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# On isomorphisms between weighted $L^p$-algebras

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*Abstract.* Let $G$ be a locally compact group and let $\omega$ be a continuous weight on $G$. In this paper, we first characterize bicontinuous biseparating algebra isomorphisms between weighted $L^p$-algebras. As a result, we extend previous results of Edwards, Parrott, and Strichartz on algebra isomorphisms between $L^p$-algebras to the setting of weighted $L^p$-algebras. We then study the automorphisms of certain weighted $L^p$-algebras on integers, applying known results on composition operators to classical function spaces.

## 1 Introduction and Preliminaries

Let $G$ be a locally compact group with a fixed Haar measure denoted by $\lambda$. By the group algebra on $G$, we mean the convolution Banach algebra $L^1(G)$ of Haar-integrable functions on $G$. It is well known that continuous unitary representations of $G$ correspond canonically to non-degenerate involutive representations of $L^1(G)$. This makes one wonder how far this correspondence goes, and, in particular, if the existence of an algebra isomorphism between two group algebras implies that the underlying groups are isomorphic. Wendel [Wen51] showed that, in general, the answer is negative, and we have to impose some constraints on the isomorphisms.

In [Kaw48], Kawada proved that if we have a bipositive algebra isomorphism between group algebras, then the underlying locally compact groups must be isomorphic. Wendel [Wen51] showed that the same conclusion holds for isometric isomorphisms between group algebras. If we assume the algebra isomorphism is of small bound instead of isometric, then as it was proved by Kalton and Wood [KW76], the existence of an algebra isomorphism of norm less than a constant $\gamma \simeq 1.2\ldots$ between the group algebras implies that the underlying topological groups must be isomorphic. Later, Wood obtained interesting results on similar questions (see [Woo96, Woo01]).

The same questions have been asked for $L^p$-convolution algebras of compact groups with $p > 1$. In [Edw65], Edwards proved that for $1 \leq p < \infty$, the existence of

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a bipositive algebra isomorphisms between $L^p$-algebras of compact groups implies that the groups are isomorphic and asked if a similar result holds for an isometric algebra isomorphism instead. Parrot [Par68] and Strichartz [Str66], independently, provided an affirmative answer to Edward's question for the case of compact groups with $1 \le p < \infty,\ p \ne 2$. When $p = 2$, as it is also mentioned in [Str66, p. 861], $L^2(G)$ does not determine $G$. In fact, if for every positive integer $k$, the cardinality of the set of equivalence classes of irreducible representations of dimension $k$ of $G$ is equal to the cardinality of the corresponding set of $H$, then $L^2(G)$ is isometrically isomorphic to $L^2(H)$.

Adding a weight can considerably enlarge the class of algebras associated with groups. Let $G$ be a locally compact group, and let $\omega : G \to \mathbb{R}^+$ be a positive continuous function. Let $L^p(G, \omega)$ denote the space of equivalence classes of measurable functions $f$ on $G$ for which $\int_G |f(x)|^p \omega(x)^p dx < \infty$. We set $\|f\|_{p,\omega} := \left( \int_G |f(x)|^p \omega(x)^p dx \right)^{1/p}$ for every $f \in L^p(G, \omega)$. Then $L^p(G, \omega)$ equipped with $\|\cdot\|_{p,\omega}$ is a Banach space. When the weight $\omega \equiv 1$, we obtain the classical $L^p$-spaces.

If $\omega(xy) \le \omega(x)\omega(y)$ for all $x, y \in G$, then $L^1(G, \omega)$ is an algebra with respect to convolution. If $G$ is not compact and $p \ne 1$, $L^p(G)$ is not an algebra for convolution. However, weighted $L^p$-spaces on non-compact groups, and in particular on any sigma-compact group [K08], can form Banach algebras. For $p > 1$, a sufficient condition for $L^p(G, \omega)$ to be a convolution algebra is $\omega^{-q} * \omega^{-q} \le \omega^{-q}$ with $q$ such that $1/p + 1/q = 1$ (see [Wer54, KM12]). Virtually all weights occurring in practice satisfy this condition.

It is then natural to ask whether the presence of a weight can enlarge the set of isomorphisms between group algebras. Surprisingly, a weighted structure is as rigid: if there exists a bipositive [GZ17] or isometric [Zad16] algebra isomorphism between $L^1(G, \omega_1)$ and $L^1(H, \omega_2)$, then the groups $G, H$ are isomorphic, and the weights are linked by a multiplicative relation.

In Section 2 of this paper, we investigate isomorphisms of weighted $L^p$-algebras on locally compact groups with $p > 1$. Our approach unites both problems of characterizing bipositive algebra isomorphism and isometric algebra isomorphism into one problem of characterizing bicontinuous biseparating algebra isomorphisms (an operator $T$ is biseparating if both $T$ and $T^{-1}$ are disjointness preserving). Doing so, we conclude that the existence of a bipositive or isometric ( $p \ne 2$) algebra isomorphism between $L^p(G, \omega_1)$ and $L^p(H, \omega_2)$, $p \ge 1$, implies that the groups $G, H$ are isomorphic, and the isomorphism is of a canonical form.

As indicated before, without restriction on the norm, there are isomorphisms of group algebras that are not expresssed in this canonical form, even if the two groups are the same. This can be observed most easily in the examples of finite groups. There is, however, a series of results that show that for certain groups, first and foremost $\mathbb{Z}$, there are very few isomorphisms of groups algebras. It follows from the Beurling–Helson theorem that every isomorphism of $\ell^1(\mathbb{Z})$ is either identical or the composition with the map $\varphi : n \mapsto -n$. In Section 3, we apply some variants of the Beurling–Helson theorem to study the automorphisms of the classical algebras $\ell^p(\mathbb{Z}, \omega)$ without restriction on the norm. We also provide some examples with fast-growing weights.

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***Notation*** In the sequel, a weight on $G$ is a positive continuous function $\omega : G \to \mathbb{R}^+$ such that $\omega(xy) \leq \omega(x)\omega(y)$, for all $x, y \in G$ and $\omega(e_G) = 1$, where $e_G$ is the identity element of $G$. By a weighted locally compact group, we mean a pair $(G, \omega)$, where $G$ is a locally compact group and $\omega$ is a weight function on $G$. Note that $L^p(G, \omega)$ as a Banach space is isometrically isomorphic to $L^p(G)$ via the mapping $L^p(G, \omega) \to L^p(G)$; $f \mapsto f\omega$. In this paper, we confine ourself to weights $\omega : G \to \mathbb{R}^+$ for which $L^p(G, \omega)$ form a Banach algebra (with convolution as multiplication).

Throughout the paper, $\chi_E$ denotes the characteristic function of a set $E$. Note that the space $L^p(G, \omega)$ contains $\chi_E$ for every set $E \subset G$ of finite measure.

Let $(G, \omega)$ be a weighted locally compact group. For $x \in G$, define the left and right translation operators by

$$l_x f(y) = f(x^{-1}y) \quad \text{and} \quad r_x f(y) = f(yx^{-1}),$$

where $f \in L^p(G, \omega)$ and $y \in G$. It follows from our assumptions on the weight that both $l_x$ and $r_x$ are bounded on $L^p(G, \omega)$, with

$$\|l_x\| \leq \omega(x) \quad \text{and} \quad \|r_x\| \leq \omega(x)\Delta(x)^{-1/p},$$

where $\Delta$ is the modular function of $G$.

For a Banach algebra $L^p(G, \omega)$, recall that a linear operator $L$ on $L^p(G, \omega)$ is a left multiplier if

$$L(f * g) = L(f) * g, \qquad f, g \in L^p(G, \omega).$$

In particular, the mapping $f \mapsto l_x f$ is a left multiplier on $L^p(G, \omega)$.

We conclude this section with some lemmas that are needed for our arguments in Sections 2. The following lemma is an analogue of [Lam58, Proposition 1], for weighted $L^p$-algebras. The proof is an easy modification of the proof of [Lam58, Proposition 1] and is therefore omitted.

***Lemma 1.1*** *A bounded operator $L : L^p(G, \omega) \to L^p(G, \omega)$ is a left multiplier if and only if $Lr_x = r_x L$, for all $x \in G$.*

The following is [K08, Lemma 4.1].

***Lemma 1.2*** *Let $(G, \omega)$ be a weighted locally compact group and let $f \in L^p(G, \omega)$ be given. The mapping $x \mapsto l_x f; G \to L^p(G, \omega)$ is continuous.*

***Lemma 1.3*** *Let $(G, \omega)$ be a weighted locally compact group. For every shrinking basis $\{U_\alpha\}$ of compact neighbourhoods of the identity $e_G$, we have that*

$$\frac{1}{\lambda(U_\alpha)} \|l_x \chi_{U_\alpha}\|_{p,\omega}^p \longrightarrow \omega^p(x).$$

**Proof** We have

$$\frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} l_x \omega^p(y) dy - \omega^p(x) = \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} \Big(\omega^p(xy) - \omega^p(x)\Big)\, dy \longrightarrow 0,$$

since $\omega^p$ is continuous. ∎

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Before we proceed with the next lemma, let us remark that, using an approximate identity of $L^p(G,\omega)$, we can see that if for some $f \in L^p(G,\omega)$,

$$h * f = 0 \qquad h \in L^p(G,\omega),$$

then we must have that $f = 0$, almost everywhere.

***Lemma 1.4*** *Suppose that $L^p(G,\omega)$ is a Banach algebra. If for some positive number $\gamma$ and $x \in G$, we have that $\gamma\, l_x$ is both a left multiplier of $L^p(G,\omega)$ and an algebra isomorphism, then $\gamma = 1$ and $x = e_G$.*

**Proof** Since $\gamma\, l_x$ is both an algebra isomorphism and a multiplier, we have that

$$\gamma\, l_x(f) * g = \gamma l_x(f * g) = \gamma\, l_x(f) * \gamma\, l_x(g), \qquad f, g \in L^p(G,\omega);$$

that is,

$$\gamma\, l_x(f) * \big(g - \gamma\, l_x(g)\big) = 0, \qquad f, g \in L^p(G,\omega).$$

So, $g - \gamma\, l_x(g) = 0$, that is, $g = \gamma\, l_x(g)$ for every $g \in L^p(G,\omega)$. In particular, for every $g \in C_c(G)$ and $y \in G$, $g(y) = \gamma\, g(xy)$. It now follows that $x = e_G$ and $\gamma = 1$. ∎

## 2 Biseparating Algebra Isomorphisms Between Weighted $L^p$-algebras

In this section, we first show that every bicontinuous biseparating algebra isomorphism $T : L^p(G,\omega_1) \to L^p(H,\omega_2)$ is a weighted composition operator. We then use this result to characterize both bipositive algebra isomorphisms ( $p \geq 1$) and isometric algebra isomorphisms ( $p \geq 1$, $p \neq 2$) between weighted $L^p$-algebras.

Throughout this section, we assume that $p \geq 1$. Let $(X, \Sigma_1, \mu)$ and $(Y, \Sigma_2, \nu)$ be measure spaces. An operator $T : L^p(X, \Sigma_1, \mu) \to L^p(Y, \Sigma_2, \nu)$ is called *disjointness preserving* if $f \cdot g = 0$ implies that $Tf \cdot Tg = 0$. The operator $T$ is called *biseparating* if $T$ is a bijection and both $T$ and $T^{-1}$ are disjointness preserving.

It is convenient to use the following terminology, as it was done in [GZ17, Definition 2.1] in a more restrictive form. Let $(G,\omega_1)$ and $(H,\omega_2)$ be weighted locally compact groups. A standard isomorphism $(\gamma,\varphi)$ from $(G,\omega_1)$ onto $(H,\omega_2)$ consists of the following data: an isomorphism of locally compact groups $\varphi : G \to H$, and a continuous character $\gamma : G \to \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ such that

$$0 < \inf_{x\in G} |\gamma(x)| \frac{(\omega_2 \circ \varphi)(x)}{\omega_1(x)} \quad \text{and} \quad \sup_{x\in G} |\gamma(x)| \frac{(\omega_2 \circ \varphi)(x)}{\omega_1(x)} < \infty.$$

Let $(G,\omega_1)$ and $(H,\omega_2)$ be weighted locally compact groups, and suppose that $(\gamma,\varphi)$ is a standard isomorphism from $(G,\omega_1)$ to $(H,\omega_2)$. Let $c \in \mathbb{R}$ be the so-called *measure adjusting-constant* (see [Wen51, Wen52]) satisfying $c = \frac{\lambda_1(E)}{\lambda_2(\varphi(E))}$ for every non-null measurable set $E \subset G$, where $\lambda_1$ and $\lambda_2$ are Haar measures on $G$ and $H$, respectively. Define $T_{\gamma,\varphi} : L^p(G,\omega_1) \to L^p(H,\omega_2)$ by

$$T_{\gamma,\varphi}(f) = c\,\gamma \circ \varphi^{-1} \cdot f \circ \varphi^{-1}.$$

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It is straightforward to check that $T_{\gamma,\varphi}$ is a well-defined, bicontinuous, biseparating algebra isomorphism. In what follows, we show that the converse is also true: every bicontinuous biseparating algebra isomorphism is of this form. The main tool for our argument is Proposition 2.1, which was inspired by the Banach–Lamperti theorem stated in [Lam58, Theorem 3.1] and also [FJ03, Theorem 3.2.5].

Before we proceed with Proposition 2.1, we need to recall some terminology. Let $(X, \Sigma_1, \mu_1)$ and $(Y, \Sigma_2, \mu_2)$ be measure spaces. A *regular set isomorphism* of $\Sigma_1$ into $\Sigma_2$, defined modulo null sets, is a mapping $\Phi$ satisfying the following conditions:

(i) $\Phi(X\backslash A) = \Phi X\backslash \Phi A$ for all $A \in \Sigma_1$;
(ii) $\Phi(\cup_1^\infty A_n) = \cup_{n=1}^\infty \Phi(A_n)$ for disjoint $A_n \in \Sigma_1$;
(iii) $\mu_2(\Phi A) = 0$ if and only if $\mu_1(A) = 0$.

The Banach–Lamperti theorem deals with isometric isomorphisms, but its proof goes by first showing that the map is disjointness preserving, what we impose in Propositions 2.1 and 2.2 as an assumption, and so the proofs follows the same lines.

***Proposition 2.1*** *Let $(X, \Sigma_1, \mu)$ be a sigma-finite measure space, let $(Y, \Sigma_2, \nu)$ be a measure space, and let $T : L^p(X, \Sigma_1, \mu) \to L^p(Y, \Sigma_2, \nu)$ be a linear disjointness preserving operator. Then there are a regular set isomorphism $\Phi$ and a function $h$ on $Y$ such that for every set $E$ of finite measure $T(\chi_E) = h \cdot \chi_{\Phi(E)}$.*

Decomposing the group $G$ into cosets by an open sigma-compact subgroup and applying the Banach–Lamperti theorem to every coset, one proves a variation of the Banach–Lamperti theorem we profit from.

***Proposition 2.2*** *Suppose that $(G, \omega)$ and $(H, \omega_2)$ are weighted locally compact groups and $T : L^p(G, \omega_1) \to L^p(H, \omega_2)$ is a linear disjointness preserving operator. Then there is a regular set isomorphism $\Phi$ between the sigma-algebras of Haar measurable subsets of $G$ and $H$ and a function $h$ on $H$ such that for every set $E \subset G$ of finite measure $T(\chi_E) = h \cdot \chi_{\Phi(E)}$.*

In the argument of the proof of Theorem 2.5, we make use of some particular left multipliers.

Recall that an operator $T : L^p(G, \omega_1) \to L^p(H, \omega_2)$ is called *positive* if for every $f \in L^p(G, \omega_1)$ such that $f \geq 0$ a.e., we have that $T(f) \geq 0$ a.e.. The operator $T$ is called *bipositive* if $T$ is a bijection and both $T$ and $T^{-1}$ are positive operators.

***Lemma 2.3*** *Every bipositive left multiplier on $L^1(G, \omega)$ is a positive multiple of a left shift.*

**Proof** Let $L$ be a bipositive left multiplier on $L^1(G, \omega)$. Then by [GZ17, Lemma 4.1], there is a positive measure $\mu \in M(G, \omega)$ such that $L(f) = \mu * f$, $f \in L^1(G, \omega)$. Since $L^{-1}$ is also a positive left multiplier, it follows that $\mu$ is an invertible positive measure with a positive inverse. Therefore, by [GZ17, Proposition 1.3.(i)], $\mu$ must be a positive multiple of a pointmass, say $\alpha\delta_x$, and therefore $L(f) = \alpha l_x(f)$, for every $f \in L^1(G, \omega)$. ∎

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***Lemma 2.4*** *Every bicontinuous biseparating left multiplier $L$ on $L^p(G,\omega)$ is a scalar multiple of a left shift.*

**Proof** Let $L : L^p(G,\omega) \to L^p(G,\omega)$ be a bicontinuous biseparating left multiplier. We will show that $L$ generates a bipositive left multiplier $L'$ on $L^1(G,\omega)$ such that for a constant $d$, we have that $\frac{1}{d}L = L'$ on characteristic functions of open sets with finite measure. By Lemma 2.3, $L'$ is then a scalar multiple of a left shift, and therefore, $L$ must be a scalar multiple of a left shift.

By Proposition 2.2, there are a regular set isomorphism $\Phi$ and a measurable function $h$ on $G$ such that for each set of finite measure $E$, we have that $L(\chi_E) = h \cdot \chi_{\Phi(E)}$. We will show that $h$ is constant.

First, note that $\Phi(G) = G$ (up to a null set); otherwise, one could pick a subset $Y \subset G\backslash\Phi(G)$ with $0 < \lambda(Y) < \infty$, and then $\chi_Y$ and $L(\chi_E)$ would have disjoint supports for every finite measure subset $E$, whence $L^{-1}(\chi_Y)$ and $\chi_E$ have disjoint supports, and thus $L^{-1}(\chi_Y) = 0$, but this contradicts the assumption that $L^{-1}$ is a bijection.

Now suppose that $h$ is not constant. Then there exist $x \in G$ and a set $E$ of nonzero finite measure such that $r_x h(y) \neq h(y)$ for $y \in E$. Moreover, we can assume that $\lambda(\Phi^{-1}(E)) < \infty$ and $\lambda(\Phi^{-1}(Ex^{-1})) < \infty$. Set $F = \Phi^{-1}(Ex^{-1}) \cup \Phi^{-1}(E)x^{-1}$; then $Ex^{-1} \subseteq \Phi(F)$ and $E \subseteq \Phi(Fx)$. For $y \in E$, we have

$$r_x L(\chi_F)(y) = L\chi_F(yx^{-1}) = h(yx^{-1})\chi_{\Phi(F)}(yx^{-1}) = h(yx^{-1}),$$
$$Lr_x\chi_F(y) = L\chi_{Fx}(y) = h(y)\chi_{\Phi(Fx)} = h(y).$$

Thus, $h(yx^{-1}) = h(y)$ for almost every $y \in E$, which is a contradiction. Therefore, there is a constant $d$ such that

$$L(\chi_E) = d\chi_{\Phi(E)} \tag{2.1}$$

for every finite measure set $E \subset G$. Let $q \geq 1$ be given. Using identity (2.1), for every finite measure set $E \subset G$, we have that

$$\|\frac{1}{d}L(\chi_E)\|_q^q = \lambda(\Phi(E)) \leq \frac{\|L\|^p}{d^p}\|\chi_E\|_q^q.$$

So, by linearity, $\frac{1}{d}L$ extends to a bounded map on simple functions on every $L^q$, and by density of simple functions, to a bounded map on every $L^q$ class, in particular on $L^1$. From (2.1), it is seen that $\frac{1}{d}L$ is bipositive, and hence by Lemma 2.3, $\frac{1}{d}L$ is a multiple of a left shift, and therefore so is $L$. ∎

***Theorem 2.5*** *Let $(G,\omega_1)$ and $(H,\omega_2)$ be weighted locally compact groups. Let $T : L^p(G,\omega_1) \to L^p(H,\omega_2)$ be a bicontinuous biseparating algebra isomorphism. Then there exists a standard isomorphism $(\gamma,\varphi)$ from $(G,\omega_1)$ to $(H,\omega_2)$, such that $T = T_{\gamma,\varphi}$.*

**Proof** Let $T : L^p(G,\omega_1) \to L^p(H,\omega_2)$ be a biseparating algebra isomorphism. For every $x \in G$, the mapping

$$Tl_xT^{-1} : L^p(H,\omega_2) \longrightarrow L^p(H,\omega_2)$$

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is a biseparating left multiplier. Therefore, by Lemma 2.4, $Tl_xT^{-1}$ is a scalar multiple of a left translation on $L^p(H,\omega_2)$. Let $\varphi : G \to H$ and $\gamma : G \to \mathbb{C}$ be such that

$$Tl_xT^{-1} = \gamma(x)l_{\varphi(x)}.$$

Let $\{U_\alpha\}_{\alpha\in I}$ be a shrinking basis of symmetric precompact neighbourhoods of identity in $H$ all contained in some fixed precompact neighbourhood of identity, say $U$. Since $T$ and $T^{-1}$ are bounded operators and $\|l_x\| \le \omega_1(x)$, we have, for every $\alpha$,

$$\|\gamma(x)l_{\varphi(x)}(\chi_{U_\alpha})\|_{p,\omega_2} = \|Tl_xT^{-1}(\chi_{U_\alpha})\|_{p,\omega_2} \le \|T\|\,\|T^{-1}\|\,\omega_1(x)\,\|\chi_{U_\alpha}\|_{p,\omega_2}.$$

Since by Lemma 1.3,

$$\frac{1}{\lambda_2(U_\alpha)}\|l_{\varphi(x)}(\chi_{U_\alpha})\|^p_{p,\omega_2} \to \omega_2^p(\varphi(x)) \quad \text{and} \quad \frac{1}{\lambda_2(U_\alpha)}\|\chi_{U_\alpha}\|^p_{p,\omega_2} \to 1,$$

we have that

$$\begin{aligned} |\gamma(x)|\,\omega_2(\varphi(x)) &= \lim_\alpha \frac{|\gamma(x)|\,\|l_{\varphi(x)}(\chi_{U_\alpha})\|_{p,\omega_2}}{\lambda_2(U_\alpha)^{\frac{1}{p}}} \\ &\le \lim_\alpha \frac{\|T\|\,\|T^{-1}\|\,\omega_1(x)\,\|\chi_{U_\alpha}\|_{p,\omega_2}}{\lambda_2(U_\alpha)^{\frac{1}{p}}} \\ &\le \|T\|\,\|T^{-1}\|\omega_1(x). \end{aligned} \tag{2.2}$$

Now let $\{V_\alpha\}_{\alpha\in J}$ be a shrinking basis of symmetric precompact neighbourhoods of identity in $G$. Since $T^{-1}(Tl_xT^{-1})T(\chi_{V_\alpha}) = l_x(\chi_{V_\alpha})$, we have

$$\begin{aligned} \|l_x(V_\alpha)\|_{p,\omega_1} &= \|T^{-1}(Tl_xT^{-1})T(\chi_{V_\alpha})\|_{p,\omega_1} \\ &\le \|T^{-1}\|\|\gamma(x)\,l_{\varphi(x)}T(\chi_{V_\alpha})\|_{p,\omega_2} \\ &\le \|T^{-1}\|\,|\gamma(x)|\,\omega_2(\varphi(x))\,\|T(\chi_{V_\alpha})\|_{p,\omega_2}. \end{aligned}$$

Using the same argument as in (2.2), we have that

$$\omega_1(x) \le \|T^{-1}\|\,\|T\|\,|\gamma(x)|\,\omega_2(\varphi(x)). \tag{2.3}$$

Thus, for the mappings $\varphi : G \to H$ and $\gamma : G \to \mathbb{C}^*$, we have that

$$0 < \inf_{x\in G}|\gamma(x)|\,\frac{(\omega_2\circ\varphi)(x)}{\omega_1(x)} \quad \text{and} \quad \sup_{x\in G}|\gamma(x)|\,\frac{(\omega_2\circ\varphi)(x)}{\omega_1(x)} < \infty.$$

We show next that $\varphi : G \to H$ is an isomorphism of locally compact groups and $\gamma : G \to \mathbb{C}^*$ is a continuous character. First, we show that $\varphi$ and $\gamma$ are homomorphisms. To see this, first note that for every $x$ and $y$ in $G$, we have that

$$\gamma(xy)\,l_{\varphi(xy)} = \gamma(x)\,\gamma(y)\,l_{\varphi(x)\varphi(y)}, \tag{2.4}$$

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since

$$\gamma(xy)\ l_{\varphi(xy)} = Tl_{xy}T^{-1} = Tl_xT^{-1}Tl_yT^{-1} = \gamma(x)\gamma(y)\ l_{\varphi(x)\varphi(y)}.$$

Let $f \in C_c(H)$ be such that $f(\varphi(xy)^{-1}) = f(\varphi(y)^{-1}\varphi(x)^{-1}) = 1$. Then using (2.4), we obtain that

$$\gamma(xy) = \gamma(xy)\ l_{\varphi(xy)}f(e_G) = \gamma(x)\gamma(y)\ l_{\varphi(x)\varphi(y)}f(e_G) = \gamma(x)\gamma(y),$$

and therefore $\gamma$ is a homomorphism. It now follows from (2.4) that $l_{\varphi(xy)} = l_{\varphi(x)\varphi(y)}$. In particular, this means that for every function $f \in C_c(H)$, we have that

$$f\big(\varphi(xy)^{-1}\big) = l_{\varphi(xy)}f(e_G) = l_{\varphi(x)\varphi(y)}f(e_G) = f\big(\varphi(y)^{-1}\varphi(x)^{-1}\big).$$

Since $G$ is a completely regular topological space, we must have that $\varphi(xy)^{-1} = (\varphi(x)\varphi(y))^{-1}$, and therefore, $\varphi$ is a homomorphism.

Now we show that $\varphi$ and $\gamma$ are continuous. To see this, suppose that $x$ in $G$ is given, and suppose that $(x_\alpha)$ is a net in $G$ converging to $x$. We first show that $\varphi(x_\alpha) \to \varphi(x)$. Let $\varphi(x)U$ be a compact neighbourhood of $\varphi(x)$. Let $W$ be a neighbourhood of the identity such that $WW^{-1} \subseteq U$. Then for every $f \in L^p(H, \omega_2)$, using Lemma 1.2 and since both $T$ and $T^{-1}$ are bounded, we have that

$$Tl_{x_\alpha}T^{-1}(f) \xrightarrow{\|\cdot\|_{p,\omega_2}} Tl_xT^{-1}(f),$$

that is

$$\gamma(x_\alpha)l_{\varphi(x_\alpha)}f \xrightarrow{\|\cdot\|_{p,\omega_2}} \gamma(x)l_{\varphi(x)}f.$$

Setting $f = \chi_W$, we see that for $\varepsilon_0 := \|\gamma(x)l_{\varphi(x)}(\chi_W)\|_{p,\omega_2}$, there is $\alpha_0$ such that

$$\|\gamma(x_\alpha)l_{\varphi(x_\alpha)}(\chi_W) - \gamma(x)l_{\varphi(x)}(\chi_W)\|_{p,\omega_2} < \varepsilon_0$$

for every $\alpha > \alpha_0$. We claim that for every $\alpha > \alpha_0$, $\varphi(x_\alpha)$ belongs to $\varphi(x)U$. Suppose, on the contrary, that for some $\alpha_1 > \alpha_0$, $\varphi(x_{\alpha_1}) \notin \varphi(x)U$. Then $\varphi(x_{\alpha_1})W \cap \varphi(x)W = \emptyset$, so the functions $l_{\varphi(x_{\alpha_1})}(\chi_W)$ and $l_{\varphi(x)}(\chi_W)$ have disjoint supports. Therefore,

$$\begin{aligned}
&\big\|\gamma(x_{\alpha_i})\ l_{\varphi(x_{\alpha_i})}(\chi_W) - \gamma(x)\ l_{\varphi(x)}(\chi_W)\big\|_{p,\omega_2} \\
&\quad = \big\|\gamma(x_{\alpha_i})\ l_{\varphi(x_{\alpha_i})}(\chi_W)\|_{p,\omega_2} + \|\gamma(x)\ l_{\varphi(x)}(\chi_W)\big\|_{p,\omega_2} \\
&\quad \geq \big\|\gamma(x)\ l_{\varphi(x)}(\chi_W)\big\|_{p,\omega_2} = \varepsilon_0,
\end{aligned}$$

a contradiction. Thus, we must have $\varphi(x_\alpha) \to \varphi(x)$.

We show next that $(\gamma(x_\alpha))$ converges to $\gamma(x)$. Suppose that on the contrary, $\gamma(x_\alpha) \not\to \gamma(x)$. Then by going through a subnet, we can assume that there is $\varepsilon_0 > 0$ such that $|\gamma(x_\alpha) - \gamma(x)| > \varepsilon_0$. Since $\varphi(x_\alpha) \to \varphi(x)$, there is a compact neighbourhood of identity $W$ such that $\varphi(x_\alpha) \in \varphi(x)WW^{-1}$ and $\lambda_2(\varphi(x_\alpha)W \cap \varphi(x)W) \geq \frac{1}{2}\lambda_2(\varphi(x)W) = \frac{1}{2}\lambda_2(W)$. Note that $\varphi(x_\alpha)W \cap \varphi(x)W$ is a nonempty compact set. Set $m := \inf\{\omega_2^p(y) : y \in \varphi(x_\alpha)W \cap \varphi(x)W\}$; then we have that

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$$\begin{aligned}\|\gamma(x_\alpha)\, l_{\varphi(x_\alpha)}(\chi_W) - \gamma(x)\, l_{\varphi(x)}(\chi_W)\|_{p,\omega}^p \\ &= \int |\gamma(x_\alpha)\chi_{\varphi(x_\alpha)W} - \gamma(x)\chi_{\varphi(x)W}|^p(y)\omega_2^p(y)\,dy \\ &\geq m \int_{\varphi(x_\alpha)W \cap \varphi(x)W} |\gamma(x_\alpha) - \gamma(x)|^p\, dy \geq \frac{m}{2}\lambda_2(W)\varepsilon_0^p,\end{aligned}$$

a contradiction. So, $(\gamma(x_\alpha))$ converges to $\gamma(x)$.

Now we show that $\varphi$ is a homeomorphism. Applying the reasoning above to $T^{-1}$, we conclude that there are continuous homomorphisms $\varphi' : H \to G$ and $\gamma' : H \to \mathbb{C}$ such that $T^{-1} l_y T = \gamma'(y) l_{\varphi'(y)}$ for every $y \in H$. Now, for each $g \in L^p(H, \omega_2)$, if we let $f := T^{-1}(g)$, we have that

$$\begin{aligned}l_y g &= T(T^{-1} l_y T f) = T(\gamma'(y) l_{\varphi'(y)} f) \\ &= T(\gamma'(y) l_{\varphi'(y)} T^{-1} g) = \gamma'(y)\gamma \circ \varphi'(y) l_{\varphi \circ \varphi'(y)} g.\end{aligned}$$

Thus, $\varphi(\varphi'(y)) = y$, for every $y$ in $H$. Similarly, we can see that $\varphi'(\varphi(x)) = x$, for every $x$ in $G$. Therefore, $\varphi$ is a bijection. Since $\varphi'$ is continuous, $\varphi$ is a homeomorphism.

Now, consider the biseparating algebra isomorphism $T_{\gamma,\varphi} : L^p(G, \omega_1) \to L^p(H, \omega_2)$. A straightforward calculation shows that for every $y$ in $H$,

$$T T_{\gamma,\varphi}^{-1} r_y T_{\gamma,\varphi} T^{-1} = r_y.$$

Therefore, by Lemma 1.1, the biseparating algebra isomorphism $T T_{\gamma,\varphi}^{-1}$ is also a left multiplier. It follows from Lemma 2.4 that there exist a constant $\zeta$ and an element $z$ in $H$ such that $T T_{\gamma,\varphi}^{-1} = \zeta l_z$. Now by Lemma 1.4, we have that $T T_{\gamma,\varphi}^{-1}$ is the identity operator on $L^p(H, \omega_2)$, and so $T = T_{\gamma,\varphi}$. ∎

We now proceed to bipositive algebra isomorphisms and isometric algebra isomorphisms.

***Proposition 2.6*** *Let $T : L^p(G, \omega_1) \to L^p(H, \omega_2)$ be an algebra isomorphism such that $T$ is bipositive, or $T$ is an isometry and $p \neq 2$; then $T$ is bicontinuous and biseparating.*

**Proof** Suppose first that $T$ is bipositive. By [AB85, Thm. 4.3], $T$ and $T^{-1}$ are bounded operators. Positivity implies that $T$ maps the subspace of real-valued functions $L^p_{\mathbb{R}}(G, \omega) \subset L^p(G, \omega)$ to itself, and on this subspace $|T(f)| = T(|f|)$ (see also [MN]). For $f, g \in L^p_{\mathbb{R}}(G, \omega)$ then $f \cdot g = 0$ implies $|f \wedge g| = 0$ and $|T(f) \wedge T(g)| = 0$, so that $T$ is disjointness preserving on $L^p_{\mathbb{R}}(G, \omega)$. It is now not hard to see that $T$ is disjointness preserving on all of $L^p(G, \omega)$. By symmetry $T^{-1}$ is also disjointness preserving and therefore $T$ is biseparating.

If $T : L^p(G, \omega_1) \to L^p(H, \omega_2)$ is a linear isometry, since Banach spaces $L^p(G, \omega_1)$ and $L^p(H, \omega_2)$ are isometric to $L^p(G)$ and $L^p(H)$, respectively, it follows from [Lam58, Corollary 2.1] that $T$ is disjointness preserving. ∎

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***Corollary 2.7*** *Let $T : L^p(G, \omega_1) \to L^p(H, \omega_2)$ be an algebra isomorphism.*

(i) *If $T$ is bipositive, then $T = T_{\gamma,\varphi}$ for some standard isomorphism $(\gamma, \varphi)$ from $(G, \omega_1)$ to $(H, \omega_2)$ with $\gamma : G \to \mathbb{R}^+$.*

(ii) *If $T$ is isometric and $p \neq 2$, then $T = T_{\gamma,\varphi}$ for some standard isomorphism, and the ratio $\omega_1/\omega_2 \circ \varphi$ is a homomorphism.*

**Proof** (i) It follows from Theorem 2.5 and Proposition 2.6 that $T = T_{\gamma,\varphi}$ for some standard isomorphism $(\gamma, \varphi)$ from $(G, \omega_1)$ to $(H, \omega_2)$. Since both $T$ and $T^{-1}$ are bipositive, the multiplier $Tl_xT^{-1}$ in the proof of Theorem 2.5 is also bipositive, and so $\gamma(x) > 0$.

(ii) It follows from Theorem 2.5 and Proposition 2.6 that $T = T_{\gamma,\varphi}$ for some standard isomorphism $(\gamma, \varphi)$ from $(G, \omega_1)$ to $(H, \omega_2)$. Since $\|T\| = \|T^{-1}\| = 1$, inequalities (2.2) and (2.3) imply that

$$\omega_1(x) \le |\gamma(x)|\, \omega_2(\varphi(x)) \le \omega_1(x).$$

Thus, $|\gamma| = \frac{\omega_1}{\omega_2 \circ \varphi}$, so that this ratio is multiplicative. ∎

## 3 Algebra Automorphisms of $\ell^p(\mathbb{Z}, \omega)$

In this section, we describe automorphisms of a family of weighted algebras $\ell^p(\mathbb{Z}, \omega)$, applying known results on composition operators and, in particular, the Beurling–Helson theorem. Set $\omega_a(n) = \max(1, |n|^a)$, $a \ge 0$. It is known that for $a \ge 1$, the weighted space $\ell^p(\mathbb{Z}, \omega_a)$ is a convolution Banach algebra (see [Ni]). For $a > b$, we have $\ell^p(\mathbb{Z}, \omega_a) \subset \ell^p(\mathbb{Z}, \omega_b)$.

***Theorem 3.1*** *Suppose that either $a \ge 1$ and $p = 1$ or $a \ge 2$ and $p > 1$. Let $T : \ell^p(\mathbb{Z}, \omega_a) \to \ell^p(\mathbb{Z}, \omega_a)$ be an algebra isomorhism. Then $T$ can be extended to a continuous operator on $\ell^p(\mathbb{Z})$, and this extension is a linear isomorphism.*

**Proof** Denote the space $\ell^p(\mathbb{Z}, \omega_a)$ by $\mathcal{L}_{p,a}$. Note that $\mathcal{L}_{p,a} \subset \mathcal{L}_{1,a-1}$ (this is checked by a direct calculation). It is known that $\mathcal{L}_{p,a}$ is an algebra with respect to convolution if $p = 1$, $a \ge 0$ and $p > 1$, $a \ge 1$. Moreover, $\mathcal{L}_{p,a}$ is a module over $\mathcal{L}_{1,a}$ for every $a \ge 0$. The spectrum of every algebra in this class is the unit circle $\mathbb{T}$ [GRS, K09]. Further, we denote by $\widehat{f}$ the Gelfand transform of $f \in \mathcal{L}_{p,a}$, $\widehat{f} : \mathbb{T} \to \mathbb{C}$. Consider the formal derivation on $\mathcal{L}_{p,a}$: $(f')_n = (n+1)f_{n+1}$. It is readily seen that $f' \in \mathcal{L}_{p,a-1}$ for every $f \in \mathcal{L}_{p,a}$, and the map $f \mapsto f'$ is bounded from $\mathcal{L}_{p,a}$ to $\mathcal{L}_{p,a-1}$.

Let us prove first that $T$ can be extended to $\mathcal{L}_{p,a-1}$. For every $z \in \mathbb{T}$, the map $f \mapsto \widehat{(Tf)}(z)$ is a character of $\mathcal{L}_{p,a}$, which is obviously nonzero, so that there exists $u_z \in \mathbb{T}$ such that $\widehat{(Tf)}(z) = \widehat{f}(u_z)$. Denote $\varphi(z) = u_z$; we get a map $\varphi : \mathbb{T} \to \mathbb{T}$. Denote by $\delta_1 \in \mathcal{L}_{p,a}$ the indicator function of 1; one can verify that $\varphi = \widehat{T\delta_1}$. This implies, in particular, that $\varphi \in \widehat{\mathcal{L}}_{p,a}$.

Since $T$ is invertible, the equation $Tf = \delta_1$ has a solution in $\mathcal{L}_{p,a}$, which has the Gelfand transform $\varphi^{-1}$, as $\widehat{(Tf)}(z) = \widehat{f}\big(\varphi(z)\big) = z$. Now since the formal derivative

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of a composition satisfies the usual formula (what can be checked by continuity and by its values on $\delta_1^n$), we have that $1/\varphi' = (\varphi^{-1})' \in \widehat{\mathcal{L}}_{p,a-1}$.

For every $f \in \mathcal{L}_{p,a}$, we have by assumption that $\widehat{Tf} = \widehat{f} \circ \varphi \in \widehat{\mathcal{L}}_{p,a}$, so $(\widehat{f} \circ \varphi)' = (\widehat{f'} \circ \varphi) \cdot \varphi' \in \widehat{\mathcal{L}}_{p,a-1}$. Since $1/\varphi' \in \widehat{\mathcal{L}}_{p,a-1}$ and this is an algebra, we also have $\widehat{f'} \circ \varphi \in \widehat{\mathcal{L}}_{p,a-1}$.

For $g = (g_n) \in \mathcal{L}_{p,a-1}$, set $f_0 = 0$ and $f_n = g_n/n$ for $n \neq 0$. We have $f \in \mathcal{L}_{p,a}$, and one calculates that $\widehat{f}'(z) = \sum n f_n z^{n-1} = \sum_{n\neq 0} g_n z^{n-1} = (\widehat{g}(z) - g_0)/z$. As $\widehat{f'} \circ \varphi \in \widehat{\mathcal{L}}_{p,a-1}$, it follows that $(\widehat{g} \circ \varphi - g_0)/\varphi \in \widehat{\mathcal{L}}_{p,a-1}$.

Since $\varphi \in \widehat{\mathcal{L}}_{p,a}$, we also have $\varphi \in \widehat{\mathcal{L}}_{1,a-1}$, so that $\widehat{g} \circ \varphi - g_0 \in \widehat{\mathcal{L}}_{p,a-1}$ for every $g \in \mathcal{L}_{p,a-1}$. The space $\widehat{\mathcal{L}}_{p,a-1}$ contains the constant function $1 = \widehat{\delta_0}$, which implies that $\widehat{g} \circ \varphi \in \widehat{\mathcal{L}}_{p,a-1}$ for every $g \in \mathcal{L}_{p,a-1}$. The operator $g \mapsto (\widehat{g} \circ \varphi)^{\vee}$ is an extension of $T$, and it is clear that it is bounded on $\mathcal{L}_{p,a-1}$ and is an algebra homomorphism.

We can apply the same reasoning to $T^{-1}$ and conclude that it extends to a bounded map on $\mathcal{L}_{p,a-1}$. On $\mathcal{L}_{p,a}$, which is a dense subspace in $\mathcal{L}_{p,a-1}$, these maps are inverse to each other, so that the extensions are inverse too, and finally, $T$ extends to an isomorphism on $\mathcal{L}_{p,a-1}$.

If $a \geq 1$ is integer, then, applying this reasoning by induction, we get the statement of the theorem for $p = 1$. If $a \geq 1$ but $a \notin \mathbb{Z}$, let $m = [a]$ be its integer part. It is known that weighted $L^p$-spaces interpolate as follows: for $\theta \in (0,1)$, the $\theta$-interpolated space between $L^p(\omega_0)$ and $L^p(\omega_1)$ is $L^p(\nu)$ with $\nu = \omega_0^{1-\theta}\omega_1^{\theta}$ (see, for example, [G]). In our case, $\mathcal{L}_{p,m}$ is $\theta$-interpolation space between $\mathcal{L}_{p,a}$ and $\mathcal{L}_{p,a-1}$ with $\theta = a - m$ that belongs to $(0,1)$. By the Riesz–Thorin theorem, $T$ and $T^{-1}$ are both bounded on $\mathcal{L}_{p,m}$, and the same reasoning as above allows us to conclude that this map is a continuous algebra isomorphism. This reduces the problem to the integer case that has been resolved already.

For $p > 1$, we arrive, by reduction to the integer case and induction, at an isomorphism of $\mathcal{L}_{p,1}$, which is of the form $g \mapsto (\widehat{g} \circ \varphi)^{\vee}$ with $\varphi \in \mathcal{L}_{p,2}$, by the initial assumption of the theorem. The above proof cannot be applied directly (with $a = 1$) to extend it to $\mathcal{L}_{p,0} = \ell^p(\mathbb{Z})$, since $\mathcal{L}_{p,0}$ is not an algebra anymore. However, this fact ($\mathcal{L}_{p,a-1}$ being an algebra) is used only to show that $\widehat{h}/\varphi' \in \widehat{\mathcal{L}}_{p,a-1}$ for all $h \in \mathcal{L}_{p,a-1}$, with $\widehat{h} = (\widehat{f} \circ \varphi)'$. In our assumptions, the inclusion $\widehat{h}/\varphi' \in \widehat{\mathcal{L}}_{p,0} = \widehat{\ell^p(\mathbb{Z})}$ holds for all $h \in \ell^p(\mathbb{Z})$, since $1/\varphi' \in \widehat{\mathcal{L}}_{p,a-1} \subset \widehat{\mathcal{L}}_{1,a-2} \subset \widehat{\ell^1(\mathbb{Z})}$. This allows us to conclude, finally, that $T$ extends to $\ell^p(\mathbb{Z})$, also as a bounded isomorphism. ∎

***Corollary 3.2*** *Suppose that either $a \geq 1$ and $p = 1$ or $a \geq 2$ and $p > 1$. Let $T : \ell^p(\mathbb{Z}, \omega_a) \to \ell^p(\mathbb{Z}, \omega_a)$ be an algebra isomorhism. Then there is $\lambda \in \mathbb{T}$ such that either $(Tf)_n = \lambda^n f_n$ or $(Tf)_n = \lambda^n f_{-n}$ for all $f \in \ell^p(\mathbb{Z}, \omega_a)$ and all $n \in \mathbb{Z}$.*

**Proof** By Proposition 3.1, $T$ extends to an isomorphism of $\ell^p$, which is a composition operator $C_\varphi$ with $\varphi^{\vee} \in \ell^p(\mathbb{Z}, \omega_a)$. If $p = 1$, it follows from the Beurling–Helson theorem that $\varphi$ has the form $\varphi(z) = z^m\lambda$ for some $m \in \mathbb{Z}$ and $\lambda \in \mathbb{T}$. Applying the same reasoning to $T^{-1}$, we conclude that $\varphi^{-1}(z) = (z/\lambda)^{1/m}$ is also a monomial in $z$, which implies that $m = \pm 1$, so that $Tf$ is as described in the statement.

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If $p > 1$ and $a \geq 2$, the result follows from a generalisation of the Beurling-Helson theorem given by Lebedev and Olevskii [LO]: if $\varphi : \mathbb{T} \to \mathbb{T}$ is a $C^1$ function of the argument and the composition operator given by $\varphi$ is bounded on $\ell^p(\mathbb{Z})$, then $\varphi$ is a monomial. In our case, $\varphi \in \ell^p(\mathbb{Z}, \omega_a)$, which implies $\varphi \in C^1$. The rest is proved as in the $\ell^1$ case. ∎

For $a = 1$ and $p > 1$, the question is open. For fast growing weights, we can observe a similar behaviour.

***Example 3.3*** Let $w$ be a weight on $\mathbb{Z}$ of one of the two forms: $w_n = e^{n^\gamma}$, $0 < \gamma < 1$, or $w_n = a^{|n|}(1 + n^2)$, $a > 1$. Then the elementary Blaschke product $b_r(z) = \frac{z-r}{1-rz}$, $r > 0$, does not define a bounded composition operator on $\ell^p(\mathbb{Z}, w)$ for any $p \neq 2$.

**Proof** *Case 1:* $w_n = e^{n^\gamma}$, $0 < \gamma < 1$. The spectrum of the algebra $\ell^p(\mathbb{Z}, w)$ is still $\mathbb{T}$, as in the unweighted case [GRS, K09]. We have $\widehat{C_{b_r}\delta_n} = b_r^n$ on $\mathbb{T}$. Set $\alpha = (1 + r)/(1 - r)$ and $k_n = [\alpha n]$. It is known [SZ] that $\check{b}_r(k_n) \geq c_r n^{-1/3}$ as $n \to \infty$, with a constant $c_r > 0$. It follows that $\|(b_r^n)^\vee\|_{p,w} = \|C_{b_r}\delta_n\|_{p,w} \geq c_r w_{k_n} n^{-1/3} \geq c_r e^{\alpha^\gamma n^\gamma - 1} n^{-1/3}$. At the same time, $\|\delta_n\|_{p,w} = w_n = e^{n^\gamma}$. Since $e^{(\alpha^\gamma - 1)n^\gamma} n^{-1/3} \to \infty$, $n \to \infty$, the operator $C_{b_r}$ cannot be bounded.

*Case 2:* $w_n = a^{|n|}(1 + n^2)$, $a > 1$. One verifies that $b_r \in \ell^p(\mathbb{Z}, w)$ if and only if $r < 1/a$. For these values of $r$ and positive $n$, we reason as above: $\|(b_r^n)^\vee\|_{p,w} = \|C_{b_r}\delta_n\|_{p,w} \geq c_r w_{k_n} n^{-1/3} \geq c_r a^{\alpha n - 1} n^{-1/3}$. At the same time, $\|\delta_n\|_{p,w} = w_n = a^n(1 + n^2)$, and we conclude that $C_{b_r}$ cannot be bounded. ∎

We conclude this section with a family of examples of nonstandard almost isometric isomorphisms on $\ell^2(\mathbb{Z}, \omega)$. We find it interesting to look at the question from the following angle. There is a detailed theory of composition operators on Hardy spaces [Sh] including their weighted variants [GP]. It provides, among others, almost isometric nonstandard operators on the spaces $\ell^2(\mathbb{Z}_+, \omega)$. Below, we show that this gives examples of nonstandard almost isometric automorphisms of the algebras $\ell^2(\mathbb{Z}, \omega)$.

***Proposition 3.4*** *Let $a > 1$ be an integer and let $\omega(n) = \max(1, |n|^a)$. For every $\varepsilon > 0$, there exists an isomorphism $T_\varepsilon : \ell^2(\mathbb{Z}, \omega) \to \ell^2(\mathbb{Z}, \omega)$ of distortion less than $1 + \varepsilon$ that is not given by a weighted automorphism of $\mathbb{Z}$.*

**Proof** It is known [Zor90] that the weighted Hardy space $H^2(\omega)$ with the same weight $(\omega_n)_{n \geq 0}$ is composition invariant with respect to any holomorphic automorphism of the unit disk. In particular, for the function $\varphi(z) = \frac{z-r}{1-rz}$, $0 < r < 1$, the norm of composition operator is bounded by $(1 + r)^\Lambda/(1 - r)^\Lambda$ with a constant $\Lambda$ depending on the weight [GP]. We will show that the operator $C_\varphi : f \mapsto (\widehat{f} \circ \varphi)^\vee$ is well defined on $\ell^2(\mathbb{Z}, \omega)$ and estimate its norm.

For $f \in \ell^2(\mathbb{Z}, \omega)$, set

$$f_+(z) = \sum_{n=1}^{\infty} f_n z^n, \quad f_-(z) = \sum_{n=1}^{\infty} \tilde{f}_{-n} z^n,$$

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then

$$\widehat{f}(z) = f_0 + f_+(z) + \overline{f_-(z)}, \quad \text{and} \quad \widehat{f} \circ \varphi(z) = f_0 + f_+ \circ \varphi(z) + \overline{f_- \circ \varphi(z)}.$$

As $f_+, f_- \in H^2(\omega)$, we have that $f_+ \circ \varphi$, $f_- \circ \varphi$ are in $H^2(\omega)$, so that

$$\begin{aligned}\widehat{f} \circ \varphi(z) &= f_0 + (f_+ \circ \varphi)(0) + \sum_{n>0} (f_+ \circ \varphi)_n z^n + \overline{(f_- \circ \varphi)(0)} + \sum_{n<0} \overline{(f_- \circ \varphi)_{-n}} z^n \\ &= \widehat{f}(\varphi(0)) + \sum_{n>0} (f_+ \circ \varphi)_n z^n + \sum_{n<0} \overline{(f_- \circ \varphi)_{-n}} z^n.\end{aligned}$$

The three summands above are orthogonal in $\ell^2(\mathbb{Z}, \omega)$; thus,

$$\begin{aligned}\|C_\varphi(f)\|_{2,\omega}^2 &= |\widehat{f}(\varphi(0))|^2 + \|f_+ \circ \varphi\|_{2,\omega}^2 + \|f_- \circ \varphi\|_{2,\omega}^2 \\ &\le |\widehat{f}(\varphi(0))|^2 + \Big(\frac{1+r}{1-r}\Big)^{2\Lambda} \Big( \|f_+\|_{2,\omega}^2 + \|f_-\|_{2,\omega}^2 \Big).\end{aligned}$$

We can estimate

$$\begin{aligned}|\widehat{f}(\varphi(0))| = |\widehat{f}(-r)| &\le |f_0| + |f_+(-r)| + |f_-(-r)| \le |f_0| + \sum_{n>0} r^n |f_n| + \sum_{n<0} r^n |f_n| \\ &\le |f_0| + r \Big( \sum_{n\neq 0} \frac{r^{2(n-1)}}{\omega(n)^2} \Big)^{1/2} \Big( \sum_{n\neq 0} |f_n|^2 \omega(n)^2 \Big)^{1/2} \\ &\le |f_0| + r\, d\, \|f - f_0\|_{2,\omega},\end{aligned}$$

where $d = \big( \sum_{n\neq 0} \omega(n)^{-2} \big)^{1/2}$. Next,

$$\begin{aligned}\|C_\varphi(f)\|_{2,\omega}^2 &\le \big( |f_0| + r\, d\, \|f - f_0\|_{2,\omega} \big)^2 + \Big(\frac{1+r}{1-r}\Big)^{2\Lambda} \Big( \|f_+\|_{2,\omega}^2 + \|f_-\|_{2,\omega}^2 \Big) \\ &= |f_0|^2 + 2rd|f_0| \cdot \|f - f_0\|_{2,\omega} + r^2 d^2 \|f - f_0\|_{2,\omega}^2 + \Big(\frac{1+r}{1-r}\Big)^{2\Lambda} \|f - f_0\|_{2,\omega}^2 \\ &\le |f_0|^2 + 2rd \|f\|_{2,\omega}^2 + \Big( r^2 d^2 + \Big(\frac{1+r}{1-r}\Big)^{2\Lambda} \Big) \|f - f_0\|_{2,\omega}^2 \\ &\le \Big( 2rd + r^2 d^2 + \Big(\frac{1+r}{1-r}\Big)^{2\Lambda} \Big) \|f\|_{2,\omega}^2.\end{aligned}$$

So, we have

$$\|C_\varphi\| \le K(r) \quad \text{where} \quad K(r) := \Big( 2rd + r^2 d^2 + \Big(\frac{1+r}{1-r}\Big)^{2\Lambda} \Big)^{1/2}.$$

The inverse operator to $C_\varphi$ is $C_\psi$ with the function $\psi(z) = \frac{z+r}{1+rz}$ and has the norm $\|C_\psi\| \le K(-r)$. Since $K(r) \to 1$, $r \to 0$, this proves the proposition. ∎

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