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How negative can $\sum_{n\le x}\frac{f(n)}{n}$ be?

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Abstract

Turán observed that logarithmic partial sums $\sum_{n\le x}f(n)/n$ of completely multiplicative functions (in the particular case of the Liouville function $f(n)=\lambda(n)$) tend to be positive. We develop a general approach to prove two results aiming to explain this phenomena.

Firstly, we show that there exist constants $C, x_0 \ge 1$, such that for any completely multiplicative function f satisfying $-1 \le f(n) \le 1$, we have

$$\sum_{n\le x}\frac{f(n)}{n} \ge -\frac{C(\log\log\log x)^2}{(\log\log x)}, \quad x \ge x_0.$$

This improves a previous bound due to Granville and Soundararajan. Secondly, we show that if f is a typical (random) completely multiplicative function $f:\mathbb{N}\to\{-1,1\}$, the probability that $\sum_{n\le x}f(n)/n$ is negative for a given large x, is $O(\exp(-\exp(\log x\cdot\log\log\log x/C\log\log x)))$. This improves on recent work of Angelo and Xu.

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1. *Introduction*

An immediate consequence of the celebrated Dirichlet's class number formula is that $L(1,\chi)>0$ for any real primitive multiplicative character χ of modulus q. It thus follows that there exists some $x_0(q)$ which may depend on q such that

$$\sum_{n\le x}\frac{\chi(n)}{n} \ge 0, \quad \text{for all} \quad x \ge x_0(q). \tag{1·1}$$

Establishing quantitative variants of (1·1) is a fundamental problem related to the existence of putative Siegel zeros. Such problems have origins in the work of Turán [**15**], who showed that if partial sums of the Liouville function satisfy

$$\sum_{n\le x}\frac{\lambda(n)}{n} \ge 0, \quad \text{for all} \quad n \ge 1, \tag{1·2}$$

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then the Riemann hypothesis is true. Haselgrove [11] proved that (1·2) is false and Borwein, Ferguson and Mossinghoff [3] have established that $x = 72,185,376,951,205$ is the smallest integer counterexample to (1·2). This naturally raises the question: why is this number so large?

As noted by Granville and Soundararajan [7], combining Haselgrove's result with quadratic reciprocity implies that for any large $x_0 > 0$ there exists a real Dirichlet character χ such that

$$\sum_{n\le x}\frac{\chi(n)}{n} < 0, \quad \text{for some} \quad x \ge x_0.$$

In particular, $x_0(q)$ in (1·1) is not bounded by a constant independent of q. If χ is a real Dirichlet character mod q, it is possible to obtain an upper bound on $x_0(q)$ in (1·1). Indeed, combining Siegel's lower bound for $L(1,\chi)$ with partial summation gives

$$\frac{1}{q^{\varepsilon}} \ll L(1,\chi) = \sum_{n\le x}\frac{\chi(n)}{n} - \left(\frac{S(x)}{x} - \int_x^{\infty}\frac{S(t)}{t^2}dt\right),$$

where

$$S(t) = \sum_{n\le t}\chi(n).$$

After applying the Pólya–Vinogradov inequality $S(t) \ll q^{1/2}\log q$, we see that one may take

$$x_0(q) \ll q^{1/2+2\varepsilon} \tag{1·3}$$

in (1·1).

In this paper, we obtain some new results which aim to quantitativley determine how negative the sums in (1·1) can get. To describe these results, we first introduce some notation from [7]. Let $\mathcal{F}, \mathcal{F}_1$ and $\mathcal{F}_0$ denote the set of completely multiplicative functions satisfying

$$-1 \le f(n) \le 1, \quad f(n) = \pm 1 \quad \text{and} \quad f(n) \in \{-1, 0, 1\}$$

respectively. A quantitative form of the above question is to establish lower bounds for

$$\delta(x) = \min_{f\in\mathcal{F}}\sum_{n\le x}\frac{f(n)}{n}, \quad \delta_i(x) = \min_{f\in\mathcal{F}_i}\sum_{n\le x}\frac{f(n)}{n}, \quad i \in \{0,1\}.$$

We clearly have

$$\delta(x) \le \delta_0(x) \le \delta_1(x), \tag{1·4}$$

and a quadratic reciprocity argument yields

$$\delta_0(x) = \min_{\substack{\chi \\ \text{quadratic character}}}\sum_{n\le x}\frac{\chi(n)}{n}.$$

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In [7], Granville and Soundararajan showed that for sufficiently large x we have

$$\delta(x) \ge -\frac{1}{(\log\log x)^{3/5}} \tag{1·5}$$

and

$$\delta_1(x) \le -\frac{c}{\log x}. \tag{1·6}$$

The inequality (1·5) is not expected to be sharp and in [7] the question of improving this result is raised. A consequence of (1·4), (1·5) and (1·6) is an upper and lower bound for each of δ, δ_0 and δ_1. We expect the lower bounds for δ, δ_0 and δ_1 to be far from the truth and understanding their asymptotic behaviour remains a mystery. Our first result is an improvement on (1·5).

THEOREM 1·1 *There exist constants* C, x_0 *such that if* $x \ge x_0$ *then*

$$\delta(x) \ge -\frac{C(\log\log\log x)^2}{(\log\log x)}.$$

Another route to explain why partial sums of multiplicative functions tend to be positive is to ask, given large x, how likely it is for

$$\sum_{n\le x}\frac{f(n)}{n}$$

to be negative. We refer the reader to [1, 2, 13] for a series of recent investigations into sign changes of partial sums of random multiplicative functions.

Let $(f(p))_{p\ prime}$ be a sequence of independent random variables taking values ± 1 with probability $1/2$ and put $f(n) = \prod_{p^k||n} f^k(p)$ for all $n \ge 1$. Given $x \ge 1$ we let $p(x)$ denote the probability that

$$\sum_{n\le x}\frac{f(n)}{n} < 0. \tag{1·7}$$

Angelo and Xu [1] have recently shown that there exists an absolute constant c such that

$$p(x) < \exp\left(-x^{\varepsilon}\right), \quad \varepsilon = \frac{c}{\log\log x}.$$

We establish a somewhat stronger result.

THEOREM 1·2 *With f as above, let p(x) denote the probability that (1·7) holds. There exists an absolute constant c such that*

$$p(x) \le \exp\left(-x^{\varepsilon}\right), \quad \varepsilon = c\frac{\log\log\log x}{\log\log x}.$$

Granville and Soundararajan [7] have shown that for any large x there is a function $f : \mathbb{N} \to \{-1, 1\}$, for which (1·7) holds. It is possible to count trivial variations of Granville and

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Soundararajan's construction [7] and this produces a bound of the form

$$\log\left(\frac{1}{p(x)}\right) \leq (1+\eta)\frac{x}{\log x},$$

for some absolute constant η. It is an interesting question to determine if there exists $\gamma \in \mathbb{R}$ such that

$$\log\left(\frac{1}{p(x)}\right) = x^{\gamma+o(1)}.$$

1·1. *Main ideas*

Roughly speaking, the proofs of both Theorem 1·1 and Theorem 1·2 start with a simple convolution identity (which is used in [7] but not in [1]) and our main novelty comes from introducing a bilinear structure allowing us to incorporate finer information on the distribution of f on large primes $x^v \leq p \leq x$. More precisely, if $g(n) = \sum_{d|n} f(d)$ then

$$\sum_{n\leq x}\frac{f(n)}{n} = \frac{1}{x}\sum_{n\leq x} g(n) + \frac{1}{x}\sum_{n\leq x} f(n)\left\{\frac{x}{n}\right\}, \tag{1·8}$$

and crucially $g(n) \geq 0$ for all $n \geq 1$. The remaining task is to produce satisfactory lower and upper bounds for the first and second terms on the right-hand side of (1·8). In the case of Theorem 1·1, arguing as in [7] one may proceed further and write, as in Lemma 2·1

$$\sum_{n\leq x}\frac{f(n)}{n} = \frac{1}{x}\sum_{n\leq x} g(n) + (1-\gamma)\frac{1}{x}\sum_{n\leq x} f(n) + O\left(\frac{1}{\log^{1/5} x}\right).$$

To upper bound the second term we use Halász type estimates due to Hall and Tenenbaum (Lemma 2·3) and more recent bounds due to Granville, Harper and Soundararajan (Lemma 2·2). At this stage we depart from the argument in [7] and key novelty in our approach lies in treating the lower bounds for the first sum. Here, rather than using a classical bound due to Hildebrand (as in [7]), we rely on a recent additive combinatorics result of Matomäki and Shao [14], which allows us to prove the main technical Proposition 3·3. Roughly speaking, either the conditions to apply Matomäki and Shao's result are satisfied or we can make a more efficient use of Halász type estimates.

We next oultine the proof of Theorem 1·2. Returning to (1·8), note that since $g(n) \geq 0$, we have

$$\frac{1}{x}\sum_{n\leq x} g(n) \geq \frac{1}{x}\sum_{p\leq x}(1+f(p)),$$

and direct application of Bernstein's inequality produces a simple lower bound

$$\frac{1}{x}\sum_{n\leq x} g(n) \gg \frac{1}{\log x}$$

with an acceptable probability. We are thus left with estimating the probability of the event

$$\sum_{n\leq x} f(n)\left\{\frac{x}{n}\right\} \ll -\frac{x}{\log x}.$$

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We do so by a direct application of the majorant principle and this reduces the problem to upper bounding very high moments of the unweighted partial sums

$$\mathbb{E}|\sum_{n\le x} f(n)|^{2q}.$$

The key observation here (see Proposition 5·6) is that since we are interested in the event where the partial sums are extremely large (in absolute value), it is more beneficial to work "close to the 1-line" rather than proceed, as in the work of Harper [**10**], close to "1/2-line" (where the main interest is when partial sums have size around $\sqrt{x}$). This is accomplished by introducing a bilinear structure which allows us to combine bounds on the density of smooth numbers with Rankin's trick to treat the contribution of large primes.

We believe that Proposition 3·3 and Proposition 5·6 are of independent interest and could be useful in future investigations.

2. *Preliminaries for the proof of Theorem 1·1*

Here we collect several key facts, based on the argument of Granville and Soundararajan [**7**]. The following is [**7**, proposition 3·1].

LEMMA 2·1. *Let* $f\in\mathcal{F}$ *and define* $g(n)=\sum_{d|n}f(d)$. *Then*

$$\sum_{n\le x}\frac{f(n)}{n}=\frac{1}{x}\sum_{n\le x}g(n)+(1-\gamma)\frac{1}{x}\sum_{n\le x}f(n)+O\left(\frac{1}{(\log x)^{1/5}}\right),$$

where γ *denotes Euler's constant.*

We will use a version of Halász's theorem in the form given by Granville and Soundararajan [**6**, Theorem 2b].

LEMMA 2·2. *Let* $f\in\mathcal{F}$ *and define* $M=M(x)$ *by*

$$\max_{|t|\ll\log x}|F(1+1/\log x+it)|:=e^{-M}(\log x),$$

where $F(s)=\sum_{n=1}^{\infty}f(n)/n^s$ *for* $\Re s>1$. *Then*

$$\sum_{n\le x}f(n)\ll\frac{(1+M)e^{-M}}{1+|\phi|}x+\frac{1}{(\log x)^{2-\sqrt{3}+o(1)}}x,$$

where $\phi=\phi_f(x)$ *is a real number in the range* $|t|\le\log x$ *where the function* $t\to|F(1+1/\log x+it)|$ *attains its maximum.*

A result of Hall and Tenenbaum [**9**] allows one to remove the dependence on t in Lemma 2·2 at the cost of a worse upper bound.

LEMMA 2·3. *For any* $f\in\mathcal{F}$ *we have*

$$\sum_{n\le x}f(n)\ll x\exp\left(-\kappa\sum_{p\le x}\frac{1-f(p)}{p}\right),$$

with $\kappa=0.32867$.

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The following estimate is well known, for a proof see [5, Equation 2.1.6].

LEMMA 2·4 *For any $x \geq 2$ and $t \in \mathbb{R}$, we have*

$$\sum_{p \leq x} \frac{\Re(p^{it})}{p} = \log\log x + O(1) \quad \text{if} \quad |t| \leq 1/\log x$$

$$\sum_{p \leq x} \frac{\Re(p^{it})}{p} = \log(1/|t|) + O(1) \quad \text{if} \quad \frac{1}{\log x} \leq |t| \leq 100$$

and

$$\left| \sum_{p \leq x} \frac{\Re(p^{it})}{p} \right| \leq \log\log(2 + |t|) + O(1) \quad \text{if} \quad |t| \geq 100.$$

3. *Lower bounds for nonnegative multiplicative functions*

We next show how ideas from [8, 14] may be used to obtain lower bounds for the function $g(n)$ defined in Lemma 2·1. The main result of this section is Proposition 3·3.

The following is due to Matomäki and Shao [14, Hypothesis P] which improves on the work of Granville, Koukoulopoulos and Matomäki [8].

LEMMA 3·1 *Fix $\lambda \in (0,1)$. If x is sufficiently large and u,v satisfy*

$$1 \leq u \leq v \leq \frac{\log x}{1000 \log\log x},$$

and $\mathcal{P}$ is a subset of the primes in $(x^{1/v}, x^{1/u}]$ with

$$\sum_{p \in \mathcal{P}} \frac{1}{p} \geq \frac{1+\lambda}{u},$$

then there exists an integer $k \in [u,v]$ such that

$$\left| \left\{ (p_1, \dots, p_k) \in \mathcal{P}^k : \frac{x}{2} \leq p_1 \dots p_k \leq x \right\} \right| \geq \pi_v \frac{x}{v^k \log x},$$

where π_v is a constant with $\pi_v = v^{-o(v)}$ as $v \to \infty$. If u is fixed and $v \geq 1000u^2/\lambda^2$, one can take $k \leq e^{-1/u} v$.

Our next result uses arguments in the spirit of [12, p. 218].

LEMMA 3·2 *Let g be a nonnegative multiplicative function satisfying*

$$g(p^j) \leq j+1 \tag{3·1}$$

for each prime power p^j. For any real numbers $x, z \geq 1$ satisfying

$$z \leq x^{1/C_1} \tag{3·2}$$

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for a sufficiently large absolute constant C_1 *we have*

$$\sum_{\substack{n\le x\\ p|n\Longrightarrow p\le z}}\frac{g(n)}{n}\gg\exp\left(\sum_{p\le z}\frac{g(p)}{p}\right).$$

Proof. Let $\delta=1/\log z$, and consider

$$\begin{aligned}\sum_{\substack{n\le x\\ p|n\Longrightarrow p\le z}}\frac{g(n)}{n}&\ge\sum_{\substack{n\ge 1\\ p|n\Longrightarrow p\le z}}\frac{g(n)}{n}-\sum_{\substack{n>x\\ p|n\Longrightarrow p\le z}}\frac{g(n)}{n}\\&\ge\prod_{p\le z}\left(1+\sum_{j\ge 1}\frac{g(p^j)}{p^j}\right)-\frac{1}{x^\delta}\sum_{\substack{n\ge 1\\ p|n\Longrightarrow p\le z}}\frac{g(n)}{n^{1-\delta}}.\end{aligned}$$

By (3·1)

$$\log\left(\sum_{\substack{n\ge 1\\ p|n\Longrightarrow p\le z}}\frac{g(n)}{n^{1-\delta}}\right)=\sum_{p\le z}\frac{g(p)}{p}+O(1)$$

which implies that for some absolute constant C

$$\sum_{\substack{n\le x\\ p|n\Longrightarrow p\le z}}\frac{g(n)}{n}\gg\exp\left(\sum_{p\le z}\frac{g(p)}{p}\right)\left(1-C\exp\left(-\frac{\log x}{\log z}\right)\right)\gg\exp\left(\sum_{p\le z}\frac{g(p)}{p}\right)$$

assuming C_1 in (3·2) is large enough. This completes the proof.

We are now in a position to establish the main result of this section.

PROPOSITION 3·3 *Let* $\varepsilon>0$ *be sufficiently small,* $f\in\mathcal{F}$ *and define* $g(n)=\sum_{d|n}f(d)$. *For* $x\ge 2$ *and* $0<\delta<1$ *which may depend on x, let* $\mathcal{P}_\delta$ *denote the set*

$$\mathcal{P}_\delta=\{p\le x\ :\text{prime},\ f(p)\ge-\delta\}.$$

Suppose that for some v satisfying

$$\frac{40000}{\varepsilon^2}\le v\le\frac{\log x}{1000\log\log x},$$

we have

$$\sum_{\substack{p\in\mathcal{P}_\delta\\ x^{1/v}\le p\le x}}\frac{1}{p}\ge 1+\varepsilon.\tag{3·3}$$

Then

$$\sum_{n\le x}g(n)\gg\varepsilon^4\left(\frac{(1-\delta)}{v}\right)^{v(1+o(1))/e}\exp\left(\sum_{p\le x}\frac{f(p)}{p}\right)x.$$

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Proof. Let $G = \sum_{n \le x} g(n)$, and define

$$\mathcal{A} = \{\, p \text{ prime} : p \le x^{1/v} \,\}, \quad \mathcal{B} = \mathcal{P}_\delta \cap (x^{1/v}, x], \tag{3·4}$$

so that

$$G \ge \sum_{\substack{a \le x^{\varepsilon/5} \\ p \mid a \Longrightarrow p \in \mathcal{A}}} g(a) \sum_{\substack{b \le x/a \\ p \mid b \Longrightarrow p \in \mathcal{B}}} g(b).$$

If $p \in \mathcal{P}_\delta$, then since f is completely multiplicative, for any integer $j \ge 1$ we have

$$|f(p^{j+1})| \le |f(p^j)|$$

and hence

$$g(p^\alpha) \ge 1 + f(p) + \sum_{2 \le j \le \alpha} f(p^j) \ge 1 - \delta$$

since

$$\sum_{2 \le j \le \alpha} f(p^j) \ge 0.$$

This implies that

$$G \ge \sum_{\substack{a \le x^{\varepsilon/5} \\ p \mid a \Longrightarrow p \in \mathcal{A}}} g(a) \sum_{\substack{b \le x/a \\ p \mid b \Longrightarrow p \in \mathcal{B}}} (1-\delta)^{\Omega(b)}, \tag{3·5}$$

where $\Omega(n)$ counts the number of prime factors (with multiplicity) of an integer n and we have used the inequality $\omega(n) \le \Omega(n)$, where $\omega(n)$ counts the number of distinct prime factors of an integer n.

Our next step is to apply Lemma 3·1 to the inner summation over b. This requires verifying for each

$$a \le x^{\varepsilon/5} \tag{3·6}$$

the following lower bound holds

$$\sum_{\substack{p \in \mathcal{P}_\delta \\ (x/a)^{1/v} \le p \le x/a}} \frac{1}{p} \ge 1 + \frac{\varepsilon}{2}. \tag{3·7}$$

By (3·3) and the fact that

$$[x^{1/v}, x/a] \subseteq [\,(x/a)^{1/v}, x/a]$$

we have

$$\sum_{\substack{p \in \mathcal{P}_\delta \\ (x/a)^{1/v} \le p \le x/a}} \frac{1}{p} \ge \sum_{\substack{p \in \mathcal{P}_\delta \\ x^{1/v} \le p \le x}} \frac{1}{p} - \sum_{\substack{p \in \mathcal{P}_\delta \\ x/a \le p \le x}} \frac{1}{p} \ge 1 + \varepsilon - \sum_{\substack{p \in \mathcal{P}_\delta \\ x/a \le p \le x}} \frac{1}{p} \tag{3·8}$$

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and by (3·6)

$$\sum_{\substack{p\in\mathcal{P}_\delta\\ x/a\le p\le x}}\frac{1}{p}\le \log\log x-\log\log(x/a)+o(1)$$
$$\le \log\log x-\log\log(x^{1-\varepsilon/5})\le\frac{\varepsilon}{4}+o(1)\cdot$$

Combining the above with (3·8) implies (3·7). By Lemma 3·1, for each a satisfying (3·6), if v satisfies

$$v\ge\frac{40000}{\varepsilon^2},\tag{3·9}$$

then there exists an integer $k\le v/e$ such that

$$\left|\left\{(p_1,\dots,p_k)\in\mathcal{P}_\delta^k:\frac{x}{2a}\le p_1\dots p_k\le\frac{x}{a}\right\}\right|\ge v^{o(v)}\frac{x}{v^k a\log x}.$$

Hence

$$\sum_{\substack{b\le x/a\\ p|b\Longrightarrow p\in\mathcal{B}}}(1-\delta)^{\Omega(b)}\ge v^{o(v)}\left(\frac{(1-\delta)}{v}\right)^k\frac{x}{a\log x}$$
$$\ge\left(\frac{(1-\delta)}{v}\right)^{v(1+o(1))/e}\frac{x}{a\log x},$$

which combined with (3·5) implies

$$G\ge\left(\frac{(1-\delta)}{v}\right)^{v(1+o(1))/e}\frac{x}{\log x}\sum_{\substack{a\le x^{\varepsilon/5}\\ p|a\Longrightarrow p\in\mathcal{A}}}\frac{g(a)}{a}.\tag{3·10}$$

Recalling (3·4)

$$\sum_{\substack{a\le x^{\varepsilon/5}\\ p|a\Longrightarrow p\in\mathcal{A}}}\frac{g(a)}{a}=\sum_{\substack{a\le x^{\varepsilon/5}\\ p|a\Longrightarrow p\le x^{1/v}}}\frac{g(a)}{a}.$$

If ε is sufficiently small then by (3·9) the condition that C_1 in (3·2) is sufficiently large is satisfied. Hence we may apply Lemma 3·2 to deduce that

$$\sum_{\substack{a\le x^{\varepsilon/5}\\ p|a\Longrightarrow p\in\mathcal{A}}}\frac{g(a)}{a}\gg\exp\left(\sum_{p\le x^{\varepsilon^2/40000}}\frac{g(p)}{p}\right).$$

Using (3·10) gives

$$G\gg\left(\frac{(1-\delta)}{v}\right)^{v(1+o(1))/e}\frac{x}{\log x}\exp\left(\sum_{p\le x^{\varepsilon^2/40000}}\frac{g(p)}{p}\right),$$

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and the result follows after noting

$$\sum_{p\le x^{\varepsilon^2/40000}}\frac{g(p)}{p}\ge\sum_{p\le x}\frac{g(p)}{p}+2\log\varepsilon+O(1)$$
$$\ge\log\log x+\sum_{p\le x}\frac{f(p)}{p}+4\log\varepsilon+O(1).$$

4. *Proof of Theorem 1·1*

First note we may assume that

$$\sum_{n\le x}\frac{f(n)}{n}\le-\frac{1}{(\log x)^{1/6}},\tag{4·1}$$

since otherwise, for any constant C there exists x_0 such that if $x\ge x_0$ then

$$\sum_{n\le x}\frac{f(n)}{n}>-\frac{1}{(\log x)^{1/6}}>-\frac{C(\log\log\log x)^2}{(\log\log x)}.$$

By (4·1) and Lemma 2·1, for sufficiently large x we have

$$\sum_{n\le x}\frac{f(n)}{n}\ge\frac{1}{x}\sum_{n\le x}g(n)+(1-\gamma)\frac{1}{x}\sum_{n\le x}f(n)-\frac{1}{2(\log x)^{1/6}}$$
$$\ge\frac{1}{x}\sum_{n\le x}g(n)+(1-\gamma)\frac{1}{x}\sum_{n\le x}f(n)+\frac{1}{2}\sum_{n\le x}\frac{f(n)}{n},$$

which implies that

$$\sum_{n\le x}\frac{f(n)}{n}\ge2\left(\frac{1}{x}\sum_{n\le x}g(n)-\frac{(1-\gamma)}{x}\left|\sum_{n\le x}f(n)\right|\right).\tag{4·2}$$

Hence from Lemma 2·2, there exists an absolute constant C_1 such that

$$\sum_{n\le x}\frac{f(n)}{n}\ge2\left(\frac{1}{x}\sum_{n\le x}g(n)-\frac{C_1(1+M)e^{-M}}{1+|t|}\right),\tag{4·3}$$

where M is defined by

$$|F(1+1/\log x+it)|=e^{-M}(\log x),\tag{4·4}$$

for some $|t|\ll\log x$. Let $\varepsilon>0$ be small, define

$$\varepsilon_1=\frac{C}{\log\log\log x}\tag{4·5}$$

for a suitable absolute constant C and put

$$\delta=1-\varepsilon_1,\quad v=\frac{\log\log x}{\log\log\log x}.\tag{4·6}$$

Consider the set

$$\mathcal{P}_\delta=\{p\le x:\ p\text{ prime},\ f(p)\ge-\delta\}.$$

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We distinguish between two cases. Either

$$\sum_{\substack{x^{1/v}\le p\le x\\ p\in\mathcal{P}_\delta}}\frac{1}{p}\le 1+\varepsilon \tag{4·7}$$

or

$$\sum_{\substack{x^{1/v}\le p\le x\\ p\in\mathcal{P}_\delta}}\frac{1}{p}> 1+\varepsilon. \tag{4·8}$$

Suppose first that (4·7) holds and consider

$$\begin{aligned}\Re\left(\sum_{p\le x}\frac{1-f(p)p^{-it}}{p}\right)&\ge \Re\left(\sum_{\substack{x^{1/v}\le p\le x\\ p\notin\mathcal{P}_\delta}}\frac{1-f(p)p^{-it}}{p}\right)+O(1)\\ &\ge \Re\left(\sum_{\substack{x^{1/v}\le p\le x\\ p\notin\mathcal{P}_\delta}}\frac{1+p^{-it}}{p}\right)-\sum_{\substack{x^{1/v}\le p\le x\\ p\notin\mathcal{P}_\delta}}\frac{|1+f(p)|}{p}+O(1).\end{aligned}$$

If $p\notin\mathcal{P}_\delta$, then recalling (4·6)

$$|1+f(p)|\le 1-\delta=\varepsilon_1,$$

which implies for such values of p

$$\begin{aligned}\Re\left(1-f(p)p^{-it}\right)&=\Re\left(1+p^{-it}\right)-\Re\left((f(p)+1)p^{-it}\right)\\ &\ge \Re\left(1+p^{-it}\right)-\varepsilon_1\end{aligned}$$

and consequently by (4·7), we have

$$\Re\left(\sum_{p\le x}\frac{1-f(p)p^{-it}}{p}\right)\ge \Re\left(\sum_{x^{1/v}\le p\le x}\frac{1+p^{-it}}{p}\right)-\varepsilon_1\log v+O(1). \tag{4·9}$$

We next consider two subcases depending on the size of t relative to x. Suppose first that

$$|t|\ge 100.$$

By Lemma 2·4

$$\left|\Re\left(\sum_{x^{1/v}\le p\le x}\frac{p^{-it}}{p}\right)\right|\le 2\log\log|t|+O(1)$$

and hence from (4·9)

$$\Re\left(\sum_{p\le x}\frac{1-f(p)p^{-it}}{p}\right)\ge(1-\varepsilon_1)\log v+O(\log\log|t|).$$

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After recalling (4·5) and (4·6), we get

$$e^{-M} = \frac{1}{\log x}|F(1+1/\log x+it)| \ll \frac{(\log|t|)^{O(1)}}{v^{1-\varepsilon_1}} \ll \frac{(\log\log\log x)(\log|t|)^{O(1)}}{\log\log x}$$

which when combined with (4·3), (4·4) and the bound

$$1+M \ll \log\log\log x,$$

gives

$$\sum_{n\le x}\frac{f(n)}{n} \ge -\frac{C(\log\log\log x)^2}{(\log\log x)}\frac{(\log|t|)^{O(1)}}{|t|} \ge -\frac{C'(\log\log\log x)^2}{(\log\log x)} \tag{4·10}$$

for some absolute constant C'. If $|t| \le 100$ then by Lemma 2·4

$$\Re\left(\sum_{x^{1/v}\le p\le x}\frac{p^{-it}}{p}\right) \ge -C'$$

for some absolute constant C'. Hence from (4·9)

$$\Re\left(\sum_{p\le x}\frac{1-f(p)p^{-it}}{p}\right) \ge (1-\varepsilon_1)\log v + O(1).$$

After recalling (4·5) and (4·6), we get

$$e^{-M} = \frac{1}{\log x}|F(1+1/\log x+it)| \ll \frac{1}{v^{1-\varepsilon_1}} \ll \frac{(\log\log\log x)}{\log\log x}$$

which when combined with (4·3), (4·4) and the fact that $g(n) \ge 0$, gives

$$\sum_{n\le x}\frac{f(n)}{n} \ge -\frac{C'(\log\log\log x)^2}{(\log\log x)} \tag{4·11}$$

for some absolute constant C'. This completes our analysis of the case when (4·7) holds. Suppose next that (4·8) holds. By (4·3) and Proposition 3·3

$$\frac{1}{x}\sum_{n\le x}g(n) \gg \varepsilon^4\left(\frac{(1-\delta)}{4}v\right)^{v(1+o(1))/e}(\log x)\exp\left(-\sum_{p\le x}\frac{1-f(p)}{p}\right).$$

Recalling (4·6), we have

$$\left(\frac{(1-\delta)}{v}\right)^{v(1+o(1))/e}\log x \gg \exp\left(-\frac{v(1+o(1))\log(v/\varepsilon_1)}{e}\right)(\log x)^{1-1/e} \gg (\log x)^{1-1/e+o(1)}.$$

Upon applying Lemma 2·3 and (4·2) we deduce

$$\sum_{n\le x}\frac{f(n)}{n} \ge C_1(\log x)^{1-o(1)}\exp\left(-\sum_{p\le x}\frac{1-f(p)}{p}\right) - C_2\exp\left(-\kappa\sum_{p\le x}\frac{1-f(p)}{p}\right),$$

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for some absolute constants C_1, C_2. Optimising the function

$$H(u) = C_1(\log x)^{1-o(1)}\exp(-u) - C_2\exp(-\kappa u)$$

subject to the restriction $u \ge 0$ implies

$$\sum_{n\le x}\frac{f(n)}{n} \ge -\frac{c}{(\log x)^{1/100}} \ge -\frac{C(\log\log\log x)}{\log\log x},$$

and combined with (4·10) we complete the proof.

5. *Preliminaries for the proof of Theorem 1·2*

We next present some preliminary results required for the proof of Theorem 1·2. The following is known as Bernstein's inequality.

LEMMA 5·1 *Let $X_1, \dots, X_n$ be a sequence of independent random variables satisfying*

$$P(|X_i| \le M) = 1, \quad \mathbb{E}(X_i) = 0, \quad 1 \le i \le n.$$

For all $t > 0$ we have

$$P\left(\sum_{i=1}^{n} X_i \ge t\right) \le \exp\left(-\frac{t^2/2}{\sum_{i=1}^{n}\mathbb{E}(X_i^2) + Mt/3}\right).$$

For $1 \le z \le w$ define

$$\Psi(w,z) = \sum_{\substack{n\le w\\ p\mid n \Longrightarrow p\le z}} 1.$$

The following is due to Canfield, Erdös and Pomerance [**4**, Corollary p. 15].

LEMMA 5·2 *Let $1 \le z \le w$ be real numbers and define*

$$u = \frac{\log w}{\log z}.$$

If

$$u \le z^{1-o(1)} \quad \textit{and} \quad u \to \infty$$

then

$$\Psi(w,z) = \frac{w}{u^{u(1+o(1))}}. \tag{5·1}$$

Recall that $\mathcal{F}_1$ denotes the space of completely multiplicative functions f satisfying $f(p) = \pm 1$ for $p \le x$. Let μ denote the counting measure on $\mathcal{F}_1$, normalised so that

$$\mu(\mathcal{F}_1) = 1$$

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and let $\mathbb{E}$ denote expectation with respect to μ. We will make considerable use of the majorant principle.

LEMMA 5·3 *Let q be an even integer and a_n and b_n be two sequences of real numbers satisfying*

$$|a_n| \leq b_n.$$

Then

$$\mathbb{E}\left[\left(\sum_{n\leq x} a_n f(n)\right)^q\right] \leq \mathbb{E}\left[\left(\sum_{n\leq x} b_n f(n)\right)^q\right].$$

Proof. Recall that $f(p) = \pm 1$ with probability $1/2$. If q is an even integer, then for any real numbers a_n we have

$$\mathbb{E}\left[\left(\sum_{n\leq x} a_n f(n)\right)^q\right] = \sum_{\substack{1\leq n_1,\cdots,n_q\leq x\\ \exists s\in\mathbb{N}\,:\, n_1n_2\cdots n_q=s^2}} a_{n_1}\cdots a_{n_q}.$$

Hence if $|a_n| \leq b_n$ then

$$\mathbb{E}\left[\left(\sum_{n\leq x} a_n f(n)\right)^q\right] \leq \sum_{\substack{1\leq n_1,\cdots,n_q\leq x\\ \exists s\in\mathbb{N}\,:\, n_1n_2\,\cdots\, n_q=s^2}} b_{n_1}\cdots b_{n_q} = \mathbb{E}\left[\left(\sum_{n\leq x} b_n f(n)\right)^q\right].$$

A particular consequence of Lemma 5·3 is as follows.

COROLLARY 5·4 *Let q be an even integer and c_n a sequence of nonnegative real numbers. For any $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$ satisfying $\mathcal{A} \subseteq \mathcal{B}$, we have*

$$\mathbb{E}\left[\left(\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} c_n f(n)\right)^q\right] \leq \mathbb{E}\left[\left(\sum_{\substack{n\leq x\\ n\in\mathcal{B}}} c_n f(n)\right)^q\right].$$

Proof. This follows from Lemma 5·3 taking

$$a_n = \begin{cases} c_n & \text{if } n\in\mathcal{A}\\ 0 & \text{otherwise}\end{cases}$$

and

$$b_n = \begin{cases} c_n & \text{if } n\in\mathcal{B}\\ 0 & \text{otherwise}\end{cases}$$

after noting the condition $a_n \leq b_n$ follows from the assumption that $\mathcal{A}\subseteq\mathcal{B}$.

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Our main input for Theorem 1·2 is the following.

COROLLARY 5·5. *For any* $M>0$*, there exists a constant* $C>0$ *such that for sufficiently large* x

$$\mu\left(\left\{f\in\mathcal{F}_1:\left|\sum_{n\le x}\left\{\frac{x}{n}\right\}f(n)\right|\ge\frac{M}{\log x}x\right\}\right)\le\exp\left(-\exp\left(\frac{(\log\log\log x)\log x}{C\log\log x}\right)\right).$$

Corollary 5·5 is a consequence of the following moment inequality.

PROPOSITION 5·6. *For any* $C>0$*, positive real number* x *sufficiently large in terms of* C *and any even integer* q*, we have*

$$\left(\mathbb{E}\left[\left(\sum_{n\le x}f(n)\right)^q\right]\right)^{1/q}\ll x^{7/8}\exp\left(\left(q\exp\left(-\frac{(\log\log\log x)\log x}{C\log\log x}\right)+2\log\log x\right)\right)+\left(\frac{x}{(\log x)^{C/4-1}}\right).$$

We first show that Proposition 5·6 implies Corollary 5·5 which in turn implies Theorem 1·2. We then prove Proposition 5·6 in Section 5·3.

5·1. *Deduction of Theorem* 1·2 *from Corollary* 5·5:

Let $g(n)=\sum_{d|n}f(d)$ and apply (1·8) to get

$$\sum_{n\le x}\frac{f(n)}{n}=\frac{1}{x}\sum_{n\le x}g(n)+\frac{1}{x}\sum_{n\le x}f(n)\left\{\frac{x}{n}\right\}.$$

By Lemma 5·1, the probability that

$$\sum_{p\le x}f(p)\le-\frac{x}{2\log x}$$

is bounded by

$$O\left(\exp\left(-\frac{x}{100\log x}\right)\right).$$

Since $g(n)\ge0$ for all $n\ge1$, this implies that for sufficiently large x the probability that

$$\frac{1}{x}\sum_{n\le x}g(n)\ge\frac{1}{x}\sum_{p\le x}(1+f(p))\ge\frac{1}{2\log x}$$

is equal to

$$1-O\left(\exp\left(-\frac{x}{100\log x}\right)\right).$$

After applying Corollary 5·5 with $M=1/2$ the desired result follows.

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5·2. *Deduction of Corollary 5·5 from Proposition 5·6*

By Lemma 5·3, for any even integer q

$$\mathbb{E}\left[\left(\sum_{n\le x}\left\{\frac{x}{n}\right\}f(n)\right)^q\right] \le \mathbb{E}\left[\left(\sum_{n\le x} f(n)\right)^q\right]. \tag{5·2}$$

We next use Markov's inequality, which states that for a nonnegative random variable X and real number $y>0$ we have

$$P(X\ge y)\le \frac{E(X)}{y}.$$

Applying this with

$$X=\left|\sum_{n\le x}\left\{\frac{x}{n}\right\}f(n)\right|^q$$

and

$$y=\left(\frac{M}{\log x}x\right)^q$$

we get

$$\mu\left(\left\{f\in\mathcal{F}_1:\ \left|\sum_{n\le x}\left\{\frac{x}{n}\right\}f(n)\right|\ge \frac{M}{\log x}x\right\}\right)\le \left(\frac{M}{\log x}x\right)^{-q}\mathbb{E}\left[\left(\sum_{n\le x}\left\{\frac{x}{n}\right\}f(n)\right)^q\right]$$

which by (5·2) implies that

$$\mu\left(\left\{f\in\mathcal{F}_1:\ \left|\sum_{n\le x}\left\{\frac{x}{n}\right\}f(n)\right|\ge \frac{M}{\log x}x\right\}\right)\le \left(\frac{M}{\log x}x\right)^{-q}\mathbb{E}\left[\left(\sum_{n\le x} f(n)\right)^q\right].$$

Applying Proposition 5·6 with the choice of parameter

$$q=\exp\left(\frac{(\log\log\log x)(\log x)}{C\log\log x}\right)$$

completes the proof of Corollary 5·5.

5·3. *Proof of Proposition 5·6*

Let

$$\varepsilon=\frac{\log\log\log x}{4C\log\log x}, \tag{5·3}$$

where C is a sufficiently large absolute constant. Since

$$\sum_{m\le x} f(m)=\sum_{\substack{n\ell\le x\\ p|n\Longrightarrow p\ge x^{\varepsilon}\\ p|\ell\Longrightarrow p<x^{\varepsilon}}} f(\ell)f(n)=\sum_{j\le \log x+1}\ \sum_{\substack{n\ell\le x\\ p|n\Longrightarrow p\ge x^{\varepsilon}\\ p|\ell\Longrightarrow p<x^{\varepsilon}\\ e^j\le \ell<e^{j+1}}} f(\ell)f(n),$$

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by Minkowski's inequality and the majorant principle to ignore the condition $n\ell \le x$ we get

$$\mathbb{E}\left[\left(\sum_{n\le x} f(n)\right)^q\right]^{1/q} \le \sum_{j\le \log x+1} \mathbb{E}\left[\left(\sum_{\substack{n\le x/e^j\\ p|n\Longrightarrow p\ge x^\varepsilon}} f(n) \sum_{\substack{\ell\le e^{j+1}\\ p|\ell\Longrightarrow p\le x^\varepsilon}} f(\ell)\right)^q\right]^{1/q}$$

and hence

$$\mathbb{E}\left[\left(\sum_{n\le x} f(n)\right)^q\right]^{1/q} \le S_1 + S_2, \tag{5·4}$$

where

$$S_1 = \sum_{j\le \log x/2} \mathbb{E}\left[\left(\sum_{\substack{n\le x/e^j\\ p|n\Longrightarrow p\ge x^\varepsilon}} f(n) \sum_{\substack{\ell\le e^{j+1}\\ p|\ell\Longrightarrow p< x^\varepsilon}} f(\ell)\right)^q\right]^{1/q} \tag{5·5}$$

and

$$S_2 = \sum_{\log x/2< j\le \log x+1} \mathbb{E}\left[\left(\sum_{\substack{n\le x/e^j\\ p|n\Longrightarrow p\ge x^\varepsilon}} f(n) \sum_{\substack{\ell\le e^{j+1}\\ p|\ell\Longrightarrow p< x^\varepsilon}} f(\ell)\right)^q\right]^{1/q}.$$

Consider first S_2. Taking a maximum over the outer summation, we see that there exists $j \ge \log x/2$ such that

$$S_2 \le (\log x)\mathbb{E}\left[\left(\sum_{\substack{n\le x/e^j\\ p|n\Longrightarrow p\ge x^\varepsilon}} f(n) \sum_{\substack{\ell\le e^{j+1}\\ p|\ell\Longrightarrow p< x^\varepsilon}} f(\ell)\right)^q\right]^{1/q}.$$

Define β by $x^\beta = e^{j+1}$, and note that $\beta \ge 1/2$. Since f is completley multiplicative, $|f(n)| \le 1$ for each $n \in \mathbb{N}$ and hence

$$\sum_{\substack{n\le x/e^j\\ p|n\Longrightarrow p\ge x^\varepsilon}} f(n) \sum_{\substack{\ell\le e^{j+1}\\ p|\ell\Longrightarrow p< x^\varepsilon}} f(\ell) \ll x^{1-\beta}\Psi(x^\beta, x^\varepsilon).$$

This implies

$$S_2 \ll (\log x)x^{1-\beta}\Psi(x^\beta, x^\varepsilon). \tag{5·6}$$

We apply Lemma 5·2 with

$$w = x^\beta, \quad z = x^\varepsilon.$$

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Define

$$u = \frac{\log w}{\log z} = \frac{\beta}{\varepsilon},$$

and note that by (5·3) $u \to \infty$ and

$$u \le \frac{10C \log\log x}{\log\log\log x} \le x^{10C \log\log\log x/\log\log x} \le x^{\varepsilon/2}.$$

Combining (5·1) and (5·6) gives

$$S_2 \ll (\log x)\left(\frac{\varepsilon}{\beta}\right)^{(1+o(1))\beta/\varepsilon} x.$$

By (5·3), if x if sufficiently large in terms of C then

$$\left(\frac{\varepsilon}{\beta}\right)^{(1+o(1))\beta/\varepsilon} \ll \frac{1}{(\log x)^C}$$

and hence

$$S_2 \ll \frac{x}{(\log x)^C}.$$

Consequently, using (5·4)

$$\mathbb{E}\left[\left(\sum_{n\le x} f(n)\right)^q\right]^{1/q} \ll S_1 + \frac{x}{(\log x)^{C-1}}. \tag{5·7}$$

We now turn our attention to estimating S_1. We recall (5·5) and write

$$S_1 = \sum_{j\le \log x/2} \mathbb{E}\left[\left(\sum_{\substack{n\le x/e^j \\ p|n \Longrightarrow p\ge x^\varepsilon \\ \ell\le e^{j+1} \\ p|\ell \Longrightarrow p<x^\varepsilon}} (\ell n^{3/4}) \frac{f(n\ell)}{n^{3/4}\ell}\right)^q\right]^{1/q}.$$

If n, ℓ satisfy conditions in the above summation, we have

$$\ell n^{3/4} \ll e^j \left(\frac{x}{e^j}\right)^{3/4} \ll x^{3/4} e^{j/4}$$

and hence Lemma 5·3 implies

$$S_1 \ll x^{3/4} \sum_{j\le \log x/2} e^{j/4} \mathbb{E}\left[\left(\sum_{\substack{n\le x/e^j \\ p|n \Longrightarrow p\ge x^\varepsilon \\ \ell\le e^{j+1} \\ p|\ell \Longrightarrow p<x^\varepsilon}} \frac{f(n\ell)}{n^{3/4}\ell}\right)^q\right]^{1/q}.$$

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By Corollary 5·4

$$\mathbb{E}\left[\left(\sum_{\substack{n\le x/e^j\\ p|n\Longrightarrow p\ge x^{\varepsilon}\\ \ell\le e^{j+1}\\ p|\ell\Longrightarrow p<x^{\varepsilon}}}\frac{f(n\ell)}{n^{3/4}\ell}\right)^q\right]^{1/q}\ll\mathbb{E}\left[\left(\sum_{\substack{n\le x\\ p|n\Longrightarrow x^{\varepsilon}\le p\le x}}\frac{f(n)}{n^{3/4}}\sum_{\substack{\ell\ge 1\\ p|\ell\Longrightarrow p<x^{\varepsilon}}}\frac{f(\ell)}{\ell}\right)^q\right]^{1/q}$$

and hence by another application of Corollary 5·4

$$\begin{aligned}S_1&\ll x^{7/8}\mathbb{E}\left[\left(\sum_{\substack{n\le x\\ p|n\Longrightarrow x^{\varepsilon}\le p\le x}}\frac{f(n)}{n^{3/4}}\sum_{\substack{\ell\ge 1\\ p|\ell\Longrightarrow p<x^{\varepsilon}}}\frac{f(\ell)}{\ell}\right)^q\right]^{1/q}\\&\ll x^{7/8}\mathbb{E}\left[\left(\prod_{x^{\varepsilon}\le p\le x}\left(1-\frac{f(p)}{p^{3/4}}\right)^{-1}\right)^q\right]^{1/q}\mathbb{E}\left[\left(\prod_{p\le x^{\varepsilon}}\left(1-\frac{f(p)}{p}\right)^{-1}\right)^q\right]^{1/q}.\end{aligned}$$

We next expand the expectation as a product to get

$$\mathbb{E}\left[\left(\prod_{x^{\varepsilon}\le p\le x}\left(1-\frac{f(p)}{p^{3/4}}\right)^{-1}\right)^q\right]\le\exp\left(O(q)\right)\mathbb{E}\left[\left(\prod_{x^{\varepsilon}\le p\le x}\left(1+\frac{f(p)}{p^{3/4}}\right)\right)^q\right].$$

Observe that

$$\mathbb{E}\left[\left(\prod_{p\le x}\left(1+\frac{f(p)}{p^{3/4}}\right)\right)^q\right]=\prod_{p\le x}\frac{1}{2}\left(\left(1+\frac{1}{p^{3/4}}\right)^q+\left(1-\frac{1}{p^{3/4}}\right)^q\right)$$

and since for any real number a

$$\frac{\exp(a)+\exp(-a)}{2}\le\exp(a^2),$$

the above implies

$$\begin{aligned}\prod_{x^{\varepsilon}\le p\le x}\frac{1}{2}\left(\left(1+\frac{1}{p^{3/4}}\right)^q+\left(1-\frac{1}{p^{3/4}}\right)^q\right)&\le\prod_{x^{\varepsilon}\le p\le x}\frac{\exp(q/p^{3/4})+\exp(-q/p^{3/4})}{2}\\&\le\prod_{x^{\varepsilon}\le p\le x}\exp\left(\frac{q^2}{p^{3/2}}\right)\\&\le\exp\left(\frac{q^2}{x^{\varepsilon/2}}\right).\end{aligned}$$

The above inequalities imply that

$$\mathbb{E}\left[\left(\prod_{x^{\varepsilon}\le p\le x}\left(1-\frac{f(p)}{p^{3/4}}\right)^{-1}\right)^q\right]\le\exp\left(\frac{q^2}{x^{\varepsilon/2}}+O(q)\right).$$

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After recalling (5·3), the latter yields

$$\mathbb{E}\left[\left(\prod_{x^{\varepsilon}\leq p\leq x}\left(1-\frac{f(p)}{p^{3/4}}\right)^{-1}\right)^{q}\right]\leq \exp\left(q^{2}\exp\left(-\frac{(\log\log\log x)(\log x)}{4C\log\log x}\right)+O(q)\right). \tag{5·8}$$

We finally deduce the bound

$$\begin{aligned}\mathbb{E}\left[\left(\prod_{p\leq x^{\varepsilon}}\left(1-\frac{f(p)}{p}\right)^{-1}\right)^{q}\right]&\ll \exp\left(O(q)+q\sum_{p\leq x^{\varepsilon}}\frac{1}{p}\right)\\&\leq \exp\left(O(q)+q\log\log x\right).\end{aligned}$$

Combining the above with (5·5) and (5·8), we get

$$S_1\ll x^{7/8}\exp\left(\left(q\exp\left(-\frac{(\log\log\log x)(\log x)}{4C\log\log x}\right)+\log\log x\right)\right),$$

and the result follows from (5·7) after renaming the constant C.

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REFERENCES

[1] R. ANGELO and M. W. XU. On a Turán conjecture and random multiplicative functions. *Quart. J. Math.* **74** (2) (2023), 767–777.
[2] M. AYMONE, W. HEAP and J. ZHAO. Sign changes of the partial sums of a random multiplicative function. *Bull. London Math. Soc.* **55** (2023), 78–89. https://doi.org/10.1112/blms.12710.
[3] P. BORWEIN, R. FERGUSON and M. MOSSINGHOFF. Sign changes in sums of the Liouville function. *Math. Comp.* **77** (2008), 1681–1694.
[4] E. R. CANFIELD, P. ERDÖS and C. POMERANCE. On a problem of Oppenheim concerning factorisatio numerorum. *J. Number Theory* **17** (1) (1983), 1–28.
[5] A. GRANVILLE and K. SOUNDARARAJAN. Multiplicative number theory; The pretentious approach. https://dms.umontreal.ca/~andrew/PDF/Book.To1.2.pdf
[6] A. GRANVILLE and K. SOUNDARARAJAN. Decay of mean values of multiplicative functions. *Canad. J. Math.* **55** 1191–1230, (2003).
[7] A. GRANVILLE and K. SOUNDARARAJAN. Negative values of truncations to $L(1,\chi)$. *Clay Mathematics Proceedings 7* (2007), 141–148.
[8] A. GRANVILLE, D. KOUKOULOPOULOS and K. MATOMÄKI. When the sieve works, *Duke Math. J.* **164** (2015), 1935–1969.
[9] R. R. HALL and G. TENENBAUM. Effective mean value estimates for complex multiplicative functions. *Math. Proc. Camb. Phil. Soc.* **110**(2), (1991), 337–351.
[10] A. Harper. Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos. *Forum Math. Pi* **8** (2020), e1, 95 pp.
[11] C. B. HASELGROVE. A disproof of a conjecture of Pólya. *Mathematika* **5** (1958), 141–145.
[12] A. HILDEBRAND. Quantitative mean value theorems for nonnegative multiplicative functions II. *Acta. Arith.* **48**(3) (1987), 209–260.

https://doi.org/10.1017/S0305004125101837 Published online by Cambridge University Press

[13] A. KALMYNIN. Quadratic characters with positive partial sums. *Preprint*. https://arxiv.org/abs/2105.06910.

[14] K. MATOMÄKI and X. SHAO. When the sieve works II. *J. Reine Angew. Math.* **763** (2020), 1–24.

[15] P. TURÁN. On some approximative Dirichlet-polynomials in the theory of the zeta-function of Riemann. *Danske Vid. Selsk. Math. Fys. Medd.* **24**(17) (1948), 3–36.

https://doi.org/10.1017/S0305004125101837 Published online by Cambridge University Press