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Integrals with a large parameter: Hilbert transforms

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Abstract

A simple and elegant method of treating one-sided Hilbert transforms

$$Hf(x) = \int_0^\infty \frac{f(t)}{t-x}dt = \lim_{\epsilon\to 0}\left(\int_0^{x-\epsilon} + \int_{x+\epsilon}^0\right)\frac{f(t)}{t-x}dt,$$

has recently been described by Wong (7). This method immediately gives the form of the expansion but some of the coefficients in the expansion are in an inconvenient form for either analytical or numerical calculation. In the present paper it is shown that these coefficients can be readily determined when the Mellin transform of $f(t)$ is known. The case $f(t) = J_0^2(t)$ is treated as an example.

1. *Introduction*

Recent work on water waves (Hulme (2)) involved one-sided Hilbert transforms like $\int_0^\infty \frac{J_0^2(t)}{t-x}\,dt$ where $J_0(t)$ is the Bessel function of order 0, and where $\int_0^\infty$ denotes the Cauchy Principal Value $\lim_{\epsilon\to 0}\left(\int_0^{x-\epsilon} + \int_{x+\epsilon}^0\right)$. Asymptotic expansions for large positive x were required. There are at least two well-known methods for finding asymptotic expansions for integrals of the form $\int_0^\infty \frac{f(t)}{t-x}\,dt$, (Bleistein and Handelsman (1); Wong (7), see also Lauwerier ((3), § 19),) but neither of these is free from difficulties when $f(t) = J_0^2(t)$. In the present paper we shall be concerned with Wong's method, the method of Bleistein and Handelsman will be briefly discussed in § 6 below. Wong's paper (7) does not cover the case $f(t) = J_0^2(t)$, but it is not difficult to generalize his method so as to obtain the form of the complete asymptotic expansion. The difficulty is, as Wong points out, that some of the coefficients (see equation (2·4) below) are in an inconvenient form for either analytical or numerical calculation. The present paper is concerned with these coefficients; it will be shown that they can be readily determined when the Mellin transform of $f(t)$ is known.

2. *The form of the expansion*

The functions $f(t)$ will be assumed to be analytic for real $t, 0 \leqslant t < \infty$. It will also be assumed that, when $t \to +\infty$,

$$f(t) \sim \sum_1^\infty \frac{a_n}{t^n} + \cos\omega t \sum_1^\infty \frac{A_n}{t^n} + \sin\omega t \sum_1^\infty \frac{B_n}{t^n}, \tag{2·1}$$

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where the coefficients a_n, A_n, B_n are regarded as known, and where $\omega > 0$. This is slightly more general than the cases treated by Wong, but his method remains applicable. Let us write $\psi_0(t) = f(t)$,

$$\psi_k(t) = f(t) - \sum_1^k \frac{a_n}{t^n} - \cos\omega t \sum_1^k \frac{A_n}{t^n} - \sin\omega t \sum_1^k \frac{B_n}{t^n} \quad (k \geqslant 1). \tag{2·2}$$

where we note that $\psi_k(t) = O(t^{-k-1})$ when $t \to \infty$. Then it is not difficult to obtain the identity

$$Hf(x) = \int_0^\infty \frac{f(t)}{t-x}dt = -\sum_1^m \frac{c_n}{x^n} - \log x \sum_1^m \frac{a_n}{x^n} + \left(\sum_1^m \frac{A_n}{x^n}\right)\int_0^\infty \frac{\cos\omega t}{t-x}dt + \left(\sum_1^m \frac{B_n}{x^n}\right)\int_0^\infty \frac{\sin\omega t}{t-x}dt + \frac{1}{x^n}\int_0^\infty \frac{t^m\psi_m(t)}{t-x}dt, \tag{2·3}$$

where $$c_n = \int_0^1 t^{n-1}\psi_{n-1}(t)\,dt + \int_1^\infty t^{n-1}\left(\psi_{n-1}(t) - \frac{a_n}{t^n}\right)dt \quad \text{when} \quad n \geqslant 1, \tag{2·4}$$

and where the last term in (2·3) is a remainder term which is discussed in detail by Wong. In the present paper we shall be concerned only with the determination of the coefficients c_n, and we shall accordingly write (2·3) in the simplified form

$$Hf(x) \sim -\sum_1^\infty \frac{c_n}{x^n} - \log x \sum_1^\infty \frac{a_n}{x^n} + \left(\sum_1^\infty \frac{A_n}{x^n}\right)\int_0^\infty \frac{\cos\omega t}{t-x}dt + \left(\sum_1^\infty \frac{B_n}{x^n}\right)\int_0^\infty \frac{\sin\omega t}{t-x}dt \quad \text{when} \quad x \to \infty. \tag{2·5}$$

We shall need the asymptotic expansions of $\int_0^\infty \frac{\cos\omega t}{t-x}dt$ and $\int_0^\infty \frac{\sin\omega t}{t-x}dt$ which are given in

LEMMA 1.
$$\int_0^\infty \frac{\cos\omega t}{t-x}dt \sim -\pi\sin\omega x - \sum_0^\infty \frac{m!}{(\omega x)^{m+1}}\cos(m+1)\frac{\pi}{2}, \tag{2·6}$$

$$\int_0^\infty \frac{\sin\omega t}{t-x}dt \sim \pi\cos\omega x - \sum_0^\infty \frac{m!}{(\omega x)^{m+1}}\sin(m+1)\frac{\pi}{2}. \tag{2·7}$$

Proof. Evidently

$$\int_0^\infty \frac{\exp(i\omega t)}{t-x}dt = \pi i\exp(i\omega x) + \int_0^\infty \frac{\exp(i\omega t)}{t-x}dt, \tag{2·8}$$

where in the last integral the contour of integration passes above the pole $t = x$. Let the contour be deformed into the positive imaginary axis on which $t = iu$, with $u > 0$. Then

$$\int_0^\infty \frac{\exp(i\omega t)}{t-x}dt = -i\int_0^\infty \frac{\exp(-\omega u)}{x-iu}du$$
$$\sim -\sum_0^\infty \frac{i^{m+1}}{x^{m+1}}\int_0^\infty u^m\exp(-\omega u)\,du, \quad \text{by Watson's Lemma ((5), §8·3)}$$
$$= -\sum_0^\infty \frac{m!}{(\omega x)^{m+1}}\exp\left(i(m+1)\frac{\pi}{2}\right).$$

The result follows from (2·8) on taking real and imaginary parts.

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3. *Determination of the coefficients* c_n

The form (2·4) of the coefficients c_n is complicated but it will now be shown that these coefficients can be readily expressed in terms of the Mellin transform

$$\mathfrak{F}(\rho) = \int_0^\infty t^{\rho-1} f(t)\, dt.$$

From our assumptions (2·1) on $f(t)$ it can be shown (see (1), sections 4·2 and 4·3) that $\mathfrak{F}(\rho)$ is analytic in the strip $0 < \operatorname{Re}\rho < 1$ and meromorphic (i.e. analytic except for poles) in the whole ρ-plane. If $a_k \neq 0$ for a certain integer value of k, then it can be shown that $\mathfrak{F}(\rho)$ has a simple pole at $\rho = k$, with residue $-a_k$; if $a_k = 0$, then $\mathfrak{F}(\rho)$ is analytic at $\rho = k$.

Let us define the numbers d_k by

$$d_k = \lim_{\rho \to k} \left(\mathfrak{F}(\rho) + \frac{a_k}{\rho - k} \right); \tag{3·1}$$

thus $d_k = \mathfrak{F}(k)$ when $a_k = 0$. Our principal results are stated in Theorem 1 and Theorem 2.

THEOREM 1. *The coefficients* c_n $(n = 1, 2, 3, \ldots)$ *are given by the equations*

$$c_n = d_n - \sum_{k=1}^{n-1} \frac{\Gamma(n-k)}{\omega^{n-k}} (A_k \cos \tfrac{1}{2}\pi(n-k) + B_k \sin \tfrac{1}{2}\pi(n-k)), \tag{3·2}$$

where in this and in later equations the series is absent when $n = 1$.

The proof of this theorem will be given later. On substituting (3·2) in (2·5) and using (2·6) and (2·7) we immediately have the following theorem.

THEOREM 2.

$$Hf(x) = \int_0^\infty \frac{f(t)}{t-x}\, dt \sim -\sum_1^\infty \frac{d_n}{x^n} - \log x \sum_1^\infty \frac{a_n}{x^n} - \pi \sin \omega x \sum_1^\infty \frac{A_n}{x^n} + \pi \cos \omega x \sum_1^\infty \frac{B_n}{x^n}, \tag{3·3}$$

where d_n *is given by* (3·1).

4. *Proof of Theorem* 1

We now proceed to the proof of Theorem 1 and begin by defining the Mellin transforms $\mathfrak{F}_1(\rho)$, $\mathfrak{F}_2(\rho)$, $M_1(\rho, n)$, $M_2(\rho, n)$ by the equations

$$\mathfrak{F}_1(\rho) = \int_0^1 t^{\rho-1} f(t)\, dt, \quad \mathfrak{F}_2(\rho) = \int_1^\infty t^{\rho-1} f(t)\, dt, \tag{4·1}$$

$$\begin{aligned} M_1(\rho, n) &= \int_0^1 t^{\rho-1} \psi_{n-1}(t)\, dt \\ &= \int_0^1 t^{\rho-1} \left(f(t) - \sum_1^{n-1} \frac{a_m}{t^m} - \cos \omega t \sum_1^{n-1} \frac{A_m}{t^m} - \sin \omega t \sum_1^{n-1} \frac{B_m}{t^m} \right) dt, \end{aligned} \tag{4·2}$$

$$\begin{aligned} M_2(\rho, n) &= \int_1^\infty t^{\rho-1} \left(\psi_{n-1}(t) - \frac{a_n}{t^n} \right) dt \\ &= \int_1^\infty t^{\rho-1} \left(f(t) - \sum_1^{n} \frac{a_m}{t^m} - \cos \omega t \sum_1^{n-1} \frac{A_m}{t^m} - \sin \omega t \sum_1^{n-1} \frac{B_m}{t^m} \right) dt. \end{aligned} \tag{4·3}$$

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We note that

$$c_n = M_1(n,n) + M_2(n,n). \tag{4.4}$$

It is readily seen that

$\mathfrak{F}_1(\rho)$ is analytic when $\mathrm{Re}\,\rho > 0$,

$\mathfrak{F}_2(\rho)$ is analytic when $\mathrm{Re}\,\rho < 1$,

$M_1(\rho, n)$ is analytic when $\mathrm{Re}\,\rho > n-1$,

and $M_2(\rho, n)$ is analytic when $\mathrm{Re}\,\rho < n+1$.

Then we can prove the following

LEMMA 2. *$\mathfrak{F}_2(\rho)$ can be continued as a meromorphic function into the half-plane $\mathrm{Re}\,\rho > 0$, and in this half-plane we have $\mathfrak{F}_1(\rho) + \mathfrak{F}_2(\rho) = \mathfrak{F}(\rho)$.*

Proof. When $0 < \mathrm{Re}\,\rho < 1$ we have $\mathfrak{F}_2(\rho) = \mathfrak{F}(\rho) - \mathfrak{F}_1(\rho)$, but $\mathfrak{F}(\rho)$ is meromorphic in the whole ρ-plane, and $\mathfrak{F}_1(\rho)$ is analytic when $\mathrm{Re}\,\rho > 0$. Thus the relation

$$\mathfrak{F}_2(\rho) = \mathfrak{F}(\rho) - \mathfrak{F}_1(\rho)$$

provides an explicit analytic continuation into the half-plane $\mathrm{Re}\,\rho > 0$, as was to be shown.

We now consider the functions $M_1(\rho, n)$ and $M_2(\rho, n)$. Consider first the function $M_1(\rho, n)$. By term-by-term integration of (4·2) we find, when $\mathrm{Re}\,\rho > n-1$, that

$$M_1(\rho,n) = \int_0^1 t^{\rho-1} f(t)\,dt - \sum_1^{n-1} a_k \int_0^1 t^{\rho-1-k}\,dt - \sum_1^{n-1} A_k \int_0^1 t^{\rho-1-k}\cos\omega t\,dt - \sum_1^{n-1} B_k \int_0^1 t^{\rho-1-k}\sin\omega t\,dt,$$

i.e.
$$M_1(\rho,n) = \mathfrak{F}_1(\rho) - \sum_1^{n-1} \frac{a_k}{\rho-k} - \sum_1^{n-1} \tfrac{1}{2}(A_k + iB_k)\int_0^1 t^{\rho-1-k} e^{-i\omega t}\,dt - \sum_1^{n-1} \tfrac{1}{2}(A_k - iB_k)\int_0^1 t^{\rho-1-k} e^{i\omega t}\,dt. \tag{4.5}$$

Consider next the function $M_2(\rho, n)$ which is evidently analytic when $\mathrm{Re}\,\rho < n+1$. However, term-by-term integration of (4·3) is permissible only in the smaller region $\mathrm{Re}\,\rho < 1$, where we find that

$$M_2(\rho,n) = \mathfrak{F}_2(\rho) + \sum_1^{n} \frac{a_k}{\rho-k} - \sum_1^{n-1} \tfrac{1}{2}(A_k + iB_k)\int_1^\infty t^{\rho-1-k} e^{-i\omega t}\,dt - \sum_1^{n-1} \tfrac{1}{2}(A_k - iB_k)\int_1^\infty t^{\rho-1-k} e^{i\omega t}\,dt. \tag{4.6}$$

In (4·6) we replace $\int_1^\infty t^{\rho-1-k} e^{-i\omega t}\,dt$ by $\int_1^{-i\infty} t^{\rho-1-k} e^{-i\omega t}\,dt$, and $\int_1^\infty t^{\rho-1-k} e^{i\omega t}\,dt$ by $\int_1^{i\infty} t^{\rho-1-k} e^{i\omega t}\,dt$; these changes in the contours of integration leave the values of the integrals unchanged when $\mathrm{Re}\,\rho < 1$. We now note that the modified integrals are

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analytic for all ρ; also, by Lemma 2, $\mathfrak{F}_2(\rho)$ can be continued into the half-plane $\mathrm{Re}\,\rho > 0$, while the sum in (4·6) is evidently meromorphic. Thus, when $\mathrm{Re}\,\rho > 0$,

$$M_2(\rho,n) = \mathfrak{F}_2(\rho)+\sum_1^n \frac{a_k}{\rho-k} - \sum_1^{n-1} \tfrac{1}{2}(A_k+iB_k)\int_1^{-i\infty} t^{\rho-1-k}e^{-i\omega t}\,dt$$
$$- \sum_1^{n-1} \tfrac{1}{2}(A_k-iB_k)\int_1^{i\infty} t^{\rho-1-k}e^{i\omega t}\,dt. \quad (4\cdot7)$$

On adding (4·5) and (4·7) and noting that $\mathfrak{F}_1(\rho)+\mathfrak{F}_2(\rho) = \mathfrak{F}(\rho)$, by Lemma 2, we see that, when $\mathrm{Re}\,\rho > n-1$,

$$M_1(\rho,n)+M_2(\rho,n) = \mathfrak{F}(\rho)+\frac{a_n}{\rho-n} - \sum_1^{n-1} \tfrac{1}{2}(A_k+iB_k)\int_1^{-i\infty} t^{\rho-1-k}e^{-i\omega t}\,dt$$
$$- \sum_1^{n-1} \tfrac{1}{2}(A_k-iB_k)\int_0^{i\infty} t^{\rho-1-k}e^{i\omega t}\,dt.$$

The integrals are obviously related to $\int_0^\infty u^{\rho-1-k}e^{-\omega u}\,du = \dfrac{\Gamma(\rho-k)}{\omega^{\rho-k}}$, in fact

$$\int_0^{i\infty} t^{\rho-1-k}e^{i\omega t}\,dt = e^{\frac{1}{2}\pi i(\rho-k)}\frac{\Gamma(\rho-k)}{\omega^{\rho-k}} \quad \text{and} \quad \int_0^{-i\infty} t^{\rho-1-k}e^{-i\omega t}\,dt = e^{-\frac{1}{2}\pi i(\rho-k)}\frac{\Gamma(\rho-k)}{\omega^{\rho-k}}.$$

Thus, when $\mathrm{Re}\,\rho > n-1$,

$$M_1(\rho,n)+M_2(\rho,n) = \mathfrak{F}(\rho)+\frac{a_n}{\rho-n} - \sum_1^{n-1}\frac{\Gamma(\rho-k)}{\omega^{\rho-k}}$$
$$\times (A_k\cos\tfrac{1}{2}\pi(\rho-k)+B_k\sin\tfrac{1}{2}\pi(\rho-k)). \quad (4\cdot8)$$

Since, by equation (4·4), $c_n = M_1(n,n)+M_2(n,n)$, Theorem 1 follows at once from equation (4·8).

As has already been noted, Theorem 2 follows immediately when the value of c_n is substituted in (2·5) and use is made of (2·6) and (2·7).

5. *An example*

We take $f(t) = J_0^2(t)$, where $J_0(t)$ is the Bessel function of order 0; this is the function mentioned in the introduction above. An earlier treatment on different lines was given by Luke (4). It is known ((5), § 7·1) that

$$J_0(t) = \left(\frac{2}{\pi t}\right)^{\frac{1}{2}}\left\{\cos(t-\tfrac{1}{4}\pi)\left(1-\frac{9}{128t^2}\right)+\sin(t-\tfrac{1}{4}\pi)\left(\frac{1}{8t}\right)+O\left(\frac{1}{t^3}\right)\right\},$$

whence
$$J_0^2(t) = \frac{1}{\pi t}-\frac{1}{8\pi t^3}-\cos 2t\left(\frac{1}{4\pi t^2}\right)+\sin 2t\left(\frac{1}{\pi t}-\frac{5}{32\pi t^3}\right)+O\left(\frac{1}{t^4}\right). \quad (5\cdot1)$$

Comparing with (2·1) we see that $\omega = 2$, and

$$\left.\begin{aligned} &a_1 = \frac{1}{\pi}, \quad a_2 = 0, \quad a_3 = -\frac{1}{8\pi};\\ &A_1 = 0, \quad A_2 = -\frac{1}{4\pi}, \quad A_3 = 0;\\ &B_1 = \frac{1}{\pi}, \quad B_2 = 0, \quad B_3 = -\frac{5}{32\pi}. \end{aligned}\right\} \quad (5\cdot2)$$

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It appears ((5), § 7·51) that no explicit expressions are known for the general coefficients A_{2m}, B_{2m+1}, but we shall see below that the coefficients a_m can be found; cf. also ((5), § 13·75).

The Mellin transform

$$\mathfrak{F}(\rho) = \int_0^\infty t^{\rho-1} J_0^2(t)\, dt = \frac{2^{\rho-1}\Gamma(\frac{1}{2}\rho)\,\Gamma(1-\rho)}{\{\Gamma(1-\frac{1}{2}\rho)\}^3}, \tag{5·3}$$

see ((5), § 13·41(2)). It is readily seen that, when $\mathrm{Re}\,\rho > 0$, $\mathfrak{F}(\rho)$ is meromorphic with simple poles at the positive odd integers $\rho = 2k+1$, and that $\mathfrak{F}(\rho)$ vanishes at the positive even integers $\rho = 2k$. By straightforward calculations we find that

$$a_{2k} = 0, \quad a_{2k+1} = \frac{1}{\pi^{\frac{3}{2}}} \cos \pi k \frac{\{\Gamma(k+\frac{1}{2})\}^3}{\Gamma(k+1)},$$

and that
$$d_{2k+1} = \lim_{\rho \to 2k+1} \left(\mathfrak{F}(\rho) + \frac{a_{2k+1}}{\rho - 2k - 1} \right)$$
$$= \frac{1}{\pi^{\frac{3}{2}}} \cos \pi k \frac{\{\Gamma(k+\frac{1}{2})\}^3}{\Gamma(k+1)} (3\psi(2k+1) - 2\psi(k+1) - 3\log 2),$$

where $\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n-1}$ when $n \geqslant 2$.

In particular, $d_1 = \frac{1}{\pi}(\gamma + 3\log 2)$ and $d_3 = -\frac{1}{8\pi}(\gamma + 3\log 2 - \frac{5}{2})$.

Thus
$$\int_0^\infty \frac{J_0^2(t)}{t-x} dt \sim -\frac{1}{\pi x}\log x - \frac{1}{\pi x}(\gamma + 3\log 2) + \frac{1}{x}\cos 2x + \frac{1}{4x^2}\sin 2x$$
$$+ \frac{1}{8\pi x^3}\log x + \frac{1}{8\pi x^3}(\gamma + 3\log 2 - \tfrac{5}{2}) - \frac{5}{32x^3}\cos 2x. \tag{5·4}$$

This expansion is valid for large x. We can find an expansion valid for small x by using the expression (6·1) below and moving the contour to the left,

$$\int_0^\infty \frac{J_0^2(t)}{t-x} dt = -\tfrac{1}{2}\pi J_0(x)\, Y_0(x) - \pi^{\frac{1}{2}} \sum_{k=0}^{\infty} \cos \pi k \frac{\Gamma(k+1)}{\{\Gamma(k+\frac{3}{2})\}^3} x^{2k+1}. \tag{5·5}$$

The expressions (5·4) and (5·5) have been computed by Dr A. Hulme for integer values x from 1 to 38 and are shown in Table 1. When $x = 5$ it is seen that the two expressions agree to within one part in 10^4, when $x = 11$ to one part in 10^5, and that the agreement improves up to $x = 30$ when the series (5·5) became numerically unsatisfactory.

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x	(I)	(II)	(I)–(II)
1	−1·0471195	−0·9632066	−0·0839129
2	−0·8882325	−0·8902668	0·0020343
3	−0·0888402	−0·0898580	0·0010177
4	−0·3419508	−0·3413297	−0·0006212
5	−0·4431789	−0·4432316	0·0000527
6	−0·0992537	−0·0993312	0·0000775
7	−0·1845895	−0·1845222	−0·0000673
8	−0·3088080	−0·3088094	0·0000014
9	−0·1006143	−0·1006333	0·0000190
10	−0·1147476	−0·1147329	−0·0000147
11	−0·2369971	−0·2369950	−0·0000021
12	−0·1025787	−0·1025857	0·0000069
13	−0·0769629	−0·0769586	−0·0000042
14	−0·1887241	−0·1887223	−0·0000018
15	−0·1046274	−0·1046304	0·0000030
16	−0·0553386	−0·0553373	−0·0000013
17	−0·1522012	−0·1522000	−0·0000013
18	−0·1059417	−0·1059431	0·0000014
19	−0·0433678	−0·0433675	−0·0000003
20	−0·1228139	−0·1228131	−0·0000009
21	−0·1059616	−0·1059623	0·0000007
22	−0·0377073	−0·0377073	0·0000000
23	−0·0985093	−0·0985087	−0·0000006
24	−0·1043742	−0·1043745	0·0000003
25	−0·0363169	−0·0363170	0·0000001
26	−0·0783077	−0·0783073	−0·0000004
27	−0·1010686	−0·1010687	0·0000001
28	−0·0377764	−0·0377766	0·0000002
29	−0·0617103	−0·0617103	0·0000001
30	−0·0960982	−0·0960968	−0·0000014
31	−0·0410171	−0·0410041	−0·0000130
32	−0·0485110	−0·0484279	−0·0000831
33	−0·0899581	−0·0896420	−0·0003161
34	−0·0458793	−0·0451326	−0·0007467
35	−0·0146145	−0·0382416	0·0236271
36	0·1653252	−0·0819880	0·2473131
37	−4·0519661	−0·0494562	−4·0025099
38	23·3353090	−0·0309329	23·3662419
39	**********	−0·0734900	**********
40	**********	−0·0534085	**********

Table 1. *Column* (I) *gives numerical values of the infinite series* (5·5), *column* (II) *gives numerical values of the finite series* (5·4). *The series* (5·5) *was found to become numerically unsatisfactory when* x *exceeds* 30.

6. *Discussion and notes*

It is not surprising that our Theorem 2 involves the Mellin transform $\mathfrak{F}(\rho)$ in a simple manner. In our argument we have used the property that $\mathfrak{F}(\rho)$ can be continued as a meromorphic function into the half-plane $\mathrm{Re}\,\rho > 0$, and one well-known proof of this property (see e.g. (1), ch. 4) is essentially our evaluation of $M_2(\rho, n)$ in reverse. In practice our result is most useful when $\mathfrak{F}(\rho)$ is known explicitly in closed form, as is often the case; see e.g. § 5 above.

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Bleistein and Handelsman (1) treat integral transforms by using the convolution theorem for Mellin transforms. The Mellin transform of $Hf(x)$ is

$$\int_0^\infty x^{\rho-1}\left(\int_0^\infty \frac{f(t)}{t-x}dt\right)dx = \int_0^\infty f(t)\left(\int_0^\infty \frac{x^{\rho-1}}{t-x}dx\right)dt$$
$$= \pi\cot\pi\rho\int_0^\infty t^{\rho-1}f(t)\,dt = \mathfrak{F}(\rho)\,\pi\cot\pi\rho,$$

whence

$$Hf(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} x^{-\rho}\mathfrak{F}(\rho)\,\pi\cot\pi\rho\,d\rho. \tag{6·1}$$

The integrand is meromorphic when $\mathrm{Re}\,\rho > 0$, with poles at some or all of the points $\rho = 1, 2, 3, \ldots$. If the contour of integration could be moved to the right then an asymptotic series involving x^{-n} and $x^{-n}\log x$ would be obtained, but not terms like $x^{-n}\cos\omega x$ and $x^{-n}\sin\omega x$. This shows that it is not usually permissible to move the contour to the right. It may however be permissible to do this after suitable terms have been subtracted from $\mathfrak{F}(\rho)$; the form of these terms can probably be found by taking the Mellin transform of the last term of equation (2·3) above but this question will not be pursued here.

The Hilbert transforms treated in the present paper are one-sided Hilbert transforms, with integration extending from $t = 0$ to $t = \infty$. A two-sided Hilbert transform can be written as the sum of two one-sided transforms,

$$\int_{-\infty}^\infty \frac{f(t)}{t-x}dt = \int_0^\infty \frac{f(t)}{t-x}dt - \int_0^\infty \frac{f(-t)}{t+x}dt,$$

when $x > 0$ (with a similar expression for $x < 0$) where the last integral is actually a Stieltjes transform which can be treated by the same method, or by the method of (1) even in some cases when this method fails for the Hilbert transform.

Wong (7) also treats functions $f(t)$ with asymptotic expansions involving non-integral powers. The methods of the present paper are readily extended to such functions. The following theorem is typical.

THEOREM 3. *Instead of* (2·1), *assume that*

$$f(t) \sim \sum_0^\infty \frac{a_n}{t^{n+\alpha}} + \cos\omega t\sum_0^\infty \frac{A_n}{t^{n+\beta}} + \sin\omega t\sum_0^\infty \frac{B_n}{t^{n+\beta}}, \quad \text{when} \quad t\to\infty, \tag{6·2}$$

where $0 < \alpha < 1$ and $0 < \beta < 1$. Then

$$\int_0^\infty \frac{f(t)}{t-x}dt \sim -\sum_1^\infty \frac{1}{x^n}\mathfrak{F}(n) + \pi\cot\pi\alpha\sum_0^\infty \frac{a_n}{x^{n+\alpha}} - \pi\sin\omega x\sum_0^\infty \frac{A_n}{x^{n+\beta}} + \pi\cos\omega x\sum_0^\infty \frac{B_n}{x^{n+\beta}}.$$

The proof is similar to the proof of Theorem 2 and will not be given. When the terms involving $\sin\omega t$ and $\cos\omega t$ are absent in (6·2) this result was obtained by Wong (6). Similar methods are applicable for small x and give the same results as (6·1) above.

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