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RESEARCH ARTICLE

Smooth surfaces with maximally many lines

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Abstract

We prove that a smooth projective surface of degree d in $\mathbb{P}^3$ contains at most $d^2(d^2-3d+3)$ lines. We characterize the surfaces containing exactly $d^2(d^2-3d+3)$ lines: these occur only in prime characteristic p and, up to choice of projective coordinates, are cut out by equations of the form $x^{p^e+1}+y^{p^e+1}+z^{p^e+1}+w^{p^e+1}=0$.

Contents

1 Introduction **1**
1.1 Characteristic zero . . . 3
1.2 Lines on quartics . . . 3
1.3 The ground field . . . 3
2 An upper bound on the number of lines of a smooth surface **4**
2.1 Maximal Picard number . . . 4
2.2 Large sets of independent curve classes . . . 4
2.3 Bounding the number of lines on smooth surfaces in $\mathbb{P}^3$ . . . 6
3 Numerical star chord pair configurations **8**
3.1 Generalized quadrangles . . . 10
3.2 Quadric configurations . . . 11
4 Surfaces with maximally many lines are extremal **13**
References **17**

1. Introduction

In this paper, we prove a sharp upper bound for the number of lines on a smooth algebraic surface in $\mathbb{P}^3$ depending only on its degree and independent of the ground field, and then we precisely characterize the surfaces that achieve this bound.

Specifically, we prove the following theorem.

Theorem 1.1. *Let $S \subseteq \mathbb{P}^3$ be a smooth algebraic surface of degree $d > 3$ over an algebraically closed field k. Then S contains at most $d^2(d^2-3d+3)$ lines. Furthermore, S contains exactly $d^2(d^2-3d+3)$ lines if and only if*

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(i) *k has characteristic $p > 0$;*
(ii) *$d = p^e + 1$ for some $e \in \mathbb{N}$; and*
(iii) *S is projectively equivalent to the Fermat surface defined by*

$$x^{p^e+1} + y^{p^e+1} + z^{p^e+1} + w^{p^e+1} = 0. \tag{1.1}$$

In this Theorem, algebraic closure is needed for the "if and only if" statement, although if the surface is *Hermitian*, the theorem holds as stated when $k = \mathbb{F}_{p^{2e}}$ by [BC66]; see Section 1.3.

Previously, an upper bound of d^4 for the number of lines on surfaces in three space was shown by Kollár [Kol15, Prop 53]. The bound given in Theorem 1.1 was discussed by Noam Elkies (without the characterization of surfaces achieving the bound) in a talk, which outlined a different and less elementary proof.[1] Little is known about *sharp* bounds beyond Theorem 1.1, although considerable effort has gone into the case of quartics (Section 1.2). Nonsharp quadratic bounds have been known for over 150 years when the field is restricted to characteristic zero (Section 1.1).

The special Fermat surfaces starring in Theorem 1.1 have extremal properties in many other contexts. For example, the cones over them are "maximally singular" in the sense that they have the smallest possible *F-pure threshold* (a singularity invariant in prime characteristic analogous to the log canonical threshold) among all reduced cone singularities of the same multiplicity [KKP+22, Thm 3.1, 4.3]. They are the only smooth surfaces (of degree $d \geq 3$) in $\mathbb{P}^3$ with the property that all smooth plane sections are isomorphic [Bea90] or with the property that the dual surface is smooth [KP91, Nom95]. They also have the maximal number of $\mathbb{F}_{p^e}$-points among surfaces of the same degree [HK13, HK15, HK16]. Notably, they have been consistently used as examples of surfaces with many lines in various settings, for example, [BC66, Seg65, SSvL10, Deg15, Kol15].

Because it is helpful to name the surfaces (1.1), we recall

Definition 1.2. Let $X = \mathbb{V}(f) \subseteq \mathbb{P}^n$ be a hypersurface over a field of characteristic p. We say that X is *extremal* if $\deg(f) = p^e + 1$ for some positive integer e and $f \in (x_0^{p^e}, x_1^{p^e}, \ldots, x_n^{p^e})$.

Extremal surfaces are classified in [KKP+22]. In particular, a smooth surface S in $\mathbb{P}^3$ is extremal if and only if S is projectively equivalent to the Fermat extremal surface (1.1); see [KKP+22, Thm 6.1], [Shi01, Prop 2.3], and [Bea90] for three different proofs. See also [HW36], [Lan56, Thm 1], [Hef85, Thm 9.10]. Using this terminology, the main result of the paper can be stated as follows:

Theorem 1.1: *A smooth surface of degree d in $\mathbb{P}^3$ contains at most $d^2(d^2 - 3d + 3)$ lines, with equality if and only if it is extremal.*

Extremal hypersurfaces are defined by p^e-bilinear forms analogously to how quadric surfaces are defined by bilinear forms; for this reason, they inherit some features of quadrics, a fact recognized by Shimada with his moniker *p-quadrics* [Shi01]. In particular, extremal surfaces are linked, via their automorphism groups, to the theory of algebraic groups over a finite field [Lan56]. For example, the famous Drinfeld curve of Deligne-Lusztig theory is extremal [DL76]. The geometry of extremal surfaces was studied in the context of *Hermitian surfaces* over $\mathbb{F}_{p^e}$ by B. Segre and Bose-Chakravarti [Seg65, BC66]; here the Frobenius map $r \mapsto r^{p^e}$ is an involution playing the role of complex conjugation in the classical theory. Their beautiful point-line geometry was developed further by Hirschfeld [Hir85], and eventually in the framework of finite incidence geometry by Payne and Thas; see [PT09].

In terms of the lines on them, extremal hypersurfaces are more like *cubics* than quadrics – note that $3^2(3^2 - 3 \cdot 3 + 3)$ is 27. The configuration of lines on an extremal surface shares properties with the configuration of lines on a cubic surface; for example, they contain configurations analogous to Schläfli's double sixes [BPRS24]. In higher dimensions, Cheng – who calls extremal hypersurfaces "*q-bics*" – shows that the Fano variety of lines on a smooth extremal threefold behaves similarly to that of cubic threefolds [Che25a], [Che25b].

[1] See https://www.birs.ca/events/2017/5-day-workshops/17w5146/videos/watch/201703150900-Elkies.html. We are grateful to Matthias Schütt for informing us of this talk.

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1.1. Characteristic zero

Of course, bounding the maximal number of lines on a smooth *complex* surface of degree d is a classical subject with a history dating back at least to Clebsch's 1861 bound of $d(11d - 24)$ [Cle61, p106]. An improved bound due to Segre [Seg43] held as the best result for eighty years until Bauer and Rams recently proved an upper bound of $11d^2 - 30d + 18$; their bound holds also in characteristic p, provided that $p > d$ [BR23]. Bauer and Rams's bound is not sharp, however, even when $d = 4$.

1.2. Lines on quartics

Rams and Schütt proved a sharp upper bound of 64 lines on a smooth quartic surface of characteristic zero or $p \geq 5$ [RS15b]. Of course, in characteristic three, Theorem 1.1 guarantees that 112 is a sharp bound on the number of lines on a smooth quartic surface, though this was first proved by Rams and Schütt [RS15a]. In characteristic two, 60 is a sharp bound on the number of lines on a smooth quartic [RS18, Deg22]. Much effort has gone into understanding lines on quartic surfaces; see also [Deg22, DIS17, DR25, GAR16, Miy09, RS14, RS15b, RS15a, RS18, Seg43, SS17, Sch82, Ven17a, Ven17b, Ven20, Ven22]. See also [RS20] for quintics.

Boissière and Sarti identify three ways to construct surfaces with many lines: those of the form $\mathbb{V}(\phi(x, y) + \psi(z, w))$, those branched over a curve in $\mathbb{P}^2$ with many total inflection points, and those with many automorphisms [BS07]. Perhaps unsurprisingly, extremal surfaces have all three of these properties. However, we do not understand well whether there can be surfaces over a fixed field of characteristic p with an abundance of lines – say, more than allowed by Bauer and Rams's bound for small d – but not as many as on an extremal surface. One question that we do not know the answer to: *are there sequences of smooth surfaces in* $\mathbb{P}^3$ *admitting sets of lines whose cardinality grows cubically in d?* The number of lines on extremal surfaces grows quartically in the degree, but for sequences of nonextremal surfaces, the number of lines is bounded by a quadratic function in the degree in every class of examples of which we are aware.

The proof of Theorem 1.1 has three main steps, which make up the remaining three sections of the paper. We prove the bound on the number of lines in Section 2 using a result of Igusa to bound the Picard number; extremal surfaces show immediately that the bound is sharp. Section 3 begins the proof of the characterization of surfaces "with maximally many lines" by considering special configurations of lines on them and invoking a combinatorial result in incidence geometry due to Thas; specifically, we identify particular configurations of lines called *numerical star chord pair configurations* that we already know to exist on extremal surfaces. Section 4 uses these configurations and some others to successively force strong restrictions on the defining equation of the surface in a particular choice of coordinates. The main idea, essentially, applies Bezout's theorem to understand when certain members in a net of lines in $\mathbb{P}^3$ lie on S.

1.3. The ground field

Throughout the paper, we work over an algebraically closed field. Of course, since a surface defined over some field k is also defined over the algebraic closure $\overline{k}$, parts (i) and (ii) of Theorem 1.1 hold for any surface over any field k with maximally many lines. Further, a surface with maximally many lines is projectively equivalent over $\bar{k}$ to the Fermat surface, but it is not necessarily projectively equivalent to the Fermat surface over k. Nor is it necessarily true that a surface defined over k for which (i), (ii), and (iii) hold has maximally many lines all defined over the field k. For example, the Fermat of degree $q + 1$ has lines which are defined over $\mathbb{F}_{q^2}$; if we were to view this surface instead as defined over $k = \mathbb{F}_q$, we would not have maximally many lines all defined over k.

For readers familiar with finite Hermitian geometry, we point out that we are *not* assuming the surface is Hermitian in Theorem 1.1; in particular, if the surface is extremal but not Hermitian – for example, if it is defined by $x^{p^e}y + y^{p^e}z + z^{p^e}w + w^{p^e}x$ – then the projective change of coordinates needed to

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achieve (iii) can *not* be defined over $\mathbb{F}_{p^{2e}}$. Indeed, the group $PGL_4(p^{2e})$ acts transitively on the set of Hermitian surfaces over k, so only Hermitian surfaces can be in the orbit of the Fermat surface. See [BPRS24, Section 2.3].

2. An upper bound on the number of lines of a smooth surface

In this section, we prove the following upper bound on the number of lines on a smooth surface of degree d, together with a restriction on the configuration of lines on a surface achieving this bound. This will establish the first part of Theorem 1.1.

Theorem 2.1. *A smooth surface S of degree d in $\mathbb{P}^3$ contains at most $d^2(d^2-3d+3)$ lines. Moreover, if S contains exactly $d^2(d^2-3d+3)$ lines, then every line on S lies in exactly d^2-2d+2 planes containing another line on S, and each such plane contains exactly d distinct lines lying on the surface.*

Remark 2.2. Smooth extremal surfaces of degree d always contain exactly $d^2(d^2-3d+3)$ lines – the maximal number possible – so the bound of Theorem 2.1 is sharp; see, for example, [BPRS24, Thm 3.3.1 (e)]. Ultimately, we will see that any smooth surface "containing maximally many lines" is extremal.

The proof of Theorem 2.1 proceeds as follows: we first use a result of Igusa to bound the Picard number of S (Proposition 2.3), then use that bound to bound the number of lines intersecting a given line (Proposition 2.10). From there, we use Bezout's theorem to conclude the result.

2.1. Maximal Picard number

The Néron-Severi group of a smooth projective variety is the group of its divisors modulo algebraic equivalence. This is a finitely generated group whose rank is called the Picard number of the variety. Equivalently, the Picard number can be defined as the rank of the group of divisors modulo numerical equivalence, since these two notions of equivalence agree on all divisors up to torsion [Mat57]. The Picard group of smooth surface $S \subseteq \mathbb{P}^3$ is torsion free in any case [SGA73, Thm 1.8], so we freely interchange algebraic and numerical equivalence in this section.

Proposition 2.3. *Let S be a smooth surface in $\mathbb{P}^3$ of degree d. Then the Picard number of S is bounded above by d^3-4d^2+6d-2.*

Remark 2.4. Extremal Fermat surfaces achieve this upper bound (as do all degree d Fermat surfaces over fields of characteristic p where d divides p^e+1) [SK79], so this bound is sharp.

Proof of Proposition 2.3. Igusa [Igu60] shows that $\rho(S) \leq \chi(S) - 2 + 4\dim(\mathrm{Pic}^0(S))$ where $\mathrm{Pic}^0(S)$ is the connected component of the identity in the Picard scheme for S and $\chi(S)$ is the "topological Euler characteristic," which in arbitrary characteristic can be defined as

$$\chi(S) = \int_S c(\mathcal{T}_S) = c_2(\mathcal{T}_S),$$

where $\mathcal{T}_S$ is the tangent bundle of S; see [Igu60, Main Theorem] or [Ful98, Ex 18.3.7(c)].

By [SGA73, 7.2, Exposé XI, Theorem 1.8], $\dim(Pic^0(S)) = 0$. The desired bound follows since $c_2(\mathcal{T}_S) = d^3-4d^2+6d$, which can be seen by applying the additivity of the total Chern character to the normal bundle sequence of S (see [Ful98, Examples 3.2.11-13]). □

2.2. Large sets of independent curve classes

As a step towards Theorem 2.1, we bound the number of lines on a surface S intersecting a fixed line on S in terms of the Picard number. In order to state the result, we need some notation.

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Notation 2.5. Consider a smooth surface $S \subseteq \mathbb{P}^3$ of degree $d \geq 3$, together with a fixed line L on S. Consider the set of all lines on S that intersect L (excluding L itself). Note that this is a finite set. Let

$$H_1, H_2, \ldots, H_s$$

be the complete list of planes in $\mathbb{P}^3$ that contain both L and some other line of S. For each, one of two possibilities holds for the plane section $H_i \cap S$. Let ℓ_i denote the number of lines on S in the plane H_i (other than L). Then either

(a) $H_i \cap S = L \cup L_{i,1} \cup \cdots \cup L_{i,d-1}$ is a union of d distinct lines[2]; or
(b) $H_i \cap S = L \cup L_{i,1} \cup \cdots \cup L_{i,\ell_i} \cup C_i$, where C_i is a degree $d - (\ell_i + 1) \geq 2$ curve in H_i, none of whose irreducible components are lines.

In Case (a), note that $\ell_i = d - 1$, whereas in case (b), $\ell_i \leq d - 3$. For each i, let

$$m_i = \begin{cases} d - 2 & \text{in Case (a);} \\ \ell_i & \text{in Case (b).} \end{cases} \tag{2.1}$$

Proposition 2.6. *Fix a line L on a smooth surface S of degree d. Let*

$$\mathcal{L}_i = \{L_{i,1}, L_{i,2}, \ldots, L_{i,m_i}\} \tag{2.2}$$

be the set of coplanar lines on S in the notation defined in Notation 2.5. Let h be an arbitrary plane section of S. The set of curves

$$\mathcal{L}_1 \cup \mathcal{L}_2 \cup \cdots \cup \mathcal{L}_s \cup \{L\} \cup \{h\} \tag{2.3}$$

represents a numerically independent set of curve classes on S.

Before proving Proposition 2.6, for convenience, we record a simple fact that we use repeatedly:

Lemma 2.7. *Three lines on a smooth surface in $\mathbb{P}^3$ intersect pairwise if and only if they are coplanar.*

Proof. Coplanar lines always intersect, so we need only prove the converse. Let L_1, L_2, and L_3 be three lines on a smooth surface $S \subseteq \mathbb{P}^3$. If their three pairwise intersection points are *distinct*, then these lines form a triangle obviously spanning a plane. If they all intersect at one point p, then for each i we have $p \in L_i \subseteq T_pS$. Since S is smooth, the tangent space T_pS is two-dimensional. Thus, the three lines are coplanar as they are contained in T_pS. □

Proof of Proposition 2.6. It suffices to show that the corresponding intersection matrix is invertible. To this end, observe that

$$h^2 = d \quad \text{and} \quad h \cdot M = 1 \tag{2.4}$$

for any line M in (2.3), including L. Likewise, since $L \subseteq H_i$ for all i, and two lines in $\mathbb{P}^3$ intersect if and only if they are coplanar,

$$L \cdot L_{i,j} = 1 \quad \text{and} \quad L_{i,j} \cdot L_{i',j'} = \delta_{ii'} \tag{2.5}$$

for all indices i, j, i', j'. Indeed, since $L_{i,j} \cap L_{i',j'} \subseteq H_i \cap H_{i'} = L$, the lines $L_{i,j}$ and $L_{i',j'}$ can only intersect at a point p on L, in which case L, $L_{i,j}$ and $L_{i',j'}$ pairwise intersect and so are coplanar by Lemma 2.7; in this case $T_pS = H_i = H_{i'}$, so $i = i'$.

[2]To see this, note that if $H_i \cap S$ contained a double line, then the Gauss map would be constant along that line, which is impossible since the Gauss map is a finite morphism. [Zak93, Theorem I.2.8]

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Finally, for any line M on S, using adjunction for M as a divisor on S, we get that $-2 = (-4+d)+M^2$ so

$$M^2 = 2 - d.$$

Order the set (2.3) so that the $L_{i,j}$ are ordered lexicographically, followed by L, then h. Then the previous calculations show that the intersection matrix for the ordered set (2.3) is

$$\begin{bmatrix} D & \vec{1} & \vec{1} \\ \vec{1}^T & 2-d & 1 \\ \vec{1}^T & 1 & d \end{bmatrix} \tag{2.6}$$

where D is a block diagonal matrix whose diagonal blocks $A_1, \ldots, A_s$ are the intersection matrices of the subsets $\mathcal{L}_1, \ldots \mathcal{L}_s$, and $\vec{1}$ is a column vector of all 1's of the appropriate size. Each A_i block satisfies

$$A_i = \mathbf{1}_{m_i \times m_i} - (d-1) I_{m_i}.$$

After a few invertible row and column operations, we can convert the matrix (2.6) to

$$\begin{bmatrix} D & \vec{0} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{0}^T & 0 & 1 \end{bmatrix}, \tag{2.7}$$

where $\vec{0}$ is a column of 0's of the appropriate size. To see that (2.7) has nonzero determinant, we compute that each block matrix A_i has determinant $(1-d)^{m_i-1}(m_i-(d-1))$, which is nonzero since $0 < m_i < d-1$.

Thus, the intersection matrix (2.6) has nonzero determinant and the set of numerical equivalence classes of curves $\bigcup_{i=1}^{s}\{L_{i,j} \mid 1 \le j \le m_i\} \bigcup \{L, h\}$ is linearly independent, as desired. □

Remark 2.8. Alternatively, we can complete the proof of Proposition 2.6 without explicitly computing $\det(A_i)$ by applying Theorem 7.1.15 in [KLOS23] to the pencil of plane sections through L on S. The set $\bigcup_{i=1}^{s}\{L_{i,j} \mid 1 \le j \le m_i\}$ is a union of components of fibers that does not completely cover any fiber, so it must be algebraically (and hence numerically) independent. This implies that the block diagonal matrix formed by $A_1, \ldots, A_r$ is invertible, so we can proceed with the rest of the proof as above.

The following corollary is immediate.

Corollary 2.9. *Fix a line L on a smooth surface S of degree d. With notation as in Notation 2.5, we have*

$$\rho(S) \ge 2 + \sum_{i=1}^{s} m_i,$$

where m_i is as defined in (2.1).

2.3. *Bounding the number of lines on smooth surfaces in* $\mathbb{P}^3$

We now use Corollary 2.9 to establish a bound on the maximal number of lines that can intersect a given line on a smooth surface, and characterize when the maximal number is achieved. Extremal surfaces show that this bound is sharp.

Proposition 2.10. *Let $S \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \ge 3$ containing a line L. Then*

(i) *At most $d^3 - 3d^2 + 4d - 2$ lines of S intersect L (not counting L itself);*
(ii) *Exactly $d^3 - 3d^2 + 4d - 2$ lines of S intersect L if and only if L lies in a configuration of lines on S formed by $d^2 - 2d + 2$ plane sections consisting of d lines of S, one of which is L.*

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Proof. In the notation of Notation 2.5, let r be the number of planes H_i satisfying (a) – that is, the number of planes H_i containing d lines of S, one of which is L. Note that r can be zero. Remembering the definition of m_i (see (2.1)), Corollary 2.9 can be rewritten as

$$\rho(S) \geq 2 + \sum_{i=1}^{s} \ell_i - r \tag{2.8}$$

where ℓ_i is the number of lines on S in H_i other than L. Putting (2.8) and Proposition 2.3 together and adding $r-2$ to both sides, we have

$$d^3 - 4d^2 + 6d - 4 + r \geq \sum_{i=1}^{s} \ell_i. \tag{2.9}$$

By definition of ℓ_i, the right-hand side of this equality is the number of lines on S that intersect L. Since $\ell_i = d-1$ for exactly r indices i, we have

$$(d-2)(d^2-2d+2) + r = d^3 - 4d^2 + 6d - 4 + r \geq r(d-1) \tag{2.10}$$

which means that

$$d^2 - 2d + 2 \geq r. \tag{2.11}$$

Combining (2.9) and (2.11), we see that

$$d^3 - 4d^2 + 6d - 4 + (d^2 - 2d + 2) \geq \sum_{i=1}^{s} \ell_i.$$

Thus, $d^3 - 3d^2 + 4d - 2$ is an upper bound on the number of lines intersecting L, as desired.

For the final statement, first note that if $r < d^2 - 2d + 2$, then

$$\sum_{i=1}^{s} \ell_i \leq \rho(S) - 2 + r < (d^3 - 4d^2 + 6d - 2) - 2 + (d^2 - 2d + 2) = d^3 - 3d^2 + 4d - 2,$$

which is the maximal possible number of lines intersecting a given line L on S. Thus, this maximum can be achieved only when $r = d^2 - 2d + 2$ – that is only when there are exactly $d^2 - 2d + 2$ planes each containing exactly d lines of S, one of which is L. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. If S does not have two skew lines, then it has at most d lines since they are all contained in a single plane. Thus, if S has more than d lines, we can choose coordinates so that S contains the skew lines $L = \mathbb{V}(x, y)$ and $M = \mathbb{V}(z, w)$. In other words, if f is the homogeneous form defining S, then $f \in (x, y) \cap (z, w)$, so that

$$f = xzf_1 + xwf_2 + yzf_3 + ywf_4, \tag{2.12}$$

where the f_i are homogenous polynomials of degree $d-2$.

Let us count the lines on S that do not intersect L. An arbitrary line N skew to L in $\mathbb{P}^3$ is cut out by two homogenous forms, h_1 and h_2 such that the ideal (x, y, h_1, h_2) is equal to (x, y, z, w). Thus, without loss of generality, we can assume h_1 and h_2 are the forms

$$a_1x + a_2y - z \quad \text{and} \quad a_3x + a_4y - w,$$

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for some scalars a_i. The line N defined by these forms lies on S if and only if $f \in (h_1, h_2)$, or equivalently, if and only if f is in the kernel of the "restriction to N" map

$$k[x, y, z, w] \to k[x, y] \quad z \mapsto a_1x + a_2y, \quad w \mapsto a_3x + a_4y.$$

Remembering that the polynomial f has the form (2.12), we see that this map sends f to a polynomial of degree d in x and y whose coefficients are polynomials of degree $d-1$ in a_1, a_2, a_3, a_4. In other words, N lies on S if and only if these degree $d-1$ polynomials in the four variables a_1, a_2, a_3, a_4 vanish. Since we know that S has only finitely many lines, Bezout's theorem for affine space implies there are at most $(d-1)^4$ solutions to these polynomials in four variables. That is, there are at most $(d-1)^4$ lines on S skew to L.

Invoking Proposition 2.10, we conclude that in total, there are at most

$$(d-1)^4 + (d^3 - 3d^2 + 4d - 2) + 1 = d^2(d^2 - 3d + 3)$$

lines on S as claimed.

Finally, note that the argument above is sharp: for any line L, if there are *not* $d^2 - 2d + 2$ distinct planes each containing d lines of S, one of which is L, then there are strictly fewer than $d^2(d^2 - 3d + 3)$ lines on S. Although we will not use this fact, it is similarly true that for any line L, if L is not skew to $(d-1)^4$ lines of S, then there are strictly fewer than $d^2(d^2 - 3d + 3)$ lines on S. □

Corollary 2.11. *If S is a smooth surface of degree $d > 3$ which contains $d^2(d^2 - 3d + 3)$ lines (the maximal possible according to Theorem 2.1), then for any two skew lines on S, there are $(d-1)^2 + 1$ lines on S intersecting both.*

Proof. Fix two skew lines L and M on S. By Theorem 2.1, L lies in $(d-1)^2 + 1$ planes containing other lines. Every line on S intersecting L must lie in one of those planes, and all lines in those planes intersect L. The line M is skew to L, so in none of those planes. This means M meets each of these planes. But M must intersect a single line in each plane by Lemma 2.7. The result follows. □

3. Numerical star chord pair configurations

The configuration of lines on an extremal surface of degree d has several strong properties. In this section, we use a combinatorial result of Thas to show that a smooth surface in 3-space with "maximally many lines" admits a special configuration of lines that we call a *numerical star chord pair configuration* that always appears on extremal surfaces. In the next section, we will use this fact to show that surfaces with the maximal number of lines *are*, in fact, extremal.

Definition 3.1. Let $S \subseteq \mathbb{P}^3$ be a surface of degree d. A **numerical star chord pair configuration** on S is a collection of d^2 lines on S that can be partitioned into d sets of d coplanar lines in *two different ways*.

The purpose of this section is to prove the following crucial ingredient in the proof of Theorem 1.1, which will follow by combining our Theorem 2.1 and Thas's Theorem 3.14:

Corollary 3.2. *Let $S \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 3$. If S contains $d^2(d^2 - 3d + 3)$ lines – the maximal possible by Theorem 2.1 – then $d-1$ is a prime power and S contains a numerical star chord pair configuration.*

Notice that Corollary 3.2 does *not* guarantee that $d-1$ is a power of the characteristic of the ground field. Indeed, Example 3.7 shows that this need not be the case. In Section 4, we will show that when $d \geq 4$, under the hypothesis of Corollary 3.2, $d-1$ is a power of the characteristic and the surface is extremal.

The term "numerical star chord pair configuration" is motivated by the following example.

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Example 3.3 [BPRS24, Section 3]. Let $X \subseteq \mathbb{P}^3$ be an extremal surface of degree d. Then whenever two lines on X intersect at a point p, the plane they span (which is necessarily T_pX) intersects X in a "star" – a union of d coplanar lines through p. This plane is called a *star plane* and the point p is called a *star point.* When $d = 3$, these are called Eckardt points. Star points are also defined in greater generality in [CC10].

If p and p' are two star points spanning a line ℓ *not* on X, then the line ℓ is said to be a *star chord* of X. Every star chord intersects X in d distinct points $p_1, \dots p_d$, all of which turn out to be star points (two of which are p and p'). The set

$$\{X \cap T_{p_1}X\} \cup \{X \cap T_{p_2}X\} \cup \cdots \cup \{X \cap T_{p_d}X\} \tag{3.1}$$

consists of d^2 lines in X partitioned into d different sets of d coplanar lines, namely the d stars with star points $p_1, \dots p_d$. On the other hand, there is another way to partition these same d^2 lines into d sets of d coplanar lines. Namely, the star planes T_pX and $T_{p'}X$ intersect in a star chord ℓ', which intersects X in d distinct star points $q_1, q_2, \dots, q_d$, none of which is any p_i. The collection of d^2 lines

$$\{X \cap T_{q_1}X\} \cup \{X \cap T_{q_2}X\} \cup \cdots \cup \{X \cap T_{q_d}X\} \tag{3.2}$$

is a *different* partition of the same d^2 lines (3.1) into d sets of d coplanar lines. As it turns out, $\cap_{i=1}^d T_{q_i}X = \ell$ and $\cap_{i=1}^d T_{p_i}X = \ell'$ so the star chord pair (ℓ, ℓ') is independent of the original choice of star points p and p'; see [BPRS24, Section 3]. We call these d^2 lines, together with the two partitions (3.1) and (3.2), a *star chord pair configuration;* it is a special case of a numerical star chord pair configuration.

Remark 3.4. Surfaces in $\mathbb{P}^3$ containing star chord pair configurations are a topic of considerable independent interest. Any surface – extremal or not – defined by $\phi(x, y) = \psi(z, w)$, where ϕ and ψ are reduced forms of the same degree, contains a star chord pair configuration [CHM95, BS07, Xia19]. See also [Hir85, p116].

Remark 3.5. Our definition of a *numerical* star chord pair configuration does *not* require that the sets of d coplanar lines making up either partition form a star, as in Example 3.3. There exist configurations of lines on smooth surfaces that are numerical but not actual star chord pair configurations; see Example 3.7 for examples on cubics. For a higher-dimensional example, consider the pencil of planes between two surfaces of degree $d \geq 4$ each defined as the union of d general planes; the generic element of the pencil is an irreducible degree d surface containing a numerical star chord pair configuration that is not a star chord pair configuration.

Remark 3.6. Suppose a smooth surface $S \subseteq \mathbb{P}^3$ of degree d admits a numerical star chord pair configuration consisting of d^2 lines on S partitioned by the planes $\{H_1, \dots, H_d\}$ and $\{K_1, \dots, K_d\}$. In this case, the d^2 lines on S are precisely $L_{i,j} = H_i \cap K_j$ as i and j run through the set $\{1, 2, \dots, d\}$. Observe that

(a) The lines $H_i \cap H_j$ (respectively $K_i \cap K_j$) do not lie on S – that is, they are *chords*;
(b) The lines $L_{i,j}$ and $L_{m,n}$ are skew if $i \neq m$ and $j \neq n$.

Indeed, (a) follows because otherwise the line $H_i \cap H_j$ would lie in both $H_i \cap S$ and $H_j \cap S$, contrary to the fact that $\{H_1, \dots, H_d\}$ *partitions* the lines. For (b), observe that $L_{i,n}$ and $L_{m,n}$ are both in the plane K_n, and that $L_{i,j}$ intersects $L_{i,n}$. If $L_{i,j}$ intersects $L_{m,n}$ as well, then $\{L_{i,n}, L_{m,n}, L_{i,j}\}$ would pairwise intersect, so $L_{i,j}$ would lie in the plane K_n by Lemma 2.7. Since $L_{i,j}$ is also in K_j, this would imply that $L_{i,j} = K_n \cap K_j$, contrary to (a).

Example 3.7. Smooth cubic surfaces always contain 27 (which is $d^2(d^2 - 3d + 3)$) lines, so by Corollary 3.2, they contain a numerical star chord pair configuration. Representing a cubic surface S as the blow up of $\mathbb{P}^2$ at six points $p_1, \dots p_6$, let E_i denote the exceptional divisors, C_i the strict transforms of conics through all p_j with $j \neq i$, and L_{ij} the strict transform of the line through p_i and p_j. Then one readily confirms that

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$$\{E_1, C_2, L_{12}\} \cup \{E_3, C_4, L_{34}\} \cup \{L_{14}, L_{23}, L_{56}\}$$
$$\{E_1, C_4, L_{14}\} \cup \{E_3, C_2, L_{23}\} \cup \{L_{12}, L_{34}, L_{56}\}$$

is a numerical star chord pair configuration on S. In fact, one can show that up to choosing the six (ordered) points to blow up, every numerical star chord pair configuration is of this type.

3.1. *Generalized quadrangles*

Generalized quadrangles were introduced by Tits in his study of simple groups of Lie type of rank 2 [Tit59], and quickly became a popular research topic in combinatorics; see [PT09] for a general reference. Our proof of Corollary 3.2 will make use of a characterization of certain generalized quadrangles due to Thas. We now briefly summarize the necessary machinery to state it.

An *incidence structure* is a triple (P, B, I), where P and B are sets (typically called "points" and "lines" respectively, but we will avoid this language as it would be confusing in our context), together with a relation $I \subseteq P \times B$ which determines when $p \in P$ is incident to $b \in B$. Two incidence structures (P, B, I) and (P', B', I') are isomorphic if there are bijections $P \to P'$ and $B \to B'$ identifying I with I'.

Definition 3.8 [PT09, p1]. A *generalized quadrangle* with parameters (s, t) is an incidence structure (P, B, I) satisfying the following conditions:

1. For all $x, x' \in P$, there is at most one $b \in B$ such that $(x, b), (x', b) \in I$.
2. For all $b, b' \in B$, there is at most one $x \in P$ such that $(x, b), (x, b') \in I$.
3. For all $(x, b) \notin I$, there exist unique $x' \in P$ and $b' \in B$ such that $(x', b), (x, b'), (x', b') \in I$.
4. For all $b \in B$, there are exactly $s + 1$ elements $x \in P$ with $(x, b) \in I$.
5. For all $x \in P$, there are exactly $t + 1$ elements $b \in B$ with $(x, b) \in I$.

Of particular interest in incidence geometry is the *Hermitian generalized quadrangle*:

Example 3.9. [PT09, 3.1.1(ii)]. Let $X = \mathbb{V}(x^{p^e+1} + y^{p^e+1} + z^{p^e+1} + w^{p^e+1}) \subseteq \mathbb{P}^3$ be the Hermitian surface (of degree $p^e + 1$) defined over $\overline{\mathbb{F}_p}$. Let P' be the set of $\mathbb{F}_{p^{2e}}$-points and let B' be the set of lines on X defined over $\mathbb{F}_{p^{2e}}$, with I the natural incidence relation of points lying on lines. Then (P', B', I') is a generalized quadrangle with parameters (p^{2e}, p^e) called the **Hermitian generalized quadrangle**.

Equivalently, we can take P' to be the set of star points and B' to be the set of lines on *any* smooth extremal surface X of degree $p^e + 1$, since up to choice of projective coordinates, X can be assumed to be the Hermitian surface $\mathbb{V}(x^{p^e+1} + y^{p^e+1} + z^{p^e+1} + w^{p^e+1})$ in which case the star points of a Hermitian surface are precisely its $\mathbb{F}_{p^{2e}}$-points; see [BPRS24, Prop 2.5.1] or [Seg67, Hir85, Shi01].

For our argument, we are interested in a *different* generalized quadrangle associated to an extremal surface:

Example 3.10. Let X be an extremal surface of degree $p^e + 1$ in $\mathbb{P}^3$, let P be the set of lines on X, and let B be the set of star planes on X (as defined in Example 3.3 or [BPRS24, Prop 2.5.1]). Defining incidence in the natural way,

$$I = \{(L, \Lambda) \mid L \subseteq \Lambda\} \subseteq P \times B,$$

(P, B, I) is a generalized quadrangle with parameters $(s, t) = (p^e, p^{2e})$. This is immediate from [BPRS24, Thm 3.3.1a] (but see also, e.g., [Hir85] or [Shi01]). In any case, it is a special case of the *a priori* more general example given by the next result:

Proposition 3.11. *Let $S \subseteq \mathbb{P}^3$ be a smooth surface of degree d containing $d^2(d^2 - 3d + 3)$ lines. Let P be the set of lines on S, and let B be the set of planes containing d lines of S. With incidence defined the natural way, (P, B, I) forms a generalized quadrangle with parameters $(d - 1, (d - 1)^2)$.*

Proof. This is a corollary of Theorem 2.1. Conditions (1) and (2) hold as any two planes share at most one line and any two lines lie in at most one plane in $\mathbb{P}^3$. Condition (3) holds because if a line L on S is

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not contained in a plane $H \in B$, then $L \cap H$ is a point on S, which must lie on a unique line $M \subseteq H \cap S$ (by Lemma 2.7); in this case, the plane spanned by L and M is in B by Theorem 2.1. Conditions (4) and (5) are the content of the second part of Theorem 2.1. □

The *dual* of an incidence structure (P, B, I) is the incidence structure $(B, P, I^\vee)$ where $(b, p) \in I^\vee$ if and only if $(p, b) \in I$. The dual of a generalized quadrangle with parameters (s, t) is a generalized quadrangle with parameters (t, s).

Proposition 3.12. *Let $X \subseteq \mathbb{P}^3$ be a smooth extremal surface. Then the following generalized quadrangles associated to X are dual:*

(i) *Example 3.10, in which P consists of lines on X and B consists of star planes on X;*
(ii) *Example 3.9, in which P' consists of star points on X and B' consists of lines on X.*

Proof. This follows immediately from the bijection between star points and star planes given by $p \leftrightarrow T_P X$ and the basic properties of the geometry of extremal surfaces proved, for example, in [BPRS24]. □

A key step in the proof of the main result of this section, Corollary 3.2, will be to show that the generalized quadrangle of Proposition 3.11 is *also* dual to the Hermitian generalized quadrangle, and therefore isomorphic to the generalized quadrangle of lines and star planes on an extremal surface (Example 3.10). To do so, we use a combinatorial characterization of the dual to the Hermitian generalized quadrangle due to Thas. To state this result, we need a few more definitions.

Definition 3.13. Let (P, B, I) be a generalized quadrangle with parameters (s, t).

(i). A *triad* is a set of three elements $x_1, x_2, x_3 \in P$ such that if $(x_i, b) \in I$, then $(x_j, b) \notin I$ for all $j \neq i$ [PT09, p. 2].
(ii). A triad $\{x_1, x_2, x_3\}$ is *3-regular* if it can be extended to a set of $s + 1$ elements $x_i \in P$ such that, whenever $y \in P$ is coincident with x_1, x_2 and x_3, then also y is coincident with $x_4, \ldots, x_{s+1}$; explicitly, this means that if for $i = 1, 2$ and 3, there exists $b_i \in B$ such that $(x_i, b_i), (y, b_i) \in I$, then also there exists $b_i \in B$ for each $i = 4, 5, \ldots, s + 1$, such that $(x_i, b_i), (y, b_i) \in I$ [PT09, p. 4], [Tha94, Page 143].

Theorem 3.14 [Tha78] [Tha94]. *Let (P, B, I) be a generalized quadrangle with parameters (q, q^2) for some positive integer $q > 1$. If every triad of (P, B, I) is 3-regular, then (P, B, I) is isomorphic to the dual of the Hermitian generalized quadrangle. In particular, q is a prime power.*

Proof. This is essentially [PT09, 5.3.2.i], where the result is attributed to Thas in [Tha78]. Note that the notation $Q(5, q)$ appearing in that statement is defined in [PT09, 3.1.1], and shown to be isomorphic to the dual of the Hermitian quadrangle in [PT09, 3.2.3], whenever q is a prime power; that q is a prime power is established in [Tha94, Corollary 3.1.4]. □

3.2. Quadric configurations

In the generalized quadrangle (P, B, I) of a surface with the maximal number of lines (defined in Proposition 3.11), triads and 3-regularity have natural geometric interpretations. For example, a *triad* of elements in P is a set of three skew lines on S. A set of three skew lines is *3-regular* if it can be extended to a set of d lines on S, each of which intersects every line on S intersecting all three initial lines. Our next step is to show that all triads in this generalized quadrangle are 3-regular. For this, we use the idea of a quadric configuration:

Definition 3.15 [BPRS24, Def 4.0.1]. Let $S \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 3$. A *quadric configuration* is a collection of $2d$ lines on S partitioned into two sets of d skew lines with the property that each line in either set meets every line of the other set.

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Equivalently, a quadric configuration is a set of $2d$ lines on S whose union is the intersection $S \cap Q$ for some smooth quadric surface Q [BPRS24, Prop 4.0.2]. By necessity, d lines of the quadric configuration lie in each ruling of Q.

Lemma 3.16. *Let $S \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 3$. If S contains $d^2(d^2 - 3d + 3)$ lines, then every set of three skew lines on S determines a quadric configuration on S.*

Proof. Any triple of skew lines in $\mathbb{P}^3$ determines a smooth quadric surface Q [Har95, 2.12], with the three lines lying in one of the rulings of Q. Note that any line in $\mathbb{P}^3$ that intersects all three lines must lie on Q, as its intersection multiplicity with the quadric Q is greater than 2. Since the intersection of Q with our degree d surface S determines a curve of bi-degree (d, d) on Q, there are at most d lines of S (all in the other ruling of Q) which intersect the given three lines. That is, at most d lines on S can intersect every line in a set of three skew lines in $\mathbb{P}^3$.

Fix a triple of skew lines $\{L_1, L_2, L_3\}$ on S, and let Q be the quadric surface they determine. To prove Lemma 3.16, we must show that the bidegree (d, d) curve $Q \cap S$ is a union of lines.

By Proposition 2.10, L_3 lies in $d^2 - 2d + 2$ planes H_i, each of which contains d lines of S. None of these H_i contain L_1 or L_2 as both are skew to L_3. Thus, L_1 and L_2 each intersect H_i in a single point, which necessarily lies on exactly one line in $H_i \cap S$ by Lemma 2.7. If L_1 and L_2 intersect the same line in every $H_i \cap S$, then the three skew lines $\{L_1, L_2, L_3\}$ would intersect $d^2 - 2d + 2$ lines in common, which is impossible since at most d lines on S can intersect all three. Thus, there is some plane H such that $L_3 \subseteq H \cap S$ such that L_1 and L_2 intersect *different* lines in $H \cap S$. Let $N_1, \ldots, N_d$ be the lines of $H \cap S$ (where N_3 also denotes L_3 for convenience), and assume without loss of generality that L_1 intersects N_1 and L_2 intersects N_2. In particular, L_1 and L_2 are skew to N_i for $3 \leq i \leq d$, and each set $\{L_1, L_2, N_i\}$ is a triple of skew lines on S.

By Corollary 2.11, there are exactly $d^2 - 2d + 2$ lines intersecting both L_1 and L_2; label them M_i. No M_i can lie in H, since no line in $H \cap S$ intersects both L_1 and L_2. Thus, each M_i intersects *exactly one* of the lines N_j on H for $1 \leq j \leq d$ by Lemma 2.7.

We claim that *exactly one M_i intersects N_1 and exactly one M_i intersects N_2*. Indeed, consider the plane spanned by L_1 and N_1 (respectively L_2 and N_2). If M_j and M_k both intersect N_1 (respectively N_2), then both are in this plane (Lemma 2.7), so meet. In this case, L_1 and L_2 are in this plane as well (again by Lemma 2.7), contradicting the fact that L_1 and L_2 are skew.

Thus, there are exactly $d^2 - 2d + 2 - 2 = d(d-2)$ lines intersecting L_1, L_2, and $\bigcup_{i=3}^{d} N_i$. Because $\{L_1, L_2, N_i\}$ is a triple of skew lines (for $3 \leq i \leq d$), at most d lines on S can intersect all three. So by the pigeonhole principle, for each $i = 3, 4, \ldots, d$, there must be exactly d lines intersecting L_1, L_2, and N_i. In particular, L_1, L_2, and L_3 intersect d distinct lines of S in common, say $M_1, \ldots, M_d$, which all lie in the same ruling of the smooth quadric surface Q determined by L_1, L_2, and $L_3 = N_3$. Thus, $\{M_1, \ldots, M_d\}$ is a set of d skew lines in $S \cap Q$.

Now, apply the same argument to the skew set $\{M_1, M_2, M_3\}$ on S. The same quadric Q is determined, together with a set of d skew lines $\{L_1, \ldots, L_d\}$ lying in $S \cap Q$ in the other ruling. So $Q \cap S$ is a union of the $2d$ lines $\{M_1, \ldots, M_d, L_1, \ldots, L_d\}$. Lemma 3.16 is proved. □

Theorem 3.17. *Let $S \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 3$ containing $d^2(d^2 - 3d + 3)$ lines. Then $d = p^e + 1$ for some prime p and positive integer e, and the generalized quadrangle of lines on S (defined in Proposition 3.11) is isomorphic to the generalized quadrangle of lines of an extremal surface (Example 3.10).*

Proof. Let (P, B, I) be the generalized quadrangle where P is the set of lines on S and B is the set of planes containing d lines of S. The parameters of this generalized quadrangle are $(d-1, (d-1)^2)$. An arbitrary triad in this generalized quadrangle is a triple $\{L_1, L_2, L_3\}$ of skew lines on S. By Lemma 3.16, any such triad determines a quadric configuration

$$S \cap Q = \{L_1, L_2, L_3, \ldots, L_d\} \bigcup \{M_1, M_2, \ldots, M_d\}$$

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on S, where the L_i's are in one of the rulings and the M_i's the other. So we can extend the triad $\{L_1, L_2, L_3\}$ to a set $\{L_1, L_2, L_3, \ldots L_d\}$ of d skew lines on S. These have the property that for any line M on S intersecting L_1, L_2, and L_3, M must also intersect $\{L_4, \ldots, L_d\}$. Since two lines on S intersect if and only if they are coplanar, this exactly says that the triad $\{L_1, L_2, L_3\}$ of the generalized quadrangle (P, B, I) is 3-regular in the sense of Definition 3.13. Thus, the hypothesis of Theorem 3.14 is satisfied, and we conclude that $d-1$ is a prime power p^e and that the incidence structure (P, B, I) is isomorphic to the dual of the Hermitian generalized quadrangle using Theorem 3.14. By Proposition 3.12, (P, B, I) is isomorphic to the generalized quadrangle of lines and star planes on an extremal surface, as described in Example 3.10. □

Proof of Corollary 3.2. Theorem 3.17 shows that $d-1$ is a prime power. Since two lines on S intersect if and only if they lie in a plane together (which then must have d lines of S by Theorem 2.1), Theorem 3.17 shows that the intersection matrix of the lines on any surface with the maximal number of lines is identical to the intersection matrix of the lines on an extremal surface (for appropriate ordering of the lines). The existence of a numerical star chord pair configuration on a surface depends only on the intersection matrix of its lines. So since extremal surfaces have (numerical) star chord pair configurations [BPRS24, Thm 3.1.4], so does S. □

4. Surfaces with maximally many lines are extremal

In this section, we complete the proof of Theorem 1.1 by proving the following:

Theorem 4.1. *Suppose $S \subseteq \mathbb{P}^3$ is a smooth surface of degree $d > 3$ containing the maximal number of lines, $d^2(d^2 - 3d + 3)$. Then S is extremal. That is, the characteristic of the ground field is $p > 0$, $d = p^e + 1$ for some $e \in \mathbb{N}$, and S is projectively equivalent (over the algebraic closure of the ground field) to the Fermat surface defined by $x^{p^e+1} + y^{p^e+1} + z^{p^e+1} + w^{p^e+1}$.*

Our strategy is to repeatedly use the numerical star chord pair configuration on S guaranteed by Corollary 3.2, together with the configurations of lines on S intersecting a pair of skew lines guaranteed by Corollary 2.11, to get successively stronger restrictions on the monomials that can appear in a defining equation f for S in an appropriate choice of coordinates. Interestingly, although Corollary 3.2 guarantees that the degree d of f must satisfy $d = p^e + 1$ for *some* prime p, it does *not* guarantee that p is the characteristic of the ground field. This will require proof.

Lemma 4.2. *Let $S \subseteq \mathbb{P}^3$ be a smooth surface of degree d containing a numerical star chord pair configuration. Then without loss of generality the defining equation of S can be taken to be*

$$f = xw \prod_{i=1}^{d-2} \ell_i + yz \prod_{i=1}^{d-2} m_i \tag{4.1}$$

where ℓ_i and m_i are distinct (up to scalar multiple) linear polynomials in x, y, z, w.

Proof. Let $\{H_i \mid i = 1, \ldots, d\}$ and $\{K_j \mid j = 1, \ldots, d\}$ be the two collections of d planes each containing d of the d^2 lines forming a numerical star chord pair configuration on S. Suppose that H_i and K_j are defined by linear forms ℓ_i and m_j, respectively. The d^2 distinct lines of the configuration are therefore $\mathbb{V}(\ell_i, m_j) = H_i \cap K_j$.

Let f be the degree d defining equation of S. Since $H_i \cap S$ is a union of the d lines $\mathbb{V}(\ell_i, m_1), \mathbb{V}(\ell_i, m_2), \ldots, \mathbb{V}(\ell_i, m_d)$, we have

$$S \cap H_i = \mathbb{V}(\ell_i, f) = \mathbb{V}(\ell_i, \prod_{j=1}^{d} m_j).$$

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So f is in the ideal $(\ell_i, \prod_{j=1}^{d} m_j)$, as this ideal is radical in $k[x, y, z, w]$. Thus, for each i, we can write

$$f = \ell_i g_i + \lambda_i \prod_{j=1}^{d} m_j$$

for some form g_i of degree $d-1$ and some nonzero constant λ_i. Replacing each linear form m_j by a suitable scalar multiple, we can assume that for each i, there is a degree $d-1$-form g_i such that

$$f - \prod_{j=1}^{d} m_j = \ell_i g_i.$$

But now the polynomial $f - \prod_{j=1}^{d} m_j$ is a degree d form in $k[x, y, z, w]$ divisible by *each* of the linear forms $\ell_1, \ell_2, \ldots, \ell_d$, so it must be $\prod_{i=1}^{d} \ell_i$ (up to nonzero scalar multiple) by the UFD property of the polynomial ring. Without loss of generality, $f = \prod_{i=1}^{d} \ell_i + \prod_{j=1}^{d} m_j$.

Finally, since the lines $\mathbb{V}(\ell_1, m_1)$ and $\mathbb{V}(\ell_2, m_2)$ are skew in $\mathbb{P}^3$ (Remark 3.6), the ideal generated by ℓ_1, ℓ_2, m_1, m_2 must be (w, x, y, z), so we can choose coordinates to make ℓ_1, ℓ_2, m_1, m_2 equal to w, x, y, z, respectively. This ensures f has the form (4.1). □

Notation 4.3. Throughout the rest of this section, we will assume $S = \mathbb{V}(f) \subseteq \mathbb{P}^3$ is a smooth surface of degree $d > 3$ that contains a numerical star chord pair configuration. By Lemma 4.2, we write f as:

$$f = xw \prod_{i=1}^{d-2} (a_i x + b_i y + c_i z + d_i w) + yz \prod_{i=1}^{d-2} (e_i x + f_i y + g_i z + h_i w). \tag{4.2}$$

Lemma 4.4. *Suppose that a smooth surface in $\mathbb{P}^3$ has defining equation of the form (4.2). Then none of the coefficients a_i, d_i, f_i, or g_i is zero for any $i = 1, \ldots, d$.*

Proof. By symmetry, it suffices to show that no a_i is zero. Suppose that $a_i = 0$. Then for $\ell_i = b_i y + c_i z + d_i w$, the point $[1 : 0 : 0 : 0]$ lies on both lines $\mathbb{V}(w, y)$ and $\mathbb{V}(\ell_i, z)$. In particular, this pair of lines is not skew, contrary to Remark 3.6(b). □

Proof of Theorem 4.1. By Corollary 3.2, S has a numerical star chord pair configuration. By Lemma 4.2 and Lemma 4.4, we may assume that the defining equation of S has the form

$$f = xw \prod_{i=1}^{d-2} (a_i x + b_i y + c_i z + d_i w) + yz \prod_{i=1}^{d-2} (e_i x + f_i y + g_i z + h_i w) \tag{4.3}$$

where none of a_i, d_i, f_i, or g_i is zero.

By Corollary 2.11, there are exactly $(d-1)^2 + 1$ lines on S intersecting both of the skew lines $\mathbb{V}(x, z)$ and $\mathbb{V}(y, w)$ of S. One of these lines is $\mathbb{V}(z, w)$. The others are all defined by

$$x = Az \quad \text{and} \quad y = Bw \tag{4.4}$$

for some scalars $A, B \in k$. Restricting the polynomial f to an arbitrary line L given by equations of the form (4.4), we get a polynomial $\overline{f}$ in two variables which is zero if and only if L lies on S. Algebraically, we can write $\overline{f}$ as the image of f under the map

$$k[x, y, z, w] \to k[z, w] \qquad x \mapsto Az, \quad y \mapsto Bw$$

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whose kernel is the ideal of L. Explicitly, in these coordinates

$$\bar{f} = Azw\prod_{i=1}^{d-2}((Aa_i + c_i)z + (b_iB + d_i)w) + Bzw\prod_{i=1}^{d-2}((e_iA + g_i)z + (Bf_i + h_i)w). \tag{4.5}$$

Since there are exactly $(d-1)^2$ lines of type (4.4) on S, there are exactly $(d-1)^2$ choices for (A, B) where $\bar{f} \in k[z, w]$ is the zero polynomial.

To analyze this, let $F_\ell \in k[A, B]$ be the coefficient of $z^\ell w^{d-\ell}$ in $\bar{f}$ as expressed in (4.5). The line L given by the equations (4.4) lies on S if and only if

$$F_1(A, B) = F_2(A, B) = \cdots = F_{d-1}(A, B) = 0. \tag{4.6}$$

In particular, there are exactly $(d-1)^2$ solutions to the equations (4.6). Looking at just F_1 and F_{d-1}, we see that

$$F_1 = A\prod_{i=1}^{d-2}(b_iB + d_i) + B\prod_{i=1}^{d-2}(f_iB + h_i) \quad \text{and} \quad F_{d-1} = A\prod_{i=1}^{d-2}(a_iA + c_i) + B\prod_{i=1}^{d-2}(e_iA + g_i).$$

As neither $\prod a_i$ nor $\prod f_i$ is zero (Lemma 4.4), the polynomials F_{d-1} and F_1 are both degree $d-1$. Furthermore, F_1 and F_{d-1} have no nontrivial common factor: viewing F_1 as a degree one polynomial in the variable A over the field $k(B)$, we see that any factor would be a polynomial in B alone, and likewise any factor of F_{d-1} would be a polynomial in A alone; hence any common factor is a scalar. Hence Bezout's theorem tells us that F_1 and F_{d-1} can have *at most* $(d-1)^2$ common zeros. On the other hand, imposing the further conditions that *all* the F_ℓ in (4.6) vanish, we have exactly $(d-1)^2$ distinct zeros. Thus, the degree $d-1$ polynomials F_1 and F_{d-1} together must cut out the full set of all $(d-1)^2$ solutions to (4.6). In particular, the ideal (F_1, F_{d-1}) must be the radical ideal describing all $(d-1)^2$ solutions to (4.6), and every other F_ℓ must be in the ideal (F_1, F_{d-1}).

The polynomial $F_\ell \in k[A, B]$ is

$$F_\ell = \sum_{\sigma\subseteq\{1,\ldots,d-2\},|\sigma|=\ell-1} A\prod_{i\in\sigma}(a_iA + c_i)\prod_{i\notin\sigma}(b_iB + d_i) + B\prod_{i\in\sigma}(e_iA + g_i)\prod_{i\notin\sigma}(f_iB + h_i). \tag{4.7}$$

When $1 < \ell < d-1$, using the fact that F_ℓ does not have an A^{d-1} or a B^{d-1} term, it can be shown that F_ℓ is the trivial linear combination of F_1 and F_{d-1} – that is, $F_\ell = 0$. (E.g., homogenize F_ℓ, F_1 and F_{d-1} to degree $d-1$ polynomials in $k[A, B, C]$ and observe that F_1 and F_{d-1} cut out $(d-1)^2$ points in the affine chart of $\mathbb{P}^2$ where $C \neq 0$.)

We now relate the coefficients of the monomials in A and B in F_ℓ to the coefficients of the monomials of x, y, z, w in f. Write f as

$$f = \sum_{j+k+m+n=d} a_{jkmn}x^j y^k z^m w^n.$$

In this notation, we have

$$\bar{f} = \sum_{j+k+m+n=d} a_{jkmn}(Az)^j(Bw)^k z^m w^n = \sum_{j+k+m+n=d} a_{jkmn}A^j B^k z^{j+m}w^{k+n}.$$

As we have defined them, F_ℓ is the coefficient of $z^\ell w^{d-\ell}$ in $\bar{f}$. In other words,

$$F_\ell = \sum_{j+m=\ell,k+n=d-\ell} a_{jkmn}A^j B^k \in k[A, B].$$

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Thus, the coefficient of A^jB^k in the polynomial F_ℓ is a_{jkmn}, for $\ell = j+m$. Since F_ℓ is the zero polynomial in A and B for $1 < \ell < d-1$, we see that $a_{jkmn} = 0$ unless $j+m = 1$ or $j+m = d-1$. In other words, f only has monomials of the form

$$\{xy^k w^{d-k-1}, y^k zw^{d-k-1}, x^j yz^{d-j-1}, x^j z^{d-j-1}w\}. \tag{4.8}$$

Note in this argument, we could have swapped the roles of y and z. In particular, looking at (4.3) and analyzing the lines on S that intersect the skew lines $\mathbb{V}(x,y)$ and $\mathbb{V}(z,w)$ (which will have the form $\mathbb{V}(x-Ay, z-Bw)$), we see that f only has monomials of the form

$$\{xz^m w^{d-m-1}, yz^m w^{d-m-1}, x^j y^{d-j-1}z, x^j y^{d-j-1}w\}. \tag{4.9}$$

Comparing (4.8) and (4.9), we see that f only contains monomials of the

$$\{xw^{d-1}, x^{d-1}w, xy^{d-2}w, xz^{d-2}w, yz^{d-1}, y^{d-1}z, x^{d-2}yz, yzw^{d-2}\}.$$

Thus, we can write

$$f = xw(A_1x^{d-2} + A_2y^{d-2} + A_3z^{d-2} + A_4w^{d-2}) + yz(B_1x^{d-2} + B_2y^{d-2} + B_3z^{d-2} + B_4w^{d-2}) \tag{4.10}$$

for some scalars A_i, B_i. Since we can also write

$$f = xw\prod_{i=1}^{d-2}(a_ix + b_iy + c_iz + d_iw) + yz\prod_{i=1}^{d-2}(e_ix + f_iy + g_iz + h_iw), \tag{4.11}$$

and no term on the left-hand side of (4.10) is divisible by yz, we can conclude that

$$\prod_{i=1}^{d-2}(a_ix + b_iy + c_iz + d_iw) = A_1x^{d-2} + A_2y^{d-2} + A_3z^{d-2} + A_4w^{d-2}. \tag{4.12}$$

The left-hand side of (4.12) is a product of *distinct* linear factors, so since the polynomial ring has unique factorization, the same is true for the right-hand side. Because $A_1 = \prod_{i=1}^{d-2} a_i \neq 0$ and $A_4 = \prod_{i=1}^{d-2} d_i \neq 0$ (Lemma 4.4), we claim this forces $A_2 = A_3 = 0$. Indeed otherwise $A_1x^{d-2} + A_2y^{d-2} + A_3z^{d-2} + A_4w^{d-2}$ is a sum of at least three nonzero diagonal terms, which is either irreducible or a power of an irreducible polynomial (the latter only in the case that $d-2$ is divisible by the characteristic of the ground field), rather than a product of distinct linear factors. Symmetrically, because $B_2, B_3 \neq 0$, we know $B_1 = B_4 = 0$. Finally, scaling the coordinates appropriately, we can assume that

$$f = x^{d-1}w + xw^{d-1} + y^{d-1}z + yz^{d-1}. \tag{4.13}$$

Since we have already seen that $d-1$ is a power of p for *some* prime p (Corollary 3.2), the surface $S = \mathbb{V}(f)$ would be extremal (see Definition 1.2 and the discussion thereafter) and the proof would be complete *if that prime p is equal to the characteristic of k*. This remains to be shown.

To this end, we consider the lines of $S = \mathbb{V}(x^{d-1}w + xw^{d-1} + y^{d-1}z + yz^{d-1})$ properly intersecting the line $L = \mathbb{V}(z,w)$. We know that there are $(d-1)(d^2-2d+2)$ lines lying on S and intersecting L. Any line M intersecting L has defining equations

$$\begin{aligned} a_1x - a_2y - a_3z - a_4w &= 0 \\ b_3z - b_4w &= 0 \end{aligned} \tag{4.14}$$

since these linear forms, together with $\{z,w\}$, must be linearly dependent. The case where $b_3 = 0$ produces the d lines of the plane section $S \cap \mathbb{V}(w)$, including L itself, and the case where $b_4 = 0$ produces

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the d lines of the plane section $S \cap \mathbb{V}(z)$, including L itself. Thus, there are $(d-1)(d^2-2d+2)-2(d-1) = d(d-1)(d-2)$ *remaining* lines M on S of the form

$$\begin{aligned} a_1x - a_2y \qquad - a_4w &= 0 \\ b_3z - b_4w &= 0 \end{aligned} \tag{4.15}$$

where $b_3, b_4 \neq 0$ on S. Assuming M is in neither plane $\mathbb{V}(w)$ nor $\mathbb{V}(z)$, note that neither a_1 nor a_2 is zero: if $a_1 = 0$, the lines M, $\mathbb{V}(z,w)$, and $\mathbb{V}(y,w)$ all pass through the point $[1:0:0:0]$, forcing them to be coplanar (Lemma 2.7), so that M would be in the plane $\mathbb{V}(w)$. Likewise, if $a_2 = 0$, the point $[0:1:0:0]$ lies on the lines M, $\mathbb{V}(x,z)$, and $\mathbb{V}(w,z)$ forcing M to be in the plane $\mathbb{V}(z)$. Without loss of generality, we assume $a_1 = b_3 = 1$.

Now, consider the restriction of forms to the line M, which we view as the map

$$k[x,y,z,w] \to k[y,w] \qquad x \mapsto a_2y + a_4w, \;\; z \mapsto b_4w.$$

The polynomial f (4.13) restricts to

$$\bar{f} = (a_2y + a_4w)^{d-1}w + (a_2y + a_4w)w^{d-1} + y^{d-1}(b_4w) + y(b_4w)^{d-1}, \tag{4.16}$$

which is the zero polynomial if and only if the line (4.15) lies on S. Since there are $d(d-1)(d-2)$ lines M on S of the form (4.15), there must be $d(d-1)(d-2)$ triples (a_2, a_4, b_4) where $\bar{f}$ is the zero polynomial. But when $d > 3$, the coefficient of $y^{d-2}w^2$ in (4.16) is the polynomial

$$(d-1)a_2^{d-2}a_4.$$

Since $a_2 \neq 0$, either a_4 or $d-1$ is zero in the field k. If $a_4 = 0$, then $\bar{f} = (a_2^{d-1} + b_4)y^{d-1}w + (a_2 + b_4^{d-1})yw^{d-1}$. Setting those two coefficients to zero, we see there are at most $(d-1)^2$ such lines. Since there are $d(d-1)(d-2)$ lines on S of the form (4.15), there must be some such lines with $a_4 \neq 0$, in which case $(d-1)$ must be zero in the field k. In other words, the characteristic of k must divide $d-1 = p^e$. We conclude that k has characteristic p, completing the proof. □

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References

[BC66] R. C. Bose and I. M. Chakravarti, 'Hermitian varieties in a finite projective space PG(N, q^2)', *Canadian J. Math.* **18** (1966), 1161–1182. MR200782

[Bea90] A. Beauville, *Sur les hypersurfaces dont les sections hyperplanes sont à module constant*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, With an appendix by David Eisenbud and Craig Huneke, pp. 121–133. MR1086884

[BPRS24] A. Brosowsky, J. Page, T. Ryan and K.E. Smith, 'Geometry of smooth extremal surfaces', *J. Algebra* **646** (2024), 376–411. MR4711041

[BR23] T. Bauer and S. Rams, 'Counting lines on projective surfaces', *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V* **24**(3) (2023), 1285–1299.

[BS07] S. Boissière and A. Sarti, 'Counting lines on surfaces', *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **6**(1) (2007), 39–52. MR2341513

[CC10] F. Cools and M. Coppens, 'Star points on smooth hypersurfaces', *J. Algebra* **323**(1) (2010), 261–286. MR2564838

[Che25a] R. Cheng, 'q-bic forms', *J. Algebra* **675** (2025), 196–236. MR4891549

[Che25b] R. Cheng, 'q-bic hypersurfaces and their Fano schemes', *Pure Appl. Math. Q.* **21**(4) (2025), 1721–1773. MR4886033

[CHM95] L. Caporaso, Joe Harris, and Barry Mazur, *How many rational points can a curve have?*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 13–31. MR1363052

[Cle61] A. Clebsch, 'Zur Theorie der algebraischen Flächen', *J. Reine Angew. Math.* **58** (1861), 93–108. MR1579142

[Deg15] A. Degtyarev, 'Lines generate the Picard groups of certain Fermat surfaces', *J. Number Theory* **147** (2015), 454–477. MR3276333

https://doi.org/10.1017/fms.2026.10243 Published online by Cambridge University Press

[Deg22] A. Degtyarev, 'Lines in supersingular quartics', *J. Math. Soc. Japan* **74**(3) (2022), 973–1019. MR4484237
[DIS17] A. Degtyarev, Ilia Itenberg, and Ali Sinan Sertöz, 'Lines on quartic surfaces', *Math. Ann.* **368**(1–2) (2017), 753–809. MR3651588
[DL76] P. Deligne and G. Lusztig, 'Representations of reductive groups over finite fields', *Ann. of Math. (2)* **103**(1) (1976), 103–161. MR393266
[DR25] A. Degtyarev and S. Rams, 'Lines on K3-quartics via triangular sets', *Discrete Comput. Geom.* **73**(3) (2025), 785–814. MR4880184
[Ful98] W. Fulton, *Intersection Theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2 (Springer-Verlag, Berlin, 1998). MR1644323
[GAR16] V. González-Alonso and S. Rams, 'Counting lines on quartic surfaces', *Taiwanese J. Math.* **20**(4) (2016), 769–785. MR3535673
[Har95] J. Harris, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 133 (Springer-Verlag, New York, 1995), A first course, Corrected reprint of the 1992 original. MR1416564
[Hef85] A. Hefez, *Duality for projective varieties*, ProQuest LLC, Ann Arbor, MI, 1985, Thesis (Ph.D.)–Massachusetts Institute of Technology. MR2941091
[Hir85] J. W. P. Hirschfeld, *Finite projective spaces of three dimensions*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1985, Oxford Science Publications. MR840877
[HK13] M. Homma and S. Jeong Kim, 'An elementary bound for the number of points of a hypersurface over a finite field', *Finite Fields Appl.* **20** (2013), 76–83. MR3015353
[HK15] M. Homma and S. Jeong Kim, 'Numbers of points of surfaces in the projective 3-space over finite fields', *Finite Fields Appl.* **35** (2015), 52–60. MR3368799
[HK16] M. Homma and S. Jeong Kim, 'The characterization of Hermitian surfaces by the number of points', *J. Geom.* **107**(3) (2016), 509–521. MR3563207
[HW36] H. Hasse and E. Witt, 'Zyklische unverzweigte Erweiterungskörper vom Primzahlgrade p über einem algebraischen Funktionenkörper der Charakteristik p', *Monatsh. Math. Phys.* **43**(1) (1936), 477–492. MR1550551
[Igu60] J.-i. Igusa, 'Betti and Picard numbers of abstract algebraic surfaces', *Proc. Nat. Acad. Sci. U.S.A.* **46** (1960), 724–726. MR118725
[KKP+22] Z. Kadyrsizova, J. Kenkel, J. Page, J. Singh, K. E. Smith, A. Vraciu and E. E. Witt, 'Lower bounds on the F-pure threshold and extremal singularities', *Trans. Amer. Math. Soc. Ser. B* **9** (2022), 977–1005. MR4498775
[KLOS23] J. Kollár, M. Lieblich, M. Olsson and W. Sawin, *What Determines an Algebraic Variety?*, Annals of Mathematics Studies, vol. 216 (Princeton University Press, Princeton, NJ, 2023). MR4648834
[Kol15] J. Kollár, 'Szemerédi-Trotter-type theorems in dimension 3', *Adv. Math.* **271** (2015), 30–61. MR3291856
[KP91] S. Kleiman and R. Piene, *On the Inseparability of the Gauss Map*, Enumerative algebraic geometry (Copenhagen, 1989), Contemp. Math., vol. 123 (Amer. Math. Soc., Providence, RI, 1991), 107–129. MR1143550
[Lan56] S. Lang, 'Algebraic groups over finite fields', *Amer. J. Math.* **78** (1956), 555–563. MR86367
[Mat57] T. Matsusaka, 'The criteria for algebraic equivalence and the torsion group', *Amer. J. Math.* **79** (1957), 53–66. MR82730
[Miy09] Y. Miyaoka, 'Counting lines and conics on a surface', *Publ. Res. Inst. Math. Sci.* **45**(3) (2009), 919–923. MR2569571
[Nom95] A. Noma, 'Hypersurfaces with smooth dual varieties', *Amer. J. Math.* **117**(6) (1995), 1507–1515. MR1363077
[PT09] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, second ed., EMS Series of Lectures in Mathematics (European Mathematical Society (EMS), Zürich, 2009). MR2508121
[RS14] S. Rams and M. Schütt, 'On quartics with lines of the second kind', *Adv. Geom.* **14**(4) (2014), 735–756. MR3276131
[RS15a] S. Rams and M. Schütt, '112 lines on smooth quartic surfaces (characteristic 3)', *Q. J. Math.* **66**(3) (2015), 941–951. MR3396099
[RS15b] S. Rams and M. Schütt, '64 lines on smooth quartic surfaces', *Math. Ann.* **362**(1–2) (2015), 679–698. MR3343894
[RS18] S. Rams and M. Schütt, 'At most 64 lines on smooth quartic surfaces (characteristic 2)', *Nagoya Math. J.* **232** (2018), 76–95. MR3866501
[RS20] S. Rams and M. Schütt, 'Counting lines on surfaces*', especially quintics*, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **20**(3) (2020), 859–890. MR4166795
[Sch82] F. Schur, 'Ueber eine besondre Classe von Flächen vierter Ordnung', *Math. Ann.* **20**(2) (1882), 254–296. MR1510168
[Seg43] B. Segre, 'The maximum number of lines lying on a quartic surface', *Quart. J. Math. Oxford Ser.* **14** (1943), 86–96. MR10431
[Seg65] B. Segre, 'Forme e geometrie hermitiane*', con particolare riguardo al caso finito*, *Ann. Mat. Pura Appl. (4)* **70** (1965), 1–201. MR213949
[Seg67] B. Segre, 'Introduction to Galois geometries', *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. Ia (8)* **8** (1967), 133–236. MR238846
[SGA73] *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, vol. Vol. 340, Springer-Verlag, Berlin-New York, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz. MR354657
[Shi01] I. Shimada, 'Lattices of algebraic cycles on Fermat varieties in positive characteristics', *Proc. London Math. Soc. (3)* **82**(1) (2001), 131–172. MR1794260

https://doi.org/10.1017/fms.2026.10243 Published online by Cambridge University Press

[SK79] T. Shioda and T. Katsura, 'On Fermat varieties', *Tohoku Math. J. (2)* **31**(1) (1979), 97–115. MR526513

[SS17] I. Shimada and T. Shioda, 'On a smooth quartic surface containing 56 lines which is isomorphic as a K3 surface to the Fermat quartic', *Manuscripta Math.* **153**(1–2) (2017), 279–297. MR3635983

[SSvL10] M. Schütt, T. Shioda and R. van Luijk, 'Lines on Fermat surfaces', *J. Number Theory* **130**(9) (2010), 1939–1963. MR2653207

[Tha78] J. A. Thas, 'Combinatorial characterizations of generalized quadrangles with parameters s = q and t = q^2', *Geom. Dedicata* **7**(2) (1978), 223–232. MR485452

[Tha94] J. A. Thas, 'Generalized quadrangles of order (s, s^2). *I, J. Combin. Theory Ser. A* **67**(2) (1994), 140–160. MR1284404

[Tit59] J. Tits, 'Sur la trialité et certains groupes qui s'en déduisent', *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13–60. MR1557095

[Ven17a] D. Cesare Veniani, 'Lines on K3 quartic surfaces in characteristic 2', *Q. J. Math.* **68**(2) (2017), 551–581. MR3667213

[Ven17b] D. Cesare Veniani, 'The maximum number of lines lying on a K3 quartic surface', *Math. Z.* **285**(3–4) (2017), 1141–1166. MR3623744

[Ven20] D. Cesare Veniani, 'Symmetries and equations of smooth quartic surfaces with many lines', *Rev. Mat. Iberoam.* **36**(1) (2020), 233–256. MR4061988

[Ven22] D. Cesare Veniani, 'Lines on K3 quartic surfaces in characteristic 3', *Manuscripta Math.* **167**(3–4) (2022), 675–701. MR4385387

[Xia19] S. Yao Xiao, 'On binary cubic and quartic forms', *J. Théor. Nombres Bordeaux* **31**(2) (2019), 323–341. MR4030910

[Zak93] F. L. Zak, *Tangents and Secants of Algebraic Varieties*, Translations of Mathematical Monographs, vol. 127 (American Mathematical Society, Providence, RI, 1993), Translated from the Russian manuscript by the author. MR1234494

https://doi.org/10.1017/fms.2026.10243 Published online by Cambridge University Press