Ergod. Th. & Dynam. Sys. (2004), **24**, 1–16
DOI: 10.1017/S0143385703000397
© *2004 Cambridge University Press*
Printed in the United Kingdom

Fractal probability distributions and transformations preserving the Hausdorff–Besicovitch dimension

SERGIO ALBEVERIO†, MYKOLA PRATSIOVYTYI‡ and GRYGORIY TORBIN§

† *Institut für Angewandte Mathematik, Universität Bonn, Wegelerstrasse 6, D-53115 Bonn, Germany; SFB 256, Bonn, BiBoS, Bielefeld, Bonn; CERFIM, Locarno and USI, Switzerland; IZKS, Bonn*
‡ *National Pedagogical University, Kyiv, Ukraine*
(e-mail: prats@ukrpost.net)
§ *National Pedagogical University, Kyiv, Ukraine*
(e-mail: torbin@imath.kiev.ua)

(*Received* 19 *January* 2002 *and accepted in revised form* 3 *September* 2003)

Abstract. This article is devoted to the development of a general theory of transformations of $\mathbb{R}^n$, called dimension-preserving (DP) transformations, which preserve the Hausdorff–Besicovitch dimension of arbitrary subsets.

The main attention is given to continuous transformations of $\mathbb{R}$ and $[0, 1]$. A class of distribution functions of random variables with independent s-adic digits is studied in detail. It is proved that any absolutely continuous function from the previously mentioned class is a DP function, despite the fact that it may have a very complicated local structure. Necessary, respectively, sufficient conditions for dimension preservation are also given for singular functions. Relations between the entropy of transformations and their DP properties are investigated.

Examples and counterexamples are provided,and some applications are discussed.

1. *Introduction*

The proof of fractality of a given subset of a metric space and the determination of its fractal dimension are often difficult problems (see, e.g., [**1, 5, 7, 10–13**]). In some cases, however, there exist simple methods for an explicit calculation of the fractal dimension: if a transformation f preserves the dimension and $f(E)$ is a self-similar set (or a set with less complicated structure than E), then it is enough to calculate the dimension of the simpler set.

The set of all metric space transformations which preserve the Hausdorff–Besicovitch dimension of every set is a group. This group includes all bi-Lipschitz transformations

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

and, therefore, all affine transformations. However, there are many other transformations which preserve the fractal dimension.

One of the main goal of the present article is a description of classes of dimension-preserving transformations of sets, called the DP transformations. We investigate the connection between the differential and DP properties of transformations and we show that the smoothness and bi-Lipschitz properties are only rough sufficient conditions for dimension preservation. We discuss relations between the entropy of the transformations and their DP properties.

For the reader's convenience in §2 we briefly recall some of the standard facts on the Hausdorff–Besicovitch and Billingsley dimensions. The interested reader might consult [**12**] for a unified approach to the many different notions of dimension.

In §3, we introduce the notions of dimension-preserving (DP for short) transformations and local dimension preserving (LDP for short) transformations of $\mathbb{R}^n$. Theorem 3 establishes the relation between these families of transformations. This section contains a few sufficient conditions for dimension preservation. In the fourth section, we study transformations which are generated by distribution functions of random variables with independent s-adic digits. Since the previously mentioned class of random variables is well investigated (see, for instance, [**4, 9, 19**]), we summarize without proofs the relevant material. Using the results of Theorem 6, we prove that any absolutely continuous function from the previously mentioned class is a DP function, despite the fact that it may have a very complicated local structure, e.g., its derivative can be infinite on an everywhere dense superfractal set N_∞ (i.e. a fractal set of $\mathbb{R}^n$ with Hausdorff–Besicovitch dimension n, for some n (see, e.g., [**15**]) and zero on another everywhere dense superfractal set N_0. As a simple corollary of Theorem 6, we obtain a method for constructing a family of singular DP functions.

Section 4 establishes the relation between the entropy of the distributions of a random variable ξ and their DP properties. In §5, we apply the results of this paper to study the fractal properties of the attractor of dynamical systems generated by some symbolic dynamics.

Problems of determination of Hausdorff–Besicovitch dimension of some sets (like repellers or attractors) arise in the theory of dynamical systems relatively often and an improvement of methods for calculation of fractal dimensions, like the ones we give in the present paper, has consequences on the development of the theory of dynamical systems with chaotic behavior [**20**].

The results of our work can be used to construct methods for the determination of fractal dimensions as well as for the creation of an axiomatic theory of fractal geometry, based on a group theoretic approach (somewhat in the spirit of F. Klein), looking at fractal geometry as the study of geometric structures which are invariant under the group of DP transformations.

2. *Hausdorff–Besicovitch dimension*

Let (M, ρ) be a metric space, E a bounded subset of M and $d(E)$ denote the diameter of the set E. Let Φ_M be a family of subsets of the space M such that for an arbitrary set

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

$E \subset M$ and, for each number $\varepsilon > 0$, there exists an at most countable ε-covering $\{E_j\}$ of E ($E_j \in \Phi_M, d(E_j) \leq \varepsilon$). Let α be a positive number.

We recall (see, e.g., [**5, 6**])) that the *α-dimensional Hausdorff measure* of a bounded set E of a metric space (M, ρ) is defined by

$$H_\alpha(E) = \lim_{\varepsilon \to 0} \left[\inf_{d(E_j) \leq \varepsilon} \left\{ \sum_j d^\alpha(E_j) \right\} \right] = \lim_{\varepsilon \to 0} m_\varepsilon^\alpha(E),$$

where the infimum is taken over all at most countable ε-coverings $\{E_j\}$ of E, $E_j \in \Phi_M$.

We note that, generally speaking, $H_\alpha(E)$ depends on the family Φ_M. If $M = \mathbb{R}^n$, then the family of all bounded sets, the family of all open sets and the family of all closed sets all give rise to the same α-dimensional Hausdorff measure (see [**16**, p. 50]). In the following we shall only deal with this α-dimensional Hausdorff measure.

Definition. The positive number

$$\alpha_0(E) = \sup\{\alpha : H_\alpha(E) = \infty\} = \inf\{\alpha : H_\alpha(E) = 0\}$$

is called the *Hausdorff–Besicovitch dimension* of the set E.

Let us recall some properties of the Hausdorff–Besicovitch dimension which we shall use in the following:

(1) $\alpha_0(E) = 0$ for any at most countable set E;

(2) if $E_1 \subset E_2$, then $\alpha_0(E_1) \leq \alpha_0(E_2)$; and

(3) $\alpha_0\left(\bigcup_n E_n\right) = \sup_n \alpha_0(E_n)$.

Let ν be a continuous probability measure on the Borel subsets of $\mathbb{R}^n$. We recall that the ν_α-Hausdorff measure of a set E is given by

$$\nu_\alpha(E) = \lim_{\varepsilon \to 0} \left[\inf_{\nu(E_j) \leq \varepsilon} \left\{ \sum_j \nu^\alpha(E_j) \right\} \right] = \lim_{\varepsilon \to 0} \nu_\varepsilon^\alpha(E) = \sup_{\varepsilon > 0} \nu_\varepsilon^\alpha(E),$$

where the infimum is taken over all at most countable $\nu - \varepsilon$-coverings $\{E_j\}$ of E where E_j is a Borel set (see, e.g., [**2**]).

Definition. The positive number

$$\alpha_\nu(E) = \sup\{\alpha : \nu_\alpha(E) = \infty\} = \inf\{\alpha : \nu_\alpha(E) = 0\}$$

is called the *Billingsley dimension of the set E* (or Hausdorff–Besicovitch dimension of the set E with respect to the measure ν).

If ν is the Lebesgue measure, then $\alpha_\nu(E)$ coincides with the Hausdorff–Besicovitch dimension of the set E.

Remark. For calculating the Hausdorff–Besicovitch dimension and the Billingsley dimension, one can use some more narrow classes of sets. In the $\mathbb{R}$-case, it is enough to use the family of all s-adic closed intervals:

$$\Delta_{\alpha_1 \ldots \alpha_k} = \left[\sum_{i=1}^k \frac{\alpha_i}{s^i}, \frac{1}{s^k} + \sum_{i=1}^k \frac{\alpha_i}{s^i} \right],$$

$k \in N, \alpha_i \in \{0, 1, \ldots, s-1\} = N_{s-1}^0$ (see, e.g., [**2, 19**]).

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

3. *Transformations preserving the fractal dimension*

Definition. A transformation f of a metric space (M, ρ) (in the sense of a bijective mapping a M into itself) is called a transformation preserving the fractal dimension, for short DP transformation, iff the Hausdorff–Besicovitch dimension of an arbitrary subset $E \subset M$ coincides with the Hausdorff–Besicovitch dimension of its image $E' = f(E)$, i.e. $\alpha_0(E) = \alpha_0(E')$.

In some articles authors considered the problem of preservation of dimension of some sets under concrete smooth transformations or mappings (see [**8, 18**]). To the best of our knowledge transformations preserving the dimension of all subsets of a metric space do not seem to have been studied systematically.

It is easy to see that *the set H of all DP transformations of a metric space (M, ρ) forms a group under the operation composition of transformations.*

It is not hard to prove the following (for proof see, e.g., [**5**]).

LEMMA 1. *Let f be a bi-Lipschitz transformation, i.e. let there exist positive constants λ_1 and λ_2 such that for all points A, B and their images $A' = f(A)$, $B' = f(B)$ one has*

$$\lambda_1 |AB| \le |A'B'| \le \lambda_2 |AB|, \tag{1}$$

then the transformation f is a DP transformation.

COROLLARY. *Any affine transformation of $\mathbb{R}^n$ is a DP transformation.*

Remark. In [**5**] an approach to fractal geometry, regarding two sets as 'the same' if there is a bi-Lipschitz mapping between them, was given. This approach is too restrictive since many simple mappings are non-bi-Lipschitz (i.e. $y = x^2$, $x \in [0, 1]$; the inversion in $\mathbb{R}^n$). As we shall show the class of DP transformations is essentially wider than the class of bi-Lipschitz transformations.

THEOREM 1. *Let f be a transformation of $\mathbb{R}^n$. If there exists an at most countable decomposition of $\mathbb{R}^n = \bigcup_i G_i$, and positive constants $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ such that for all points $A \in G_i$, $B \in G_i$ and their images $A' = f(A)$, $B' = f(B)$, one has*

$$\lambda_1^{(i)} |AB| \le |A'B'| \le \lambda_2^{(i)} |AB|,$$

then the transformation f is a DP transformation.

Proof. Let E be an arbitrary subset of $\mathbb{R}^n$. Then $E = \bigcup_i [E \cap G_i]$. Using Lemma 1, we have

$$\alpha_0(E \cap G_i) = \alpha_0(f(E \cap G_i)) \quad \text{for every } i.$$

Due to property (3) of the Hausdorff–Besicovitch dimension, one has

$$\alpha_0(E) = \alpha_0\left(\bigcup_i [E \cap G_i]\right) = \sup_i \alpha_0(E \cap G_i)$$
$$= \sup_i \alpha_0(f(E \cap G_i)) = \alpha_0\left(\bigcup_i f([E \cap G_i])\right) = \alpha_0(f(E)). \qquad \square$$

COROLLARY. *The inversion in $\mathbb{R}^n$ is a DP transformation.*

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

Definition. Let $O(x_0, \varepsilon)$ be an ε-neighbourhood of the point x_0. The positive number

$$\alpha_0(x_0, E) = \lim_{\varepsilon \to 0} \alpha_0(E \cap O(x_0, \varepsilon)) = \inf_{\varepsilon > 0} \alpha_0(E \cap O(x_0, \varepsilon)),$$

is called the *local Hausdorff–Besicovitch dimension of the set* E at the point x_0.

We note that if $\varepsilon_1 < \varepsilon_2$, then $\alpha_0(E \cap O(x_0, \varepsilon_1)) \leq \alpha_0(E \cap O(x_0, \varepsilon_2))$. Therefore, the limit in the previous definition always exists, and the local *Hausdorff–Besicovitch dimension* of the set E can be determined at any point x. Moreover, $\alpha_0(x_0, E) = \lim_{n \to \infty} \alpha_0(E \cap O(x_0, b_n))$ for any sequence $\{b_n\}$: $b_n \to 0$ $(n \to \infty)$. It is easy to see that $0 \leq \alpha_0(x_0, E) \leq \alpha_0(E)$.

LEMMA 2. *Let $U_n(x_0)$ be a sequence of open sets containing x_0 and $d(U_n(x_0)) \to 0$ $(n \to \infty)$. Then the number $\alpha_*(x_0, E) = \lim_{n \to \infty} \alpha_0(E \cap U_n(x_0))$ coincides with the local Hausdorff–Besicovitch dimension of the set E at the point x_0.*

Proof. For every open set $U_n(x_0)$ there exist two positive numbers a_n and $c_n = 2d(U_n(x_0))$ such that

$$O(x_0, a_n) \subset U_n(x_0) \subset O(x_0, c_n).$$

Therefore, $\alpha_0(E \cap O(x_0, a_n)) \leq \alpha_0(E \cap U_n(x_0)) \leq \alpha_0(E \cap O(x_0, c_n))$.

If $d(U_n(x_0)) \to 0$, then $c_n \to 0$ and $a_n \to 0$, and we have

$$\lim_{n \to \infty} \alpha_0(E \cap O(x_0, a_n)) \leq \lim_{n \to \infty} \alpha_0(E \cap U_n(x_0)) \leq \lim_{n \to \infty} \alpha_0(E \cap O(x_0, c_n))$$

and

$$\alpha_0(x_0, E) \leq \alpha_*(x_0, E) \leq \alpha_0(x_0, E).$$

Thus, $\alpha_*(x_0, E) = \alpha_0(x_0, E)$. □

Remark. Lemma 2 remains true if $U_n(x_0)$ denotes the closure of an open set containing x_0.

THEOREM 2. *For each $E \subset \mathbb{R}^n$, there exists a point x^* such that*

$$\alpha_0(x^*, E) = \alpha_0(E),$$

moreover, if E is uncountable, then the point x^* is a limit point of the set E.

Proof. If $\alpha_0(E) = 0$, then the assertion of the theorem is obvious.

Let $\alpha_0(E) > 0$ and K_0 be the n-dimensional cube with diameter equal to a and containing the bounded set E. The set K_0 can be written in the form

$$K_0 = \bigcup_{i=1}^{2^n} K_i,$$

where K_i are congruent closed cubes. It is easy to see that the diameter of the cube K_i equals $a/2$ and that there exists a cube K_{m_1} such that $\alpha_0(E) = \alpha_0(E \cap K_{m_1})$.

Analogously, the cube K_{m_1} can be written in the form

$$K_{m_1} = \bigcup_{i=1}^{2^n} K_{m_1 i},$$

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

where $K_{m_1 i}$ are congruent closed cubes. It is easy to see that the diameter of the cube $K_{m_1 i}$ equals $a/2^2$ and there exists a cube $K_{m_1 m_2}$ such that $\alpha_0(E) = \alpha_0(E \cap K_{m_1}) = \alpha_0(E \cap K_{m_1 m_2})$ and so on.

Thus we have a sequence of embedded closed cubes $K_{m_1 m_2 \ldots m_l}$, $l \in N$, such that

$$\alpha_0(E) = \alpha_0(E \cap K_{m_1 m_2 \ldots m_l}).$$

It is also easy to see that $E \cap K_{m_1 m_2 \ldots m_l} \neq \emptyset$, and $d(K_{m_1 m_2 \ldots m_l}) = a/2^l \to 0$, if $l \to \infty$. Then there exists a unique point $x^* \in K_{m_1 m_2 \ldots m_l}$ for all $l \in N$. Using the definition of the local Hausdorff–Besicovitch dimension of the set E at the point x^*, we have $\alpha_0(E) = \alpha_0(x^*, E)$.

The point x^* is a limit point of the set E, because, for all $l \in N$, there exist infinitely many points of the set E in the cube $K_{m_1 m_2 \ldots m_l}$. □

COROLLARY. *The Hausdorff–Besicovitch dimension of the set E can be determined as follows:*

$$\alpha_0(E) = \max_{x \in E^{\mathrm{cl}}} \alpha_0(E, x),$$

where E^{cl} stands for the closure of the set E.

Definition. A transformation of $\mathbb{R}^n$ is called a transformation preserving the local fractal dimension (for short an LDP transformation), iff the local Hausdorff–Besicovitch dimension of an arbitrary subset $E \subset \mathbb{R}^n$ at any point x coincides with the local Hausdorff–Besicovitch dimension of its image $E' = f(E)$ at the point $x' = f(x)$, i.e.

$$\alpha_0(E, x) = \alpha_0(E', x').$$

THEOREM 3. *Let f be a continuous transformation of $\mathbb{R}^n$. Then the transformation f is a DP transformation iff the transformation f is an LDP transformation.*

Proof. Let f be a DP transformation of $\mathbb{R}^n$, $E \subset \mathbb{R}^n$ an arbitrary set, $x_0 \in \mathbb{R}^n$ an arbitrary point. If x_0 is an isolated point of the set E, then x_0' is an isolated point of E'. In this case, $\alpha_0(E, x) = \alpha_0(E', x') = 0$. If x_0 is a limit point of E, then x_0 is a limit point of E', because f is a continuous transformation.

Let us consider a sequence $O(x_0, b_n)$ with $\lim_{n\to\infty} b_n = 0$ and let $U_n(x_0')$ be the image of $O(x_0, b_n)$ under f. Since f is a homeomorphic DP transformation, it follows that $U_n(x_0')$ is an open set, $\alpha_0(E \cap O(x_0, b_n)) = \alpha_0(E' \cap U_n(x_0'))$ and $\lim_{n\to\infty} d(U_n(x_0')) = 0$. Therefore, $\lim_{n\to\infty} \alpha_0(E \cap O(x_0, b_n)) = \lim_{n\to\infty} \alpha_0(E' \cap U_n(x_0'))$ and $\alpha_0(E, x) = \alpha_0(E', x')$.

To prove the inverse assertion, we assume that f is an LDP transformation of $\mathbb{R}^n$, i.e. $\alpha_0(E, x) = \alpha_0(E', x')$, for all $E \subset \mathbb{R}^n$, for all $x \in \mathbb{R}^n$. Using the previous theorem, we conclude that there exists a point $x_* \in E^{\mathrm{cl}}$ such that $\alpha_0(E) = \max_{x \in E^{\mathrm{cl}}} \alpha_0(E, x) = \alpha_0(E, x_*)$. Due to the fact that f is a continuous LDP transformation, the point $x_*' = f(x_*)$ is a limit point of the set E' and

$$\alpha_0(E) = \max_{x \in E^{\mathrm{cl}}} \alpha_0(E, x) = \alpha_0(E, x_*) = \alpha_0(E', x_*') = \max_{x' \in (E')^{\mathrm{cl}}} \alpha_0(E', x') = \alpha_0(E').$$

Therefore, f is a DP transformation. □

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

THEOREM 4. *Let f be a transformation of $\mathbb{R}^n$. If for each points $x_0 \in \mathbb{R}^n$ the finite limit*

$$\lim_{|x_0x|\to 0} \frac{|f(x_0)f(x)|}{|x_0x|} = A(x_0) > 0 \tag{2}$$

exists, then f is a DP transformation.

Proof. From (2), it follows that f is a continuous transformation. We shall prove that f preserves the local dimension. Let $E \subset \mathbb{R}^n$ and let x_0 be an arbitrary point of E. If x_0 is an isolated point of E, then $\alpha_0(E, x_0) = \alpha_0(E', x_0') = 0$, where $x_0' = f(x_0)$.

Let now x_0 be a limit point of the set E.

Let fix ε_0 such that $A(x_0) - \varepsilon_0 > 0$. From (2) it follows, that for chosen ε_0, there exists a $\delta_0 = \delta_0(\varepsilon_0)$ such that

$$|x_0x| < \delta_0 \Rightarrow \left| \frac{|f(x_0), f(x)|}{|x_0x|} - A(x_0) \right| < \varepsilon_0 \Leftrightarrow$$

$$(A(x_0) - \varepsilon_0) \cdot |x_0x| < |f(x_0), f(x)| < (A(x_0) + \varepsilon_0) \cdot |x_0x|, \quad \text{for all } x \in O(x_0, \delta).$$

Then, using Lemma 1, we conclude that, for any subset $G \subset O(x_0, \delta_0)$,

$$\alpha_0(G) = \alpha_0(G'), \quad \text{where } G' = f(G).$$

Let us consider a sequence $O(x_0, b_n)$, $b_n < \delta_0$, $b_n \to 0$. Let $U_n(x_0)$ be the image of $O(x_0, b_n)$. Since $O(x_0, b_n) \subset O(x_0, \delta)$, we have

$$\alpha_0(E \cap O(x_0, b_n)) = \alpha_0(E' \cap U_n(x_0)).$$

Due to the continuity and bijectivity of f, we have $b_n \to 0 \Leftrightarrow d(U_n(x_0)) \to 0$ and

$$\alpha_0(E, x_0) = \lim_{n\to\infty} \alpha_0(E \cap O(x_0, b_n)) = \lim_{n\to\infty} \alpha_0(E' \cap U_n(x_0)) = \alpha_0(E', x_0').$$

So, f is an LDP transformation. Therefore, f is a DP transformation. □

Remark. In [**8**] a weaker result was proven. Namely, if f is strictly monotonic and differentiable on $[0, 1]$ and $0 < \lambda_1 < f'(x) < \lambda_2 < \infty$ for all $x \in [0, 1]$, then for any subset $E \subset [0, 1]$ one has $\alpha_0(f(E)) = \alpha_0(E)$. The result was applied to the calculation of the Hausdorff–Besicovitch dimension of set M_L of Markov numbers. It was proven that M_L is a superfractal set.

Remark. It is easy to generalize many of the previously mentioned results about DP transformations of $\mathbb{R}^n$ to arbitrary metric spaces.

4. *Transformations preserving dimension and distributions of random variables with independent s-adic digits*

Every transformation of $\mathbb{R}$ (i.e. one-to-one mapping of $\mathbb{R}$ into itself) is an unbounded function. If this function is continuous, then it is a strictly monotonic function. In the following we shall only deal with bounded sets and, therefore, we shall consider transformations of $[0, 1] \subset \mathbb{R}$. The following assertion is obvious.

LEMMA 2. *A continuous function is a transformation of $[0, 1]$ iff it is either a strictly increasing distribution function F on $[0, 1]$ ($F(0) = 0$, $F(1) = 1$) or it is of the form $1 - F$, with F a strictly increasing distribution function.*

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

Because of this reason in the following we shall study the DP properties of continuous distribution functions.

Let s be a fixed natural number, $s > 1$. Each real number $x \in [0, 1]$ admits a decomposition of the form

$$x = \frac{\alpha_1}{s^1} + \frac{\alpha_2}{s^2} + \cdots + \frac{\alpha_k}{s^k} + \cdots,$$

where $\alpha_k \in \{0, 1, \ldots, s-1\} = N_{s-1}^0$. We shall use the notation

$$x = \Delta_{\alpha_1(x)\ldots\alpha_k(x)\ldots},$$

where $\alpha_k(x)$ is called the kth s-adic digits of the number x.

Remark. Some real numbers have two s-adic representations. Such numbers are called s-adic rational. The set of such numbers is countable.

Let

$$\xi = \sum_{k=1}^{\infty} \frac{\eta_k}{s^k} \tag{3}$$

be a random variable (r.v.) with independent s-adic digits η_k having the following distributions:

η_k	0	1	...	$s-1$
p_{ik}	p_{0k}	p_{1k}	...	$p_{(s-1)k}$

where $p_{ik} \geq 0$, $\sum_{i=1}^{s-1} p_{ik} = 1$, $k \in N$.

The distribution of r.v. ξ is uniquely defined by the matrix $P^* = \|p_{ik}\|$ and is concentrated on $[0, 1]$. The structure and the properties (including fractal properties) of the distributions of this class of r.v. are well studied (see, e.g., [**4, 9, 19**]). The first essential results in this direction were obtained by Salem in [**17**]. Criteria for singularity and absolutely continuity of a random variable with independent s-adic digits were obtained by Chatterji in [**4**]. In [**9**] Marsaglia reproved the same results in an independent way.

LEMMA 3. [**19**] *The distribution function F_ξ of the r.v. ξ can be written in the following form*

$$F_\xi(x) = \beta_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left[\beta_{\alpha_k(x)} \cdot \prod_{j=1}^{k-1} p_{\alpha_j(x)j} \right],$$

where $\beta_{\alpha_k(x)} = \sum_{i=0}^{\alpha_k(x)-1} p_{ik}$, $k \in N$.

COROLLARY 1. *The distribution function F_ξ is a strictly increasing function if and only if all elements of the matrix P^* are positive numbers.*

COROLLARY 2. [**19**] *If F_ξ is differentiable at x, then its derivative can be written in the form*

$$F'_\xi(x) = \prod_{k=1}^{\infty} \{s \cdot p_{\alpha_k(x)k}\}.$$

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

THEOREM 5. [**9**] *Let F_ξ be a distribution function of a random variable ξ with independent s-adic digits and assume there are no zeros in the matrix P^*.*

The distribution of the r.v. ξ has pure type and

(1) *is pure discrete iff*

$$\prod_{k=1}^{\infty} \max_{0\le i\le s-1} \{p_{ik}\} > 0; \tag{4}$$

(2) *is pure absolutely continuous iff*

$$\sum_{k=1}^{\infty} \left\{ \sum_{i=0}^{s-1} (1 - s \cdot p_{ik})^2 \right\} < \infty; \tag{5}$$

(3) *is pure singular iff both the infinite product* (4) *and the series* (5) *diverge.*

THEOREM 6. *Let $p'_k = \min_i p_{ik}$, $p''_k = \max_i p_{ik}$. If*

$$\lim_{k\to\infty} \sum_{j=1}^{k} \ln p'_j \Big/ -k \ln s = \lim_{k\to\infty} \sum_{j=1}^{k} \ln p''_j \Big/ -k \ln s = 1, \tag{6}$$

then F_ξ is a DP function on [0, 1].

Proof. Let $\Delta^*_{\alpha_1(x)\ldots\alpha_k(x)}$ be the image of the s-adic segment $\Delta_{\alpha_1(x)\ldots\alpha_k(x)}$ under the F_ξ-transformation of [0, 1], i.e. $\Delta^*_{\alpha_1(x)\ldots\alpha_k(x)} = F_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})$. Let μ_ξ be a probability measure corresponding to the distribution of the r.v. ξ and λ stands for Lebesgue measure. We have $\lambda(\Delta_{\alpha_1(x)\ldots\alpha_k(x)}) = s^{-k}$ and

$$\mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)}) = P\{\xi \in \Delta_{\alpha_1(x)\ldots\alpha_k(x)}\} = \lambda(\Delta^*_{\alpha_1(x)\ldots\alpha_k(x)}) = \prod_{j=1}^{k} \{p_{\alpha_j(x)j}\}.$$

Let

$$V_k(x) := \lim_{k\to\infty} \frac{\ln \mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \lambda(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} = \frac{\sum_{j=1}^{k} \ln p_{\alpha_j(x)j}}{-k \ln s}. \tag{7}$$

Since $p'_j \le p_{\alpha_j(x)j} \le p''_j$, we have

$$V_{k*} := \sum_{j=1}^{k} \ln p''_j \Big/ -k \ln s \le V_k(x) \le \sum_{j=1}^{k} \ln p'_j \Big/ -k \ln s =: V^*_k.$$

By assumption, $\lim_{k\to\infty} V_{k*} = \lim_{k\to\infty} V^*_k = 1$. Therefore, for any $x \in [0, 1]$, $\lim_{k\to\infty} V_k(x) = 1$ and, as a consequence of the Billingsley theorem (see, e.g., [**2**, p. 142]), for an arbitrary set $E \subset [0, 1]$ we have $\alpha_\lambda(E) = \alpha_\mu(E)$. Since $\alpha_\lambda(E) = \alpha_0(E)$ and $\alpha_\mu(E) = \alpha_0(F_\xi(E))$, we obtain the equality $\alpha_0(E) = \alpha_0(F_\xi(E))$. So, F_ξ is a DP function. □

COROLLARY 1. *If $\lim_{k\to\infty} p_{ik} = 1/s$ (for all $i \in N^0_{s-1}$), then F_ξ is a DP function on* [0, 1] .

COROLLARY 2. *If F_ξ is an absolutely continuous distribution function, then F_ξ is a DP function on* [0, 1].

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

Proof. If F_ξ is a distribution function of an absolutely continuously distributed random variable ξ, then (5) holds. Hence $\lim_{k\to\infty} p_{ik} = 1/s$ and the previous corollary applies to this case. □

COROLLARY 3. *There exist singular distribution functions F_ξ preserving the fractal dimension.*

Proof. Let us consider a r.v. ξ with the following distributions of η_k ($s = 2$):

$$\begin{array}{cccc} \eta_k & & 0 & & 1 \\ p_{ik} & \dfrac{1}{2} + \dfrac{1}{\sqrt{k+4}} & & \dfrac{1}{2} - \dfrac{1}{\sqrt{k+4}} \end{array}.$$

Since

$$\prod_{k=1}^{\infty} \max_{0\le i\le 1}\{p_{ik}\} = \prod_{k=1}^{\infty}\left\{\frac{1}{2} + \frac{1}{\sqrt{k+4}}\right\} = 0 \text{ and } \sum_{k=1}^{\infty}\left\{\sum_{i=0}^{s-1}(1-s\cdot p_{ik})^2\right\} = \sum_{k=1}^{\infty}\frac{8}{k+4} = \infty,$$

we see that F_ξ is a singular function. Obviously, condition (6) holds. Therefore, F_ξ preserves the fractal dimension. □

Remark 1. Of course, condition (6) and the condition of Corollary 1 are not equivalent. For instance, let $s = 2$,

$$p_{0k} = \begin{cases} \frac{1}{2}, & \text{if } k \ne 10^m, m \in N, \\ \frac{1}{3}, & \text{if } k = 10^m, m \in N; \end{cases} \qquad p_{1k} = 1 - p_{0k}.$$

It is easy to see that $\lim_{k\to\infty} p_{ik}$ does not exist and condition (6) holds. Since

$$\sum_{k=1}^{\infty}[(1-2p_{0k})^2 + (1-2p_{1k})^2] = \sum_{m=1}^{\infty}[(1-2p_{0,10^m})^2 + (1-2p_{1,10^m})^2] = \infty,$$

we see that F_ξ is a singular DP function.

Remark 2. The assertion of the latter corollary is surprising, because, for an arbitrary singular distribution function F, there exists a set A such that $\lambda(A) = 0$ and $\lambda(F(A)) = 1$. It is easy to see that $\alpha_0(F(A)) = 1$. If F is singular and if F is a DP function, then $\alpha_0(A) = 1$ and the corresponding distribution is superfractal.

Most authors considered the case of strictly increasing functions F such that both sets

$$N_\infty = \{x : F'(x) = +\infty\} = \left\{x : \lim_{\varepsilon\to 0}\frac{F(x+\varepsilon) - F(x-\varepsilon)}{2\varepsilon} = +\infty\right\},$$

and

$$N_0 = \{x : F'(x) = 0\}$$

were either empty or at most finite.

However, for every singular DP function F_ξ the sets N_∞ and N_0 have the cardinality of the continuum. Moreover, $\lambda(N_0) = 1$ and $\alpha_0(N_\infty) > 0$ (excluding the anomalously fractal distributions).

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

We shall now see that even in such a simple class as the class of the distribution functions F_ξ of the r.v. ξ with independent s-adic digits, there exist absolutely continuous DP functions with unexpectedly complicated differential structure.

PROPOSITION. *There exists an absolutely continuous DP function with the following properties:*

$$\alpha_0(N_\infty) = 1,$$
$$\alpha_0(N_0) = 1,$$
the sets N_∞ and N_0 are dense in $[0, 1]$.

Proof. We shall use the 5-adic representation of real numbers

$$x = \sum_{k=1}^{\infty} \frac{i_k}{5^k} = \Delta_{i_1(x)\ldots i_k(x)\ldots},$$

where $i_k \in \{0, 1, 2, 3, 4\} = N_4^0$.

Let us consider the set

$$M_{\alpha\beta} = \Bigg\{ x = \Delta \underbrace{\alpha\beta\gamma_{1,1}\gamma_{2,1}}_{\text{first group}} \underbrace{\alpha\alpha\beta\beta\gamma_{1,2}\gamma_{2,2}\ldots\gamma_{8,2}}_{\text{second group}} \underbrace{\alpha\alpha\alpha\alpha\beta\beta\beta\beta\gamma_{1,3}\gamma_{2,3}\ldots\gamma_{24,3}}_{\text{third group}} \ldots$$
$$\ldots \underbrace{\overbrace{\alpha\alpha\ldots\alpha\alpha}^{2^{k-1}}\overbrace{\beta\beta\ldots\beta\beta}^{2^{k-1}}\overbrace{\gamma_{1,k}\gamma_{2,k}\ldots\gamma_{k\cdot 2^k,k}}^{k\cdot 2^k}}_{k\text{th group}} \ldots \Bigg\},$$

where α and β are some fixed integers from the set $\{1, 3\}$, and γ_{ik} are arbitrary integers from the set N_4^0. So, the kth group in the definition of $M_{\alpha\beta}$ includes 2^{k-1} fixed integers α, 2^{k-1} fixed integers β, and $k \cdot 2^k$ integers γ_{ik}, taking the values from the set N_4^0 independently.

LEMMA 4. *For any previously mentioned fixed integers α and β, the set $M_{\alpha\beta}$ is a superfractal set, i.e.*

$$\alpha_0(M_{\alpha\beta}) = 1 \quad \textit{and} \quad \lambda(M_{\alpha\beta}) = 0.$$

Proof. The proof of this lemma is contained in [**14**].

Let A be the set of the position number of the symbols γ_{ik} in the set $M_{\alpha\beta}$, i.e.

$$A = \{3, 4, 9, 10, \ldots, 16, 25, 26, \ldots, 48, \ldots\}.$$

Let us consider the r.v. $\xi = \sum_{k=1}^{\infty} \eta_k/5^k$, where independent random variables η_k have the following distributions

$$\begin{matrix} \eta_k & 0 & 1 & 2 & 3 & 4 \\ p_{ik} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{matrix}, \quad \text{if } k \in A;$$

and

$$\begin{matrix} \eta_k & 0 & & 1 & & 2 & & 3 & & 4 \\ p_{ik} & \frac{1}{5} & \frac{1}{5} & - & \frac{1}{10\cdot n_k} & \frac{1}{5} & \frac{1}{5} & + & \frac{1}{10\cdot n_k} & \frac{1}{5} \end{matrix}, \quad \text{if } k \notin A,$$

where n_k is an order number of the integer $k \notin A$ in the set $N\backslash A \equiv N_{\alpha\beta}$.

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

Let us consider the set

$$M_{33} = \{x = \Delta_{33\gamma_{1,1}\gamma_{2,1}3333\gamma_{1,2}\gamma_{2,2}\ldots\gamma_{8,2}33333333\gamma_{1,3}\gamma_{2,3}\ldots\gamma_{24,3}\ldots}\}.$$

We shall prove that if $x' \in M_{33}$, then $x' \in N_\infty$.

Let $x' = \sum_{k=1}^{\infty} i_k(x')/5^k = \Delta_{i_1(x')\ldots i_k(x')\ldots}$.

One of the derivative numbers of the distribution function F_ξ can be calculated as follows:

$$D_1(x') = \prod_{k=1}^{\infty}[5p_{i_k(x')k}] = \prod_{k\in N_{\alpha\beta}}[5p_{i_k(x')k}] = \prod_{l=1}^{\infty}\left[5\left(\frac{1}{5}+\frac{1}{10l}\right)\right] = +\infty.$$

We shall prove that all other derivative numbers are also equal infinity. Let $\{[a_m, b_m]\}$ be a sequence of embedded closed intervals such that

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = x' \quad \text{and } x' \in [a_m, b_m] \quad \text{for all } m \in N.$$

For any $m \in N$, there exists a number k_m such that $\Delta_m = \Delta_{i_1(x')\ldots i_{k_m}(x')} \subset [a_m, b_m]$, and $\Delta_{i_1(x')\ldots i_k(x')} \subsetneq [a_m, b_m]$, if $k < k_m$. We shall set $\Delta F_\xi(\Delta_m) = P\{\xi \in \Delta_m\}$.

There exist two segments Δ' and Δ'' of $(k_m - 1)$th rank with the following properties:

(1) $[a_m, b_m] \subset (\Delta' \cup \Delta'')$;

(2) $\Delta_m \subset \Delta'$ or $\Delta_m \subset \Delta''$;

(3) $|\Delta_m| = 5^{-k_m}$, $|\Delta'| = |\Delta''| = 5^{-k_m+1}$.

Therefore,

$$\frac{F(b_m) - F(a_m)}{b_m - a_m} \geq \frac{\Delta F(\Delta_m)}{b_m - a_m} \geq \frac{\Delta F(\Delta_m)}{2|\Delta'|} = \frac{\Delta F(\Delta_m)}{10|\Delta_m|} = \frac{1}{10}\prod_{k=1}^{k_m}[5p_{i_k(x')k}],$$

and

$$\lim_{m\to\infty}\frac{F(b_m) - F(a_m)}{b_m - a_m} \geq \frac{1}{10}\lim_{m\to\infty}\prod_{k=1}^{k_m}[5p_{i_k(x')k}] = \frac{1}{10}\prod_{k=1}^{\infty}[5p_{i_k(x')k}] = +\infty.$$

So, $x' \in N_\infty$ and $\alpha_0(N_\infty) = 1$.

The set M_{33} is a nowhere dense set. However, the following assertion is true: if $x' \neq x$ and there exists a number $n_0 \in N$ such that $i_k(x') = i_k(x)$ for $k > n_0$, then $x' \in N_\infty$ iff $x \in N_\infty$. Therefore, the set N_∞ is dense in $[0, 1]$.

We consider the set

$$M_{11} = \{x = \Delta_{11\gamma_{1,1}\gamma_{2,1}1111\gamma_{1,2}\gamma_{2,2}\ldots\gamma_{8,2}11111111\gamma_{1,3}\gamma_{2,3}\ldots\gamma_{24,3}\ldots}\}.$$

It is not difficult to see that $M_{11} \subset N_0$. But as $\alpha_0(M_{11}) = 1$, so also does $\alpha_0(N_0) = 1$.

The set N_0 is dense in $[0, 1]$, because the fact '$x \in N_0$' does not depend on the first s-adic digits of x.

It is easy to see that

$$\sum_{k=1}^{\infty}\left\{\sum_{i=0}^{s-1}(1 - s\cdot p_{ik})^2\right\} = \sum_{k\notin A}\{(1 - 5\cdot p_{1k})^2 + (1 - 5\cdot p_{3k})^2\} = \sum_{l=1}^{\infty}\frac{1}{2l^2} < \infty.$$

Therefore, F_ξ is an absolutely continuous function. Using the first corollary of the latter theorem we conclude that F_ξ is a DP function. □

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

This example illustrates also the following facts:

(1) the dimension of the set of non-differentiability of an absolutely continuous function can be greater than the one of a singular function. For the previously constructed function we have $\alpha_0(N_\infty) = 1$ and for the classical Cantor singular function we have $\alpha_0([0, 1]\backslash N_0) = \ln 2/\ln 3$ (cf. e.g., [**3**]);

(2) 'good' differential properties of F_ξ are not necessary for the preservation of a fractal dimension.

To obtain necessary conditions for dimension preservation, we investigate the relations between the entropy of transformations and their DP properties.

Let $H_k = -\sum_{i=0}^{s-1} p_{ik} \ln p_{ik}$ be the entropy of the r.v. ξ_k and let

$$H_\xi := \lim_{k\to\infty} \frac{H_1 + H_2 + \cdots + H_k}{k \ln s}.$$

THEOREM 7. *If F_ξ is a DP function on* $[0, 1]$, *then* $H_\xi = 1$.

Proof. Let A_ξ be the set of all possible 'supports' of the distribution of the random variable ξ, i.e.

$$A_\xi = \{E : E \in \mathcal{B}, \mathbb{P}_\xi(E) = 1\}.$$

The number $\alpha_0(\xi) = \inf_{E\in A_\xi}\{\alpha_0(E)\}$ is called the dimension of the distribution of the random variable ξ. In [**4**] it has been proved that $\alpha_0(\xi) = H_\xi$. If $H_\xi = a < 1$, then there exists a set $E \subset A_\xi$ such that $a \le \alpha_0(E) < 1$. Since $\mathbb{P}_\xi(E) = 1$, we conclude that $\alpha_0(F_\xi(E)) = 1 \ne \alpha_0(E)$, which contradicts the assumption of Theorem 7. □

COROLLARY. *The distribution function F_ξ of r.v. ξ with independent identically distributed s-adic digits is a DP function iff $p_i = 1/s$ for all $i \in N_0^{s-1}$.*

Proof. If $p_i = 1/s$ for all $i \in N_{s-1}^0$, then we conclude from Corollary 1 of Theorem 6 that F_ξ is a DP function.

For any $j \in N$, $H_j = -\sum_{i=0}^{s-1} p_i \ln p_i$ and, therefore, $H_\xi = H_j/\ln s$. It is well known that $H_j \le \ln s$ and $H_j = \ln s \Leftrightarrow p_0 = p_1 = \cdots = p_{s-1} = 1/s$. If F_ξ is a DP function, then $H_\xi = 1$ and, therefore, $p_i = 1/s$ for all $i \in N_{s-1}^0$. □

5. *Some applications to dynamical systems*

Finally, we shall apply these results to study the fractal properties of some dynamical systems.

Let $s > 1$ be an arbitrary natural number and let $Q^* = \|q_{ik}\|$ be a matrix with the following properties:

(1) $q_{ik} > 0$, for all $i \in N_{s-1}^0$, for all $k \in \mathbb{N}$;

(2) $\sum_{i=0}^{s-1} q_{ik} = 1$, for all $k \in \mathbb{N}$;

(3) $\prod_{k=1}^{\infty} \max_i q_{ik} = 0$.

We shall denote by

$$\Delta^*_{\alpha_1 \ldots \alpha_k \ldots} \tag{8}$$

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

the number

$$y = \beta_{\alpha_1} + \sum_{k=2}^{\infty} \left[\beta_{\alpha_k} \prod_{i=1}^{k-1} p_{\alpha_i i} \right], \quad \text{where } \beta_{\alpha_k} = \sum_{i=0}^{\alpha_k - 1} p_{ik}. \tag{9}$$

It is easy to check that the number y is the value of the distribution function F_ξ of the previous r.v. ξ with independent s-adic digits and $P^* = Q^*$. Notation (8) is called the Q^*-representation of y and α_k is called the kth Q^*-symbol of y. Let us explain the Q^*-representation of real numbers in a geometric way. The Q^*-representation of a real number $y \in [0, 1]$ is generated by the following sequence of partitions of different ranks:

$$[0, 1] = \Delta_0^* \cup \Delta_1^* \cup \cdots \cup \Delta_{s-1}^*, \quad \text{where } |\Delta_i^*| = q_{i1}, i \in N_{s-1}^0;$$
$$\Delta_{\alpha_1}^* = \Delta_{\alpha_1 0}^* \cup \Delta_{\alpha_1 1}^* \cup \cdots \cup \Delta_{\alpha_1 (s-1)}^*, \quad \text{where } |\Delta_{\alpha_1 i}^*| = q_{\alpha_1 1} \cdot q_{i2}; \ldots$$
$$\Delta_{\alpha_1 \ldots \alpha_k}^* = \Delta_{\alpha_1 \ldots \alpha_k 0}^* \cup \Delta_{\alpha_1 \ldots \alpha_k 1}^* \cup \cdots \cup \Delta_{\alpha_1 \ldots \alpha_k (s-1)}^*,$$
$$\text{where } |\Delta_{\alpha_1 \ldots \alpha_k i}^*| = q_{i(k+1)} \cdot \prod_{j=1}^{k} q_{\alpha_j j}.$$

For any sequence $\{\alpha_k\}$, $\alpha_k \in N_{s-1}^0$, there is a sequence of imbedded closed intervals $\Delta_{\alpha_1}^* \supset \Delta_{\alpha_1 \alpha_2}^* \supset \cdots \supset \Delta_{\alpha_1 \ldots \alpha_k}^* \supset \cdots$ contracting to a unique point $y = \bigcap_{k=1}^{\infty} \Delta_{\alpha_1 \ldots \alpha_k}^*$.

Conversely, for any point $x \in [0, 1)$ there exists a sequence of embedded intervals $\Delta_{\alpha_1}^* \supset \Delta_{\alpha_1 \alpha_2}^* \supset \cdots \supset \Delta_{\alpha_1 \ldots \alpha_k}^* \supset \ldots$ containing x. Obviously, a point $x \in [0, 1)$ has a unique Q^*-representation, if x is not an end-point of any closed interval $\Delta_{\alpha_1 \ldots \alpha_k}^*$. There exists a countable set of points having two Q^*-representations:

$$\Delta_{\alpha_1 \ldots \alpha_{k-1} \alpha_k 00\ldots}^* \quad \text{and} \quad \Delta_{\alpha_1 \ldots \alpha_{k-1} (\alpha_k - 1)(s-1)(s-1)\ldots}^*.$$

Points from the latter set are called Q^*-rationals.

Let V be a fixed subset of N_{s-1}^0, $V \neq N_{s-1}^0$, $|V| \geq 2$, and let $W = N_{s-1}^0 \setminus V$. We shall study the dynamical system $([0, 1), f)$, where

$$f(x) = f(\Delta_{\alpha_1(x) \ldots \alpha_k(x) \ldots}^*)$$
$$= \begin{cases} \Delta_{\alpha_1 \ldots \alpha_k \ldots}^*, & \text{if } \alpha_k \in V, \text{ for all } k \in \mathbb{N}; \\ \Delta_{\alpha_1 \ldots \alpha_{m-1} \alpha_{m+1} \alpha_{m+2} \ldots}^*, & \text{if } \alpha_j \in V, \text{ for all } j \in \{1, 2, \ldots, m-1\},\ \alpha_m \in W. \end{cases}$$

If x is Q^*-rational point, then we define f via a Q^*-representation of x having infinite number of zeros.

It is easy to see that any point of the Cantor-like set

$$C[Q^*, V] = \{x : x = \Delta_{\alpha_1 \ldots \alpha_k \ldots}^*, \alpha_k \in V\}$$

is an invariant point for f. If a number $x = \Delta_{\beta_1 \ldots \beta_k \ldots}^*$ contains infinitely many symbols from V, then the trajectory $\{f^{(k)}(x)\}$ converges to a point from $C[Q^*, V]$. For any point $x \in C[Q^*, V]$, there exists a set $S_x \subset [0, 1)$ such that for every $y \in S_x$ $f^{(k)}(y) \to x$. The set S_x has the cardinality of the continuum.

The set

$$C[Q^*, m, W] = \{x : x = \Delta_{\alpha_1 \ldots \alpha_m \alpha_{m+1} \ldots \alpha_{m+i} \ldots}^*;$$
$$\alpha_j \in V, \forall j \in \{1, \ldots, m\};\ \alpha_{m+i} \in W, \forall i \in N\}$$

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

is invariant under f. If the Q^*-representation of x contains exactly m symbols from V, then $\rho(f^{(k)}(x), C[Q^*, m, W]) \to 0$ if $k \to \infty$. The set $C[Q^*, _W] = \bigcup_{m=0}^{\infty} C[Q^*, m, W]$ is also an invariant set under f.

$\lim_{k\to\infty} \rho(f^{(k)}(x), C[Q^*, _W]) = 0$ iff the Q^*-representation of x contains only finitely many symbols from V. If $|W| \geq 2$, then for any natural number k_0 there exist a point $x \in C[Q^*, _W]$ such that x has a k_0-periodic trajectory.

So, the closed set $A = C[Q^*, V] \cup C[Q^*, _W]$ contracts the trajectory of any point $x \in [0, 1)$ and A is the attractor of this dynamical system.

The fractal properties of the set A depend on the matrix Q^*, which, in fact, induces a metric on the space of all sequences $(\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots)$, $\alpha_k \in N_{s-1}^0$. For some classes of matrices Q^* the Hausdorff–Besicovitch dimension of A can be easily calculated by using these results. In particular, if the matrix $Q^* = \|p_{ik}\|$ satisfies the conditions of Theorem 6, then the distribution function F_ξ is a DP function, and, therefore,

$$\alpha_0(C[Q^*, V]) = \alpha_0(F_\xi^{-1}(C[Q^*, V])) = \alpha_0\left(\left\{x : x = \sum_{k=1}^{\infty} \frac{\alpha_k}{s^k}, \alpha_k \in V\right\}\right) = \log_s |V|,$$

and

$$\alpha_0(C[Q^*, _W]) = \alpha_0(C[Q^*, W]) = \log_s |W|,$$

because

$$C[Q^*, _W] = \bigcup_{m=0}^{\infty} C[Q^*, m, W]$$

and

$$\alpha_0(C[Q^*, m, W]) = \alpha_0(F_\xi^{-1}(C[Q^*, _W])) = \log_s |W|.$$

So, the Hausdorff–Besicovitch dimension of the attractor of this dynamical system is equal to $\max\{\log_s |V|, \log_s |W|\}$.

Let us note that the following non-compact everywhere dense sets are also invariant under f:

$$M[Q^*, \{V, \nu_i\}] = \left\{x : x = \Delta^*_{\alpha_1(x)\ldots\alpha_k(x)\ldots}, \alpha_k(x) \in V, \lim_{k\to\infty} \frac{N_i(x, k)}{k} = \nu_i,\ i \in N_{s-1}^0\right\},$$

where

$$N_i(x, k) = \sum_{j=1}^{k} \beta_j(x) \quad \text{with } \beta_j(x) = \begin{cases} 0, & \text{if } \alpha_j(x) \neq i, \\ 1, & \text{if } \alpha_j(x) = i. \end{cases}$$

The set $M[Q^*, \{V, \nu_i\}]$ consists of those $x \in C[Q^*, V]$ for which the asymptotic relative frequency of a digit 'i' ($i \in V$) in the Q^*-representation is ν_i. If the matrix Q^* satisfies the condition of Theorem 6, then

$$\alpha_0(M[Q^*, \{V, \nu_i\}]) = \sum_{i\in V} \nu_i \ln \nu_i \Big/ -\ln s,$$

because the image of the latter set under F_ξ^{-1} is a well-known set which consists of the points whose s-adic expansion contains the digit 'i' with the relative frequency ν_i.

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press

Acknowledgements. This work was partly supported by DFG 436 UKR 113/53, INTAS 00-257, UFFI 01.07/00081 and SFB-611 projects. The authors would like to express their gratitude to the referee for helpful remarks and comments. The last two named authors gratefully acknowledged the hospitality of the Institute of Applied Mathematics and of the IZKS of the University of Bonn.

REFERENCES

[1] C. Bandt. Deterministic fractals and fractal measures. School on measure theory and real analysis (Grado, 1991). *Rend. Istit. Mat. Univ. Trieste* **23**(1) (1991), 1–40.

[2] P. Billingsley. *Ergodic Theory and Information*. John Wiley & Sons, New York, 1965.

[3] A. S. Besicovitch. Sets of points of non-differentiability of absolutely continuous functions and divergence of Fejer sums (On linear set of fractal dimensions III). *Math. Ann.* **110** (1934), 331–335.

[4] S. D. Chatterji. Certain induced measures and the fractional dimensions of their 'supports'. *Z. Wahrschein.* **3** (1964), 184–192.

[5] K. J. Falconer. *Fractal Geometry*. Wiley, Chichester, 1995.

[6] F. Hausdorff. Dimension und äußeres Maß. *Math. Ann.* **79** (1918), 157–179.

[7] R. Kaufman. On Hausdorff dimension of projections. *Mathematika* **15** (1968), 153–155.

[8] P. G. Kogoniya. On the structure of a set of Markov numbers. *Proc. Tbil. Math. Inst.* **XIX** (1953), 121–133.

[9] G. Marsaglia. Random variables with independent binary digits. *Ann. Math. Statist.* **42** (1971), 1922–1929.

[10] P. Mattila. Hausdorff dimension, orthogonal projections and intersections with planes. *Ann. Acad. Sci. Fennicae* A **1** (1975), 227–244.

[11] U. Mosco. Invariant field metrics and dynamical scalings on fractals. *Phys. Rev. Lett.* **79** (1997), 4067–4070.

[12] Ya. B. Pesin. Dimension theory in dynamical systems. Contemporary views and applications. *Chicago Lectures in Mathematics*. University of Chicago Press, Chicago, IL, 1997.

[13] M. V. Pratsiovytyi. *Fractal Approach to Investigations of Singular Distributions*. National Pedagogical University, Kyiv, 1998.

[14] M. V. Pratsiovytyi. Fractal, superfractal and anomalously fractal distribution of random variables with a fixed infinite set of independent n-adic digits. *Exploding Stochastic Laws (Festschrift in Honour of 70th Birthday of Acad.)*. Eds. A. V. Skorokhod and Yu. V. Borovskikh. VSP (1995), pp. 409–416.

[15] M. V. Pratsiovytyi and G. M. Torbin. Superfractality of the set of real numbers, which have no frequency of the n-adic digits and fractal probability distributions. *Ukr. Math. J.* **47** (1995), 971–975.

[16] C. A. Rogers. *Hausdorff Measures*. Cambridge University Press, London, 1970.

[17] R. Salem. On some singular monotonic functions which are strictly increasing. *Trans. Amer. Math. Soc.* **53** (1943), 423–439.

[18] T. D. Sauer and J. A. Yorke, Are the dimensions of a set and its image equal under typical smooth functions? *Ergod. Th. & Dynam. Sys.* **17** (1997), 941–956.

[19] A. F. Turbin and M. V. Pratsiovytyi. *Fractal Sets, Functions, and Distributions*. Naukova Dumka, Kiev, 1992.

[20] F. Schweiger. *Ergodic Theory of Fibred Systems and Metric Number Theory*. The Clarendon Press, New York, 1995.

https://doi.org/10.1017/S0143385703000397 Published online by Cambridge University Press