}_\pi \\ \varphi &\longmapsto f_1^* I =: F_\pi(\varphi, I) \end{aligned} \tag{3.5}$$

which locally parametrises all integrable complex structures in the same diffeomorphism class of I that are fibrewise cscK with respect to the fixed ω. Its differential at the origin is

$$\begin{aligned} \mathrm{d}_0 F_\pi(\dot{\varphi}) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J_t = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^* I = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{L}_{\xi_t} I = \mathcal{L}_{(I\,\mathrm{grad}^\omega \dot{\varphi})_\mathcal{V}} I = \\ &= -\frac{1}{2}\bar{\partial}(\mathrm{grad}^\omega \dot{\varphi})_\mathcal{V}, \end{aligned}$$

where, again by Proposition 2.11, $\dot{\varphi} \in C^\infty(E(\omega, I)) \oplus C^\infty(B)$.

Now let $v_0 \in V_\pi$ be the deformation of the complex structure which generates the family $Y \to B$ and let Φ be the relative Kuranishi map (2.9). Then we can extend the map (3.5) to a map

$$\begin{aligned} F'_\pi : \mathcal{K}_E(I) &\to \widetilde{H}^1_\mathcal{V} \\ \varphi &\mapsto f_1^* v_0 =: F'_\pi(\varphi, I, v_0). \end{aligned}$$

Its differential at the origin computed at $\dot{\varphi} \in C^\infty(E(\omega, I)) \oplus C^\infty(B)$ is

$$\mathrm{d}_0 F'_\pi(\dot{\varphi}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^* v_0 = \mathcal{L}_{(I\,\mathrm{grad}^\omega \dot{\varphi})_\mathcal{V}} v_0. \tag{3.6}$$

Definition 3.9. We denote by $\mathcal{P}$ the set of triples $(\varphi, x, v) \in C^\infty(M) \times TV_\pi$ such that $\varphi \in \mathcal{K}_E(\Phi(x))$ and x is $K_\pi^\mathbb{C}$-polystable. The optimal symplectic connection operator is the map

$$\begin{aligned} \mathcal{G} : \mathcal{P} &\to C^\infty(X) \\ (\varphi, x, v) &\mapsto p_{E(\varphi,x)}(\Theta(\omega, F_\pi(\varphi, \Phi(x)))) + \frac{\lambda}{2} \nu_{\varphi,x}\big(F'_\pi(\varphi, \Phi(x), v)\big). \end{aligned}$$

In this expression $E(\varphi, x)$ is the vector bundle of fibre holomorphy potentials with respect to the Kähler structure $(\omega, F_\pi(\varphi, \Phi(x)))$ and $\nu_{\varphi,x}$ is the map (2.10) computed with respect to the complex structure $F_\pi(\varphi, \Phi(x))$.

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We now compute the differential of $\mathcal{G}$ along the φ-variable computed at $(0,0,v_0)$. We write

$$\mathcal{G}(\varphi,x,v)=\mathcal{G}_1(\varphi,x)+\mathcal{G}_2(\varphi,x,v),$$

and we split the computation into two separate lemmas.

Lemma 3.10. *Let $\varphi\in C^\infty(E(\omega,I))\oplus C^\infty(B)$. The differential along the first variable of $\mathcal{G}_1$ is*

$$D_1\mathcal{G}_1|_{(0,0,v_0)}(\varphi)=-\frac{1}{2}\mathcal{R}^*\mathcal{R}(\varphi).$$

Proof. Let $\{J_t\}$ be a family of relatively cscK complex structures compatible with ω, such that $J_0=I$ and consider the scalar curvature of (ω_k,J_t),

$$\mathrm{Scal}(\omega_k,J_t)=\mathrm{Scal}(\omega_k,I)+t\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Scal}(\omega_k,J_t)+O(t^2). \tag{3.7}$$

Let $\alpha=\partial_{t=0}J_t$, and define

$$Q_k(\alpha):=\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{Scal}(\omega_k,J_t).$$

From Lemma 2.3 we obtain

$$Q_k(\alpha)=\mathrm{Im}\Big((g_k)^{p\bar q}\,\nabla_p\nabla_a\alpha^a_{\bar q}\Big)=\mathrm{Re}\Big((\omega_k)^{p\bar q}\,\nabla_p\nabla_a\alpha^a_{\bar q}\Big).$$

We need to compute the sub-leading order term of Q_k along the differential of the map F_π (3.5)

Let us consider $\alpha=\bar\partial(\mathrm{grad}^\omega\varphi_E)^{1,0}_{\mathcal{V}}$, where $\varphi_E\in C^\infty_E(X)$ is a fibrewise holomorphy potential. Then

$$Q_k(\alpha)=k^{-1}\mathrm{Re}(\mathcal{R}^*\mathcal{R}(\varphi_E))+O\Big(k^{-2}\Big), \tag{3.8}$$

where $\mathcal{R}(\varphi_E)=\bar\partial_B\mathrm{grad}_{\mathcal{V}}\varphi_E$ and the adjoint is computed with respect to $\omega_F+\omega_B$. As explained in §2.3, the operator $\mathcal{R}^*\mathcal{R}$ can actually be seen as $p_E\circ\mathcal{L}_1$ restricted to $\mathcal{C}^\infty(E(\omega,I))$. Its kernel consists of fibrewise holomorphy potentials which are global holomorphy potentials on X with respect to ω_k.

A local coordinate expression for α is

$$\alpha=\bar\partial(Y_{\varphi_E})^{1,0}_{\mathcal{V}}=\partial_{\bar z^j}\Big(\omega_F^{a\bar b}\partial_{\bar w^b}\varphi_E\Big)\partial_{w^a}\otimes\mathrm{d}\bar z^j=\nabla_{\bar j}\Big(\omega_F^{a\bar b}\nabla_{\bar b}\varphi_E\Big)\partial_{w^a}\otimes\mathrm{d}\bar z^j.$$

In particular, since the potential φ_E is a fibrewise holomorphy potential, the component of α with the covariant index of vertical type vanishes. Therefore, using Lemma 2.7,

$$\begin{aligned}Q_k(\alpha)&=k^{-1}\mathrm{Re}\Big(\omega_B^{i\bar j}\,\nabla_i\nabla_a\nabla_{\bar j}\Big(\omega_F^{a\bar b}\nabla_{\bar b}\varphi_E\Big)\Big)+O\Big(k^{-2}\Big)=\\&=k^{-1}\mathrm{Re}\Big(\omega_B^{i\bar j}\,\omega_F^{a\bar b}\,\nabla_i\nabla_a\nabla_{\bar j}\nabla_{\bar b}\varphi_E\Big)+O\Big(k^{-2}\Big)=\\&=k^{-1}\mathrm{Re}(\mathcal{R}^*\mathcal{R}(\varphi_E))+O\Big(k^{-2}\Big).\end{aligned}$$

A similar computation also holds if we consider, instead of a potential φ in $C^\infty(E(\omega,I))$, a map $\varphi\in C^\infty(E(\omega,I))\oplus C^\infty(B)$, so that $\alpha=\bar\partial(Y_\varphi)^{1,0}_{\mathcal{V}}$. This choice amounts to considering an element in the image of the differential of the map F_π (3.5). In this case

$$Q_k(\alpha)=\mathrm{Re}(\mathcal{L}_0\varphi)+\frac{1}{k}\mathrm{Re}(\mathcal{L}_1\varphi)+O\Big(k^{-2}\Big),$$

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where $\mathcal{L}_0(\varphi) = \mathcal{L}_0(\varphi_B) = 0$, since φ_B is constant when restricted to a fibre, so we obtain the same equation as (3.8). To obtain the correct coefficient in the claimed expression note that

$$\mathrm{d}_0 F_\pi(\varphi) = -\frac{1}{2}\alpha.$$

□

Lemma 3.11. *Let $\varphi \in C^\infty(E(\omega, I)) \oplus C^\infty(B)$. The differential along the first variable of $\mathcal{G}_2$ is*

$$D_1\mathcal{G}_2|_{(0,0,\nu_0)}(\varphi) = -\mathcal{A}^*\mathcal{A}(\varphi).$$

Proof. We compute

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nu_t\left(f_t^*\nu_0\right),$$

where ν_t is the map ν computed with respect to the complex structure $f_t^*\Phi(0)$. Now, f_t is the isotopy of the vector field $\xi_t = (I\mathrm{grad}^{\omega_t}\dot{\varphi}_t)_{\mathcal{V}}$, where $\dot{\varphi}_t$ is in $C^\infty(E_t) \oplus C^\infty(B)$ and $\dot{\varphi}_0 = \varphi$. In the expression of ξ_t we are fixing the complex structure I and varying the Kähler form ω_t. In particular, ξ_t is a fibrewise holomorphic vector field with respect to I. This implies that $f_t^*I = I$, so $\nu_t = \nu$. Therefore

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nu_t\left(f_t^*\nu_0\right) = \mathrm{d}_{\nu_0}\nu\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^*\nu_0\right).$$

Using the expression (3.6) we obtain

$$\mathrm{d}_{\nu_0}\nu\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^*\nu_0\right) = \mathrm{d}_{\nu_0}\nu\left(\mathcal{L}_{(I\mathrm{grad}^\omega\varphi)_{\mathcal{V}}}\nu_0\right).$$

The right-hand side can be written as $-\mathcal{A}^*\mathcal{A}(\varphi)$ following the description (2.14). The minus sign follows from the relation

$$(I\mathrm{grad}^\omega\varphi)_{\mathcal{V}} = -\nabla_{\mathcal{V}}\varphi,$$

where $\nabla_{\mathcal{V}}$ is the vertical Riemannian gradient. □

We define the operator

$$\begin{aligned}\mathcal{G} : \mathcal{P}^{2,l} &\longrightarrow W^{2,l-2}(X)\\ (\varphi, x, v) &\longmapsto p_{E(x,\varphi)}(\Theta(\omega, F_\pi(\varphi, \Phi(x)))) + \frac{\lambda}{2}\nu_{\varphi,x}\left(F'_\pi(\varphi, \Phi(x), v)\right),\end{aligned} \tag{3.9}$$

where $\mathcal{P}^{2,l}$ is the space defined in Definition 3.9, but the functions are considered to be in the Sobolev space $W^{2,l}(X)$ instead of smooth.

Proposition 3.12. *Let $\pi_X : X \to B$ be a holomorphic submersion with a fibrewise cscK structure (ω, I) and let $\pi_Y : Y \to B$ be a deformation of $X \to B$ with complex structure J. Let V_π be the Kuranishi space based at I and let $\nu_0 \in V_\pi$ represent the complex structure J. Assume that*

(i) *(ω, J) is an optimal symplectic connection;*
(ii) *the relative automorphism group*

$$\mathrm{Aut}(\pi_Y) := \{f \in \mathrm{Aut}(Y, H_Y) \mid f \circ \pi = \pi\}$$

is discrete.
(iii) *the group $\mathrm{Aut}(X_b, H_b)$ is independent of B, and will be denoted G.*

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Then for any small deformation v of v_0 in V_π^+ there exists $x \in V_\pi$ such that $(\omega, \Phi(x))$ is relatively cscK, and a Kähler potential φ such that

$$\omega + i\partial\bar{\partial}\varphi$$

is an optimal symplectic connection with respect to the complex structure J' corresponding to v, where the ∂, $\bar{\partial}$ operators are with respect to $\Phi(x)$.

Proof. The proof consists of proving that the operator 3.9 is an elliptic operator with a trivial kernel so that we can apply the implicit function theorem. We note that this can be done even though V_π^+ may be a singular complex space. Indeed, let $\mathcal{X} \to B \times V_\pi^+$ be a family of holomorphic submersions such that the fibre over $0 \in V_\pi^+$ is $X \to B$. By the Kuranishi theorem [30, p.149] we can locally consider a smooth trivialisation of the family over V_π^+ such that the complex structures of the fibrations $\mathcal{X}_x \to B \times \{x\}$ form a smooth family $\{J_s\}$.

As before, let I denote the complex structure of X. As we assume that (ω, I, v_0) is an optimal symplectic connection, $\mathcal{G}(0, 0, v_0) = 0$. The derivative with respect to the first component, given by Lemma 3.10 and 3.11, is

$$\mathrm{d}_1\mathcal{G}_{(0,0,v_0)}(\varphi) = -\mathcal{R}^*\mathcal{R}(\varphi) - \lambda\mathcal{A}^*\mathcal{A}(\varphi) = -\widehat{\mathcal{L}}(\varphi).$$

The hypothesis on the automorphism group implies that the kernel of the linearisation is empty, so the implicit function theorem guarantees that there exists a path

$$(\varphi(x, v), x, v) \in \mathcal{P}^{2,l}$$

such that locally around $(0, 0, v_0)$

$$\mathcal{G}(\varphi(x, v), x, v) = 0. \tag{3.10}$$

The function $\varphi(x, v)$ is smooth by the standard theory of regularity of solutions to elliptic partial differential equations, applied to the operator $\mathcal{G}$ [2, Theorem 41]. Therefore, solutions to (3.10) produce optimal symplectic connections. □

3.3. The moduli space of optimal symplectic connections

Let $(Y, H_Y) \to B$ be a relatively K-semistable holomorphic submersion with a degeneration to a relatively cscK fibration $(X, H_X) \to B$. Assume that we have a relatively Kähler metric (ω, J) on Y that is an optimal symplectic connection. Let W be the subset of the Kuranishi space V_π that corresponds to fibrations satisfying the hypotheses of Proposition 3.12. Then W is an open subset of the locally closed analytic subvariety V_π^+ described in §3.1 by Proposition 3.12.

Lemma 3.13 [18, Corollary to Proposition 2]. *The group* $\mathrm{Aut}(\pi_Y)$ *is a subgroup of* $\mathrm{Aut}(Y, H_Y)$ *with finitely many connected components.*

In particular, under our assumption $\mathrm{Aut}(\pi_Y)$ is a finite discrete group. Let V_{π_Y} be the Kuranishi space of the fibration π_Y and let

$$\tau : V_{\pi_Y} \to V_\pi$$

be the map given by completeness of the Kuranishi space. If we denote $\tau^{-1}W =: W_Y$, then W_Y is a locally closed subvariety of V_{π_Y}.

Let $\mathcal{Y} \to B \times W_Y$ be the Kuranishi family of fibrations which admit an optimal symplectic connection, with central fibration $Y \to B$. The quotient

$$W_Y / \mathrm{Aut}(\pi_Y) \tag{3.11}$$

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is a local complex space and it is Hausdorff since it is the quotient of a variety by a finite group. We now explain that we can glue the quotients (3.11) to obtain a global Hausdorff moduli space $\mathcal{M}$ of fibrations that admit an optimal symplectic connection.

Remark 3.14. The moduli space $\mathcal{M}$ depends on the group $G = \mathrm{Aut}_0(X_b, H_b)$. In other words, $\mathcal{M}$ parametrises fibrations $\pi_Y : Y \to B$ such that they have a discrete relative automorphism group and such that they degenerate to a relatively cscK fibration whose fibres have G as their automorphism group.

The following result builds on [22, Proposition 6.5] and [21, Lemma 3.8].

Lemma 3.15. *Let $\mathcal{Y}$ and $\mathcal{Y}'$ over W_Y be two families of fibrations that admit an optimal symplectic connection. The group of isomorphisms between $\mathcal{Y}$ and $\mathcal{Y}'$ that preserve the fibration structure, denoted* $\mathrm{Isom}_{W_Y}(\mathcal{Y}, \mathcal{Y}', B)$*, is proper over W_Y.*

Proof. Let $y_t \to \bar{y}$ be a convergent sequence in W_Y and consider a family of isomorphisms $f_t : \mathcal{Y}_{y_t} \to \mathcal{Y}'_{y_t}$ preserving the projection onto B. Such isomorphisms are fibrewise isometries with respect to the underlying fibrewise Riemannian metrics. Therefore there exists a subsequence $\{f_{t_k}\}$ which converges in the C^m-topology to a fibrewise C^m-isometry $f : \mathcal{Y}_{\bar{y}} \to \mathcal{Y}'_{\bar{y}}$ [21, Lemma 3.8]. Moreover, f is a biholomorphism because it is the limit of biholomorphic maps. Therefore the convergence takes place in $\mathrm{Isom}_{W_Y}(\mathcal{Y}, \mathcal{Y}', B)$, which is then proper over W_Y. □

The global definition of $\mathcal{M}$ relies on the following lemma, whose proof follows from [1, XI.6].

Lemma 3.16. *Let $\mathcal{Y} \to B \times W_Y$ be a family of fibrations that admit an optimal symplectic connection. For any points $y, w \in W_Y$, the fibrations $Y_y \to B$ and $Y_w \to B$ are isomorphic if and only if there exists an element $g \in \mathrm{Aut}(\pi_Y)$ such that $g(w) = y$.*

Proof. We can prove the lemma for the entire relative Kuranishi space V_{π_Y}. The statement follows from:

(i) For any $y \in V_{\pi_Y}$, the automorphism group $\mathrm{Aut}(\pi_{Y_y})$ is contained in $\mathrm{Aut}(\pi_Y)$;
(ii) For any $y \in V_{\pi_Y}$ there exists an $\mathrm{Aut}(\pi_{Y_y})$-invariant open neighbourhood $\mathcal{U}_y$ such that any isomorphism between fibres of $\mathcal{Y}|_{\mathcal{U}_y} \to \mathcal{U}_y$ is induced by an element of $\mathrm{Aut}(\pi_{Y_y})$.

We begin by proving (ii). Assume by contradiction that there exist two sequences $\{y_n\}$ and $\{w_n\}$ both converging to y. Assume that there exist g_n in $\mathrm{Aut}(\pi_{Y_y}) \setminus \mathrm{Aut}(\pi_Y)$ such that $g_n \cdot w_n = y_n$. Then by Lemma 3.15 there exists $g \in \mathrm{Aut}(\pi_{Y_y})$ such that $g_n \to g$. Up to replacing g_n with $g_n g^{-1}$ and w_n with $g^{-1} w_n$ we may assume that g is the identity of $\mathrm{Aut}(\pi_{Y_y})$. Consider the map

$$\begin{aligned} \mathrm{Aut}(\pi_Y) \times V_{\pi_Y} &\to \widetilde{\mathrm{H}}^1_{\mathcal{V}} \\ (\eta, y) &\mapsto \eta \cdot y. \end{aligned}$$

By Theorem 2.12, this is a local biholomorphism at (id, y). So we have $F(\mathrm{id}, y_n) = F(g_n, w_n)$. Hence $g_n = \mathrm{id}$, a contradiction.

The claim (i) then follows exactly as in [1, p.204]. We report the proof for completeness. Set

$$I = \{(y, g) \in W_Y \times \mathrm{Aut}(\pi_{Y_y})\}.$$

Then I is equal to $\mathrm{Isom}(\mathcal{Y}, \mathcal{Y}, B)$. Up to shrinking W_Y we can assume that any connected component I' of I intersects $\{0\} \times \mathrm{Aut}(\pi_Y)$. Let $\widehat{I} = \{(y, g) \in I' \mid g \in \mathrm{Aut}(\pi_Y)\}$. Then $\widehat{I}$ is nonempty and Zariski-closed. It follows from (ii) that $\widehat{I}$ contains an open set. Then $\widehat{I} = I'$, which concludes the proof. □

To glue the charts (3.11), we use the completeness of the Kuranishi space. Let $Y_1 \to B$ be another relatively K-semistable fibration which admits an optimal symplectic connection and is close to $Y \to B$. Then the Kuranishi theorem gives a map

$$\tau : V_{\pi_{Y_1}} \to V_{\pi_Y}$$

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such that a family $\mathcal{Y}_1 \to B \times V_{\pi_{Y_1}}$ is isomorphic to the pull-back via τ of $\mathcal{Y} \to B \times V_{\pi_Y}$. Consider the composition of this map with the inclusion $i : W_{Y_1} \to V_{\pi_{Y_1}}$:

$$\tau \circ i : W_{Y_1} \to V_{\pi_Y}.$$

Let $y_1 \in W_{Y_1}$. Then y_1 represents a fibration with an optimal symplectic connection, that is, the image of y_1 via the Kuranishi map is a complex structure J_{y_1} such that (ω, J_{y_1}) is an optimal symplectic connection. Therefore, there is a representative y of J_{y_1} in V_{π_Y} which belongs to W_Y, so

$$\alpha := \tau \circ i : W_{Y_1} \to W_Y.$$

Lemma 3.16 allows us to pass to the quotient and obtain an isomorphism

$$\widetilde{\alpha} : W_{Y_1}/\mathrm{Aut}(\pi_{Y_1}) \to W_Y/\mathrm{Aut}(\pi_Y), \tag{3.12}$$

which is uniquely determined (while α itself might not be). Indeed, the inverse is constructed by reverting the roles of Y and Y_1. Therefore, we can use it to glue the local charts to give $\mathcal{M}$ the structure of a complex space.

Proposition 3.17. *The space $\mathcal{M}$ is a Hausdorff complex space with at most countably many connected components.*

Proof. The countability follows from [21, Theorem 7.3]. We prove the Hausdorff property. Let $\mathcal{Y} \to B \times W_Y$ and $\mathcal{Y}_1 \to B \times W_{Y_1}$ be two families of fibrations which admit an optimal symplectic connection. Let $y_t \to \bar{y}$ be a sequence in W_Y and $y_{1t} \to \bar{y}_1$ be a sequence in W_{Y_1} and assume that Y_{y_t} is isomorphic to $Y_{1,y_{1t}}$ as fibrations over B. Following the proof of [18, Proposition 10], we show that $Y_{\bar{y}} \to B$ is isomorphic to $Y_{1,\bar{y}_1}$. Let

$$\widehat{W} = W_Y \times W_{Y_1},$$

and let $\widehat{\mathcal{Y}} \to B \times \widehat{W}$ and $\widehat{\mathcal{Y}}_1 \to B \times \widehat{W}$ be the pull-back of $\mathcal{Y}$ and $\mathcal{Y}_1$ using the projections of $\widehat{W}$ onto the first and second component respectively. Consider

$$\Sigma = \{(y, y_1) \in \widehat{W} \mid Y_y \to B \text{ is isomorphic to } Y_{1,y_1} \to B\}.$$

It follows from the properness of $\mathrm{Isom}(\widehat{\mathcal{Y}}, \widehat{\mathcal{Y}}_1, B)$ that Σ is a locally closed analytic subvariety of $\widehat{W}$. Therefore $(\bar{y}, \bar{y}_1) \in \Sigma$, which concludes the proof. □

We have proved the following.

Corollary 3.18. *There exists a Hausdorff complex space $\mathcal{M}$ which parametrises holomorphic submersions over a fixed base admitting an optimal symplectic connection, with discrete relative automorphism group and fixed vertical automorphism group of the relatively cscK degeneration.*

Remark 3.19. The hypothesis that the vertical automorphism group of the relatively cscK degeneration is fixed is key in the definition of an optimal symplectic connection and in the construction of §3.1, and removing it would require a deeper, and challenging, generalisation of the theory of optimal symplectic connections.

Another natural generalisation of Corollary 3.18 would be to consider the moduli space of fibrations whose group of relative automorphisms is not discrete. In this case, we expect that the local charts of the moduli space should be given by a GIT quotient of the complex space of deformations of Y which also admit an optimal symplectic connection by the group $\mathrm{Aut}(\pi_Y)$, reflecting the behaviour of the moduli space of cscK manifolds constructed by Dervan-Naumann [8], and using the techniques introduced by the author in [41] on the moment map interpretation of the optimal symplectic connection equation.

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Remark 3.20 (Examples). The projectivisation of a stable, simple holomorphic vector bundle is represented by a point of our moduli space: in this sense, $\mathcal{M}$ generalises the moduli space of stable holomorphic vector bundles.

More generally, we can consider an isotrivial fibration, that is, a fibration whose fibres are all holomorphic, of which projectivised vector bundles are a special case. McCarthy constructed an example [33, Example 3.9] of an isotrivial fibration $X \to B$ which admits an optimal symplectic connection, whose model fibre X_0 is given by the Mukai-Umemura threefold. Such a fibration has a trivial group of relative automorphisms, so it is represented by a point in our moduli space $\mathcal{M}$.

Deformations of such a fibration, obtained by deforming the model fibre X_0 as described in [15, §5.3], could generate other examples. In general, however, constructing explicit examples of fibrations admitting an optimal symplectic connection is challenging, because most cases where the relatively K-polystable degeneration is known and can be described explicitly have singular fibres.

4. A Weil-Petersson type Kähler metric

In this section, we define a Kähler metric on the moduli space of fibrations admitting an optimal symplectic connection. We do so by describing a relative version of the theory of Weil-Petersson type metrics developed by Fujiki and Schumacher for cscK manifolds [22, Sections 8, 9]. In particular, we first define the Weil-Petersson metric locally on the sets W_Y and subsequently we extend the definition to the charts $W_Y/\mathrm{Aut}(\pi_Y)$ and then to the moduli space $\mathcal{M}$. To define the Kähler metric, we give $\mathcal{M}$ the induced reduced structure.

We start by recalling that the notions of a smooth Hermitian metric and of a smooth Kähler metric are well posed on a complex space [22, Definitions 1.1, 1.2]. Consider a family of holomorphic submersions that admit an optimal symplectic connection, denoted by $\mathcal{Y} \to B \times W_Y$, and let $Y \to B$ be the central fibration. Let ω_t be also the optimal symplectic connection on $Y_t \to B$. For each k sufficiently large, $\omega_{t,k} = \omega_t + k\omega_B$ is a Kähler form on Y_t, and we denote by $\omega_{t,F}$ its purely vertical part, which does not depend on k. Let $\mathcal{Y} \to W_Y$ be the composition with the second projection.

From Theorem 2.12, for each $t \in W_Y$ there is an injective map

$$\mathrm{d}_t\Phi : T_tW_Y \hookrightarrow \widetilde{H}^1_{\mathcal{V}}(Y_t) \subseteq \Omega^{0,1}(\mathcal{V}^{1,0}_{Y_t}) \tag{4.1}$$

that identifies a vector α in the Zariski tangent space T_tW_Y with a $(0,1)$-form valued in the $(1,0)$-tangent bundle of Y_t, which we will also denote by α. The map (4.1) is equivariant with respect to the action of $\mathrm{Aut}(\pi_Y)$. Therefore we can define an inner product on T_tW_Y by pulling back the L^2-product on $\Omega^{0,1}(\mathcal{V}^{1,0}_{Y_t})$ induced by the Hermitian metric associated to $\omega_{t,F} + \omega_B$. For any $\alpha, \beta \in T_tW_Y$, its imaginary part is given by

$$\Omega_t(\alpha,\beta) := \langle \alpha, \beta \rangle_{\omega_{t,F}+\omega_B} = \int_{Y_t} \Lambda_{\omega_{t,F}+\omega_B} \mathrm{Tr}_{\omega_{t,F}}(\alpha\overline{\beta})\omega_t^m \wedge \omega_B^n, \tag{4.2}$$

where we denote by Λ the contraction of the covariant part and by Tr the trace of the contravariant part. We give the following definition.

Definition 4.1. The *relative Weil-Petersson metric* on W_Y, denoted by Ω_{WP}, is the two-form $\{\Omega_t\}_{t\in W_Y}$.

Using the compatibility of the deformations with the Kähler form, we write the trace in coordinates as

$$\Lambda_{\omega_{t,F}+\omega_B}\mathrm{Tr}_{\omega_{t,F}}(\alpha\overline{\beta}) = \alpha^a_{\bar{q}}\overline{\beta^b_{\bar{p}}}(\omega_{t,F})_{a\bar{b}}(\omega_{t,F}+\omega_B)^{\bar{q}p} = \alpha^a_{\bar{b}}\overline{\beta^b_{\bar{a}}} + \Lambda_{\omega_B}\mathrm{Tr}_{\omega_{t,F}}(\alpha\overline{\beta}).$$

Therefore the integral (4.2) can be written as the sum

$$\int_{Y_t} \alpha^a_{\bar{b}}\overline{\beta^b_{\bar{a}}}\omega_t^m \wedge \omega_B^n + \int_{Y_t} \Lambda_{\omega_B}\mathrm{Tr}_{\omega_{t,F}}(\alpha\overline{\beta})\omega_t^m \wedge \omega_B^n.$$

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Remark 4.2. The first term can be split over Y_t as

$$\int_B \left(\int_{Y_{t,b}} \alpha^a_{\bar{b}} \overline{\beta^b_{\bar{a}}} \omega_t^m \right) \omega_B^n. \tag{4.3}$$

In particular, it vanishes when α and β each restrict to the trivial deformation on the fibres. This is the case when $\mathcal{Y} \to B \times W_Y$ is a family of holomorphic submersions with rigid fibres, for example. In particular, in the case of projectivised stable vector bundles, we recover the Weil-Petersson metric studied by Schumacher–Toma [42, p. 103].

We next prove that (4.2) is closed and positive definite. Consider for each $t \in W_Y$ a k-dependent inner product on $\widetilde{H}^1_{\mathcal{V}}(Y_t)$, defined using the Kähler form $\omega_{t,k}$ of Y_t:

$$\Omega_{k,t}(\alpha, \beta) := \int_{Y_t} \langle \alpha, \beta \rangle_{\omega_{t,k}} \omega_{t,k}^{n+m} = \int_{Y_t} \Lambda_{\omega_{t,k}} \mathrm{Tr}_{\omega_{t,k}}(\alpha \overline{\beta}) \omega_{t,k}^{n+m}.$$

The collection $\{\Omega_{k,t}\} =: \Omega_k$ is the Weil-Petersson type Hermitian metric defined by Fujiki-Schumacher [22, Definition 7.2] for any family of smooth polarised varieties. Using Lemma 2.7, we can write in local holomorphic coordinates

$$\langle \alpha, \beta \rangle_{\omega_{t,k}} = \alpha^a_{\bar{q}} \overline{\beta^b_{\bar{p}}} (\omega_{t,k})_{a\bar{b}} (\omega_{t,k})^{\bar{q}p} = \alpha^a_{\bar{b}} \overline{\beta^b_{\bar{a}}} + k^{-1} \Lambda_{\omega_B} \mathrm{Tr}_{\omega_{t,F}}(\alpha \overline{\beta}) + O\left(k^{-2}\right).$$

The expansion in powers of k of $\omega_{t,k}$ reads

$$\omega_{t,k}^{n+m} = k^n \omega_t^m \wedge \omega_B^n + \frac{k^{n-1}}{nm} \omega_t^{m+1} \wedge \omega_B^{n-1} + O\left(k^{n-2}\right).$$

Then

$$\begin{aligned} \Omega_{k,t}(\alpha, \beta) = k^n \int_{Y_t} \alpha^a_{\bar{b}} \overline{\beta^b_{\bar{a}}} \omega_t^m \wedge \omega_B^n + k^{n-1} \Bigg[\int_{Y_t} \alpha^a_{\bar{b}} \overline{\beta^b_{\bar{a}}} \omega_t^{m+1} \wedge \omega_B^{n-1} \\ + \int_{Y_t} \Lambda_{\omega_B} \mathrm{Tr}_{\omega_{t,F}}(\alpha \overline{\beta}) \omega_t^m \wedge \omega_B^n \Bigg] + O\left(k^{n-2}\right). \end{aligned} \tag{4.4}$$

Then the two terms in the sum (4.2) are the first and third coefficients of this expansion.

We next describe a fibre integral formula for the Weil-Petersson metric on W_Y. Let $\omega_{\mathcal{Y},k}$ be the relatively Kähler metric on W_Y such that its restriction to each Y_t is the Kähler metric $\omega_t + k\omega_B$, where ω_t is an optimal symplectic connection on $Y_t \to B$. Let also $\rho_{\mathcal{Y},k}$ be the curvature of the Hermitian structure induced by $\omega_{\mathcal{Y},k}$ on the relative anticanonical bundle

$$-K_{\mathcal{Y}/W_Y} = \bigwedge^{n+m} \mathcal{V}_{\mathcal{Y}/W_Y},$$

where $\mathcal{V}_{\mathcal{Y}/W_Y}$ denotes the vertical tangent bundle of the fibration $\mathcal{Y} \to W_Y$. We first recall the following result [22, Theorem 8.1].

Proposition 4.3. *The k-dependent $(1,1)$-form Ω_k can be written as a fibre integral over the map $\mathcal{Y} \to W_Y$:*

$$\Omega_k = -\int_{\mathcal{Y}/W_Y} \rho_{\mathcal{Y},k} \wedge \omega_{\mathcal{Y},k}^{n+m} + \frac{1}{n+m+1} \int_{\mathcal{Y}/W_Y} \mathrm{Scal}_{\mathcal{V}}(\omega_{\mathcal{Y},k}) \omega_{\mathcal{Y},k}^{n+m+1}. \tag{4.5}$$

By expanding Ω_k in powers of k, we can find a fibre integral formula for the Weil-Petersson metric (4.2). Since the base B is fixed, the relative metric $\omega_{\mathcal{Y},k}$ can be written as

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$$\omega_{\mathcal{Y},k} = \widehat{\omega} + k\omega_B,$$

where the restriction of $\widehat{\omega}$ to each Y_t is the optimal symplectic connection ω_t. Then $\rho_{\mathcal{Y},k}$ can be expanded in powers of k as

$$\begin{aligned}\rho_{\mathcal{Y},k} &= i\partial\bar{\partial}\log\left(k^n\widehat{\omega}^m \wedge \omega_B^n + O\left(k^{n-1}\right)\right)\\ &= i\partial\bar{\partial}\log\left(\widehat{\omega}^m \wedge \omega_B^n\right) + O\left(k^{-1}\right)\\ &= i\partial\bar{\partial}\log\det(\widehat{\omega}) + i\partial\bar{\partial}\log\det(\omega_B) + O\left(k^{-1}\right),\end{aligned}$$

where the second line follows from the fact that $\widehat{\omega}^m \wedge \omega_B^n$ is a volume form. The k^{-1}-term is exact, because the two volume forms $\omega_{\mathcal{Y},k}^{m+n}$ and $\widehat{\omega}^m \wedge \omega_B^n$ both induce the class $c_1(-K_{\mathcal{Y}/W_Y})$. Moreover, $\det(\widehat{\omega})$ is the relative determinant of $\widehat{\omega}$ and $i\partial\bar{\partial}\log\det(\widehat{\omega})$ is the curvature of the Hermitian metric induced by $\widehat{\omega}$ on the relative anticanonical bundle $-K_{\mathcal{Y}/B\times W_Y}$. To compute the expansion of the vertical scalar curvature, we use the expression [40, Proposition 5.4] for the scalar curvature of ω_k, which we report here.

Proposition 4.4. *The scalar curvature of ω_k on the submersion $Y \to B$ admits an expansion*

$$\mathrm{Scal}(\omega_k) = \widehat{S}_b + k^{-1}\left(\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\alpha + p_E(\Theta(\omega))\right) + O\left(k^{-3/2}\right)$$

where:

(i) *$\widehat{S}_b$ is the average scalar curvature of the fibres;*
(ii) *α is the Weil-Petersson metric on B induced by the relatively cscK degeneration $X \to B$.*

Since $\widehat{\omega}$ is an optimal symplectic connection when restricted to each Y_t, using the proposition 4.4 the vertical scalar curvature of $\omega_{\mathcal{Y},k}$ admits an expansion as

$$\mathrm{Scal}_{\mathcal{V}}(\omega_{\mathcal{Y},k}) = \widehat{S}_b + k^{-1}\left(\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\widehat{\alpha}\right) + O\left(k^{-2}\right),$$

where $\widehat{\alpha}$ is a closed two-form on B. Then the leading order term of (4.5) is

$$I_0 = -\int_{\mathcal{Y}/W_Y} i\partial\bar{\partial}\log\det(\widehat{\omega}) \wedge \widehat{\omega}^m \wedge \omega_B^n + \frac{1}{n+m+1}\int_{\mathcal{Y}/W_Y} \widehat{S}_b\widehat{\omega}^{m+1} \wedge \omega_B^n.$$

The sub-leading order term is the sum of the four integrals

$$\begin{aligned}I_1 &= -\int_{\mathcal{Y}/W_Y} i\partial\bar{\partial}\log\det(\widehat{\omega}) \wedge \widehat{\omega}^{m+1} \wedge \omega_B^{n-1}, \qquad (4.6)\\ I_2 &= \frac{1}{n+m+1}\int_{\mathcal{Y}/W_Y} \widehat{S}_b\widehat{\omega}^{m+2} \wedge \omega_B^{n-1},\\ I_3 &= -\frac{1}{n}\int_{\mathcal{Y}/W_Y} \mathrm{Scal}(\omega_B)\widehat{\omega}^{m+1} \wedge \omega_B^n,\\ I_4 &= \frac{1}{n+m+1}\int_{\mathcal{Y}/W_Y} \left(\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\widehat{\alpha}\right)\widehat{\omega}^{m+1} \wedge \omega_B^n.\end{aligned}$$

We can use this expansion to prove a fibre integral formula in our setting.

Lemma 4.5. *The Weil-Petersson metric Ω_{WP} (4.2) can be written as the fibre integral*

$$I_0 + I_2 + I_3 + I_4. \qquad (4.7)$$

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Proof. It follows from [22, Lemma 8.5] applied to the family $\mathcal{Y} \to B \times W_Y$ that the two-form defined as the collection of the integrals

$$\int_{Y_t} \alpha^a_{\bar{b}} \overline{\beta^b_{\bar{a}}} \omega_t^{m+1} \wedge \omega_B^{n-1}$$

is equal to I_1. Indeed, working locally in $D \subset B \times W_Y$, Fujiki and Schumacher prove that one can trivialise the family $\mathcal{Y}$ over D as $Y_{0,0} \times D$ such that the horizontal distribution induced by $\widehat{\omega}$ is preserved. Then, given a family $\{\beta_t\}$ of vertical deformations of the complex structure of $Y_{0,0}$ which represent the family $Y_{0,0} \times D \to D$, [22, Lemma 8.5] gives the equality

$$i\partial\bar{\partial} \log \det(\widehat{\omega}) = \mathrm{Tr}(\partial_t \beta_t|_{t=0} \overline{\partial_t \beta_t}|_{t=0}),$$

where $\partial_t \beta_t$ is the map (4.1). □

Lemma 4.6. *The two-form Ω_{WP} is closed and positive-definite on W_Y.*

Proof. Since the map (4.1) is injective, the integral (4.2) is positive. To prove closedness, we show that the terms I_0, I_2, I_3 and I_4 in Lemma 4.5 are closed. The terms I_0 and I_2 are closed because they are the fibre integrals of a closed form. The term I_4 can be written as

$$\int_{\mathcal{Y}/W_Y} (\rho_B - \widehat{\alpha}) \wedge \widehat{\omega}^{m+1} \wedge \omega_B^{n-1},$$

where ρ_B is the Ricci form of ω_B. In particular $(\rho_B - \widehat{\alpha}) \wedge \omega_B^{n-1}$ is a top-degree form on B, hence it is closed. So I_4 is closed. Analogously, I_3 is closed. □

Let $h_{WP}(W_Y)$ be the Hermitian metric on the tangent bundle to W_Y induced by the two-form Ω_{WP}.

Theorem 4.7. *The Hermitian metric $h_{WP}(W_Y)$ induces a global Kähler metric on the moduli space $\mathcal{M}$.*

Proof. The theorem follows from the fact that the action of the finite group $\mathrm{Aut}(\pi_Y)$ is induced by an automorphism of the Kuranishi family, so the Weil-Petersson metric $h_{WP}(W_Y)$ is invariant for the action of $\mathrm{Aut}(\pi_Y)$. Therefore, it defines a metric on the quotient $W_Y/\mathrm{Aut}(\pi_Y)$. □

Remark 4.8. We have defined a Weil-Petersson metric on W_Y that is independent of the adiabatic parameter k. This is reasonable from the point of view of describing the moduli space of fibrations using the optimal symplectic connection alone, which is only relatively Kähler. However, it is possible that a different kind of Weil-Petersson-type metric could be defined by taking a sequence of Kähler metrics that depend on k, where the adiabatic construction plays a bigger role.

4.1. The determinant line bundle for the Weil-Petersson metric

We construct a line bundle on W_Y, and hence on the moduli space, such that the Weil-Petersson metric represents its first Chern class. To do so, we appeal to the theory of Deligne pairings [16], [17], [4, §1]. Let $M \to B$ be a flat, projective morphism between complex analytic spaces of relative dimension d and consider $d+1$ line bundles $L_0, \ldots, L_d$ on M. The push-forward of the intersection product of $L_0, \ldots, L_d$ is an isomorphism class of line bundles on B, represented by the cohomology class

$$\int_{M/B} c_1(L_0) \wedge \cdots \wedge c_1(L_d). \tag{4.8}$$

The *Deligne pairing* of $L_0, \ldots, L_d$, denoted by $\langle L_0, \ldots, L_d \rangle_{M/B}$, is a canonical choice of a line bundle on B such that (4.8) is its first Chern class. The construction is symmetric, multilinear and functorial. Moreover, if $h_0, \ldots, h_d$ are Hermitian metrics on $L_0, \ldots, L_d$ respectively, the theory provides a metric

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$\langle h_0, \dots, h_d \rangle_{M/B}$ on $\langle L_0, \dots, L_d \rangle_{M/B}$. Denoting by $\omega_0, \dots, \omega_d$ the curvature forms of $h_0, \dots, h_d$ respectively, the curvature of $\langle h_0, \dots, h_d \rangle_{M/B}$ is given by the fibre integral

$$\int_{M/B} \omega_0 \wedge \cdots \wedge \omega_d. \tag{4.9}$$

The fibre integral formula for the Weil-Petersson metric on W_Y of Lemma 4.5 is a special case of the expression (4.9). To describe it as the curvature form of a line bundle on W_Y, we first recall the following result of Fujiki and Schumacher for the k-dependent Weil-Petersson metric [22, Theorem 11.1], which we write using the Deligne pairing notation.

Proposition 4.9. *The k-dependent Weil-Petersson type Kähler metric Ω_k represents the first Chern class of the line bundle*

$$\langle -K_{\mathcal{Y}/W_Y}, \mathcal{L}_k^{n+m} \rangle_{\mathcal{Y}/W_Y} - \frac{1}{n+m+1} \frac{-K_Y \cdot (H_Y + kL)^{n+m-1}}{(H_Y + kL)^{n+m}} \langle \mathcal{L}_k^{n+m+1} \rangle_{\mathcal{Y}/W_Y}, \tag{4.10}$$

where the constant

$$\widehat{S}_Y := \frac{-K_Y \cdot (H_Y + kL)^{n+m-1}}{(H_Y + kL)^{n+m}}$$

is the average scalar curvature of Y, and hence of each Y_t, with respect to the metric $\omega + k\omega_B$.

We use Proposition 4.9 to prove the following.

Proposition 4.10. *There exists a line bundle $\mathcal{D}(Y)$ on W_Y whose first Chern class is represented by the Weil-Petersson metric (4.2), given as the sum of the line bundles*

$$\begin{aligned}
\mathcal{D}_0(Y) &= -\langle -K_{\mathcal{Y}/B\times W_Y}, \widehat{\mathcal{H}}^m, L^n \rangle_{\mathcal{Y}/W_Y}, \\
\mathcal{D}_2(Y) &= \frac{1}{n+m+1} \frac{-K_{Y/B} \cdot L^n \cdot H_Y^{m-1}}{L^n \cdot H_Y^m} \Big(\langle \widehat{\mathcal{H}}^{m+1}, L^n \rangle_{\mathcal{Y}/W_Y} + \langle \widehat{\mathcal{H}}^{m+2}, L^{n-1} \rangle_{\mathcal{Y}/W_Y} \Big), \\
\mathcal{D}_3(Y) &= -\langle -K_B, \widehat{\mathcal{H}}^{m+1}, L^{n-1} \rangle_{\mathcal{Y}/W_Y}, \\
\mathcal{D}_4(Y) &= \frac{\big(-K_{Y/B} \cdot H_Y^{m-1} \cdot L^n\big)\big(H_Y^{m+1} \cdot L^{n-1}\big)}{(n+m+1)(H_Y^m \cdot L^n)^2} \langle \widehat{\mathcal{H}}^{m+1}, L^n \rangle_{\mathcal{Y}/W_Y},
\end{aligned}$$

defined using the Deligne pairing.

Proof. Let $\widehat{\mathcal{H}} \to \mathcal{Y}$ be the relatively ample line bundle induced by each relative polarisation $H_t \to Y_t$. More precisely, we consider the fibration $\mathcal{Y} \to W_Y$ as the composition of

$$\mathcal{Y} \to B \times W_Y \to W_Y$$

so $\widehat{\mathcal{H}}$ is relatively ample with respect to the first projection. Moreover, since the base B is fixed, we can pull-back L to $\mathcal{Y}$ and consider it as a line bundle on $\mathcal{Y}$. So we can define a relatively ample line bundle $\mathcal{L}_k$ over $\mathcal{Y}$ as

$$\mathcal{L}_k = \widehat{\mathcal{H}} \otimes kL, \tag{4.11}$$

whose fibre over Y_t is $\mathcal{L}_k|_t = H_t \otimes kL$. Its first Chern class contains the relative metric $\omega_{\mathcal{Y},k}$, where $\widehat{\omega}$ is in $c_1(\widehat{\mathcal{H}})$. We define a line bundle $\mathcal{D}(Y)$ on W_Y by using the fibre integral formula (4.7) and the expansion in powers of k of the line bundle (4.10). Expanding in k the intersection product $\mathcal{L}_k^{n+m}, \mathcal{L}_k^{n+m+1}$ and the expression (4.10) we obtain that the Weil-Petersson metric Ω_{WP} represents the first Chern class of the line bundle $\mathcal{D}(Y)$ given as the tensor product of the line bundles $\mathcal{D}_0(Y), \mathcal{D}_2(Y), \mathcal{D}_3(Y), \mathcal{D}_4(Y)$

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above. Indeed, by expanding the expression (4.10) in powers of k we see that the sum of the leading order term and the sub-leading order term is given by

$$\mathcal{D}_0(Y) + \mathcal{D}_1(Y) + \mathcal{D}_2(Y) + \mathcal{D}_3(Y) + \mathcal{D}_4(Y),$$

where $\mathcal{D}_1(Y)$ is given by

$$\mathcal{D}_1(Y) = -\langle -K_{\mathcal{Y}/B\times W_Y}, \widehat{\mathcal{H}}^{m+1}, L^{n-1}\rangle_{\mathcal{Y}/W_Y}.$$

However, its first Chern class is represented by the term (4.6), which does not appear in the fibre integral formula for the relative Weil-Petersson metric of Lemma 4.5. This concludes the proof. □

We have constructed a line bundle $\mathcal{D}_Y$ on W_Y whose first Chern class is the Weil-Petersson metric Ω_{WP}. Let now $p : W_Y \to \mathcal{M}$ be the composition of the maps

$$W_Y \to W_Y/\mathrm{Aut}(\pi_Y) \hookrightarrow \mathcal{M},$$

where the first map is the quotient by the group action and the second map is the inclusion of a local chart in the moduli space. It follows from Lemma 3.16 that the orbits of the $\mathrm{Aut}(\pi_Y)$-action on W_Y correspond to isomorphic manifolds in the family $\mathcal{Y} \to W_Y$. Therefore the line bundle $\mathcal{D}(Y)$ is invariant for the action of $\mathrm{Aut}(\pi_Y)$ and thus descends to a line bundle $\widetilde{\mathcal{D}}(Y)$ on the quotient $W_Y/\mathrm{Aut}(\pi_Y)$.

Theorem 4.11. *There exists a line bundle $\mathcal{F}$ on the moduli space $\mathcal{M}$ such that its restriction to each chart $W_Y/\mathrm{Aut}(\pi_Y)$ is isomorphic to $\widetilde{\mathcal{D}}(Y)$.*

Proof. Let $W_{Y_1}/\mathrm{Aut}(\pi_{Y_1})$ and $W_Y/\mathrm{Aut}(\pi_Y)$ be two local charts of $\mathcal{M}$ with nonempty intersection. Then using completeness of the Kuranishi space there exists an isomorphism $\widetilde{\alpha} : W_{Y_1}/\mathrm{Aut}(\pi_{Y_1}) \to W_Y/\mathrm{Aut}(\pi_Y)$ (3.12) which preserves the relative polarisation (4.11) and the submersions onto the base B. By functoriality of the Deligne pairings, the pull-back $\widetilde{\alpha}^*\widetilde{\mathcal{D}}(Y_1)$ is then isomorphic to $\widetilde{\mathcal{D}}(Y)$. Therefore, on the intersection of $W_Y/\mathrm{Aut}(\pi_Y)$ and $W_{Y_1}/\mathrm{Aut}(\pi_{Y_1})$ there is a morphism of line bundles $\chi : \widetilde{\mathcal{D}}(Y_1) \xrightarrow{\sim} \widetilde{\mathcal{D}}(Y)$. Let $\varphi_Y : \widetilde{\mathcal{D}}(Y) \to W_Y/\mathrm{Aut}(\pi_Y) \times \mathbb{C}$ and $\varphi_{Y_1} : \widetilde{\mathcal{D}}(Y_1) \to W_{Y_1}/\mathrm{Aut}(\pi_{Y_1}) \times \mathbb{C}$ be local trivialisations and, on the intersection,

$$\psi_{Y_1Y} := \varphi_Y \circ \chi \circ \varphi_{Y_1}^{-1}.$$

The map ψ_{Y_1Y}, viewed as a function on $\mathbb{C}$, is invertible, with inverse ψ_{YY_1}. Indeed, the map χ is an isomorphism because $\widetilde{\alpha}$ is. The same argument proves that the cocycle condition holds. □

The following corollary is a consequence of Theorem 4.11 and of [22, Proposition 1.7].

Corollary 4.12. *Any compact analytic subspace of $\mathcal{M}$ is projective.*

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