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Absolutely non-computable predicates and functions in analysis†

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In the representation approach (TTE) to computable analysis, the representations of an algebraic or topological structure for which the basic predicates and functions become computable are of particular interest. There are, however, many predicates (like equality of real numbers) and functions that are *absolutely non-computable*, that is, not computable for any representation. Many of these results can be deduced from a simple lemma. In this article we prove this lemma for multi-representations and apply it to a number of examples. As applications, we show that various predicates and functions on computable measure spaces are absolutely non-computable. Since all the arguments are topological, we prove that the predicates are not relatively open and the functions are not relatively continuous for any multi-representation.

1. Introduction

In the representation approach to Computable Analysis (TTE) (Weihrauch 2000), computability on finite words $w \in \Sigma^*$ and infinite sequences $p \in \Sigma^\omega$ of symbols from a finite alphabet Σ is defined explicitly, for example, by Turing machines. Computability on other sets M is introduced via naming systems, that is, notations $\nu :\subseteq \Sigma^* \to M$ or multi-representations $\delta : \Sigma^\omega \rightrightarrows M$, where finite or infinite sequences of symbols serve as 'concrete names' of the 'abstract' objects $x \in M$. The concept of computability induced on M depends crucially on the naming system. For a mere set M there is no criterion to select from the set of all naming systems the relevant or interesting ones. If, however, a structure on M is given, for example, a topology or an algebra, the naming systems for which the functions and predicates from the structure become computable are of particular interest. There are examples of structures for which there is exactly one (up to equivalence) naming system with this property (Hertling 1999).

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On the other hand, there are functions and predicates that become computable for no representation – we call them 'absolutely non-computable'. For example, there is no representation δ of the real numbers for which equality is (δ,δ)-decidable, no representation of the closed subsets of Euclidean space for which intersection is $(\delta,\delta,\psi^<)$-computable and no representation δ of a non-separable metric space for which the distance is $(\delta,\delta,\rho_>)$-computable (Weihrauch 2000). In this paper we prove a number of other results of this type.

The above theorems of the form 'there is no representation such that ...' are concerned with binary predicates or functions. We prove as a central lemma that for multi-representations γ,δ, every (γ,δ)-open set is a *countable* union of Cartesian products. Applying this lemma, we prove for a variety of sets that they are not (γ,δ)-open, and for some binary functions that they cannot be (γ,δ,ψ)-continuous for any multi-representations γ,δ and ψ.

Since no choice of multi-representation can make them effective, we call such relations and functions 'absolutely non-effective'. We then obtain as immediate consequences the fact that the sets are not (γ,δ)-r.e. or (γ,δ)-decidable, and the functions are not (γ,δ,ψ)-computable for any multi-representations γ,δ and ψ. Up to the derivation of these consequences, the results in this paper are purely topological and require only the topological framework of TTE (Type 2 Theory of Effectivity) (Weihrauch 2000).

In Section 2 we summarise some basic definitions from TTE. In Section 3 we prove the central lemma on countable union of products and derive a number of 'non-continuities' as consequences. In Section 4 we consider a σ-algebra $\mathscr{A}$ of measurable sets with infinite measure that is generated by a σ-finite ring. We present a number of binary relations that cannot be (γ,δ)-open, and binary functions that cannot be (γ,δ,ψ)-continuous for any representations γ,δ of the set $\mathscr{A}$. This paper generalises and extends Ding *et al.* (2007).

2. Concepts from computable analysis

In this section we summarise some basic definitions from TTE – see Weihrauch (2000) for details. Let Σ be a sufficiently large finite alphabet of symbols, and let Σ^* be the set of all finite strings and Σ^ω be the set of all infinite sequences over Σ. For $w\in\Sigma^*$, let $w\Sigma^\omega\subseteq\Sigma^\omega$ be the set of all $p\in\Sigma^\omega$ such that w is a prefix of p. A partial function $f:\subseteq Y_1\times\ldots\times Y_k\to Y_0$ $(Y_0,\ldots,Y_k\in\{\Sigma^*,\Sigma^\omega\})$ is *computable* if it can be computed by a *Type-2 machine*, that is, a Turing machine with k input tapes and a one-way output tape for possibly one-way infinite inscriptions. For $(y_1,\ldots,y_k)\in Y_1\times\ldots\times Y_k$, we have $f(y_1,\ldots,y_k)=y_0$ if and only if on input $(y_1,\ldots,y_k)$ the machine halts with result $y_0\in\Sigma^*$ (for $Y_0=\Sigma^*$) or the machine computes forever and writes $y_0\in\Sigma^\omega$ (for $Y_0=\Sigma^\omega$). A set $Z\subseteq Y_1\times\ldots\times Y_k$ is *recursively enumerable* (r.e.) if $Z=\mathrm{dom}(f)$ for some computable function $f:\subseteq Y_1\times\ldots\times Y_k\to\Sigma^*$. Let $\iota:\Sigma^*\to\Sigma^*$, $a_1\ldots a_k\mapsto 110a_10\ldots0a_k011$ be the 'wrapping function'. The 'pairing function' $\pi:\Sigma^\omega\times\Sigma^\omega\to\Sigma^\omega$,

$$\pi(a_0a_1\ldots,b_0b_1\ldots):=a_0b_0a_1b_1\ldots\quad(a_i,b_i\in\Sigma),$$

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is bijective and computable and the projections of its inverse are computable. As usual, we write $\langle p, q\rangle = \pi(p, q)$. The straightforward generalisation of π to more than two arguments is also denoted by $\langle\ \rangle$.

On Σ^* we consider the discrete topology. On Σ^ω we consider the *Cantor topology* generated by the base $\{w\Sigma^\omega \mid w \in \Sigma^*\}$, and on $(\Sigma^\omega)^k$ its product topology generated by the base $\{v_1\Sigma^\omega \times \ldots \times v_k\Sigma^\omega \mid v_1, \ldots, v_k \in \Sigma^*\}$. Every partial computable function $f :\subseteq Y_1 \times \ldots \times Y_k \to Y_0$ is continuous. Every r.e. set $Z \subseteq Y_1 \times \ldots \times Y_k$ is open.

A *multi-function* is a triple $f = (X, R_f, Y)$ with $R_f \subseteq X \times Y$, written $f : X \rightrightarrows Y$. As usual, for $A \subseteq X$ and $B \subseteq Y$, we have $f[A] = \{y \in Y \mid (\exists x \in A)(x, y) \in R_f\}$ and $f^{-1}[B] = \{x \in X \mid (\exists y \in B)(x, y) \in R_f\}$. For $x \in X$, let $f(x) := f[\{x\}]$, with $\mathrm{graph}(f) := R_f$ the graph and $\mathrm{dom}(f) := f^{-1}[Y]$ the domain of f. A partial function $f :\subseteq X \to Y$ is treated as a multi-function for which every set $f(x)$ contains at most one element.

A *notation* is a surjective function $\nu :\subseteq \Sigma^* \to M$. Let $\nu_{\mathbb{N}}$ and $\nu_{\mathbb{Q}}$ be canonical notations of the natural and rational numbers, respectively. A *representation* is a surjective function $\delta :\subseteq \Sigma^\omega \to M$. Let ρ be the standard representation of the set of real numbers $\mathbb{R}$ (Weihrauch 2000) and let $\overline{\rho}_>$ be the upper representation of $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ defined by $\overline{\rho}_>(p) := \inf\{\nu_{\mathbb{Q}}(w) \mid \iota(w)$ is a subword of p and $w \in \mathrm{dom}(\nu_{\mathbb{Q}})\}$. Let $\rho_>$ be the restriction of $\overline{\rho}_>$ to $\mathbb{R}$ (Weihrauch 2000). A *multi-representation* is a surjective multi-function $\delta : \Sigma^\omega \rightrightarrows M$. A *naming system* is a notation, a representation or a multi-representation. Notations and representations are special multi-functions.

For naming systems $\delta_i : Y_i \rightrightarrows M_i$, a partial function $h :\subseteq Y_1 \times \ldots \times Y_k \to Y_0$ *realises* a partial function $f :\subseteq M_1 \times \ldots \times M_k \to M_0$ with respect to $(\delta_1, \ldots, \delta_k, \delta_0)$ if and only if

$$f[\delta_1(y_1) \times \ldots \times \delta_k(y_k)] \subseteq \delta_0(h(y_1, \ldots, y_k))$$

for all $(y_1, \ldots, y_k) \in Y_1 \times \ldots \times Y_k$ (Schröder 2003; Weihrauch 2008). The function f is $(\delta_1, \ldots, \delta_k, \delta_0)$-*continuous (-computable)* if and only if it has a continuous (computable) $(\delta_1, \ldots, \delta_k, \delta_0)$-realisation.

A set $X \subseteq M_1 \times \ldots \times M_k$ is called $(\delta_1, \ldots, \delta_k)$-*open* (-r.e.) if and only if there is some open set (r.e. set) $U \subseteq Y_1 \times \ldots \times Y_k$ such that

$$(x_1, \ldots x_k) \in X \iff (y_1, \ldots, y_k) \in U \tag{1}$$

whenever $(x_1, \ldots x_k) \in \delta(y_1) \times \ldots \times \delta(y_k)$. X is said to be $(\delta_1, \ldots, \delta_k)$-decidable if X and its complement are $(\delta_1, \ldots, \delta_k)$-r.e.

For multi-representations $\delta_1, \ldots, \delta_k$, the tupling $[\delta_1, \ldots, \delta_k] :\subseteq \Sigma^\omega \to M_1 \times \ldots \times M_k$ is defined by $[\delta_1, \ldots, \delta_k]\langle y_1, \ldots, y_k\rangle := \delta_1(y_1) \times \ldots \times \delta_k(y_k)$. In this paper we will need the definitions mainly for the cases $k = 1$ or $k = 2$. For representations $\gamma :\subseteq \Sigma^\omega \to M$, $\delta :\subseteq \Sigma^\omega \to M'$ and $\psi :\subseteq \Sigma^\omega \to M''$, and for $X \subseteq M \times M'$ and $f :\subseteq M \times M' \to M''$, we have (Weihrauch 2000):

$$X \text{ is } (\gamma, \delta)\text{-decidable} \implies X \text{ is } (\gamma, \delta)\text{-r.e.} \implies X \text{ is } (\gamma, \delta)\text{-open}, \tag{2}$$

$$X \text{ is } (\gamma, \delta)\text{-PP} \iff X \text{ is } [\gamma, \delta]\text{-PP (for PP} \in \{\text{decidable, r.e., open}\}), \tag{3}$$

$$f \text{ is } (\gamma, \delta, \psi)\text{-PP} \iff f \text{ is } ([\gamma, \delta], \psi)\text{-PP (for PP} \in \{\text{computable, continuous}\}). \tag{4}$$

We will apply these properties without further discussion. We use $\delta|^N$ to denote the restriction of δ to range N.

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Lemma 2.1. Let $\delta : \Sigma^\omega \rightrightarrows M$ and $\delta' : \Sigma^\omega \rightrightarrows M'$ be multi-representations and $X \subseteq M$ be δ-open.

1 If $f : M' \to M$ is (δ', δ)-continuous, then $f^{-1}[X]$ is δ'-open.
2 For any $N \subseteq M$, $X \cap N$ is $\delta|^N$-open.

Proof.

1 Since X is δ-open, there is some open set $U \subseteq \Sigma^\omega$ such that for all $(p, x) \in \mathrm{graph}(\delta)$, $x \in X \iff p \in U$. Since f is (δ', δ)-continuous, there is a continuous function $h : \mathrm{dom}(\delta') \to \Sigma^\omega$ such that $(q, y) \in \mathrm{graph}(\delta') \Rightarrow (h(q), f(y)) \in \mathrm{graph}(\delta)$. Suppose $(q, y) \in \mathrm{graph}(\delta')$. Then $(h(q), f(y)) \in \mathrm{graph}(\delta)$ and $y \in f^{-1}[X] \iff f(y) \in X \iff h(q) \in U \iff q \in h^{-1}[U]$. Since U is open and h is continuous (with respect to the induced topology on $\mathrm{dom}(\delta')$), we have $h^{-1}[U]$ is open in $\mathrm{dom}(\delta')$, and thus $h^{-1}[U] = V \cap \mathrm{dom}(\delta')$ for some open $V \subseteq \Sigma^\omega$. Therefore, $y \in f^{-1}[X] \iff q \in V$ whenever $(q, y) \in \mathrm{graph}(\delta')$, hence $f^{-1}[X]$ is δ'-open.
2 The embedding $f : N \to M$ is $(\delta|^N, \delta)$-continuous (realised by the identity on Σ^ω). Furthermore, $f^{-1}[X] = X \cap N$. Finally, apply Part 1 to $\delta' := \delta|^N$. □

3. Absolutely non-open sets

The main results of this paper are based on the following observation.

Lemma 3.1. Let $\gamma : \Sigma^\omega \rightrightarrows M$ and $\delta : \Sigma^\omega \rightrightarrows M'$ be multi-representations. Then for every non-empty (γ, δ)-open set $X \subseteq M \times M'$ there are sequences $(U_k)_{k\in\mathbb{N}}$ and $(V_k)_{k\in\mathbb{N}}$ of non-empty sets such that

$$X = \bigcup_{k\in\mathbb{N}} U_k \times V_k \,. \tag{5}$$

Proof. Since X is (γ, δ)-open, there is some open set $W \subseteq \Sigma^\omega \times \Sigma^\omega$ such that for all $(p, x) \in \mathrm{graph}(\gamma)$ and all $(q, y) \in \mathrm{graph}(\delta)$, we have $((x, y) \in X \iff (p, q) \in W)$. Since W is open and $X \neq \emptyset$, there are words u_k, v_k such that $W = \bigcup_{k\in\mathbb{N}} u_k\Sigma^\omega \times v_k\Sigma^\omega$. Define $U_k \subseteq M$ and $V_k \subseteq M'$ by $U_k := \gamma[u_k\Sigma^\omega]$ and $V_k := \delta[v_k\Sigma^\omega]$.

Then for all $x \in M$ and $y \in M'$,

$$\begin{aligned}
(x, y) \in X &\iff (\exists\, p, q)((p, x) \in \mathrm{graph}(\gamma) \wedge (q, y) \in \mathrm{graph}(\delta) \wedge (p, q) \in W) \\
&\iff (\exists k)(\exists\, p, q)((p, x) \in \mathrm{graph}(\gamma) \wedge (q, y) \in \mathrm{graph}(\delta) \wedge \\
&\qquad\qquad p \in u_k\Sigma^\omega \wedge q \in v_k\Sigma^\omega) \\
&\iff (\exists k)(x \in U_k \wedge y \in V_k) \\
&\iff (x, y) \in \bigcup_{k\in\mathbb{N}} U_k \times V_k \,.
\end{aligned}$$

□

By Equation (5), X is the *countable* union of Cartesian products. The lemma can be generalised straightforwardly to more than 2 factors.

Using Lemma 3.1, we will prove that several sets are not (γ, δ)-open and functions f are not (γ, δ, ψ)-continuous for *all* multi-representations γ, δ and ψ. By Formulas (2, 3, 4), these results have a number of obvious implications, which we will not mention repeatedly. Many applications of Lemma 3.1 follow from the following lemma.

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Lemma 3.2.

1 Let $\sim \subseteq M \times M$ be an equivalence relation such that $M/_\sim$ is uncountable. Then there are no multi-representations γ, δ of M such that $\sim$ is (γ, δ)-open.
2 Let $\sim \subseteq M \times M$ be an equivalence relation such that the set $M/_\sim$ of equivalence classes is uncountable. Let $\prec \subseteq M \times M$ be a relation on M such that

$$a \prec a \quad \text{and} \quad (a \prec b \quad \text{and} \quad b \prec a \implies a \sim b) \tag{6}$$

for all $a, b \in M$. Then there are no multi-representations γ, δ of M such that $\prec$ is (γ, δ)-open.

Proof. We prove Part 2 first:

2 Assume $\prec = \bigcup_{k \in \mathbb{N}} U_k \times V_k$. If $(x,x),(y,y) \in U_k \times V_k$, then $(x,y) \in U_k \times V_k$ and $(y,x) \in U_k \times V_k$, and hence $x \sim y$ by (6). Since, by (6), every (x,x) must be in some $U_i \times V_i$, we have $M/_\sim$ is countable, which gives a contradiction.
1 Apply Part 2 to $\prec := \sim$. □

Let $A \Delta B := (A \setminus B) \cup (B \setminus A)$ be the symmetric difference of the sets A and B. The following theorem summarises some more concrete results.

Theorem 3.3.

1 For an uncountable set M there are no multi-representations γ, δ such that $\mathrm{Eq} = \{(x,y) \mid x, y \in M, \quad x = y\}$ is (γ, δ)-open.
2 For the set $\mathbb{R}$ of real numbers there are no multi-representations γ, δ such that $\mathrm{Leq} := \{(x,y) \mid x \leqslant y\}$ is (γ, δ)-open.
3 For any uncountable collection $\mathscr{S}$ of sets there are no multi-representations γ, δ such that $\mathrm{Sub} := \{(A,B) \mid A, B \in \mathscr{S}, \quad A \subseteq B\}$ is (γ, δ)-open.
4 For any collection $\mathscr{S}$ of sets containing uncountably many singletons (that is, the one-element sets) there are no multi-representations γ, δ such that $\mathrm{Nei} := \{(A,B) \mid A, B \in \mathscr{S}, \quad A \cap B \neq \varnothing\}$ is (γ, δ)-open.
5 For each of the following binary relations T on $2^{\mathbb{N}}$:
   — $A \cap B = \varnothing$
   — $A \cap B$ is finite
   — $A \setminus B = \varnothing$
   — $A \setminus B$ is finite
   — $A \Delta B = \varnothing$
   — $A \Delta B$ is finite

   there are no multi-representations γ, δ of $2^{\mathbb{N}}$ such that T is (γ, δ)-open.

Proof.

1 Eq is an equivalence relation. Apply Lemma 3.2.1.
2 Apply Lemma 3.2.2 with $\leqslant$ and $=$ for $\prec$ and $\sim$, respectively.
3 Apply Lemma 3.2.2 with $\subseteq$ and $=$ for $\prec$ and $\sim$, respectively.

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4 If Nei is (γ,δ)-open, the uncountably many pairs $(\{x\},\{x\})$ must be elements of a countable union $\bigcup_i X_i \times Y_i$. If $(\{x\},\{x\}) \in X_i \times Y_i$ and $(\{y\},\{y\}) \in X_i \times Y_i$, then $(\{x\},\{y\}) \in X_i \times Y_i$, hence $x = y$. Therefore, each $X_i \times Y_i$ contains at most one pair $(\{x\},\{x\})$. But there are uncountably many such pairs.

5 Let $\pi : \mathbb{N}^2 \to \mathbb{N}$ be a bijection. For $i \in \mathbb{N}$ let $C_i := \{\pi(i,j) \mid j \in \mathbb{N}\}$, and for $L \subseteq \mathbb{N}$ $(L \neq \varnothing, \mathbb{N})$, let $A_L := \bigcup_{i\in L} C_i$. Then all the A_L constitute an uncountable family of infinite sets such that A_K and A_L differ by an infinite set if $K \neq L$. If T is (γ,δ)-open, by Lemma 3.1 it can be written as $T = \bigcup_{i\in\mathbb{N}} X_i \times Y_i$.

— $A \cap B = \varnothing$:
For each L, we have $A_L \cap A_L^c = \varnothing$. If $(A_K, A_K^c), (A_L, A_L^c) \in X_i \times Y_i$, then $A_K \cap A_L^c = \varnothing$ and $A_L \cap A_K^c = \varnothing$, hence $A_K = A_L$. Therefore, each $X_i \times Y_i$ contains at most one pair (A_L, A_L^c). But the set of such pairs is uncountable.

— $A \cap B$ is finite:
Proceed as above and observe that $A_K \cap A_L^c = \varnothing$ if $A_K \cap A_L^c$ is finite.

— $A \setminus B = \varnothing$:
For each L, we have $A_L \setminus A_L = \varnothing$. If $(A_K, A_K), (A_L, A_L) \in X_i \times Y_i$, then $A_K \setminus A_L = \varnothing$ and $A_L \setminus A_K = \varnothing$, hence $A_K = A_L$. Therefore, each $X_i \times Y_i$ contains at most one pair (A_L, A_L). But the set of such pairs is uncountable.

— $A \setminus B$ is finite:
Proceed as above and observe that $A_K \setminus A_L = \varnothing$ if $A_K \setminus A_L$ is finite.

— $A\Delta B = \varnothing$:
For each L, we have $A_L \Delta A_L = \varnothing$. If $(A_K, A_K), (A_L, A_L) \in X_i \times Y_i$, we have $A_K \Delta A_L = \varnothing$, hence $A_K = A_L$. Therefore, each $X_i \times Y_i$ contains at most one pair (A_L, A_L). But the set of such pairs is uncountable.

— $A\Delta B$ is finite:
Proceed as above and observe that $A_K \Delta A_L = \varnothing$ if $A_K \Delta A_L$ is finite. □

Theorem 3.3.1 applies, for example, to the real numbers (Weihrauch 2000, Theorem 4.1.16), to the powerset $2^{\mathbb{N}}$ of $\mathbb{N}$, to the various subset spaces of Euclidean space $\mathbb{R}^n$ and to spaces of continuous functions (Weihrauch 2000, Chapters 5 and 6). Notice, however, that *inequality* may become r.e., for example, for the standard representation ρ of the real numbers, $\{(x,y) \in \mathbb{R}^2 \mid x \neq y\}$ is (ρ,ρ)-r.e. Theorem 3.3.3 can be applied to $2^{\mathbb{N}}$ and to the open, the closed and the compact subsets of $\mathbb{R}^{\mathbb{N}}$. Theorem 3.3.4 can be applied to the closed and the compact subsets of $\mathbb{R}^{\mathbb{N}}$.

Every represented metric space with continuous distance is separable, that is, it has a countable dense subset. The proofs in Weihrauch (1993) and the proof of Lemma 8.1.1 in Weihrauch (2000) essentially use Lemma 3.1. Here we prove a slight generalisation.

Theorem 3.4. Let γ,δ be representations of a metric space (M,d) such that the distance $d : M \times M \to \mathbb{R}$ is $(\gamma,\delta,\rho_<)$-continuous. Then the space is separable.

Proof. For each $n \in \mathbb{N}$ define an open subset of Σ^ω by

$$U_n := \bigcup \{u\iota(v)\Sigma^\omega \mid u,v \in \Sigma^*,\ v \in \mathrm{dom}(\nu_{\mathbb{Q}}) \ \text{ and } \ \nu_{\mathbb{Q}}(v) < 2^{-n}\},$$

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recalling that ι is the 'wrapping function', and let $W_n := \{x \in \mathbb{R} \mid x < 2^{-n}\}$. Then $\rho_>^{-1}[W_n] = U_n \cap \mathrm{dom}(\rho_>)$, hence W_n is $\rho_>$-open. By assumption, we have that d is $([\gamma, \delta], \rho_>)$-continuous. By Lemma 2.1.1, $d^{-1}[W_n]$ is (γ, δ)-open and hence by Lemma 3.1 there are (non-empty) sets $X_k^n, Y_k^n \subseteq M$ such that

$$\{(a,b) \in M \times M \mid d(a,b) < 2^{-n}\} = d^{-1}[W_n] = \bigcup_{k \in \mathbb{N}} X_k^n \times Y_k^n .$$

For every $n, k \in \mathbb{N}$, choose some $x_k^n \in X_k^n$. Suppose $x \in M$ and $n \in \mathbb{N}$. Then for some k, we have $(x,x) \in X_k^n \times Y_k^n$, so $(x_k^n, x) \in X_k^n \times Y_k^n$, that is, $d(x_k^n, x) < 2^{-n}$. Therefore, the countable set $\{x_k^n \mid n, k \in \mathbb{N}\}$ is dense in (M,d). □

Notice that d is $(\gamma, \delta, \rho_>)$-continuous if d is (γ, δ, ρ)-continuous.

By Theorem 5.1.3 in Weihrauch (2000), intersection on the closed subsets of $\mathbb{R}^n$ is not $(\psi, \psi, \psi_<)$-continuous. We can conclude this from the observation that the set of non-empty closed subsets of $\mathbb{R}^n$ is $\psi_<$-open. More generally, from Theorem 3.3.4, we get the following corollary.

Corollary 3.5. Let ψ be a representation of a collection $\mathscr{S}$ of sets containing uncountably many singletons such that $\{A \in \mathscr{S} \mid A \neq \varnothing\}$ is ψ-open. Then there are no multi-representations γ, δ such that intersection on $\mathscr{S}$ is (γ, δ, ψ)-continuous.

Proof. Suppose that intersection on $\mathscr{S}$ is $([\gamma, \delta], \psi)$-continuous. Since Nem $:= \{A \in \mathscr{S} \mid A \neq \varnothing\}$ is ψ-open, we have $\cap^{-1}[\mathrm{Nem}] = \{(A,B) \in \mathscr{S} \times \mathscr{S} \mid A \cap B \neq \varnothing\} = \mathrm{Nei}$ is (γ, δ)-open by Lemma 2.1.1. But this is false by Theorem 3.3.4. □

The corollary is applicable to the set of closed subsets of an uncountable T_1-space since for every point x the set $\{x\}$ is closed (Engelking 1989).

4. Applications to computable measure theory

Incomputability of the set-theoretical operators was discussed in Wu (2005). Here we have some extended results with completely new proofs applying the central lemmas (Lemmas 3.1 and 3.2).

Computable measure spaces were introduced in Wu and Weihrauch (2006). We recall the main definitions. As references for measure theory, see Bauer (1974), Bauer (2001) and Cohn (1980).

A *ring* in a set Ω is a class $\mathscr{R}$ of subsets of Ω such that $\varnothing \in \mathscr{R}$ and $A \cup B \in \mathscr{R}$ and $A \setminus B \in \mathscr{R}$ if $A, B \in \mathscr{R}$. A *σ-algebra* in Ω is a class $\mathscr{A}$ of subsets of Ω such that $\Omega \in \mathscr{A}$ and $\Omega \setminus A \in \mathscr{A}$ if $A \in \mathscr{A}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$ if $A_1, A_2, \ldots \in \mathscr{A}$. For any system $\mathscr{E}$ of subsets of Ω, let $\sigma(\mathscr{E})$ be the smallest σ-algebra in Ω containing $\mathscr{E}$. A *premeasure* on a ring $\mathscr{R}$ is a function $\mu : \mathscr{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ such that $\mu(\varnothing) = 0$, $\mu(A) \geqslant 0$ for $A \in \mathscr{R}$ and

$$\mu\Big(\bigcup_{i=1}^{\infty} A_n\Big) = \sum_{i=1}^{\infty} \mu(A_n)$$

if $A_1, A_2, \ldots \in \mathscr{A}$ are pairwise disjoint and $\bigcup_{i=1}^{\infty} A_n \in \mathscr{A}$. A premeasure μ on a σ-algebra $\mathscr{A}$ is called a *measure*. In this case, $(\Omega, \mathscr{A}, \mu)$ is called a *measure space*.

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Definition 4.1 (Wu and Weihrauch 2006). A *computable measure space* is a quintuple $\mathscr{M} = (\Omega, \mathscr{A}, \mu, \mathscr{R}, \alpha)$ such that:
1 $(\Omega, \mathscr{A}, \mu)$ is a measure space.
2 $\mathscr{R}$ is a countable ring such that $\mathscr{A} = \sigma(\mathscr{R})$, $\Omega = \bigcup \mathscr{R}$, and $\mu(A) < \infty$ for all $A \in \mathscr{R}$.
3 $\alpha :\subseteq \Sigma^* \to \mathscr{R}$ is a notation of $\mathscr{R}$ with recursive domain such that union and difference are (α, α, α)-computable and μ is (α, ρ)-computable on $\mathscr{R}$.

Since our negative results concern only continuity, we do not need the third axiom on computability. Therefore, in the following, let $\mathscr{M} = (\Omega, \mathscr{A}, \mu, \mathscr{R})$ be a measure space such that Parts 1 and 2 hold true. Since most cases of non-effectivity occur for sets with infinite measure, we assume $\mu(\Omega) = \infty$. Let $\mathscr{A}_0$ be the set of all $A \in \mathscr{A}$ with finite measure and let $\mathscr{A}_{\infty\infty}$ be the set of all $A \in \mathscr{A}$ such that A and its complement A^c have infinite measure.

By Definition, $\Omega = \bigcup \mathscr{R}$. The whole set Ω can also be exhausted by a sequence of pairwise disjoint ring elements each of which has measure $\mu(D) > 1$ and by a sequence of pairwise disjoint algebra elements each of which has measure $\mu(G) = \infty$.

Lemma 4.2.
1 There is a sequence $(D_i)_{i\in\mathbb{N}}$ of ring elements such that
$$\mu(D_i) > 1 \text{ and } D_i \cap D_j = \varnothing \text{ for } i \neq j \text{ and } \textstyle\bigcup_i D_i = \Omega.$$
2 There is a sequence $(G_i)_{i\in\mathbb{N}}$ of algebra elements such that
$$\mu(G_i) = \infty \text{ and } G_i \cap G_j = \varnothing \text{ for } i \neq j \text{ and } \textstyle\bigcup_i G_i = \Omega.$$
3 For $L \subseteq \mathbb{N}$, let $G_L := \bigcup_{i\in L} G_i$. Then
$$\mu(G_L) = 0 \text{ for } L = \varnothing, \mu(G_L) = \infty \text{ for } L \neq \varnothing \tag{7}$$
$$G_L \in \mathscr{A}_{\infty\infty} \quad \text{for} \quad L \notin \{\varnothing, \mathbb{N}\} \tag{8}$$
$$G_L^c = G_{L^c} \quad \text{and} \quad G_K \diamond G_L = G_{K\diamond L} \quad \text{for binary boolean operations } \diamond\,. \tag{9}$$

Proof.
1 Let $B_0, B_1, \ldots$ be a numbering of $\mathscr{R}$. Define $C_i := B_i \setminus (B_0 \cup \ldots \cup B_{i-1})$. Then for $j < i$, we have $B_j \cap C_i = \varnothing$, so $C_j \cap C_i = \varnothing$. By induction, $B_0 \cup \ldots \cup B_i = C_0 \cup \ldots \cup C_i$. Therefore, $\bigcup_i C_i = \bigcup_i B_i = \Omega$. There are numbers $0 = i_0 < i_1 < i_2 < \ldots$ such that $\mu \bigcup \{C_m \mid i_k \leqslant m < i_{k+1}\} > 1$ for all k. Define $D_k := \bigcup \{C_m \mid i_k \leqslant m < i_{k+1}\}$.
2 Let $\pi : \mathbb{N}^2 \to \mathbb{N}$ be a bijection. Define $G_i := \bigcup_{j\in\mathbb{N}} D_{\pi(i,j)}$ with D_i from Part 1.
3 This statement is obvious. □

The proof can be changed easily to a computable version of the lemma, which, however, we do not need in this paper.

Since our ring $\mathscr{R}$ is countable, the cardinality of the generated σ-algebra $\mathscr{A} = \sigma(\mathscr{R})$ is not greater than the cardinality of Σ^ω. Therefore, the set $\mathscr{A}$ has a representation. Several concrete representations of $\mathscr{A}$ were defined in Wu and Ding (2005) and Wu and Ding (2006), and the computability of boolean functions and measure on sets were then discussed with respect to these.

In the following theorem we list a number of properties that are not open for any multi-representations of measurable sets. For measurable sets A, B, let $A \subset B$ almost everywhere (a.e.) if and only if $\mu(B \setminus A) > 0$ and $\mu(A \setminus B) = 0$.

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Theorem 4.3. Let $E \in \mathscr{A}$ such that $\mu(E) < \infty$. In each of the following 13 cases of $Q(A, B)$, there are no multi-representations γ, δ of $\mathscr{A}$ such that

$$X := \{(A, B) \mid A, B \in \mathscr{A}, \quad Q(A, B)\} \text{ is } (\gamma, \delta)\text{-open :}$$

1 $A \subset B$ a.e.
2 — $\mu(A \setminus B) = 0$
— $\mu((A \setminus B) \setminus E) = 0$
— $A \setminus B \subset E$ a.e.
— $\mu(A \setminus B) < \infty$.
3 — $\mu(A \Delta B) = 0$
— $\mu((A \Delta B) \setminus E) = 0$
— $A \Delta B \subset E$ a.e.
— $\mu(A \Delta B) < \infty$.
4 — $\mu(A \cap B) = 0$
— $\mu((A \cap B) \setminus E) = 0$
— $A \cap B \subset E$ a.e.
— $\mu(A \cap B) < \infty$.

The statement holds accordingly for multi-representations γ, δ of $\mathscr{A}_{\infty\infty}$ and the sets $Y := \{(A, B) \mid A, B \in \mathscr{A}_{\infty\infty}, \quad Q(A, B)\}$.

Proof. First we consider multi-representations γ, δ of $\mathscr{A}_{\infty\infty}$.

Let $\Gamma := \{L \subseteq \mathbb{N} \mid 0 \notin L,\ 1 \notin L,\ 2 \in L\}$. Then by Lemma 4.2.3, $G_K, G_{K\cup\{0\}} \in \mathscr{A}_{\infty\infty} \subseteq \mathscr{A}$ for all $K \in \Gamma$. In the following, for each of the relations $Q(A, B)$, we assume that it is (γ, δ)-open and hence can be written as $\bigcup_{k\in\mathbb{N}} U_k \times V_k$.

1 $A \subset B$ a.e.:

By Lemma 4.2.3, for $K, L \in \Gamma$,

$$\begin{aligned}
&(G_K, G_{K\cup\{0\}}) \in U_k \times V_k \quad \text{and} \quad (G_L, G_{L\cup\{0\}}) \in U_k \times V_k \\
&\qquad \Longrightarrow (G_L, G_{K\cup\{0\}}) \in U_k \times V_k \quad \text{and} \quad (G_K, G_{L\cup\{0\}}) \in U_k \times V_k \\
&\qquad \Longrightarrow \mu(G_L \setminus G_{K\cup\{0\}}) = 0 \quad \text{and} \quad \mu(G_K \setminus G_{L\cup\{0\}}) = 0 \qquad (10) \\
&\qquad \Longrightarrow L \subseteq K \quad \text{and} \quad K \subseteq L \\
&\qquad \Longrightarrow K = L.
\end{aligned}$$

Since for every $K \in \Gamma$ we have $G_K \subset G_{K\cup\{0\}}$ a.e., and hence there is some k such that $(G_K, G_{K\cup\{0\}}) \in U_k \times V_k$, there are only countably many $K \in \Gamma$, which gives a contradiction.

2 — $\mu(A \setminus B) = 0$:

Copy the proof for '$A \subset B$ a.e.'.

— $\mu((A \setminus B) \setminus E) = 0$:

Replace (10) in the proof of Part 1 by

$$\mu((G_L \setminus G_{K\cup\{0\}}) \setminus E) = 0 \quad \text{and} \quad \mu((G_K \setminus G_{L\cup\{0\}}) \setminus E) = 0$$

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and notice that by (7), $\mu(G_N \setminus E) = 0$ for $N = \varnothing$ and $= \infty$ otherwise.

— $A \setminus B \subset E$ a.e.:

Copy the proof for '$\mu((A \setminus B) \setminus E) = 0$'.

— $\mu(A \setminus B) < \infty$:

Copy the proof for '$A \subset B$ a.e.' and use (7).

3 — $\mu(A \Delta B) = 0$:

By Lemma 4.2.3, for $K, L \in \Gamma$,

$$\begin{aligned}(G_K, G_K) \in U_k \times V_k \ \text{ and } \ (G_L, G_L) \in U_k \times V_k \\ \Longrightarrow (G_L, G_K) \in U_k \times V_k \\ \Longrightarrow \mu(G_L \Delta G_K) = 0 \\ \Longrightarrow L = K\,.\end{aligned} \tag{11}$$

Since for every $K \in \Gamma$ there is some k such that $(G_K, G_K) \in U_k \times V_k$, there are only countably many $K \in \Gamma$, which gives a contradiction.

— $\mu((A \Delta B) \setminus E) = 0$:

Replace (11) in the proof for '$\mu(A \Delta B) = 0$' by '$\mu((G_L \Delta G_K) \setminus E) = 0$'.

— $A \Delta B \subset E$ a.e.:

Copy the proof for '$\mu((A \Delta B) \setminus E) = 0$'.

— $\mu(A \Delta B) < \infty$:

Copy the proof for '$\mu(A \Delta B) = 0$'.

4 — $\mu(A \cap B) = 0$:

By Lemma 4.2.3, for $K, L \in \Gamma$,

$$\begin{aligned}(G_K, G_K^c) \in U_k \times V_k \ \text{ and } \ (G_L, G_L^c) \in U_k \times V_k \\ \Longrightarrow (G_L, G_K^c) \in U_k \times V_k \ \text{ and } \ (G_K, G_L^c) \in U_k \times V_k \\ \Longrightarrow \mu(G_{L \cap K^c}) = 0 \ \text{ and } \ \mu(G_{K \cap L^c}) = 0 \\ \Longrightarrow L \subseteq K \ \text{ and } \ K \subseteq L \\ \Longrightarrow K = L\,.\end{aligned}$$

Since for every $K \in \Gamma$ there is some k such that $(G_K, G_K^c) \in U_k \times V_k$ there are only countably many $K \in \Gamma$, which gives a contradiction.

— Last three cases:

Modify the proof for '$\mu(A \cap B) = 0$' accordingly (see the proofs for the cases '$A \setminus B$').

Finally, by Lemma 2.1.2 and (3,4) the 13 relations cannot be (γ, δ)-open for any multi-representations γ, δ of $\mathscr{A}$. □

For a reasonable representation of sets of a measure space, the boolean functions and the measure should be computable. We show that this cannot be achieved in general.

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Theorem 4.4. Let $\psi :\subseteq \Sigma^\omega \to \mathscr{A}$ be a multi-representation such that the measure μ is $(\psi, \overline{\rho}_>)$-continuous. Then for any representations γ, δ of $\mathscr{A}$:

1 Intersection is not (γ, δ, ψ)-continuous on $\mathscr{A}_{\infty\infty}$.
2 Intersection is not (γ, δ, ψ)-continuous.
3 Difference is not (γ, δ, ψ)-continuous on $\mathscr{A}_{\infty\infty}$.
4 Difference $A \setminus B$ is not (γ, δ, ψ)-continuous.
5 Union and complement are not simultaneously continuous with respect to ψ.

Proof.

1 Assume that intersection restricted to $\mathscr{A}_{\infty\infty}$ is (γ, δ, ψ)-continuous. Then the function $f : (A, B) \mapsto \mu(A \cap B)$ is $(\gamma, \delta, \overline{\rho}_>)$-continuous, and hence $([\gamma, \delta], \overline{\rho}_>)$-continuous on $D := \mathscr{A}_{\infty\infty} \times \mathscr{A}_{\infty\infty}$ by (4), and hence $([\gamma, \delta]|^D, \overline{\rho}_>)$-continuous. The set $\mathbb{R} \subseteq \overline{\mathbb{R}}$ is $\overline{\rho}_>$-open (Weihrauch 2000). By Lemma 2.1.1, $f^{-1}(\mathbb{R}) = \{(A, B) \in D \mid \mu(A \cap B) < \infty\}$ is $[\gamma, \delta]|^D$-open, and hence (γ', δ')-open by (3) (where γ' and δ' are the restrictions of γ and δ to $\mathscr{A}_{\infty\infty}$, respectively. This is false by Theorem 4.3.
2 This follows immediately from Part 1.
3 This is similar to Parts 1 and 2.
4 This is similar to Parts 1 and 2.
5 Suppose that union and complement on $\mathscr{A}$ are continuous with respect to ψ. Since $A \cap B = (A^c \cup B^c)^c$, intersection on $\mathscr{A}$ is (ψ, ψ, ψ)-continuous, which is impossible by Part 2. □

Usually in measure theory sets that differ by a set of measure 0 are not distinguished. For $A, B \in \mathscr{A}$, let $A \equiv B \iff \mu(A \Delta B) = 0$. Then $\equiv$ is an equivalence relation. If $A \equiv B$ and $A' \equiv B'$, then $A^c \equiv B^c$ and $A \diamond A' \equiv B \diamond B'$ for every binary boolean operation $\diamond$. Therefore, on the set $\mathscr{A}/_\equiv$ of equivalence classes, the complement and the binary boolean functions are well defined by $(A/_\equiv)^c := (A^c)/_\equiv$ and $A/_\equiv \diamond B/_\equiv := (A \diamond B)/_\equiv$. Furthermore, a measure $\mu/_\equiv$ on $\mathscr{A}/_\equiv$ is well defined by $\mu/_\equiv(A/_\equiv) := \mu(A)$.

For every representation $\delta' :\subseteq \Sigma^\omega \to \mathscr{A}/_\equiv$, we can define a multi-representation $\delta : \Sigma^\omega \rightrightarrows \mathscr{A}$ by $\delta[\{p\}] := \delta'(p)$. From Theorem 4.3, we can deduce a corresponding theorem for $\mathscr{A}/_\equiv$.

Theorem 4.5. Let $e \in \mathscr{A}/_\equiv$ such that $\mu(e) < \infty$. In each of the following 13 cases of $Q'(a, b)$, there are no representations γ', δ' of $\mathscr{A}/_\equiv$ such that

$$X' := \{(a, b) \mid a, b \in \mathscr{A}/_\equiv,\ \ Q'(a, b)\} \text{ is } (\gamma', \delta')\text{-open :}$$

1 $a \subset b$ a.e.
2 — $\mu(a \setminus b) = 0$
— $\mu((a \setminus b) \setminus e) = 0$
— $a \setminus b \subset e$ a.e.
— $\mu(a \setminus b) < \infty$.
3 — $\mu(a \Delta b) = 0$
— $\mu((a \Delta b) \setminus e) = 0$

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— $a \Delta b \subset e$ a.e.

— $\mu(a \Delta b) < \infty$.

4 — $\mu(a \cap b) = 0$

— $\mu((a \cap b) \setminus e) = 0$

— $a \cap b \subset e$ a.e.

— $\mu(a \cap b) < \infty$.

The statement holds accordingly for representations γ', δ' of $\mathscr{A}_{\infty\infty}/_{\equiv}$ and the sets $Y := \{(a,b) \mid a,b \in \mathscr{A}_{\infty\infty}/_{\equiv}, \quad Q'(a,b)\}$.

Proof. For each of the predicates Q, we have $Q'(A/_{\equiv}, B/_{\equiv}) \iff Q(A,B)$. Suppose $X' := \{(a,b) \mid a,b \in \mathscr{A}/_{\equiv}, \quad Q'(a,b)\}$ is (γ', δ')-open. Then there is some $U \subseteq \Sigma^\omega \times \Sigma^\omega$ such that

$$(A/_{\equiv} = \gamma'(p) \text{ and } B/_{\equiv} = \delta'(q)) \implies (Q'(A/_{\equiv}, B/_{\equiv}) \iff (p,q) \in U).$$

Suppose $A \in \gamma(p)$ and $B \in \delta(q)$. Then $(A/_{\equiv} = \gamma'(p)$ and $B/_{\equiv} = \delta'(q))$, hence $Q'(A/_{\equiv}, B/_{\equiv}) \iff (p,q) \in U$. We also get $Q(A,B) \iff (p,q) \in U$. Therefore, X is (γ, δ)-open by (1).

By Theorem 4.3, the predicates Q' cannot be (γ', δ')-open. □

Corollary 4.6. Theorem 4.4 holds accordingly for representations of the space $\mathscr{A}/_{\equiv}$.

Proof. The proof is straightforward. □

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