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A DECISION PROCEDURE FOR PROBABILITY CALCULUS WITH APPLICATIONS

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Abstract. A decision procedure (PrSAT) for classical (Kolmogorov) probability calculus is presented. This decision procedure is based on an existing decision procedure for the theory of real closed fields, which has recently been implemented in *Mathematica*. A *Mathematica* implementation of PrSAT is also described, along with several applications to various non-trivial problems in the probability calculus.

1. Probability calculus: Axiomatic presentation. The standard presentation of probability calculus is that of Kolmogorov (1956). His presentation is axiomatic and couched in set-theoretic terms. I will use Kolmogorov's axioms here, but I will couch them in logical rather than set-theoretic terms. This choice is merely conventional, but it will make subsequent sections of the paper easier to digest.

I will be concerned only with finite probability models (i.e., probability models over finite Boolean algebras). As such, I begin with a *sentential language* $\mathcal{L}_n$, containing n atomic sentences $\{A_1, \ldots, A_n\}$ and all wffs (well-formed formulas) that can be constructed from them using the standard truth-functional connectives. I will use '$\top$' ('$\bot$') to denote an arbitrary tautology (contradiction) of $\mathcal{L}_n$, '$\dashv\models$' to denote the metatheoretic relation of tautological entailment in $\mathcal{L}_n$ ('$\vDash$' will denote tautological equivalence in $\mathcal{L}_n$), and lowercase italic letters $p, q, r, \ldots$ will be metavariables ranging over sentences of $\mathcal{L}_n$. With this background in place, we have the following definition.

Definition. *A probability model* $\mathfrak{M}_n = \langle \mathcal{L}_n, \Pr \rangle$ *consists of a sentential language* $\mathcal{L}_n$ *together with a unary unconditional probability function* $\Pr(\cdot)\colon \mathcal{L}_n \mapsto \mathbb{R}$ *satisfying the following axioms of Kolmogorov (1956).*

1. *For all* $p \in \mathcal{L}_n$, $\Pr(p) \geq 0$.
2. *For all* $p \in \mathcal{L}_n$, *if* $p \dashv\models \top$, *then* $\Pr(p) = 1$.
3. *For all* $p, q \in \mathcal{L}_n$, *if* $p \,\&\, q \vDash \bot$, *then* $\Pr(p \vee q) = \Pr(p) + \Pr(q)$.

Again following Kolmogorov (1956), we also define a binary conditional probability function $\Pr(\cdot \mid \cdot)\colon \mathcal{L}_n \times \mathcal{L}_n \mapsto \mathbb{R}$, *in terms of* $\Pr(\cdot)$, *as follows:*

4. *For all* $p, q \in \mathcal{L}_n$, *if* $\Pr(q) \neq 0$, *then* $\Pr(p \mid q) \stackrel{\text{def}}{=} \dfrac{\Pr(p \,\&\, q)}{\Pr(q)}$.

In the next section, I will show how such probability models can, alternatively, be couched entirely in simple *algebraic* terms. This will be the key to our subsequent decision procedure for probability calculus.

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2. Probability calculus: Algebraic presentation. In this section, I explain how any probability model $\mathfrak{M}_n$ can be given a simple algebraic representation using what I will call a *stochastic truth-table*. Here is a simple example. Let us take a sentential language $\mathcal{L}_3$ with 3 atomic sentences $\{A, B, C\}$. An ordinary truth-table for such a language has $2^3 = 8$ rows. Each row (i) of the truth-table corresponds to a state description (s_i) of the language $\mathcal{L}_3$. And, every truth-functional proposition definable in terms of $\{A, B, C\}$ can be expressed as a disjunction of said state descriptions. So far, this is all just elementary classical sentential calculus with which we are all intimately familiar. Now, a *stochastic* truth-table for $\mathfrak{M}_3 = \langle \mathcal{L}_3, \Pr \rangle$ is a truth-table with an additional column containing real numbers corresponding to the values assigned by Pr to each state description of $\mathcal{L}_3$. Thus, for $\mathfrak{M}_3$, we have:

A	B	C	State Descriptions (s_i)	$\Pr(s_i) = \mathfrak{s}_i$
T	T	T	$A \,\&\, B \,\&\, C = s_1$	$\Pr(s_1) = \mathfrak{s}_1$
T	T	F	$A \,\&\, B \,\&\, \sim C = s_2$	$\Pr(s_2) = \mathfrak{s}_2$
T	F	T	$A \,\&\, \sim B \,\&\, C = s_3$	$\Pr(s_3) = \mathfrak{s}_3$
T	F	F	$A \,\&\, \sim B \,\&\, \sim C = s_4$	$\Pr(s_4) = \mathfrak{s}_4$
F	T	T	$\sim A \,\&\, B \,\&\, C = s_5$	$\Pr(s_5) = \mathfrak{s}_5$
F	T	F	$\sim A \,\&\, B \,\&\, \sim C = s_6$	$\Pr(s_6) = \mathfrak{s}_6$
F	F	T	$\sim A \,\&\, \sim B \,\&\, C = s_7$	$\Pr(s_7) = \mathfrak{s}_7$
F	F	F	$\sim A \,\&\, \sim B \,\&\, \sim C = s_8$	$\Pr(s_8) = \mathfrak{s}_8$

Now, just as each proposition p expressible in $\mathcal{L}_3$ is equivalent to some disjunction $\bigvee_i s_i$ of state descriptions of $\mathcal{L}_3$, each unconditional probability term $\Pr(p)$ of the language of $\mathfrak{M}_3$ can be expressed as a sum $\sum_i \mathfrak{s}_i$ of the real numbers assigned by Pr to those state descriptions of $\mathcal{L}_3$ that make up the disjunction of state descriptions which expresses p.[1] Moreover, each conditional probability term $\Pr(p \mid q)$ of the language of $\mathfrak{M}_3$ can be expressed as a ratio of the sums corresponding to $\Pr(p \,\&\, q)$ and $\Pr(q)$, respectively. This gives us a faithful translation from the language of $\mathfrak{M}_3$ into the language of simple real algebra (viz., high-school algebra). Here are some examples of translations of probability terms in the language of $\mathfrak{M}_3$ into terms of high-school algebra:

- $\Pr(A) = \Pr(s_1 \vee s_2 \vee s_3 \vee s_4) = \mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_3 + \mathfrak{s}_4$.
- $\Pr(B \,\&\, \sim C) = \Pr(s_2 \vee s_6) = \mathfrak{s}_2 + \mathfrak{s}_6$.
- $\Pr(A \mid B \equiv C) = \dfrac{\Pr(A \,\&\, (B \equiv C))}{\Pr(B \equiv C)} = \dfrac{\Pr(s_1 \vee s_4)}{\Pr(s_1 \vee s_4 \vee s_5 \vee s_7)} = \dfrac{\mathfrak{s}_1 + \mathfrak{s}_4}{\mathfrak{s}_1 + \mathfrak{s}_4 + \mathfrak{s}_5 + \mathfrak{s}_7}$.

Our only remaining task is to demonstrate that this procedure really does yield a general algebraic characterization of the concept of a (Kolmogorov) probability model $\mathfrak{M}_n$. There

[1] Note added in proof: in the case where $p \dashv\models \bot$, the corresponding set of state description will be the empty set Ø. In this degenerate case, we just stipulate that the empty disjunction gets assigned probability zero under all algebraic translations. That is, if $p \dashv\models \bot$, then the term '$\Pr(p)$' will be translated to '0'.

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are two things that need to be shown here. First, we need to show that if we are given a *concrete* probability model (which assigns real numbers to each of the $\mathfrak{s}_i$), then the translation procedure yields an algebraic characterization of that concrete probability model. And, second, that, in the absence of specified numerical values for all the real variables $\mathfrak{s}_i$, we can use our algebraic translations to reason *generally* about probability models $\mathfrak{M}_n$.

For the first part, consider a concrete probability model $\mathfrak{M}_n$, and note that our translation procedure has two steps. First, it goes from $\Pr(p)$ to $\Pr(\bigvee_i s_i)$, where $\bigvee_i s_i$ is tautologically equivalent to p. That step is Kosher, since it is a theorem of the (Kolmogorov) probability calculus (hence, true for *all* probability models) that $\Pr(p) = \Pr(q)$ if $p \dashv\models q$. The second step of our translation procedure goes from $\Pr(\bigvee_i s_i)$ to $\sum_i \mathfrak{s}_i$. This step is also Kosher, since the s_i are mutually exclusive and finite additivity is one of the axioms of (Kolmogorov) probability calculus. Finally, since our translations of unconditional probability terms (for a concrete probability model) are Kosher, it follows that our translations of conditional probability terms will also be Kosher (for a concrete model), since translated conditional probabilities are defined in terms of translated unconditional probabilities in the appropriate way.

For the second part, no numerical values are assigned to the $\mathfrak{s}_i$, and we want to use our translations to reason *generally* about *arbitrary* probability models of size n. Here we just need to impose two simple general algebraic constraints on the real variables $\mathfrak{s}_i$ that appear in a generic (nonnumerical) stochastic truth-table for a model of size n:

- $\mathfrak{s}_1 \geq 0, \mathfrak{s}_2 \geq 0, \ldots, \mathfrak{s}_{2^n} \geq 0.$
- $\sum_{i=1}^{2^n} \mathfrak{s}_i = 1.$

It is easy to verify that these two algebraic constraints ensure that the Kolmogorov axioms are respected (in general) by our translation. To see why, note that these two constraints ensure the following:

(i) $\Pr(p)$ will always (and for all p) get translated to a high-school algebra expression that is guaranteed (for all assignments of real numbers to the $\mathfrak{s}_i$) to be nonnegative. This follows from the first constraint on the $\mathfrak{s}_i$'s (that they are all nonnegative) and the fact that $\Pr(p)$ is always translated as a sum of $\mathfrak{s}_i$'s.

(ii) $\Pr(\top)$ will always get translated to an expression that is algebraically identical (in high-school algebra) to the number 1. This follows from the translation setup and the second constraint on the $\mathfrak{s}_i$'s.

(iii) Provided that p and q are mutually exclusive, the terms $\Pr(p \vee q)$ and $\Pr(p) + \Pr(q)$ will always (and for all p and q) get translated to expressions that are algebraically identical (in high-school algebra). This does not even depend on the two constraints on the $\mathfrak{s}_i$'s. It is guaranteed by the nature of the translation scheme (and algebra).

Finally, since our translation is *generally* Kosher for unconditional probability terms, it will also be *generally* Kosher for conditional probability terms, since conditional probability translations are constructed in the appropriate way out of unconditional probability translations.[2]

With this translation technique in hand, we can now transform any set of statements in probability calculus into a corresponding set of statements in high-school algebra. By a *statement of probability calculus*, I mean an algebraically well-formed statement, which

[2] See (Paris, 1995, pp. 13–14) for a more rigorous mathematical presentation of this algebraic representation of probability functions over sentential languages.

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is an equation, an inequation, or an inequality containing terms of the language of a probability model $\mathfrak{M}_n$ and/or symbols denoting real numbers (i.e., real constants), where these probabilistic and/or algebraic terms can be combined using the standard algebraic operations of multiplication, division, addition, and subtraction. We also allow such simple statements of probability calculus to be combined using logical connectives. Here are a few example statements of probability calculus:

- $\Pr(A \mid B) \cdot \Pr(B \mid C) \geq \dfrac{\Pr(A \mid B \,\&\, C)}{\Pr(B \vee C)}$.
- $\Pr(X) - \Pr(Y)^2 = \frac{1}{2}$.
- $\Pr(C \mid A \vee \sim B) + 2 \cdot \Pr(B) \neq \frac{1}{\pi}$.
- $\Pr(X \mid Y) > \Pr(X) \vee \Pr(X) < \Pr(Y)$.

This definition of a statement of probability calculus is rather broad, but it does rule out (in addition to statements that are not algebraically or logically well-formed) statements involving *quantification* over sentence letters and/or real variables. I enforce this restriction here because (a) it makes the presentation of our translation methods and our decision procedure much simpler and (b) most applications of probability calculus (indeed, almost all applications with which I am familiar) do not (essentially) involve such quantification. In any event, the methods described here can be extended to this more general class of statements. But I will not explain such extensions in this paper.

A set S of statements of probability calculus is *satisfiable* just in case there exists a concrete probability model $\mathfrak{M}$ on which all members of S are true. If S is unsatisfiable, this means that there is no such model, i.e., a contradiction can be deduced from the members of S using only the Kolmogorov axioms for probability calculus. A *decision procedure for probability calculus* is an algorithm that takes as input an arbitrary finite set S of statements of probability calculus. If S is satisfiable, then a decision procedure must output a concrete model $\mathfrak{M}$ on which all members of S are true, and if S is unsatisfiable, then a decision procedure must tell us that there are no such models. In the next section, I will explain how the algebraic translation procedure described above allows us to produce a decision procedure for probability calculus based on an existing decision procedure for high-school algebra.

3. PrSAT: A decision procedure for probability calculus.

3.1. A high-level description of PrSAT. The first step in constructing PrSAT is writing the translation algorithm, which transforms any finite set S of statements in probability calculus into a finite set S' of statements in simple high-school algebra, in accordance with the procedure above.[3] Once this is done, we see immediately that the problem of giving a decision procedure for probability calculus is just the problem of giving a decision procedure for the satisfiability of sets of statements in high-school algebra. Specifically, the

[3] Here I am greatly indebted to Jason Alexander for implementing a very elegant and fully general version of this translation procedure in *Mathematica*. Before Jason came along, I had hard coded the translation procedure in *Mathematica* for models of size $n = 2, 3, 4$ (I did not even bother doing $n = 5$!). See Fitelson (2001b, Appendix) for some early implementations of this kind. The new *Mathematica* implementation of PrSAT (available on my PrSAT Web site: http://fitelson.org/PrSAT/) uses Jason's techniques to generate (canonical) stochastic truth-tables and translations on-the-fly, from arbitrary sets S of statements in probability calculus.

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problem of deciding whether a set S of statements in probability calculus (in the language of $\mathfrak{M}_n$) is satisfiable is just the problem of deciding whether the following set of statements of high-school algebra is satisfiable:

$$S' \cup \left\{ \mathfrak{s}_1 \geq 0, \ldots, \mathfrak{s}_{2^n} \geq 0, \sum_{i=1}^{2^n} \mathfrak{s}_i = 1 \right\}.$$

In other words, if we can find a procedure for deciding whether the union of our translated set S' and the set containing the pair of algebraic constraints on our $\mathfrak{s}_i$ is satisfiable, then we are done.

Happily, there is an existing decision procedure for statements of high-school algebra, and so all we need to do is use our translation algorithm, plus this existing decision procedure, and we will have our decision procedure for probability calculus. In other words, we have:

1. Translate the set S of statements of probability calculus (in the language of $\mathfrak{M}_n$) into the set S' of high-school algebra statements, in accordance with the translation procedure described above.
2. Form the union $S' \cup \mathcal{C}_n$, where $\mathcal{C}_n = \{\mathfrak{s}_1 \geq 0, \ldots, \mathfrak{s}_{2^n} \geq 0, \sum_{i=1}^{2^n} \mathfrak{s}_i = 1\}$ is the set of algebraic constraints on the $\mathfrak{s}_i$, which ensures that they are basic *probabilities* corresponding to state descriptions of $\mathcal{L}_n$.
3. Consult the existing decision procedure for sets of high-school algebra statements (see below) as to the satisfiability of $S' \cup \mathcal{C}_n$.[4]
4. If the decision procedure for high-school algebra says that $S' \cup \mathcal{C}_n$ is unsatisfiable, then report that S is unsatisfiable. And, if the decision procedure returns a model $\mathfrak{M}'_n$ on which all members of $S' \cup \mathcal{C}_n$ are true, then translate $\mathfrak{M}'_n$ back into the language of $\mathfrak{M}_n$, and this will yield a probability model on which all members of S are true. □

This general procedure, now known as PrSAT, has been implemented in *Mathematica* (which has the requisite decision procedure for high-school algebra as a built-in function – see below), and it has been used to solve many nontrivial problems in probability calculus. I will describe some example applications of PrSAT in the next section. But, first, I should say something about the existing decision procedure for high-school algebra (now built-in to *Mathematica*), which has been treated as a 'black box' in our high-level description of PrSAT.

3.2. Decidability in the theory of real closed fields. The theory of real closed fields (TRCF) is a vast generalization of high-school algebra. It is a first-order theory, which allows for arbitrary quantification over statements of arbitrary complexity that can be expressed using real variables and constants, the standard algebraic operations, equality and inequality signs, and logical connectives. The theory of high-school algebra is a tiny fragment of TRCF, which only involves existential (or universal) quantification over such (sets of) statements. High-school algebra does *not* involve *mixed* (∀∃ or ∃∀) quantification.

[4] Actually, the latest versions of PrSAT do some other tricks at this stage of the high-level description of the procedure to simplify and optimize the input set. Also, the latest version of PrSAT contains a random search option (implemented by Ben Blum), which can be tried before sending the problem to the decision procedure (i.e., before this step). See the Directions for Future Research section for discussion of this added functionality.

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This is a *significant* difference in expressive power. To get a sense of the power of full TRCF, note (e.g.) that the definition of a limit of any high-school algebra term can be defined in TRCF. Here is an example:

$$\lim_{x\to\infty} \frac{3x^2-5x+7}{7x^2-2x+11} \stackrel{\text{def}}{=} \iota\lambda\left[(\forall\epsilon>0)(\exists\delta)(\forall x>\delta)\left(-\epsilon<\frac{3x^2-5x+7}{7x^2-2x+11}-\lambda<\epsilon\right)\right].$$

Of course, we do not need the full power of the full TRCF for our present purposes. But, as it turns out, even this much more powerful theory is decidable. In the late 1920s, Alfred Tarski proved that the full TRCF is decidable (Feferman & Feferman, 2004). Tarski's quantifier elimination algorithm for TRCF was not properly published until 1951 (Tarski, 1951). And, even after the publication of Tarski's method, over 20 years went by before there was significant further work on the problem or any real applications of such decision procedures. One reason for this lag is that Tarski's algorithm is extremely computational complex. In general, its complexity is not bounded by any tower of exponentials.[5] In the early 1970s, Collins (1975) came up with a much more efficient decision procedure for TRCF based on the methods of cylindrical algebraic decomposition (CAD). The complexity of Collins' CAD algorithm is (merely!) double exponential in the number of real variables contained in the set of TRCF sentences. It has since been shown that double-exponential complexity in the number of real variables is a *lower bound* for *any* general decision procedure for TRCF (Davenport & Heintz, 1988).[6] In the 1990s, various practical improvements to various parts of the CAD algorithm were developed by Hong (1990, 1992), McCallum (1998), Brown (2001b,a), and others. Ultimately, these efforts have led to the computer program QEPCAD. An up-to-date C implementation of QEPCAD is now freely available for download (Hong, 2006). In 2003, *Mathematica* (Wolfram, 2003) was released with a suite of QEPCAD-like functions built in to its main kernel. Strzebonski (2000b,a) was (and still is) responsible for these *Mathematica* functions. Specifically, *Mathematica* has a function called FindInstance, which will take as input a finite set S' of statements of high-school algebra, and it will return an assignment of real numbers

[5] See Feferman & Feferman (2004) for a detailed accounting of the fascinating and complicated history of Tarskian decision procedures for TRCF. And, see Caviness & Johnson (1998) for all technical details about various decision procedures for TRCF and their computational complexities. The details of these methods are beyond the scope of this paper, which is primarily concerned with communicating the basic ideas behind PrSAT and its applications.

[6] Since the number of real variables in our problems grows exponentially in n (2^n), the complexity of PrSAT, using any decision procedure for the full TRCF, will be (at least) *triple* exponential in n! This means that (naively) solving problems of size $n > 4$ is not (generally) practical. Some heuristics for speeding up PrSAT's model finding are illustrated in http://fitelson.org/pm.nb, which is this paper's companion *Mathematica* notebook. There, all the problems in the final section of this paper are quickly solved using PrSAT, with the help of a simple heuristic (which serves to reduce the effective number of real variables in the problem). That heuristic technique can also be used to quickly find models for problems involving 4 or even 5 atomic sentence letters. It is crucial that today's computers are *much* more powerful than even those of the late 1990s, when QEPCAD began to be applied to nontrivial problems. For instance, all of the 'practically intractable' application problems reported in Brown's 1998 CAD applications paper (Brown, 2001b) can be solved (with *Mathematica*) in under 15 min on today's PCs (even on our laptops!). This is one of the reasons that PrSAT is (nowadays) able to solve so many nontrivial problems. In the latest version of PrSAT, there is also a random search feature, which greatly speeds up model finding in many (satisfiable) examples. See the Directions for Future Research section for further discussion.

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to all the real variables in S' on which all the members of S' are true (if there is one). If the set S' is unsatisfiable, it will return the empty set { }. This is precisely the function that PrSAT uses to decide the satisfiability of its sets $S' \cup \mathcal{C}_n$ of high-school algebra statements (from step 3 of our high-level description of PrSAT, above). As we will see in the next section, the *Mathematica* implementation of PrSAT is quite powerful, as it is able to solve many nontrivial problems in probability calculus. But, before we get to these applications, I want to say one more thing about decision procedures for TRCF and its fragments.

Since high-school algebra only involves existential quantification over sets of simple algebraic claims, a natural question to ask is whether there are algorithms more efficient than QEPCAD for just the *pure existential fragment* of TRCF. *Theoretically*, the answer to this question is 'Yes'. But, *practically*, the answer is less clear. That is, theoretically, it can be shown that the complexity of the decision problem for the existential fragment of TRCF is 'only' *single* exponential, as opposed to double exponential for the full TRCF (Basu *et al.*, 1996). However, when one looks at the existing implementations of these algorithms, one finds that *in practice* they tend to be *much slower* than QEPCAD, except on very large problems that are practically intractable anyway (Hong, 1991). However, recently, there has been some progress on improving the implementations of these pure existential decision procedures.[7] This is an interesting area for future research and development.

4. Some applications of PrSAT. This section contains several nontrivial problems from probability calculus (some of which are also of philosophical interest) that can be solved rather easily with the *Mathematica* implementation of PrSAT.[8]

4.1. A well-known problem in probability calculus. We begin with a well-known problem in probability calculus, which is discussed in most contemporary introductory texts on probability (Feller, 1968, p. 116). Consider a language $\mathcal{L}_3$ with three atomic sentences $\{A, B, C\}$. The question is that if A, B, and C are *pairwise* probabilistically independent, does it follow that they are *mutually* probabilistically independent? That is, do the statements (1)–(3) in probability calculus entail statement (4)?

1. $\Pr(A \,\&\, B) = \Pr(A) \cdot \Pr(B)$.
2. $\Pr(A \,\&\, C) = \Pr(A) \cdot \Pr(C)$.
3. $\Pr(B \,\&\, C) = \Pr(B) \cdot \Pr(C)$.
4. $\Pr(A \,\&\, B \,\&\, C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$.

An equivalent question is whether the set S consisting of (1), (2), (3), and the denial of (4) is *unsatisfiable*. To illustrate how our translation procedure works on a real example, we

[7] Galen Huntington has been working on more practically efficient decision procedures for the pure existential fragment of TRCF. Indeed, he has made significant progress on this front. He now has an algorithm that solves all of Hong's challenge problems Hong (1991) in a matter of minutes (as opposed to *eons*, which is what previous algorithms would have taken). Galen's work will soon appear in his thesis (Huntington, 2008).

[8] All the problems discussed in this section have been solved with the *Mathematica* implementation of PrSAT, which can be freely downloaded from the PrSAT Web site, at http://fitelson.org/PrSAT/. The PrSAT Web site also contains documentation and further examples. A *Mathematica* notebook containing all the examples discussed here can be downloaded from http://fitelson.org/pm.nb. That notebook also describes a heuristic technique for speeding up model searches, as well as Blum's random search add-on (which often finds models very quickly), which has recently been added to PrSAT. See the Directions for Future Research section for discussion.

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can use the stochastic truth-table for $\mathfrak{M}_3$ above to translate (1), (2), (3), and the denial of (4) into the following set S' of high-school algebra statements in terms of $\mathfrak{s}_1$–$\mathfrak{s}_8$:

$1'.\ \mathfrak{s}_1 + \mathfrak{s}_2 = (\mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_3 + \mathfrak{s}_4) \cdot (\mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_5 + \mathfrak{s}_6).$
$2'.\ \mathfrak{s}_1 + \mathfrak{s}_3 = (\mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_3 + \mathfrak{s}_4) \cdot (\mathfrak{s}_1 + \mathfrak{s}_3 + \mathfrak{s}_5 + \mathfrak{s}_7).$
$3'.\ \mathfrak{s}_1 + \mathfrak{s}_5 = (\mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_5 + \mathfrak{s}_6) \cdot (\mathfrak{s}_1 + \mathfrak{s}_3 + \mathfrak{s}_5 + \mathfrak{s}_7).$
$4'.\ \mathfrak{s}_1 \neq (\mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_3 + \mathfrak{s}_4) \cdot (\mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_5 + \mathfrak{s}_6) \cdot (\mathfrak{s}_1 + \mathfrak{s}_3 + \mathfrak{s}_5 + \mathfrak{s}_7).$

Following our procedure, we need to determine whether the set $S' \cup \mathcal{C}_3$ is satisfiable, where $\mathcal{C}_3$ is the following set of 9 algebraic statements:

$$\{\mathfrak{s}_1 \geq 0, \mathfrak{s}_2 \geq 0, \mathfrak{s}_3 \geq 0, \mathfrak{s}_4 \geq 0, \mathfrak{s}_5 \geq 0, \mathfrak{s}_6 \geq 0, \mathfrak{s}_7 \geq 0, \mathfrak{s}_8 \geq 0,$$
$$\mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_3 + \mathfrak{s}_4 + \mathfrak{s}_5 + \mathfrak{s}_6 + \mathfrak{s}_7 + \mathfrak{s}_8 = 1\}.$$

PrSAT can find *many* assignments to the $\mathfrak{s}_i$ on which all members of $S' \cup \mathcal{C}_3$ are true. So the set S is satisfiable. Hence, the answer to the question is 'No' – pairwise probabilistic independence of three statements does *not* entail their *mutual* independence. Before reporting a nice model found by PrSAT, it is worth noting that all the answers to this problem that I have seen in the textbooks – see, e.g., Feller (1968, p. 116) – involve models that assign probability zero to at least one state description of $\mathcal{L}_3$, owing to some kind of logical dependencies between atomic sentences. It is interesting to note that PrSAT can easily (and automatically) find models that are *regular* – i.e., that assign nonzero probability to all state descriptions of $\mathcal{L}_3$. Indeed, PrSAT can find regular models such that the $\mathfrak{s}_i$'s have equal denominators (i.e., regular 'urn models'). Here is a nice PrSAT model on which all members of $S' \cup \mathcal{C}_3$ are true (think in terms of sampling from an urn containing 2000 balls, with properties A, B, and C, distributed according to the $\mathfrak{s}_i$):

A	B	C	State Descriptions (s_i)	$\Pr(s_i) = \mathfrak{s}_i$
T	T	T	$A \,\&\, B \,\&\, C = s_1$	$\Pr(s_1) = \mathfrak{s}_1 = \frac{1}{2000}$
T	T	F	$A \,\&\, B \,\&\, \sim C = s_2$	$\Pr(s_2) = \mathfrak{s}_2 = \frac{19}{2000}$
T	F	T	$A \,\&\, \sim B \,\&\, C = s_3$	$\Pr(s_3) = \mathfrak{s}_3 = \frac{19}{2000}$
T	F	F	$A \,\&\, \sim B \,\&\, \sim C = s_4$	$\Pr(s_4) = \mathfrak{s}_4 = \frac{161}{2000}$
F	T	T	$\sim A \,\&\, B \,\&\, C = s_5$	$\Pr(s_5) = \mathfrak{s}_5 = \frac{19}{2000}$
F	T	F	$\sim A \,\&\, B \,\&\, \sim C = s_6$	$\Pr(s_6) = \mathfrak{s}_6 = \frac{161}{2000}$
F	F	T	$\sim A \,\&\, \sim B \,\&\, C = s_7$	$\Pr(s_7) = \mathfrak{s}_7 = \frac{161}{2000}$
F	F	F	$\sim A \,\&\, \sim B \,\&\, \sim C = s_8$	$\Pr(s_8) = \mathfrak{s}_8 = \frac{1459}{2000}$

On this model, we have the following:

1. $\Pr(A \,\&\, B) = \frac{1}{100} = \frac{1}{10} \cdot \frac{1}{10} = \Pr(A) \cdot \Pr(B).$
2. $\Pr(A \,\&\, C) = \frac{1}{100} = \frac{1}{10} \cdot \frac{1}{10} = \Pr(A) \cdot \Pr(C).$
3. $\Pr(B \,\&\, C) = \frac{1}{100} = \frac{1}{10} \cdot \frac{1}{10} = \Pr(B) \cdot \Pr(C).$
4. $\Pr(A \,\&\, B \,\&\, C) = \frac{1}{2000} \neq \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} = \Pr(A) \cdot \Pr(B) \cdot \Pr(C).$

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Next, we will discuss some examples motivated by philosophical (specifically, philosophy of science) applications of probability calculus.

4.2. Three problems from Bayesian confirmation theory. According to Bayesian confirmation theory, an evidential statement E is said to *confirm* a hypothesis statement H, relative to a probability model $\mathfrak{M} = \langle \mathcal{L}, \Pr \rangle$, just in case E and H are positively correlated under the probability function Pr, i.e., just in case $\Pr(H \mid E) > \Pr(H)$. Moreover, there are many Bayesian measures $\mathfrak{c}(H, E)$ of the *degree* to which E confirms H (Fitelson, 1999). Here are a few measures that have been used:

$$\mathsf{d}(H, E) \stackrel{\text{def}}{=} \Pr(H \mid E) - \Pr(H),$$

$$\mathsf{l}(H, E) \stackrel{\text{def}}{=} \frac{\Pr(E \mid H) - \Pr(E \mid \sim H)}{\Pr(E \mid H) + \Pr(E \mid \sim H)},$$

$$\mathsf{s}(H, E) \stackrel{\text{def}}{=} \Pr(H \mid E) - \Pr(H \mid \sim E).$$

All these measures respect the *qualitative* Bayesian definition, since they are all positive when E confirms H, negative when E disconfirms H, and zero when E and H are independent. Moreover, all three of these measures are defined on a $[-1, 1]$ scale. But these measures disagree radically on *comparative* claims (Fitelson, 1999) of the form $\mathfrak{c}(H, E) \geq \mathfrak{c}(H', E')$. The following condition is assumed by most commentators to be a *desideratum* for Bayesian measures of degree of confirmation (Joyce, 2003):

($\star$) If $\Pr(H \mid E_1) \geq \Pr(H \mid E_2)$, then $\mathfrak{c}(H, E_1) \geq \mathfrak{c}(H, E_2)$.

Indeed, it was simply assumed by many that ($\star$) *must* be satisfied by *any* Bayesian measure. An early prototype of `PrSAT` was able to show that, while d and l satisfy ($\star$), s *violates* it (Fitelson, 2001b)! This result came as a surprise to many Bayesian confirmation theorists. If you run `PrSAT` on the d-instance or the l-instance of ($\star$), it will tell you (correctly) that there are no models on which these instances of ($\star$) are false. However, if you run `PrSAT` on the s-instance of ($\star$), it will find a countermodel. Here is one such `PrSAT` model, which is regular and has equal denominators (think of an urn with 5120 balls and 3 properties E_1, E_2, and H):

E_1	E_2	H	State Descriptions (s_i)	$\Pr(s_i) = \mathfrak{s}_i$
T	T	T	$E_1 \,\&\, E_2 \,\&\, H = s_1$	$\Pr(s_1) = \mathfrak{s}_1 = \frac{669}{5120}$
T	T	F	$E_1 \,\&\, E_2 \,\&\, \sim H = s_2$	$\Pr(s_2) = \mathfrak{s}_2 = \frac{291}{5120}$
T	F	T	$E_1 \,\&\, \sim E_2 \,\&\, H = s_3$	$\Pr(s_3) = \mathfrak{s}_3 = \frac{127}{5120}$
T	F	F	$E_1 \,\&\, \sim E_2 \,\&\, \sim H = s_4$	$\Pr(s_4) = \mathfrak{s}_4 = \frac{193}{5120}$
F	T	T	$\sim E_1 \,\&\, E_2 \,\&\, H = s_5$	$\Pr(s_5) = \mathfrak{s}_5 = \frac{1539}{5120}$
F	T	F	$\sim E_1 \,\&\, E_2 \,\&\, \sim H = s_6$	$\Pr(s_6) = \mathfrak{s}_6 = \frac{1341}{5120}$
F	F	T	$\sim E_1 \,\&\, \sim E_2 \,\&\, H = s_7$	$\Pr(s_7) = \mathfrak{s}_7 = \frac{225}{5120}$
F	F	F	$\sim E_1 \,\&\, \sim E_2 \,\&\, \sim H = s_8$	$\Pr(s_8) = \mathfrak{s}_8 = \frac{735}{5120}$

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Strictly speaking, Bayesian confirmation is a four-place relation between E, H, a corpus of background knowledge K, and a probability model $\mathfrak{M} = \langle \mathcal{L}, \Pr \rangle$. So our three measures should really be defined as:

$$\mathsf{d}(H, E \mid K) \;\; \Pr(H \mid E \,\&\, K) - \Pr(H \mid K),$$

$$\mathsf{l}(H, E \mid K) \;\; \frac{\Pr(E \mid H \,\&\, K) - \Pr(E \mid {\sim}H \,\&\, K)}{\Pr(E \mid H \,\&\, K) + \Pr(E \mid {\sim}H \,\&\, K)},$$

$$\mathsf{s}(H, E \mid K) \;\; \Pr(H \mid E \,\&\, K) - \Pr(H \mid {\sim}E \,\&\, K).$$

For many results – like the ones concerning ($\star$) above – the background corpus K plays no role, and so it may be suppressed (for simplicity). But, in many cases, the background corpus plays a crucial role. One set of important results in confirmation theory depends essentially on the background corpus. These results have to do with the concept of *independent evidence* defined in Fitelson (2001a, 2001b) in the following way.

- E_1 and E_2 are said to be *confirmationally independent regarding* H, according to a measure $\mathfrak{c}$ just in case both $\mathfrak{c}(H, E_1 \mid E_2) = \mathfrak{c}(H, E_1)$ and $\mathfrak{c}(H, E_2 \mid E_1) = \mathfrak{c}(H, E_2)$. Here, $\mathfrak{c}(H, E)$ can be thought of unconditionally, as above, or it can be thought of as $\mathfrak{c}(H, E \mid \top)$.

Intuitively, if E_1 and E_2 are confirmationally independent regarding H, then they do not 'interact' in the support they provide for H. Following Peirce, we have argued (Fitelson, 2001b) that in such cases (provided that each of E_1 and E_2 individually confirms H), the degree to which the conjunction E_1 & E_2 confirms H should be greater than the degree to which either conjunct *alone* confirms H. Specifically, we should have the following:

(†) If each of E_1 and E_2 individually confirms H and if $\mathfrak{c}(H, E_1 \mid E_2) = \mathfrak{c}(H, E_1)$ and $\mathfrak{c}(H, E_2 \mid E_1) = \mathfrak{c}(H, E_2)$, then $\mathfrak{c}(H, E_1 \,\&\, E_2) > \mathfrak{c}(H, E_2)$.

It seems to me that (†) should be a desideratum for any measure of degree of confirmation $\mathfrak{c}(H, E \mid K)$. Interestingly (and, again, surprisingly, to my mind), an early prototype of PrSAT was able to establish that, while measures d and l satisfy (†), s *violates* it. If you run PrSAT on the d-instance or the l-instance of (†), it will tell you (correctly) that there are no models on which these instances of (†) are false. However, if you run PrSAT on the s-instance of (†), it will find a countermodel. Indeed, PrSAT is able to find a single model that shows *both* that the s-instance of ($\star$) is false *and* that the s-instance of (†) is false *simultaneously*. We have already seen such a model: the model reported above for ($\star$) *also* serves as a countermodel to the s-instance of (†). To see this, note that the following facts obtain on this model:

- $\Pr(H \mid E_1) = \frac{199}{320} > \frac{23}{40} = \Pr(H \mid E_2) > \Pr(H) = \frac{1}{2}$.

- $$\mathsf{s}(H, E_1) = \Pr(H \mid E_1) - \Pr(H \mid {\sim}E_1) = \frac{199}{320} - \frac{147}{320} = \frac{13}{80}$$
$$< \mathsf{s}(H, E_2) = \Pr(H \mid E_2) - \Pr(H \mid {\sim}E_2) = \frac{23}{40} - \frac{11}{40} = \frac{3}{10}.$$

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– $\mathsf{s}(H, E_1 \mid E_2) = \Pr(H \mid E_1 \,\&\, E_2) - \Pr(H \mid {\sim}E_1 \,\&\, E_2) = \frac{223}{320} - \frac{171}{320} = \frac{13}{80}$
$= \mathsf{s}(H, E_1) = \Pr(H \mid E_1) - \Pr(H \mid {\sim}E_1) = \frac{199}{320} - \frac{147}{320} = \frac{13}{80}.$

– $\mathsf{s}(H, E_2 \mid E_1) = \Pr(H \mid E_2 \,\&\, E_1) - \Pr(H \mid {\sim}E_2 \,\&\, E_1) = \frac{223}{320} - \frac{127}{320} = \frac{3}{10}$
$= \mathsf{s}(H, E_2) = \Pr(H \mid E_2) - \Pr(H \mid {\sim}E_2) = \frac{23}{40} - \frac{11}{40} = \frac{3}{10}.$

– $\mathsf{s}(H, E_1 \,\&\, E_2) = \Pr(H \mid E_1 \,\&\, E_2) - \Pr(H \mid {\sim}(E_1 \,\&\, E_2)) = \frac{223}{320} - \frac{1891}{4160} = \frac{63}{260}$
$< \mathsf{s}(H, E_2) = \Pr(H \mid E_2) - \Pr(H \mid {\sim}E_2) = \frac{23}{40} - \frac{11}{40} = \frac{3}{10}.$

Our third and final class of problems from Bayesian confirmation theory involves Hempel's infamous raven paradox. This is also a case in which nontrivial new theorems and nontrivial new models were found by PrSAT. I will not get into the philosophical background behind the raven paradox here. Rather, I will just briefly (and at a high level) explain the purely formal part of contemporary Bayesian approaches to the paradox and how PrSAT allowed us to discover a new and improved approach. See Fitelson & Hawthorne (to appear) for all the philosophical and technical details. The formal part of standard Bayesian resolutions of the raven paradox rests on the following four statements of probability calculus:

1. $\Pr(B \mid R \,\&\, H) = 1.$
2. $\Pr({\sim}B) > \Pr(R).$
3. $\Pr(B \mid H) = \Pr(B).$
4. $\Pr(R \mid H) = \Pr(R).$

From these assumptions, it is possible to prove the following:

5. $\Pr(H \mid R \,\&\, B) > \Pr(H \mid {\sim}R \,\&\, {\sim}B).$

Indeed, the main formal component of Bayesian resolutions of the raven paradox involves describing sets of sufficient conditions for (5). The usual set of sufficient conditions is (1)–(4). But these assumptions are controversial, because they have undesirable consequences, such as:

6. $\Pr(H \mid {\sim}R \,\&\, {\sim}B) > \Pr(H).$
7. $\Pr(H \mid {\sim}R \,\&\, B) < \Pr(H).$

So if we could find sufficient conditions for (5) that were weaker than (1)–(4) – so that they do not entail (6) and (7) – that would be an improvement on the standard Bayesian approaches. Note that conditions (1) and (2) are nonnegotiable, as they are either logically implied by the setup of the raven paradox or otherwise uncontroversial. As such, the challenge is to find conditions C such that: (a) the conditions (1), (2), and C jointly entail (5); (b) C is strictly weaker than the conjunction (3) & (4); and (c) the conditions (1), (2), and C do not jointly entail either (6) or (7). Fitelson & Hawthorne (to appear) have recently reported that the following condition C will serve as a replacement for (3) and (4):

(C) $\Pr(H \mid R) \geq \Pr(H \mid {\sim}B).$

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We discovered this condition by using PrSAT to examine various conditions that are strictly weaker than the conjunction (3) & (4). It is easy to see that (C) satisfies (b), since (3) & (4) entails $\Pr(H \mid R) = \Pr(H \mid \sim B) = \Pr(H)$, which entails (C), but (C) does not entail $\Pr(H \mid R) = \Pr(H \mid \sim B)$ or $\Pr(H \mid R) = \Pr(H)$ or $\Pr(H \mid \sim B) = \Pr(H)$. The hard part of the problem is establishing (a). PrSAT told us that (a) was true, but it took awhile for us to find an axiomatic proof (PrSAT does not output proof objects). James Hawthorne eventually discovered an axiomatic proof of (a), which is reported in Fitelson & Hawthorne (to appear). As for (c), PrSAT gives us concrete models which show that (1), (2), and (C) do not entail either (6) or (7). In fact, here is a single PrSAT model that shows *both* things *simultaneously*:

B	H	R	State Descriptions (s_i)	$\Pr(s_i) = \mathfrak{s}_i$
T	T	T	$B \,\&\, H \,\&\, R = s_1$	$\Pr(s_1) = \mathfrak{s}_1 = \frac{125}{2650}$
T	T	F	$B \,\&\, H \,\&\, \sim R = s_2$	$\Pr(s_2) = \mathfrak{s}_2 = \frac{30}{2650}$
T	F	T	$B \,\&\, \sim H \,\&\, R = s_3$	$\Pr(s_3) = \mathfrak{s}_3 = \frac{80}{2650}$
T	F	F	$B \,\&\, \sim H \,\&\, \sim R = s_4$	$\Pr(s_4) = \mathfrak{s}_4 = \frac{21}{2650}$
F	T	T	$\sim B \,\&\, H \,\&\, R = s_5$	$\Pr(s_5) = \mathfrak{s}_5 = 0$
F	T	F	$\sim B \,\&\, H \,\&\, \sim R = s_6$	$\Pr(s_6) = \mathfrak{s}_6 = \frac{105}{2650}$
F	F	T	$\sim B \,\&\, \sim H \,\&\, R = s_7$	$\Pr(s_7) = \mathfrak{s}_7 = \frac{51}{2650}$
F	F	F	$\sim B \,\&\, \sim H \,\&\, \sim R = s_8$	$\Pr(s_8) = \mathfrak{s}_8 = \frac{1128}{2650}$

To see this, note that, on the above model, we have the following.

(1) $\Pr(B \mid R \,\&\, H) = 1$.
(2) $\Pr(\sim B) = \frac{9}{10} > \frac{1}{10} = \Pr(R)$.
(C) $\Pr(H \mid R) = \frac{125}{256} = \Pr(H \mid \sim B)$.
~(6) $\Pr(H \mid \sim R \,\&\, \sim B) = \frac{375}{751} < \frac{1}{2} = \Pr(H)$.
~(7) $\Pr(H \mid \sim R \,\&\, B) = \frac{10}{17} > \frac{1}{2} = \Pr(H)$.

4.3. Examples from Howard Sobel's 'Lotteries and Miracles'. In his recent paper 'Lotteries and Miracles', Sobel (2006) reports many interesting new theorems of probability calculus, which he puts to good use for the purpose of reconstructing (and critiquing) various arguments concerning testimonial evidence regarding the possibility of miracles. One important class of results (in a section devoted to a particular Humean claim about such testimony) involves the following five claims of probability calculus (using a slightly different notation):

1. $\Pr(T) < 1/2$.
2. $\Pr(T \mid W) > 1/2$.
3. $\Pr(W \mid T) > 1/2$.
4. $\Pr(T \mid \sim W) < \Pr(W)$.
5. $\Pr(T \mid W) - \Pr(T \mid \sim W) > \Pr(\sim W) - \Pr(W)$.

Sobel's analysis led him to the following three questions:

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- Do (1)–(3) entail (4)?
- Do (1)–(3) entail (5)?
- Do (1)–(3) entail the *disjunction* (4) ∨ (5)?

Sobel (e-mail correspondence, May 2006) asked me if I could answer any of these questions. Of course, I immediately plugged them into PrSAT and found answers to all these questions very quickly. The answers are 'No', 'No', and 'Yes', respectively. That is, PrSAT found (i) models in which (1)–(3) are all true but (4) is false, (ii) models in which (1)–(3) are all true but (5) is false, and (iii) *no* models in which (1)–(3) are all true but the disjunction (4) ∨ (5) is false. Sobel reports the PrSAT models for (i) and (ii) in his paper. Since PrSAT does not produce *proofs* of theorems, the task remained (as in our raven paradox example above) to find an axiomatic proof of the disjunction (4) ∨ (5) from (1)–(3). And, just as in the case of our new ravens theorem, James Hawthorne (e-mail correspondence, May 2006) was able to find an axiomatic proof.

These examples should be sufficient to illustrate the power and usefulness of PrSAT in real mathematical and philosophical applications of probability calculus. I encourage readers to try PrSAT for themselves. In the final section, I will discuss three kinds of extensions to (and/or generalizations of) PrSAT that suggest directions for future research.

5. Directions for future research. There are three main extensions/improvements to PrSAT that are being investigated. First, there is the extension (mentioned above) to the broader class of statements that allow for (nontrivial) *quantification* over both sentence letters and real variables of the language of probability calculus. Such an extension would allow us to work with statements that are now beyond the scope of PrSAT.

The second alteration to PrSAT involves trying out different (asymptotically single-exponential) decision procedures for the existential fragment of TRCF (on the simple class of statements considered in this paper). As I mentioned, Hong's (1991) investigation of these alternatives to QEPCAD in the early 1990s indicated that no practical gain in performance would be had by making such a switch (except on problems of prohibitive size). Of course, Hong did not investigate all possible decision procedures for existential TRCF. And it now appears that there is good reason for optimism on this front.[9]

Finally, a more radical departure from the current approach would be to consider alternative *incomplete* or *nondeterministic* model finding (or SAT) techniques. For instance, various *local search* SAT algorithms (Schuurmans & Southey, 2001) have proved to be very powerful for other sorts of SAT problems. Ben Blum has implemented a simple random search add-on for PrSAT, which has linear complexity in the number of state descriptions.[10]

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[9] As I mentioned above, Galen Huntington has succeeded in implementing a much more efficient algorithm for this purpose, which will be reported in his thesis (Huntington, 2008).

[10] While attending my probability and induction course at Berkeley in spring 2005, Ben Blum (then a first-year CS student) implemented a prototype local search add-on to our *Mathematica* implementation of PrSAT (as his project for the course). It is now included in the current version of PrSAT, which (by default) tries a few random search runs before giving up and sending the problem to the decision procedure. See the companion *Mathematica* notebook that goes along with this paper (at http://fitelson.org/pm.nb) for illustrations.

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