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A QUEUE WITH POSITIVE AND NEGATIVE ARRIVALS GOVERNED BY A MARKOV CHAIN

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We consider a single-server queue with exponential service time and two types of arrivals: positive and negative. Positive customers are regular ones who form a queue and a negative arrival has the effect of removing a positive customer in the system. In many applications, it might be more appropriate to assume the dependence between positive arrival and negative arrival. In order to reflect the dependence, we assume that the positive arrivals and negative arrivals are governed by a finite-state Markov chain with two absorbing states, say 0 and 0′. The epoch of absorption to the states 0 and 0′ corresponds to an arrival of positive and negative customers, respectively. The Markov chain is then instantly restarted in a transient state, where the selection of the new state is allowed to depend on the state from which absorption occurred.

The Laplace–Stieltjes transforms (LSTs) of the sojourn time distribution of a customer, jointly with the probability that the customer completes his service without being removed, are derived under the combinations of service disciplines FCFS and LCFS and the removal strategies RCE and RCH. The service distribution of phase type is also considered.

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1. INTRODUCTION

We consider a queue with two types of arrival: positive and negative. Positive customers are ordinary ones who, upon arrival, join a queue with the intention of being served. Each arrival of a negative customer removes a positive customer in the system, if any present, and then disappears immediately; otherwise the negative customer does not affect the system behavior. The queue with negative arrivals is called a *G*-queue. There is an extensive body of research on the *G*-queue and its applications since the notion of negative customer has been introduced by Gelenbe [5]. For the survey and comprehensive references, see Artalejo [1].

In many applications, it might be more appropriate to assume the dependence between positive arrival and negative arrival. For example, in the computer system, electronic mails or files can be regarded as ordinary customers and the ones infected by virus may be regarded as negative customers. In this case, more e-mails induce more infected mail.

To our knowledge, the *G*-queue in which positive and negative arrivals are dependent has not been addressed previously. In order to reflect the dependence, we assume that the positive arrivals and negative arrivals are governed by a finite-state Markov chain with two absorbing states, say 0 and 0′. The epoch of absorption to the states 0 and 0′ corresponds to an arrival of positive and negative customers, respectively. The Markov chain is then instantly restarted in a transient state, where the selection of the new state is allowed to depend on the state from which absorption occurred.

By using the first passage time argument, we derive sojourn time distribution under the combinations of two different types of removal policy, RCE (removal of customer at the end of the queue) and RCH (removal of the customer at the head of the system—i.e., the one in service), and two service disciplines, FCFS (first-come-first-served) and preemptive LCFS (last-come-first-served).

As related works with our model, Harrison and Pitel [7] derived the Laplace–Stieltjes (LSTs) of the sojourn time distributions, jointly with the probability that the tagged customer is not removed by the negative customer in the $M/M/1$ *G*-queue under the combinations of various queuing disciplines and removal strategies. Shin [15] dealt with the waiting time distribution in the $M/M/c$ *G*-queue and in the Markovian *G*-queue with batch arrival and batch removal [16]. The negative customer that removes all of the customers in the system upon its arrival is called the disaster or catastrophe. The queue and queuing network with catastrophes are introduced and analyzed by Chao [3]. For the waiting time or workload distribution in the queue with catastrophes, see, for instance, Boucherie and Boxma [2], Dudin and Nishimura [4], and Jain and Sigman [10,11].

This article is organized as follows. We describe the model in detail and derive the queue length distribution in Section 2. The LSTs of sojourn time distributions under combinations of service disciplines FCFS and LCFS and removal strategies RCH and RCE are derived in Sections 3–6. In Section 7, the sojourn time distributions for the queue with the service time distribution of phase type are presented.

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2. MODEL DESCRIPTION AND PRELIMINARY RESULTS

In this section, we describe the model in detail and derive the queue length distribution at an arbitrary time, and in Sections 3–6, we describe the arrival epochs of positive customers.

We consider a single-sever queue with exponential service time $\tilde{S}$ with rate μ and two types of arrival: positive and negative. For the description of the arrival process, we follow the same line as the description of MAP (Markovian arrival process) in Lucantoni, Meier-Hellstern, and Neuts [12]. Consider a continuous-time Markov chain on the state space $\{0,0',1,2,\ldots,m\}$, where 0 and $0'$ are absorbing states and $\{1,2,\ldots,m\}$ are transient. The arrival process is constructively defined as follows. When the chain enters a transient state i, $1 \le i \le m$, it stays for an exponential time with parameter λ_i. At the end of that sojourn, there are three possible transitions. With probabilities p_{ij}, $1 \le j \le m$, the chain enters the absorbing state 0 and is instantaneously restarted in the transient state j and a positive arrival occurs. With probabilities q_{ij}, $1 \le j \le m$, the chain enters the absorbing state $0'$ and is immediately restarted in the transient state j and a negative arrival occurs. With probability r_{ij}, $1 \le j \le m$, $j \ne i$, the chain enters the transient state j and no arrival occurs. We define the matrices D_k, $k = -1,0,1$, of size m by $[D_0]_{ii} = -\lambda_i$ and $[D_0]_{ij} = \lambda_i r_{ij}$, $i \ne j$, and $[D_1]_{ij} = \lambda_i p_{ij}$ and $[D_{-1}]_{ij} = \lambda_i q_{ij}$, $1 \le i \le m$, $1 \le j \le m$. It is easily seen that $D = D_{-1} + D_0 + D_1$ is an infinitesimal generator of a continuous-time Markov chain. The arrival process of positive and negative customers described earlier is a special kind of marked Markovian arrival process (MMAP) with representation (D_0, D_{-1}, D_1) introduced by He and Neuts [9]. We assume that D is irreducible and $D_0\mathbf{1} \ne \mathbf{0}$, where $\mathbf{1}$ is the column vector whose components are all 1's. Let $\boldsymbol{\pi}$ be the stationary probability vector of D; that is, $\boldsymbol{\pi}D = 0$ and $\boldsymbol{\pi}\mathbf{1} = 1$.

Now, we consider the counting process for positive and negative customers. Let $N^+(t)$ and $N^-(t)$ be the number of positive arrivals and negative arrivals, respectively, in time $(0, t]$ and $J(t)$ be the arrival phase at time t. Let

$$P_{jj'}(n_1, n_2, t) = P\big(N^+(t) = n_1, N^-(t) = n_2, \\ J(t) = j' \,|\, N^+(0) = 0, N^-(0) = 0, J(0) = j\big)$$

be the (j, j') element of the $m \times m$ matrix $P(n_1, n_2, t)$. Then, $P(n_1, n_2, t)$ satisfies the Kolmogorov equation; for $n_1 \ge 0$, $n_2 \ge 0$, and $t \ge 0$,

$$\frac{d}{dt} P(n_1, n_2, t) = P(n_1, n_2, t)D_0 + P(n_1 - 1, n_2, t)D_1 + P(n_1, n_2 - 1, t)D_{-1},$$

with $P(-1, n_2, t) = P(n_1, -1, t) = 0$ and $P(0,0,0) = I$, the identity matrix. The matrix generating function $P^*(z, w, t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} z^{n_1} w^{n_2} P(n_1, n_2, t)$ is given by

$$P^*(z, w, t) = \exp[D_0 + zD_1 t + wD_{-1} t], \qquad |z| \le 1, |w| \le 1, t \ge 0. \tag{2.1}$$

It is noted that $(N^+, J) = \{(N^+(t), J(t)), t \ge 0\}$, $(N^-, J) = \{(N^-(t), J(t)), t \ge 0\}$ and $(N, J) = \{(N^+(t) + N^-(t), J(t)), t \ge 0\}$ are counting processes corresponding to the

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Markovian arrival processes (e.g., see Lucantoni et al. [12]) with representations $(D_0 + D_{-1}, D_1)$, $(D_0 + D_1, D_{-1})$, and $(D_0, D_1 + D_{-1})$ respectively.

Following the procedure of He and Neuts [9] (or Narayana and Neuts [13]), the covariance of $N^+(t)$ and $N^-(t)$ with initial distribution $\boldsymbol{\pi}$ of arrival phase is given by

$$\begin{aligned}\mathrm{Cov}(N^+(t), N^-(t)) = {} & [\boldsymbol{\pi} D_1(\mathbf{1}\cdot\boldsymbol{\pi} - D)^{-1} D_{-1}\mathbf{1} + \boldsymbol{\pi} D_{-1}(\mathbf{1}\cdot\boldsymbol{\pi} - D)^{-1} D_1 \mathbf{1}]t \\ & - 2(\boldsymbol{\pi} D_1 \mathbf{1})(\boldsymbol{\pi} D_{-1}\mathbf{1})t \\ & - \boldsymbol{\pi} D_1 (I - \exp[Dt])(\mathbf{1}\cdot\boldsymbol{\pi} - D)^{-2} D_{-1}\mathbf{1} \\ & - \boldsymbol{\pi} D_{-1}(I - \exp[Dt])(\mathbf{1}\cdot\boldsymbol{\pi} - D)^{-2} D_1 \mathbf{1}.\end{aligned}$$

Remark 1: If $m = 1$, then $D_0 = -\lambda$, $D_1 = \lambda p$, and $D_{-1} = \lambda(1-p)$ with $0 < p < 1$ and, hence, the generating function $P^*(z, w, t)$ in (2.1) becomes

$$P^*(z, w, t) = e^{-\lambda p(1-z)t} e^{-\lambda(1-p)(1-w)t}.$$

In this case, $N^+(t)$ and $N^-(t)$ are independent. Thus, the dependency between $N^+(t)$ and $N^-(t)$ can occur only in the case of $m > 1$.

Remark 2: If a positive arrival and negative arrival occur according to independent MAPs with representations (C_1, D_1) and (C_2, D_2), respectively, where C_i and D_i are square matrices of order m_i, $i = 1, 2$, then the arrival processes of positive and negative customers are MMAPs with representation $(C_1 \oplus C_2, D_1 \otimes I, I \otimes D_2)$, where $\otimes$ and $\oplus$ denote the Kronecker product and Kronecker sum (see Graham [6]), respectively.

Let $\xi(t)$ be the number of positive customers in the system at time t. Then, the process $(\xi, J) = \{(\xi(t), J(t)), t \geq 0\}$ is a quasi-birth-and-death (QBD) process with generator Q of the block tridiagonal matrix form

$$Q = \begin{pmatrix} D_{-1} + D_0 & D_1 & & & & O \\ D_{-1} + \mu I & D_0 - \mu I & D_1 & & & \\ & D_{-1} + \mu I & D_0 - \mu I & D_1 & & \\ & & D_{-1} + \mu I & D_0 - \mu I & D_1 & \\ & & & \ddots & \ddots & \ddots \\ O & & & & & \end{pmatrix}. \tag{2.2}$$

Note that by the memoryless property of exponential distribution, the stationary queue length distribution in this system is invariant under the removal strategies and service disciplines FSFS and LCFS. Neuts [14] provided the detailed analysis of the QBD process and the rest of this section is immediately obtained from there.

The necessary and sufficient condition for the QBD process (ξ, J) to be positive recurrent is that

$$\boldsymbol{\pi} D_{-1}\mathbf{1} + \mu > \boldsymbol{\pi} D_1 \mathbf{1}. \tag{2.3}$$

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The stationary distribution $\boldsymbol{x} = [\boldsymbol{x_0}, \boldsymbol{x_1}, \ldots]$ with $\boldsymbol{x_i} = (x_{i1}, \ldots, x_{im})$, $i = 0,1,2,\ldots$, of Q is given by

$$\boldsymbol{x_i} = \boldsymbol{x_0} R^i, \qquad i = 0,1,2,\ldots, \tag{2.4}$$

with

$$\boldsymbol{x_0}((I+R)D_{-1} + D_0 + \mu R) = 0 \qquad \text{and} \qquad \boldsymbol{x_0}(I-R)^{-1}\mathbf{1} = 1,$$

where the $m \times m$ matrix R is the minimal nonnegative solution of the matrix equation

$$D_1 + R(D_0 - \mu I) + R^2(D_{-1} + \mu I) = 0. \tag{2.5}$$

Now, we consider the queue length distribution at the arrival epoch for later use. Let T_k be the epoch of the kth positive arrival and let $\xi_k = \xi(T_k - 0)$ and $J_k = J(T_k + 0)$, $k = 0,1,2,\ldots$ with $T_0 = 0$. Then, the transition probability matrix $\tilde{P}$ of $\{(\xi_k, J_k), k = 0,1,2,\ldots\}$ is of the form

$$\tilde{P} = \begin{pmatrix} \tilde{B}_0 & \tilde{A}_0 & & & \\ \tilde{B}_1 & \tilde{A}_1 & \tilde{A}_0 & & \\ \tilde{B}_2 & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.6}$$

where

$$\tilde{B}_k = [-(D_0 - \mu I)^{-1}(D_{-1} + \mu I)]^{k+1}[-(D_0 + D_{-1})^{-1}D_1], \qquad k = 0,1,2,\ldots,$$
$$\tilde{A}_k = [-(D_0 - \mu I)^{-1}(D_{-1} + \mu I)]^{k}[-(D_0 - \mu I)^{-1}D_1], \qquad k = 0,1,2,\ldots.$$

Under stability condition (2.3), the invariant probability vector $\boldsymbol{y} = [\boldsymbol{y_0}, \boldsymbol{y_1}, \ldots]$ with $\boldsymbol{y_i} = (y_{i1}, \ldots, y_{im})$, $i = 0,1,2,\ldots$, of $\tilde{P}$ is given by

$$\boldsymbol{y_i} = \boldsymbol{y_0} \tilde{R}^i, \qquad i = 0,1,2,\ldots, \tag{2.7}$$

with

$$\boldsymbol{y_0} B[\tilde{R}] = \boldsymbol{y_0} \qquad \text{and} \qquad \boldsymbol{y_0}(I - \tilde{R})^{-1}\mathbf{1} = 1,$$

where $\tilde{R}$ is the minimal nonnegative solution of the matrix equation

$$\tilde{R} = \sum_{n=0}^{\infty} \tilde{R}^n \tilde{A}_n \tag{2.8}$$

and $B[\tilde{R}] = \sum_{n=0}^{\infty} \tilde{R}^n \tilde{B}_n$.

3. RCH WITH FCFS DISCIPLINE

Let W denote the time period the tagged customer spends in the system from the epoch of arrival to the epoch of his service completion. We assume that W is infinite

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if the tagged customer is removed from the system before its service completion. Denote by (ξ_a, J_a) the pair of the number of customers in the system immediately before the tagged customer's arrival instant and the arrival phase immediately after the tagged customer's arrival. Under the RCH removal strategy, upon a negative arrival, a positive customer in service is removed and, hence, the customers behind the tagged customer do not affect the sojourn time of the tagged customer. When $\xi_a = n$, the sojourn time of the tagged customer is the time period that $n+1$ customers consecutively leave the system. In order for the tagged customer to complete his service, his service time has to be less than the interarrival time denoted by T^- of negative customers. It is easily seen that the matrix function $F^-(x) = (F^-_{jj'}(x))$ with (j, j') component

$$F^-_{jj'}(x) = P(T^- \le x, J(T^-) = j' \,|\, J(0) = j)$$

is given by

$$F^-(x) = (I - \exp[(D_0 + D_1)x])[-(D_0 + D_1)]^{-1} D_{-1}. \tag{3.1}$$

Thus, we have

$$\begin{aligned} W_{0j}(x) &\triangleq P(W \le x | \xi_a = 0, J_a = j) \\ &= P(\tilde{S} \le x, \tilde{S} < T^- | \xi_a = 0, J_a = j) \\ &= \mu \left[\int_0^x \exp((D_0 + D_1 - \mu I)t)\, dt\, \mathbf{1} \right]_j \\ &= \mu [(I - \exp((D_0 + D_1 - \mu I)x))(\mu I - D_0 - D_1)^{-1}\, \mathbf{1}]_j, \end{aligned} \tag{3.2}$$

where $[\cdot]_j$ is the jth component of the vector $[\cdot]$. The LST $\widetilde{W}_0(s)$ of the column m-vector $W_0(x) = (W_{01}(x), \ldots, W_{0m}(x))^t$ is given by

$$\widetilde{W}_0(s) = \mu(sI - (D_0 + D_1 - \mu I))^{-1}\, \mathbf{1}. \tag{3.3}$$

By conditioning the first departure time of positive customers and using the law of total probability, we have that for $n \ge 1$,

$$\begin{aligned} W_{nj}(x) &\triangleq P(W \le x | \xi_a = n, J_a = j) \\ &= \left[\int_0^x \exp((D_0 + D_1 - \mu I)t)(D_{-1} + \mu I) W_{n-1}(x - t)\, dt \right]_j \end{aligned} \tag{3.4}$$

and, hence, the LST $\widetilde{W}_n(s)$ of $W_n(x) = (W_{n1}(x), \ldots, W_{nm}(x))^t$ is given by

$$\widetilde{W}_n(s) = (sI - (D_0 + D_1 - \mu I))^{-1}(D_{-1} + \mu I)\widetilde{W}_{n-1}(s), \qquad n = 1, 2, \ldots. \tag{3.5}$$

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Finally, we get from (3.3) and (3.5) that for $n \geq 0$,

$$\widetilde{W}_n(s) = \mu[(sI - (D_0 + D_1 - \mu I))^{-1}(D_{-1} + \mu I)]^n (sI - (D_0 + D_1 - \mu I))^{-1}\mathbf{1} \quad \textbf{(3.6)}$$

and

$$\widetilde{W}(s) = \boldsymbol{y_0}\,\tilde{U}(s)(sI - (D_0 + D_1 - \mu I))^{-1}\mu\mathbf{1}, \qquad s \geq 0, \quad \textbf{(3.7)}$$

where

$$\tilde{U}(s) = \sum_{n=0}^{\infty} \tilde{R}^n[(sI - (D_0 + D_1 - \mu I))^{-1}(D_{-1} + \mu I)]^n.$$

It is noted that $\tilde{U}(s)$ satisfies the matrix equation

$$\tilde{U}(s) = I + \tilde{R}\tilde{U}(s)[sI - (D_0 + D_1 - \mu I)]^{-1}(D_{-1} + \mu I). \quad \textbf{(3.8)}$$

It follows from Neuts [14, pp. 132–138] that the inverse transform $U(x)$ of $\tilde{U}(s)$ is given by

$$U(x) = I + \tilde{R}(V^o - \Theta(x))(D_{-1} + \mu I), \qquad x \geq 0, \quad \textbf{(3.9)}$$

where the matrix V^o is the solution of the matrix equation

$$V^o(D_0 + D_1 - \mu I) + \tilde{R}V^o(D_{-1} + \mu I) = -I$$

and $\Theta(x)$ is the solution of the matrix-differential equation

$$\Theta'(x) = \Theta(x)(D_0 + D_1 - \mu I) + \tilde{R}\Theta(x)(D_{-1} + \mu I)$$

with the initial condition $\Theta(0) = V^o$. Thus, we obtain

$$W(x) = \int_0^x \boldsymbol{y_0}\,U(x - t)\exp[(D_0 + D_1 - \mu I)t]\mu\mathbf{1}\,dt, \qquad x \geq 0. \quad \textbf{(3.10)}$$

It can be seen from (3.10) that the sojourn time W is the sum, whose distribution function is $\boldsymbol{y_0}U(x)$, of the time period that the tagged customer is in queue and the time period that the customer in service completes his service without being removed.

The probability that the tagged customer successfully completes his service time without being removed is

$$\widetilde{W}(0) = \boldsymbol{y_0}\,U[-(D_0 + D_1 - \mu I)]^{-1}\mu\mathbf{1}, \quad \textbf{(3.11)}$$

where $U = \tilde{U}(0) = U(\infty)$ is given by the solution of the matrix equation

$$U = I + \tilde{R}U[-(D_0 + D_1 - \mu I)]^{-1}(D_{-1} + \mu I).$$

The conditional mean sojourn time given that the tagged customer completes his service time without being removed is $\overline{W}/\widetilde{W}(0)$, where $\overline{W} = -W'(0)$ is given from (3.7) and (3.8) by

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$$\overline{W} = \boldsymbol{y_0}\bar{U}[-(D_0 + D_1 - \mu I)]^{-1}\mu\mathbf{1} + \boldsymbol{y_0}U[-(D_0 + D_1 - \mu I)]^{-2}\mu\mathbf{1} \qquad \textbf{(3.12)}$$

and $\bar{U} = -\tilde{U}'(0)$ is the solution of the matrix equation

$$\bar{U} = \tilde{R}\bar{U}[-(D_0 + D_1 - \mu I)]^{-1}(D_{-1} + \mu I) + \tilde{R}U[-(D_0 + D_1 - \mu I)]^{-2}(D_{-1} + \mu I).$$

4. RCH WITH PREEMPTIVE LCFS DISCIPLINE

Here, the number of customers ahead of the tagged customer on his arrival is zero and the queue behind him does not affect his sojourn time. Thus, the sojourn time distribution is the same as that of the distribution of the absorption time T_a to the level $\tilde{\mathbf{0}}$ of the Markov chain (ξ^*, J^*) starting at level $\mathbf{0}$ with generator

$$Q^* = \begin{array}{c} \\ \tilde{\mathbf{0}} \\ \tilde{\mathbf{0}}' \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \tilde{\mathbf{0}} & \tilde{\mathbf{0}}' & \mathbf{0} & \mathbf{1} & \mathbf{2} & \cdots \end{array} \\ \begin{pmatrix} O & O & O & O & & \\ O & O & O & O & & \\ \mu I & D_{-1} & D_0 - \mu I & D_1 & & \\ O & O & D_{-1} + \mu I & D_0 - \mu I & D_1 & \\ O & O & & D_{-1} + \mu I & D_0 - \mu I & \\ & & & & \ddots & \ddots \end{pmatrix} \end{array}. \qquad \textbf{(4.1)}$$

Let $W_{nj}(x) = P(T_a \le x | \xi^*(0) = n, J^*(0) = j)$ and $W_n(x) = (W_{n1}(x), \ldots, W_{nm}(x))^t$, $n \ge 0$. Then, the sojourn time distribution $W(x)$ is given by

$$W(x) = \tilde{\boldsymbol{\pi}} W_0(x), \qquad \textbf{(4.2)}$$

where $\tilde{\boldsymbol{\pi}} = \boldsymbol{y_0}(I - \tilde{R})^{-1}$ is the distribution of the phase at the arrival epoch of positive customers in equilibrium. Taking the conditional probability on the first time of level change of the the Markov chain (ξ^*, J^*) and applying the law of total probability, we have that

$$W_0(x) = \mu \int_0^x \exp((D_0 - \mu I)t)\mathbf{1}\, dt + \int_0^x \exp((D_0 - \mu I)t) D_1 W_1(x - t)\, dt \qquad \textbf{(4.3)}$$

and, hence,

$$\widetilde{W}_0(s) = [sI - (D_0 - \mu I)]^{-1}\,\mathbf{1}\mu + [sI - (D_0 - \mu I)]^{-1} D_1 \widetilde{W}_1(s). \qquad \textbf{(4.4)}$$

Let $G_{jj'}(x)$ be the conditional probability that the QBD process (ξ^*, J^*) starting in the state $(1, j)$ at time $t = 0$ reaches the set $\mathbf{0}$ for the first time no later than x and $(0, j')$ is the first state in the set $\mathbf{0}$ to be visited. It is known (see, e.g., Neuts [14]) that the LST $\tilde{G}(s)$ of the matrix $G(x) = (G_{jj'}(x))$ is the minimal nonnegative solution of the matrix equation

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$$\tilde{G}(s) = [sI - (D_0 - \mu I)]^{-1}[(D_{-1} + \mu I) + D_1\tilde{G}^2(s)].$$

Noting that

$$\widetilde{W}_n(s) = \tilde{G}^n(s)\widetilde{W}_0(s), \qquad n = 1,2,\ldots,$$

we have from (4.2) and (4.4) that

$$\widetilde{W}(s) = \boldsymbol{y_0}(I - \tilde{R})^{-1}[sI - (D_0 - \mu I) - D_1\tilde{G}(s)]^{-1}\mathbf{1}\mu. \tag{4.5}$$

Simple calculation yields that the probability that the tagged customer completes his service is

$$\widetilde{W}(0) = \boldsymbol{y_0}(I - \tilde{R})^{-1}[-(D_0 - \mu I + D_1 G)]^{-1}\,\mathbf{1}\mu$$

and the conditional expectation of the sojourn time given that the the tagged customer completes his service is $\overline{W}/\widetilde{W}(0)$, with

$$\overline{W} = -\widetilde{W}'(0) = \boldsymbol{y_0}(I - \tilde{R})^{-1}[-(D_0 - \mu I + D_1 G)]^{-2}(I + D_1\bar{G})\,\mathbf{1}\mu,$$

where $G = \tilde{G}(0+)$ and $\bar{G} = -\tilde{G}'(0)$ are the minimal nonnegative solutions of the matrix equations

$$(D_{-1} + \mu I) + (D_0 - \mu I)G + D_1G^2 = 0,$$
$$(D_0 - \mu I)\bar{G} + D_1(G\bar{G} + \bar{G}G) + G = 0.$$

5. RCE WITH FCFS DISCIPLINE

Under the RCE removal strategy, a negative arrival removes a customer from the end of the queue. When $\xi_a = n$, the tagged customer completes his service without being removed if the service time of $n+1$ customers is less than the first passage time from level **1** to level **0** of the the process (X, J) with generator generator $\tilde{Q}$ of the block matrix form

$$\tilde{Q} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \cdots \end{array} \\ \begin{pmatrix} O & O & O & O & O & \cdots \\ D_{-1} & D_0 & D_1 & & & \\ & D_{-1} & D_0 & D_1 & & \\ & & D_{-1} & D_0 & D_1 & \\ & O & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}. \tag{5.1}$$

Let $\tilde{S}_n$ be the time needed to serve n consecutive customers. Because the service time distribution is exponential with parameter μ, the probability density function $s_n(t)$ of $\tilde{S}_n$ is given by

$$s_n(t) = \mu e^{-\mu t}\frac{(\mu t)^{n-1}}{(n-1)!}, \qquad t \geq 0.$$

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Let T be the first time that the Markov process (X, J) with generator $\tilde{Q}$ reaches the level $\mathbf{0}$ starting at level $\mathbf{1}$ and $T_{jj'}(x) = P(T \le x, J(T) = j' | (X(0), J(0)) = (1, j))$, the conditional probability that (X, J) starting $(1, j)$ at time $t = 0$ reaches the set $\mathbf{0}$ for the first time no later than time x with the $(0, j')$ is the first state in level $\mathbf{0}$ to be visited. It is easily seen from Neuts [14] that the LST $\tilde{T}(s)$ of the matrix $T(x) = (T_{jj'}(x))$ is the minimal nonnegative solution of the matrix equation

$$\tilde{T}(s) = (sI - D_0)^{-1}(D_{-1} + D_1 \tilde{T}^2(s)), \qquad s \ge 0. \tag{5.2}$$

From the above discussion, we have that

$$\begin{aligned} W_{nj}(x) &= P(W \le x | \xi_a = n, J_a = j) \\ &= P(\tilde{S}_{n+1} \le x, T > \tilde{S}_{n+1} | \xi_a = n, J_a = j) \\ &= \int_0^x \mu e^{-\mu t} \frac{(\mu t)^n}{n!} \left(1 - \sum_{j'=1}^{m} T_{jj'}(t)\right) dt. \end{aligned} \tag{5.3}$$

Let $W_n(x)$ be the column m-vector whose jth component is $W_{nj}(x)$. Then, the sojourn time distribution $W(x) = P(W \le x)$ and its LST $\widetilde{W}(s)$ are given by

$$\begin{aligned} W(x) &= \boldsymbol{y_0} \sum_{n=0}^{\infty} \tilde{R}^n W_n(x) \\ &= \boldsymbol{y_0} \int_0^x \mu \exp(-\mu(I - \tilde{R})t)(\mathbf{1} - T(t)\mathbf{1})\, dt \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \widetilde{W}(s) &= \boldsymbol{y_0} \int_0^{\infty} \mu \exp(-[sI + \mu(I - \tilde{R})]t)(\mathbf{1} - T(t)\mathbf{1})\, dt \\ &= \mu \boldsymbol{y_0} (sI + \mu(I - \tilde{R}))^{-1} [I - \tilde{T}(sI + \mu(I - \tilde{R}))]\mathbf{1}, \end{aligned} \tag{5.5}$$

where $\tilde{T}(M) = \int_0^{\infty} \exp(-Mx)\, dT(x)$ for the $m \times m$ matrix M.

6. RCE WITH PREEMPTIVE LCFS DISCIPLINE

We assume that the tagged customer arrives at time $t = 0$ and $(\xi(0), J(0)) = (n, j)$ and let $\xi_a(t)$ and $\xi_b(t)$ be the number of customers ahead and behind of the tagged customer, respectively. Let $(\tilde{0}, j)$ represent the state that the tagged customer completes his service, the phase at the service completion time is $j, j = 1, 2, \ldots, m$, and $\tilde{\mathbf{0}} = \{(\tilde{0}, 1), \ldots, (\tilde{0}, m)\}$. We also denote by $(\tilde{0}', j)$ the state that the tagged customer is removed by a negative arrival before its service completion, the phase is j, $j = 1, 2, \ldots, m$ at that time, and $\tilde{\mathbf{0}}' = \{(\tilde{0}', 1), \ldots, (\tilde{0}', m)\}$. Let T be the first time that the system states are in $\tilde{\mathbf{0}}$ or $\tilde{\mathbf{0}}'$. Consider a Markov chain $\Xi = \{\Xi(t), t > 0\}$ on the state space $E = \tilde{\mathbf{0}} \cup \tilde{\mathbf{0}}' \cup \{(k, i, j),\ k = 0, 1, 2, \ldots,\ 0 \le i \le n,\ 1 \le j \le m\}$ with $\Xi(t) = (\xi_a(t), \xi_b(t), J(t))$ for $0 \le t < T$ and $\Xi(t) \in \tilde{\mathbf{0}} \cup \tilde{\mathbf{0}}'$ for $t \ge T$, and initial state

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$\xi_a(0) = 0$, $\xi_b(0) = n$. Setting $\boldsymbol{k}_n = \{(\boldsymbol{k},\boldsymbol{0}),\ldots,(\boldsymbol{k},\boldsymbol{n})\}$, $(\boldsymbol{k},\boldsymbol{i}) = \{(k,i,1),(k,i,2),\ldots,(k,i,m)\}$, $0 \le i \le n$, it is easily seen that the generator of Ξ is of the form

$$Q_n^* = \begin{array}{c} \\ \tilde{\boldsymbol{0}} \\ \tilde{\boldsymbol{0}}' \\ \boldsymbol{0}_n \\ \boldsymbol{1}_n \\ \boldsymbol{2}_n \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \tilde{\boldsymbol{0}} & \tilde{\boldsymbol{0}}' & \boldsymbol{0}_n & \boldsymbol{1}_n & \boldsymbol{2}_n & \cdots \end{array} \\ \begin{pmatrix} O & O & O & O & O & \cdots \\ O & O & O & O & O & \cdots \\ C^* & B^* & A_1^* & A_0^* & O & \cdots \\ O & B & A_2^* & A_1^* & A_0^* & \\ O & B & O & A_2^* & A_1^* & \ddots \\ \vdots & \vdots & \vdots & & \ddots & \ddots \end{pmatrix} \end{array}, \tag{6.1}$$

where $A_0^* = I_{n+1} \otimes D_1$, $A_2^* = \mu I_{(n+1)m}$, and

$$A_1^* = \begin{pmatrix} D_0 - \mu I & & & & O \\ D_{-1} & D_0 - \mu I & & & \\ & \ddots & \ddots & & \\ & & & D_{-1} & D_0 - \mu I \\ O & & & & \end{pmatrix}$$

are $m(n+1) \times m(n+1)$ matrices and $B^* = \boldsymbol{e}_1 \otimes D_{-1}$ and $C^* = \mu \boldsymbol{1} \otimes I_m$, where $\boldsymbol{e}_1 = (1,0,\ldots,0)^t$ is the column $(n+1)$-vector. It is noted that the distribution of W under the RCE with preemptive LCFS discipline is the same as that of the absorption time of the process Ξ to the state $\tilde{\boldsymbol{0}}$ starting at $(0,n,j)$ before reaching the state $\tilde{\boldsymbol{0}}'$.

Let $V_{k,i}^{(n)}(x)$ be the $m \times m$ matrix whose component $[V_{k,i}^{(n)}(x)]_{jj'}$ is the conditional probability that the Markov chain Q_n^* absorbs in $(\tilde{0},j')$ starting at (k,i,j) no later than x and let $V_k^{(n)}(x)$ denote the $m(n+1) \times m$ matrix whose ith block matrix is $[V_{k,i}^{(n)}(x)]$, $i = 0,1,\ldots,n$. It is noted that

$$\begin{aligned} W_n(x) &= (W_{n1}(x),\ldots,W_{nm}(x))^t \\ &= V_{0,n}^{(n)}(x)\,\boldsymbol{1}, \end{aligned}$$

with $W_{nj}(x) = P(W \le x | \xi_a = n, J_a = j)$, $j = 1,2,\ldots,m$. The LST $\tilde{V}_k^{(n)}(s)$ is given by

$$\tilde{V}_0^{(n)}(s) = (sI - A_1^*)^{-1}(C^* + A_0^* \tilde{V}_1^{(n)}(s)) \tag{6.2}$$

and

$$\tilde{V}_1^{(n)}(s) = \tilde{G}_n(s)\tilde{V}_0^{(n)}(s), \tag{6.3}$$

where $\tilde{G}_n(s)$ is the LST of the first hitting time distribution of the Markov chain Q_n^* to the level $\boldsymbol{0}_n$ starting at level $\boldsymbol{1}_n$, and $\tilde{G}_n(s)$ is the minimal nonnegative solution of the matrix equation

$$\tilde{G}_n(s) = (sI - A_1^*)^{-1}(A_2^* + A_0^* \tilde{G}_n^2(s)) \tag{6.4}$$

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or, equivalently,

$$\tilde{G}_n(s) = (sI - A_1^* - A_0^* \tilde{G}_n(s))^{-1}\mu. \tag{6.5}$$

Thus, we have from (6.2), (6.3), and (6.5) that

$$\begin{aligned}\tilde{V}_0^{(n)}(s) &= (sI - A_1^* - A_0^* \tilde{G}_n(s))^{-1}C \\ &= \tilde{G}_n(s)(\mathbf{1} \otimes I_m).\end{aligned} \tag{6.6}$$

Writing the matrix $\tilde{G}_n(s)$ in the block partitioned form

$$\tilde{G}_n(s) = \begin{pmatrix} G_{00}^{(n)}(s) & G_{01}^{(n)}(s) & \dots & G_{0n}^{(n)}(s) \\ G_{10}^{(n)}(s) & G_{11}^{(n)}(s) & \dots & G_{1n}^{(n)}(s) \\ \vdots & \vdots & & \vdots \\ G_{n0}^{(n)}(s) & G_{n1}^{(n)}(s) & \dots & G_{nn}^{(n)}(s) \end{pmatrix},$$

where each $G_{ij}^{(n)}(s)$ is $m \times m$ matrix, we have that

$$\tilde{W}_n(s) = \sum_{j=0}^{n} G_{nj}^{(n)}(s)\mathbf{1}. \tag{6.7}$$

7. SOJOURN TIMES FOR THE QUEUE WITH PHASE-TYPE SERVICE DISTRIBUTIONS

In this section, we present the sojourn time distributions for the queue with phase-type service distributions under the combinations of service disciplines, FCFS and LCFS preemptive repeat with resampling, and removal strategies RCH and RCE. Following the same approach used in Sections 2–6, the earlier results can be extended to the case of phase-type service distributions. Hence, we just point out the changes in the analysis and results instead of repeating all the procedures.

We assume that the service time distribution is of phase type with representation $(\boldsymbol{\beta}, S)$, where S is the square matrix of order ν with negative diagonal elements and $S^0 = -S\mathbf{1} \geq \mathbf{0}$ and $S^0 \neq \mathbf{0}$. Then, the evolution of the number of customers, arrival phase, and service phase are represented by the QBD process on the state space

$$\mathcal{S} = \{(0, j),\ 1 \leq j \leq m\} \cup \{(i, j, k),\ i \geq 1,\ 1 \leq j \leq m,\ 1 \leq k \leq \nu\}.$$

The index $i \geq 1$ denotes the number of customers in the system; the index j, $1 \leq j \leq m$, represents the phase of arrival process, whereas the index k, $1 \leq k \leq \nu$, indicates the phase of the service in course. Labeling the states in the lexicographic order, that is,

$$(0,1), \dots, (0, m), (1,1,1), (1,2,1), \dots, (1,2, m), \dots,$$

the generator Q of the QBD process corresponding to (2.2) is given by

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$$Q = \begin{pmatrix} B_0 & A_0' & & & \\ B_1 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & \ldots \\ & & \vdots & \vdots & \end{pmatrix}. \tag{7.1}$$

The stationary distribution $\boldsymbol{y} = [\boldsymbol{y_0}, \boldsymbol{y_1}, \cdots]$ at the arrival epoch is given by the same formula as (2.7), with the matrices $\tilde{A}_k$ and $\tilde{B}_k$ corresponding to $\tilde{P}$ in (2.6) as

$$\tilde{B}_k = (-A_1^{-1}A_2)^k(-A_1^{-1}B_1)(-B_0^{-1}A_0'), \qquad k = 0,1,2,\ldots,$$

$$\tilde{A}_k = (-A_1^{-1}A_2)^k(-A_1^{-1}A_0), \qquad k = 0,1,2,\ldots.$$

The matrix components of Q are given in Table 1 for each combination of removal strategies and service disciplines.

Now, we present the changes and results for each combination of removal strategies and service disciplines.

RCH, FCFS Discipline

$$\widetilde{W}_n(s) = \{[sI_{m\nu} - (D_0 + D_1) \oplus S]^{-1}(D_{-1} \otimes (\mathbf{1}\cdot\boldsymbol{\beta}) + I_m \otimes (S^0\cdot\boldsymbol{\beta}))\}^n$$
$$\times [sI_{m\nu} - (D_0 + D_1) \oplus S]^{-1}(\mathbf{1} \otimes S^0), \qquad n \geq 0,$$

$$\widetilde{W}(s) = \boldsymbol{y}_0\tilde{U}(s)[sI_{m\nu} - (D_0 + D_1) \oplus S]^{-1}(\mathbf{1} \otimes S^0),$$

where

$$\tilde{U}(s) = \sum_{n=0}^{\infty} \tilde{R}^n\{[sI_{m\nu} - (D_0 + D_1) \oplus S]^{-1}(D_{-1} \otimes (\mathbf{1}\cdot\boldsymbol{\beta}) + I_m \otimes (S^0\cdot\boldsymbol{\beta}))\}^n.$$

TABLE 1. Matrix Components of Q in (7.1)

	RCH-FCFS	RCH-LCFS	RCE-FCFS	RCE-LCFS	Matrix Size
B_0	$D_0 + D_{-1}$	$D_0 + D_{-1}$	$D_0 + D_{-1}$	$D_0 + D_{-1}$	$m \times m$
B_1	$D_{-1} \otimes \mathbf{1} + I \otimes S^0$	$D_{-1} \otimes \mathbf{1} + I \otimes S^0$	$D_{-1} \otimes \mathbf{1} + I \otimes S^0$	$D_{-1} \otimes \mathbf{1} + I \otimes S^0$	$m\nu \times m$
A_0'	$D_1 \otimes \boldsymbol{\beta}$	$D_1 \otimes \boldsymbol{\beta}$	$D_1 \otimes \boldsymbol{\beta}$	$D_1 \otimes \boldsymbol{\beta}$	$m \times m\nu$
A_0	$D_1 \otimes I$	$D_1 \otimes (\mathbf{1}\cdot\boldsymbol{\beta})$	$D_1 \otimes I$	$D_1 \otimes (\mathbf{1}\cdot\boldsymbol{\beta})$	$m\nu \times m\nu$
A_1	$D_0 \oplus S$	$D_0 \oplus S$	$D_0 \oplus S$	$D_0 \oplus S$	$m\nu \times m\nu$
A_2	$D_{-1} \otimes (\mathbf{1}\cdot\boldsymbol{\beta})$ $+I \otimes (S^0\cdot\boldsymbol{\beta})$	$D_{-1} \otimes (\mathbf{1}\cdot\boldsymbol{\beta})$ $+I \otimes (S^0\cdot\boldsymbol{\beta})$	$D_{-1} \oplus (S^0\cdot\boldsymbol{\beta})$	$D_{-1} \oplus (S^0\cdot\boldsymbol{\beta})$	$m\nu \times m\nu$

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RCH, LCFS with preemptive repeat with resampling

Observing that the generator Q^* corresponding to (4.1) is given by

$$Q^* = \begin{array}{c} \\ \tilde{\mathbf{0}} \\ \tilde{\mathbf{0}}' \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} \tilde{\mathbf{0}} & \tilde{\mathbf{0}}' & \mathbf{0} & \mathbf{1} & \mathbf{2} & \cdots \end{array} \\ \begin{pmatrix} O & O & O & O & O & \cdots \\ O & O & O & O & O & \cdots \\ I \otimes S^0 & D_{-1} \otimes \mathbf{1} & A_1 & A_0 & O & \cdots \\ O & O & A_2 & A_1 & A_0 & \cdots \\ O & O & O & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \end{array}, \tag{7.2}$$

we have that

$$\widetilde{W}_0(s) = (sI_{m\nu} - A_1)^{-1}(\mathbf{1} \otimes S^0) + (sI_{m\nu} - A_1)^{-1}A_0\widetilde{W}_1(s),$$

$$\widetilde{W}(s) = \boldsymbol{y}_0(I - \tilde{R})^{-1}(sI_{m\nu} - A_1 - A_0\tilde{G}(s))^{-1}(\mathbf{1} \otimes S^0),$$

where $\tilde{G}(s)$ is the minimal nonnegative solutions of the matrix equation

$$\tilde{G}(s) = (sI_{m\nu} - A_1)^{-1}(A_2 + A_0\tilde{G}^2(s)).$$

RCE, FCFS Discipline

Noting that the nth convolution of the phase-type distribution with representation $(\boldsymbol{\beta}, S)$ is also the phase-type distribution with representation, say $(\boldsymbol{\beta_n}, S_n)$, we have that

$$W_n(x) = \int_0^x (\mathbf{1} - T(t)\mathbf{1}) \otimes [\exp(S_{n+1}t)S_{n+1}^0]\,dt, \qquad n = 0,1,2,\ldots,$$

where the LST $\tilde{T}(s)$ of $T(t)$ is given by (5.2).

RCE, LCFS with preemptive repeat with resampling

In this case, the formulas for $\widetilde{W}_n(s)$, $\widetilde{V}_0^{(n)}(s)$, and $\tilde{G}_n(s)$ are not changed, but the matrix components of Q_n^* corresponding to (6.1) are given by

$$C^* = \mathbf{1}_{n+1} \otimes I_m \otimes (S^0 \cdot \boldsymbol{\beta}), \qquad B^* = \boldsymbol{e}_1 \otimes D_{-1} \otimes I_\nu,$$

$$A_0^* = I_{n+1} \otimes D_1 \otimes (\mathbf{1} \cdot \boldsymbol{\beta}), \qquad A_2^* = I_{n+1} \otimes I_m \otimes (S^0 \cdot \boldsymbol{\beta}),$$

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and

$$A_1^* = \begin{pmatrix} D_0 \oplus S & & & O \\ D_{-1} \otimes I_\nu & D_0 \oplus S & & \\ & \ddots & \ddots & \\ O & & D_{-1} \otimes I_\nu & D_0 \oplus S \end{pmatrix}.$$

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