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Generalized θ operators on unitary Shimura varieties

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Abstract. The main result of this article is the construction of a new class of weight shifting operators, similar to the theta operators of de Shalit and Goren (2019, *Algebra Number Theory*, 13, 1829–1877), Eischen et al. (2021, *Algebra Number Theory*, 15, 1469–1504), and others, which are defined on the lower Ekedahl–Oort strata of the geometric special fiber of unitary Shimura varieties of signature $(n-1,1)$ at a good prime p, split in the reflex field E, which we assume to be quadratic imaginary. These operators act on certain graded sheaves which are obtained from the arithmetic structure of the EO strata, in particular the p-rank on each stratum. We expect these operators to have applications to the study of Hecke-eigensystems of $(\mathrm{mod}\, p)$ modular forms and generalizations of the weight part of Serre's conjecture.

1 Introduction

1.1 Motivation

The theory of the classical theta operator was instrumental in the proof of the weight part of Serre's modularity conjecture of [14]. Edixhoven's proof relied, in particular, on the study of the *θ-cycles* of Tate and Jochnowitz, introduced in [26]. Since then, much work has been devoted to extending the construction of this operator to other Shimura varieties, with an eye toward generalizations of Serre's conjecture, or to gain insight in the Langlands programme $(\mathrm{mod}\, p)$ in a broader sense.

Interesting results have been obtained in the case of Hilbert modular varieties, starting with the work of Katz, in [31]. Following Katz's construction of *partial Hasse invariants*, Andreatta and Goren, in [2], constructed *partial theta operators* and described their kernels and effects on the *weight filtration*. These results have subsequently been improved upon and generalized further (see, for instance, [13]).

In [40], Yamauchi constructed theta operators for Siegel modular forms, in degree 2, and managed to study their theta cycles.

In [9], de Shalit and Goren, building on their previous work [7], constructed $(\mathrm{mod}\, p)$ and p-adic theta operators on certain unitary Shimura varieties.

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At the same time, Eischen, Mantovan, and others, in a series of papers, [15–17], constructed theta operators on PEL Shimura varieties of types A and C. Their approach is independent from that of de Shalit and Goren and it uses geometric techniques which, unlike the more classical theory, do not rely on q-expansions, or more general Fourier–Jacobi and Serre–Tate expansions.

Most of these works leave the problem of studying theta cycles largely open and, where results are obtained, they seem to depend on the specific context. We believe that, in order to understand theta cycles in greater generality, one could benefit from considering new theta or "theta-like" operators, which produce more general weight shifts. Our goal is to present the construction of a new class of *generalized theta operators* that seem to produce exactly the weight shifts that one would expect from a representation-theoretic viewpoint. Our theory of generalized theta operators ties in neatly with the theory of generalized Hasse invariants of [5, 21].

1.2 Notations and conventions

We fix E a quadratic imaginary extension of $\mathbb{Q}$ and p an odd, rational prime, split in E. We write $\mathbb{F}$ for a given algebraic closure of $\mathbb{F}_p$. We choose, once and for all, a preferred embedding $\sigma\colon E \to \mathbb{C}$, so that $\mathrm{Hom}(E,\mathbb{C}) = \{\sigma, \overline{\sigma}\}$, and an element $i = \sqrt{-1} \in \mathbb{C}$. Let $\delta_{E/\mathbb{Q}}$ denote the unique generator of the different ideal $\mathfrak{D}_{E/\mathbb{Q}}$ with positive imaginary part (with respect to our choices of σ and i). In particular, we have the discriminant $D = D_{E/\mathbb{Q}} = -\mathrm{N}_{E/\mathbb{Q}}(\delta_{E/\mathbb{Q}}) = -\delta_{E/\mathbb{Q}}\overline{\delta}_{E/\mathbb{Q}} = -|\delta_{E/\mathbb{Q}}|^2$. If we also fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$, we obtain

$$\mathrm{Hom}(E,\mathbb{C}) \cong \mathrm{Hom}(E,\overline{\mathbb{Q}}_p) \cong \mathrm{Hom}(\mathcal{O}_E, \mathcal{O}_{\overline{\mathbb{Q}}_p}^{\mathrm{ur}}) \cong \mathrm{Hom}(\mathcal{O}_E/(p),\mathbb{F}),$$

the last two isomorphisms depending on the fact that p is unramified in $\mathcal{O}_E$. The last identification induces on $\{\sigma,\overline{\sigma}\}$ an action of the Frobenius automorphism $\mathbb{F} \to \mathbb{F}$ by post-composition. Since p is split, this action is trivial. We also write $\mathcal{O}_{E,\mathrm{ur}} \coloneqq \mathcal{O}_E[1/2D]$. We fix $n \geq 3$ an integer, which will denote throughout the article the relative dimension of the abelian schemes parameterized by the moduli spaces under consideration.

Unless otherwise specified, we assume that all the schemes we work with are locally Noetherian.

1.3 The elliptic case

We sketch here the construction of the theta operator in the classical case of modular curves, following [30]. This construction is the prototype on which most generalizations are based.

Let $N \geq 5$ be an integer prime to p. Let $Y_1(N)$, or simply Y, be the modular curve of level $\Gamma_1(N)$ over $\mathbb{F}$. It is a smooth, affine, connected curve over $\mathbb{F}$, which comes equipped with a universal elliptic curve $\pi\colon \mathcal{E} \to Y_1(N)$. From this universal object, we obtain the invertible sheaf $\underline{\omega} = \pi_*\left(\Omega^1_{\mathcal{E}/Y}\right)$, the so-called *Hodge sheaf*. One can consider a projective compactification $Y_1(N) \subset X_1(N)$, which we simply denote by X, and extend $\underline{\omega}$ to X (see, for instance, [12]).

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In this setting, *modular forms with coefficients in* $\mathbb{F}$ are the elements of the graded $\mathbb{F}$-algebra $\mathrm{M}(N) = \oplus_k \mathrm{M}_k(N)$, with $\mathrm{M}_k(N) \coloneqq H^0(X, \underline{\omega}^k)$. The space of *cusp forms* of weight k is $\mathrm{S}_k(N) \coloneqq H^0(X, \underline{\omega}^k(-C)) \subseteq M_k(N)$, where $C = X \backslash Y$ is the cuspidal divisor. On this algebra, one can define as usual an action of the Hecke operators T_l, for $l \neq p$ any prime, and hence an action of the Hecke algebra they generate.

Since we are working in characteristic p, over Y, we can consider the Verschiebung morphism $V\colon \mathcal{E}^{(p)} \to \mathcal{E}$, which by pullback defines $V\colon \underline{\omega} \to \underline{\omega}^{(p)} \cong \underline{\omega}^p$ and hence a section $h \in H^0(Y, \underline{\omega}^{p-1})$. This section extends to a form $h \in M_{p-1}(N)$, which is called the *Hasse invariant*. It vanishes with simple zeros precisely at the supersingular points $Y^{\mathrm{ss}} \subset Y \subset X$ and its q-expansion at the cusps is identically 1. The complement $Y^{\mathrm{ord}} = Y \backslash Y^{\mathrm{ss}}$, called the ordinary locus, is a dense open subset of Y.

A key observation in Katz's geometric construction of the classical theta operator is that, over the ordinary locus, there is a natural splitting of the Hodge filtration

$$0 \longrightarrow \underline{\omega} \longrightarrow H^1_{\mathrm{dR}}(\mathcal{E}/Y) \longrightarrow \underline{\omega}^{\vee} \longrightarrow 0. \tag{H}$$

Write $H = H^1_{\mathrm{dR}}(\mathcal{E}/Y) = R^1\pi_*\Omega^{\bullet}_{\mathcal{E}/Y}$ for the (relative) de Rham cohomology of π. Let $F\colon H^{(p)} \to H$ be the morphism obtained by pulling back via the relative Frobenius $F\colon \mathcal{E} \to \mathcal{E}^{(p)}$. Katz shows that, over Y^{ord}, $\mathcal{U} = \mathrm{im}(F)$ is a complement of the subsheaf $\underline{\omega} \subset H$, providing $H \cong \underline{\omega} \oplus \mathcal{U}$, which is called the *unit-root splitting* (of (H)). While this splitting cannot be naturally extended to Y, the projection parallel to it, $p_{\mathrm{ur}}\colon H \to \underline{\omega}$, does extend to Y upon multiplication by the Hasse invariant. This underlies Katz's construction of the theta operator. We present here a reformulation of these facts which leads naturally to the generalizations we want to discuss. The open Y^{ord} is the locus where $V|_{\underline{\omega}}\colon \underline{\omega} \to \underline{\omega}^{(p)}$ is an isomorphism. Thus, on Y^{ord}, we may consider the composition

$$H \xrightarrow{V} \underline{\omega}^{(p)} \xrightarrow{V|_{\underline{\omega}}^{-1}} \underline{\omega}. \tag{P}$$

It is easy to see that the map $H \to \underline{\omega}$ from (P) is precisely p_{ur}. In particular, the morphism $h \cdot p_{\mathrm{ur}}\colon H \to \underline{\omega}^p$ can be written as

$$H \xrightarrow{V} \underline{\omega}^p \xrightarrow{V|_{\underline{\omega}}^{-1}} \underline{\omega} \xrightarrow{h = V|_{\underline{\omega}}} \underline{\omega}^p, \tag{hP}$$

which is simply the surjection $V\colon H \to \underline{\omega}^p$ and extends from Y^{ord} to the entire modular curve. More generally, one can look at $\mathrm{Sym}^k(p_{\mathrm{ur}})$, where Sym^k is the k-th symmetric power. The locally free sheaf $\mathrm{Sym}^k(H)$ admits a descending filtration $F^i(\mathrm{Sym}^k(H))$, afforded by (H), and one can show that the morphism

$$h \cdot \mathrm{Sym}^k(p_{\mathrm{ur}})|_{F^{k-1}(\mathrm{Sym}^k H)}\colon F^{k-1}(\mathrm{Sym}^k H) \to \underline{\omega}^{k+p-1}$$

extends from Y^{ord} to Y. This is important, because on $\mathrm{Sym}^k(H)$ and over Y, one can define the Gauss–Manin connection

$$\nabla\colon \mathrm{Sym}^k(H) \longrightarrow \mathrm{Sym}^k(H) \otimes \Omega^1_{Y/\mathbb{F}},$$

which satisfies a general transversality property that implies $\nabla(\underline{\omega}^k) \subseteq F^{k-1}(\mathrm{Sym}^k(H)) \otimes \Omega^1_{Y/\mathbb{F}}$. All of this is used by Katz to define the theta operator

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as the composition

$$\theta\colon \underline{\omega}^k \xrightarrow{\nabla} F^{k-1}(\mathrm{Sym}^k(H)) \otimes \Omega^1_{Y/\mathbb{F}} \xrightarrow{(h\cdot \mathrm{Sym}^k(p_{\mathrm{ur}}))\otimes \underline{\mathrm{ks}}^{-1}} \underline{\omega}^{k+p+1},$$

where $\underline{\mathrm{ks}}\colon \underline{\omega}^2 \to \Omega^1_{Y/\mathbb{F}}$ is the Kodaira–Spencer isomorphism. A more detailed inspection of θ, for instance, via q-expansions and the q-expansion principle, shows that it extends to an operator over X. Taking global sections gives rise to $\theta\colon \mathrm{M}_k(N) \to \mathrm{S}_{k+p+1}(N)$, the theta operator on modular forms. This is a derivation of the algebra $\mathrm{M}(N)$ of modular forms of degree $p+1$. By this, we mean that for two modular forms f, g, we have $\theta(fg) = f\theta(g) + \theta(f)g$, which is a cusp form of degree $p+1+\deg(fg)$. The operator θ is h-linear, in the sense that $\theta(h) = 0$. Moreover, one can show that $T_l\,\theta = l\,\theta\,T_l$. In particular, if f is an eigenform, so is $\theta(f)$.

1.4 The Picard case

We briefly sketch the construction of ordinary and generalized theta operators in the special case of *Picard modular surfaces* to illustrate the main ideas involved.

First, let us set up some notation. Write S for the geometric special fiber of the Picard modular surface over $\mathbb{F}$ with some neat, p-hyperspecial level. Roughly speaking, this is a moduli space of polarized abelian schemes of relative dimension 3 endowed with an action of $\mathcal{O}_E$, the ring of integers of E. For more details on the moduli problem, see the next section. We have a universal object $\pi\colon A \to S$ and, as before, we can consider the Hodge sheaf $\underline{\omega} = \pi_*(\Omega^1_{A/S})$. In this case, $\underline{\omega}$ is locally free of rank 3 and, under the induced action of $\mathcal{O}_E$, it decomposes into the direct sum of two locally free sheaves $\mathcal{P} = \underline{\omega}_\sigma$ and $\mathcal{L} = \underline{\omega}_{\overline{\sigma}}$, the subscripts indicating that $\mathcal{O}_E$ acts on $\underline{\omega}_\tau$ via $\tau \in \{\sigma, \overline{\sigma}\}$. We assume that $\mathcal{P}$ and $\mathcal{L}$ have ranks 2 and 1, respectively. We also consider the sheaf $\delta = \delta_\sigma \cong \det \mathcal{P} \otimes \mathcal{L}^{-1}$. We write $S^\mu \subseteq S$ for the *ordinary locus*, the maximal stratum in the Ekedahl–Oort stratification of S, which is a dense open in S. The complement $S^{\mathrm{no}} = S \backslash S^\mu$ is called the *non-ordinary locus* and is given by the disjoint union of the *almost ordinary locus* S^{ao}, the EO stratum of dimension 1, and the *core locus* S^{core}, the EO stratum of dimension 0. We define an automorphic weight to be a couple $(\underline{k}, w)$, where $\underline{k} = (k_1 \geq k_2) \in \mathbb{Z}^2$, $w \in \mathbb{Z}$, and the corresponding automorphic sheaf to be

$$\underline{\omega}^{\underline{k},w} \coloneqq \mathrm{Sym}^{k_1-k_2}(\mathcal{P}) \otimes \det(\mathcal{P})^{k_2} \otimes \delta^w.$$

Analogously to (H), we have a Hodge filtration on $H = H^1_{\mathrm{dR}}(A/S)$, of which we can take components according to the action of E:

$$\text{(HP)} \qquad 0 \longrightarrow \mathcal{P} \longrightarrow H_\sigma \longrightarrow \mathcal{L}^\vee \longrightarrow 0,$$

$$\text{(HL)} \qquad 0 \longrightarrow \mathcal{L} \longrightarrow H_{\overline{\sigma}} \longrightarrow \mathcal{P}^\vee \longrightarrow 0.$$

Katz's explicit construction of the unit-root splitting in the elliptic case carries over and provides a natural splitting of (HP) and (HL) over S^μ. Like in Section 1.3, we may reinterpret this unit-root splitting in terms of the Verschiebung morphism. Let $V\colon H \to H^{(p)}$ be the pullback of the Verschiebung morphism of the universal abelian scheme $A \to S$. We consider its CM components $V_\sigma\colon H_\sigma \to H^{(p)}_\sigma$, $V_{\overline{\sigma}}\colon H_{\overline{\sigma}} \to H^{(p)}_{\overline{\sigma}}$,

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whose images are $\mathcal{P}^{(p)}, \mathcal{L}^p$, respectively. Over S^μ, the restrictions $V_\sigma|_{\mathcal{P}}: \mathcal{P} \to \mathcal{P}^{(p)}$, $V_{\overline{\sigma}}|_{\mathcal{L}}: \mathcal{L} \to \mathcal{L}^p$ are isomorphisms and we can use them to define

$$p_{\mathrm{ur},\sigma}: H_\sigma \xrightarrow{V_\sigma} \mathcal{P}^{(p)} \xrightarrow{V_\sigma|_{\mathcal{P}}^{-1}} \mathcal{P},\ p_{\mathrm{ur},\overline{\sigma}}: H_{\overline{\sigma}} \xrightarrow{V_{\overline{\sigma}}} \mathcal{L}^p \xrightarrow{V_{\overline{\sigma}}|_{\mathcal{L}}^{-1}} \mathcal{L}.$$

These morphisms give the unit-root splitting in this case. Like in the classical case, we cannot extend this splitting naturally to the non-ordinary locus $S^{\mathrm{no}} = S \backslash S^\mu$, essentially because $V: \underline{\omega} \to \underline{\omega}^{(p)}$ is not invertible on S^{no}, but we can use the Hasse invariant to clear this obstruction. In this case, the Hasse invariant is the section $h \in H^0(S, (\det \underline{\omega})^{p-1})$, which is obtained as the determinant of the morphism $V: \underline{\omega} \to \underline{\omega}^{(p)}$. The section h is nowhere-vanishing on S^μ, identically 0 on S^{no} and it splits as the product $h = h_\sigma \cdot h_{\overline{\sigma}}$, where $h_\sigma \in H^0(S, (\det \mathcal{P})^{p-1})$, $h_{\overline{\sigma}} \in H^0(S, \mathcal{L}^{p-1})$, are obtained from $\det V_\sigma$ and $V_{\overline{\sigma}}$, respectively. Both h_σ and $h_{\overline{\sigma}}$ vanish with simple zeros on S^{no}. Just like in the elliptic case, the morphism $h_{\overline{\sigma}} \cdot p_{\mathrm{ur},\overline{\sigma}}: H_{\overline{\sigma}} \to \mathcal{L}^p$ can be extended from S^μ to S, since it simply coincides with $V: H_{\overline{\sigma}} \to \mathcal{L}^p$. We can also extend

$$h_\sigma \cdot p_{\mathrm{ur},\sigma}: H_\sigma \to \mathcal{P} \otimes (\det \mathcal{P})^{p-1}$$

from S^μ to S, even if the rank of $\mathcal{P}$ is greater than 1. In fact, the product $h_\sigma \cdot V^{-1}: \mathcal{P}^{(p)} \to \mathcal{P} \otimes (\det \mathcal{P})^{p-1}$ is the *adjugate* V^{adj}, of the morphism $V: \mathcal{P} \to \mathcal{P}^{(p)}$ which, like V, is defined on the whole of S. As a result, the extension of $h_\sigma \cdot p_{\mathrm{ur},\sigma}$ is the composition $V|_{\mathcal{P}}^{\mathrm{adj}} \circ V: H_\sigma \to \mathcal{P} \otimes (\det \mathcal{P})^{p-1}$. Considering V^{adj} is a key idea of [15]. In Lemma 6.1, we prove a general result for extending morphisms from S^μ to S, which allows us to consider the correct analog of the above construction to define the theta operator on $\underline{\omega}^{\underline{k},w}$, for a general weight $(\underline{k}, w)$. Lemma 6.1 is also used crucially in our definition of generalized theta operators. In the Picard case, we define, for $(\underline{k}, w)$ with $k_2 \geq 0$, a morphism $h_\sigma \cdot (p_{\mathrm{ur}})^{\underline{k},w}: H^{\underline{k},w} \to \underline{\omega}^{\underline{k},w} \otimes (\det \mathcal{P})^{p-1}$, where

$$h_\sigma \cdot (p_{\mathrm{ur}})^{\underline{k},w} = h_\sigma \cdot \mathrm{Sym}^{k_1-k_2}(p_{\mathrm{ur},\sigma}) \otimes \mathrm{Sym}^{k_2}(\wedge^2(p_{\mathrm{ur},\sigma})) \otimes \mathrm{id}_\delta,$$
$$H^{\underline{k},w} = \mathrm{Sym}^{k_1-k_2}(H_\sigma) \otimes \mathrm{Sym}^{k_2}(\wedge^2(H_\sigma)) \otimes \delta.$$

While $h_\sigma \cdot (p_{\mathrm{ur}})^{\underline{k},w}$ itself does not extend from S^μ to S, a relevant restriction of this morphism does, by Lemma 6.1.

As in the elliptic case, we have natural filtrations $\mathcal{P} \subset H_\sigma$ and $\mathcal{L} \subset H_{\overline{\sigma}}$, with respect to which the Gauss–Manin connection $\nabla: H \to H \otimes \Omega^1_{S/\mathbb{F}}$ satisfies a natural transversality property. Unlike in the classical case, the Kodaira–Spencer morphism $\underline{\mathrm{KS}}: \underline{\omega} \otimes \underline{\omega} \to \Omega^1$ is not an isomorphism, but its restriction $\underline{\mathrm{ks}} = \underline{\mathrm{KS}}_\sigma: \mathcal{P} \otimes \mathcal{L} \to \Omega^1_{S/\mathbb{F}}$ is. With our conventions for the automorphic sheaves, this becomes the isomorphism $\underline{\mathrm{ks}}: \mathcal{P} \otimes \det(\mathcal{P}) \otimes \delta^{-1} \to \Omega^1_{S/\mathbb{F}}$. We can use all of this to define the operator θ_1 as the following composition:

$$\theta_1: \underline{\omega}^{\underline{k},w} \xrightarrow{\nabla} H^{\underline{k},w} \otimes \Omega^1_{S/\mathbb{F}} \xrightarrow{\underline{\mathrm{ks}}^{-1}} H^{\underline{k},w} \otimes \mathcal{P} \otimes \det(\mathcal{P}) \otimes \delta^{-1}$$
$$\xrightarrow{h_\sigma \cdot (p_{\mathrm{ur}})^{\underline{k},w} \otimes \mathrm{id}} \underline{\omega}^{\underline{k},w} \otimes \mathcal{P} \otimes \det(\mathcal{P})^p \otimes \delta^{-1} \longrightarrow \underline{\omega}^{\underline{k}+(p+1,p),w-1}.$$

This produces a weight shift of the form $((p+1, p), -1)$, that is, mostly in the direction of $\det \mathcal{P} \cong \mathcal{L}$. To obtain a different weight shift, we have to generalize the projection $h_\sigma \cdot$

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$(p_{\mathrm{ur}})^{\underline{k},w}$. This does not seem to be possible on S. To remedy this, in this work, we use the structure theory of the Ekedahl–Oort stratification to obtain such generalizations and construct theta operators on lower strata. Let us discuss some of the ingredients involved in this new construction.

On S^{no}, we have a short exact sequence

$$0 \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}_\mu \longrightarrow 0, \tag{F}$$

where $\mathcal{P}_0 \coloneqq \ker(V_\sigma : \mathcal{P} \to \mathcal{P}^{(p)})$ and the quotient $\mathcal{P}_\mu$ are invertible sheaves. Setting $H_\mu = H_\sigma/\mathcal{P}_0$, we have a short exact sequence on S^{no}

$$0 \longrightarrow \mathcal{P}_\mu \longrightarrow H_\mu \longrightarrow \mathcal{L}^\vee \longrightarrow 0, \tag{HP'}$$

which is analogous to (HP). By general properties of the GM connection, we have that $\nabla(\mathcal{P}_0) \subseteq \mathcal{P}_0 \otimes \Omega^1_{S^{\mathrm{no}}/\mathbb{F}}$, which implies that ∇ induces a connection $\nabla : H_\mu \to H_\mu \otimes \Omega^1_{S/\mathbb{F}}$. The Verschiebung morphism induces a map $V : H_\mu \to H_\mu^{(p)}$, the image of which is $\mathcal{P}_\mu^p$. The almost-ordinary locus S^{ao} can be characterized as the locus in S^{no} where the restriction $V|_{\mathcal{P}_\mu} : \mathcal{P}_\mu \to \mathcal{P}_\mu^p$ is an isomorphism. Hence, we can use an idea similar to our reinterpretation of the unit-root splitting to construct a splitting of (HP'), by defining

$$p_{\mathrm{ur},2} : H_\mu \xrightarrow{V} \mathcal{P}_\mu^p \xrightarrow{V|_{\mathcal{P}_\mu}^{-1}} \mathcal{P}_\mu.$$

We can also consider a *partial generalized Hasse invariant* $A_2 \in H^0(S^{\mathrm{no}}, \mathcal{P}_\mu^{p-1})$, corresponding to $V : \mathcal{P}_\mu \to \mathcal{P}_\mu^p$. The vanishing locus of A_2 is precisely $S^{\mathrm{no}} \backslash S^{\mathrm{ao}}$, the core locus of S. We can show that while the morphism $p_{\mathrm{ur},2}$ does not extend from S^{ao} to S^{no}, the map $A_2 \cdot p_{\mathrm{ur},2} : H_\mu \to \mathcal{P}_\mu^p$ does. More generally, the morphisms $A_2 \cdot \mathrm{Sym}^k(p_{\mathrm{ur},2}) : \mathrm{Sym}^k(H_\mu) \to \mathcal{P}_\mu^{k+p-1}$, for $k \geq 0$, extend to S^{no} when restricted to a relevant subsheaf of $\mathrm{Sym}^k(H_\mu)$. With this partial unit-root splitting, we can define a new differential operator on the graded sheaves

$$\mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}) = \mathrm{Sym}^{k_1-k_2}(\mathcal{P}_0 \oplus \mathcal{P}_\mu) \otimes (\mathcal{P}_0 \otimes \mathcal{P}_\mu)^{k_2} \otimes \delta^w,$$

which correspond to a filtration on $\underline{\omega}^{\underline{k},w}$ induced by (F). This *generalized theta operator* will have the form

$$\theta_2 : \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}) \longrightarrow \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k}+(p+1,1),w-1}),$$

thus producing the weight shift $(p+1,1)$, mostly in the direction of $\mathcal{P}$, we were looking for.

This construction works more generally on unitary Shimura varieties of signature $(n-1,1)$, $n \geq 3$, where it gives rise to a family of generalized theta operators defined on various EO strata of the Shimura variety. In future work, we plan to extend these results to even more general Shimura varieties. The main result is as follows.

Theorem 1.1 *Let $1 \leq r < n$ be an integer and $(\underline{k}, w)$ an automorphic weight with $k_{n-1} \geq 0$. There exists a differential operator*

$$\theta_r : \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w}) \longrightarrow \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k}+\underline{\Delta}_r,w-1}),$$

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defined on the (closure of the) Ekedahl–Oort stratum $\overline{S}_{K,w_r}$*, see Section 5, with*

$$\underline{\Delta}_r = (p+1, p, \ldots, p, 1, \ldots, 1),$$

where exactly the last $r-1$ *entries are* 1*. The operator* θ_r *satisfies the following properties:*

(1) *The operator* θ_r *is* A_r*-linear, that is,* $\theta_r(A_r) = 0$*, where* A_r *is the partial Hasse invariant defined in Section 5.2.*
(2) *The operator* θ_r *commutes with the action of prime-to-p Hecke operators.*
(3) *Let* $f \in H^0(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w}))$ *and write it as* $f = \sum_{\underline{a}} f_{\underline{a}}$*, for the decomposition described in Section 6.2. If* $r = n-1, n-2$*, then* $\theta_r(f)$ *is divisible by the Hasse invariant* A_r *if and only if for each component* $f_{\underline{a}}$ *either* $A_r \mid f_{\underline{a}}$ *or* $p \mid a_1$*.*

2 Unitary Shimura varieties of signature $(n-1,1)$

We keep here the notations and conventions adopted in Section 1.2. In particular, E is a fixed quadratic imaginary extension of $\mathbb{Q}$ and $n \geq 3$ an integer.

2.1 The PEL datum

We work with the integral PEL datum, in the sense of [33, Chapter 5], defined as follows:

(1) The simple $\mathbb{Q}$-algebra B is E, with $\mathcal{O}_E$ as its maximal $\mathbb{Z}$-order.
(2) The positive involution * on E is the complex conjugation. The fixed field F_0 is $\mathbb{Q}$.
(3) We let $V \cong E^n$ and $\Lambda = \mathcal{O}_E^n$, the canonical $\mathcal{O}_E$-lattice, with canonical basis $\{e_1, e_2, \ldots, e_n\} \subset \Lambda$.
(4) Take the Hermitian pairing $(\cdot,\cdot) : V \times V \to E$ of signature $(n-1,1)$ given by the diagonal matrix $I_{n-1,1} = \mathrm{diag}(1, \ldots, 1, -1)$, with respect to the canonical basis on V. It restricts to a perfect pairing $(\cdot,\cdot) : \Lambda \times \Lambda \to \mathcal{O}_E$. Then, $\langle \cdot,\cdot \rangle := \mathrm{T}_{E/\mathbb{Q}}(\delta_{E/\mathbb{Q}}^{-1}(\cdot,\cdot))$ is a perfect alternating $\mathbb{Q}$-linear pairing such that $\langle \alpha u, v \rangle = \langle u, \overline{\alpha} v \rangle$, $u, v \in V$, $\alpha \in \mathcal{O}_E$, whose restriction to Λ induces a perfect $\mathbb{Z}$-linear pairing $\Lambda \times \Lambda \to \mathbb{Z}$. By adjunction, $(\cdot,\cdot)$ defines an involution * of $\mathrm{End}_E(V)$, which restricts to an involution of $\mathrm{End}_{\mathcal{O}_E}(\Lambda)$, compatible with the conjugation on $E \subset \mathrm{End}_E(V)$. With respect to the canonical basis, this involution is given by $M \mapsto I_{n-1,1}\, {}^t\overline{M} I_{n-1,1}$.
(5) We have isomorphisms $\mathrm{End}_E(V) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{End}_{E \otimes_{\mathbb{Q}} \mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{M}_n(\mathbb{C})$, afforded by the canonical basis and σ. We take $h : \mathbb{C} \to \mathrm{End}_E(V) \otimes_{\mathbb{Q}} \mathbb{R}$, via these isomorphisms, to be the map of $\mathbb{R}$-algebras defined by $z \mapsto \mathrm{diag}(z, \ldots, z, \overline{z})$. Then, $(\Lambda, h, \langle \cdot,\cdot \rangle)$ is an *integral polarized Hodge structure*.

These data satisfy some properties which we now recall. The Hodge structure on $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C} = V_1 \oplus V_2$ induced by h can be described explicitly. Consider the primitive idempotents

$$e_\sigma = \frac{1}{2}(1 \otimes 1 + \delta_{E/\mathbb{Q}} \otimes \delta_{E/\mathbb{Q}}^{-1}), \quad e_{\overline{\sigma}} = \frac{1}{2}(1 \otimes 1 - \delta_{E/\mathbb{Q}} \otimes \delta_{E/\mathbb{Q}}^{-1}),$$

in the ring $E \otimes_{\mathbb{Q}} E \subset E \otimes_{\mathbb{Q}} \mathbb{C}$. Then

$$V_1 = \mathrm{Span}_{\mathbb{C}} \langle e_\sigma \cdot e_1, \ldots, e_\sigma \cdot e_{n-1}, e_{\overline{\sigma}} \cdot e_n \rangle, \; V_2 = \mathrm{Span}_{\mathbb{C}} \langle e_{\overline{\sigma}} \cdot e_1, \ldots, e_{\overline{\sigma}} \cdot e_{n-1}, e_\sigma \cdot e_n \rangle.$$

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Both V_1 and V_2 are defined over E, in the sense of [34, Definition 1.1.2.7]. In fact, they are integral: $e_\sigma, e_{\overline{\sigma}} \in \mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}_{E,\mathrm{ur}}$ and we can decompose $\Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{E,\mathrm{ur}} \cong \Lambda_1 \oplus \Lambda_2$, with $\Lambda_i \otimes_{\mathcal{O}_{E,\mathrm{ur}}} E \cong V_i$, $i = 1, 2$. This shows that E is reflex field of the datum $(E, {}^*, V, \langle \cdot, \cdot \rangle, h)$ (see [33, Chapter 5]).

Consider the action of any $\alpha \in \mathcal{O}_E$ on Λ_1 and V_1. We have the characteristic polynomial

$$p_\alpha(X) \coloneqq \det(X - \alpha|_{V_1}) = (X - \sigma(\alpha))^{n-1}(X - \overline{\sigma}(\alpha)) \in \mathcal{O}_E[X].$$

These polynomials are necessary to express Kottwitz's determinant condition (see [33, Chapter 5] and [34, Section 1.3.4]).

Remark 2.1 (CM decomposition) We have decompositions of $V \otimes_{\mathbb{Q}} E$ and $\Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{E,\mathrm{ur}}$ given by $e_\sigma, e_{\overline{\sigma}}$,

$$V \otimes_{\mathbb{Q}} E \cong V_\sigma \oplus V_{\overline{\sigma}} \coloneqq e_\sigma(V \otimes_{\mathbb{Q}} E) \oplus e_{\overline{\sigma}}(V \otimes_{\mathbb{Q}} E),$$
$$\Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{E,\mathrm{ur}} \cong \Lambda_\sigma \oplus \Lambda_{\overline{\sigma}} \coloneqq e_\sigma(\Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{E,\mathrm{ur}}) \oplus e_{\overline{\sigma}}(\Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{E,\mathrm{ur}}).$$

We call this the *CM decomposition* of V and Λ. More generally, we have CM decompositions for $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}_{E,\mathrm{ur}}$-module or sheaves over $\mathcal{O}_{E,\mathrm{ur}}$-schemes with a linear $\mathcal{O}_E$-action.

2.2 The reductive group

From $(\mathcal{O}_E, \bar{\cdot}, \Lambda, \langle \cdot, \cdot \rangle, h)$, we obtain a group scheme $\mathbf{G}$ over $\mathbb{Z}$ whose R-points, for R a commutative ring, are

$$\begin{aligned}\mathbf{G}(R) = \mathbf{GU}(\Lambda, (\cdot, \cdot))(R) &\coloneqq \{g \in \mathrm{End}_{\mathcal{O}_E \otimes R}(\Lambda \otimes_{\mathbb{Z}} R) \mid gg^* = \nu(g) \in R^\times\} \\ &= \{(g, \nu(g)) \in \mathrm{End}_{\mathcal{O}_E \otimes R}(\Lambda_R) \times R^\times \mid (gu, gv) = \nu(g)\,(u, v)\,,\ \forall u, v \in \Lambda_R\}.\end{aligned}$$

The morphism $\nu \colon \mathbf{G} \to \mathbb{G}_m$ is the *similitude factor*. The kernel of ν is denoted $\mathbf{G}_1$. We have the following.

Lemma 2.2 *Let $s \colon \mathrm{Spec}\, k \to \mathrm{Spec}\, \mathbb{Z}[1/2D]$ be a morphism with k an algebraically closed field. We have natural isomorphisms of group schemes:*

$$\mathbf{G}_{\mathcal{O}_{E,\mathrm{ur}}} \cong \mathrm{GL}_n \times_{\mathcal{O}_{E,\mathrm{ur}}} \mathbb{G}_m,\ \mathbf{G}_{1,\mathcal{O}_{E,\mathrm{ur}}} \cong \mathrm{GL}_n$$

and similarly for $\mathbf{G}_s$ and $\mathbf{G}_{1,s}$. In particular, $\mathbf{G}$ and $\mathbf{G}_1$ are reductive over $\mathrm{Spec}\, \mathbb{Z}[1/2D]$.

Moreover, under the isomorphism $\mathbf{G}_{\mathcal{O}_{E,\mathrm{ur}}} \cong \mathrm{GL}_n \times \mathbb{G}_m$, the Levi subgroup H of the parabolic fixing the filtration $\Lambda_1 \subseteq \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{E,\mathrm{ur}} \cong \Lambda_1 \oplus \Lambda_2$ corresponds to $\mathrm{GL}_{n-1} \times \mathrm{GL}_1 \times \mathbb{G}_m$. The similarly defined Levi H_1 of $\mathbf{G}_1$ is isomorphic after base change to $\mathrm{GL}_{n-1} \times \mathrm{GL}_1$.

Proof This follows from standard arguments. ∎

The real points of $\mathbf{G}, \mathbf{G}_1$ give the classical unitary groups $\mathbf{G}(\mathbb{R}) = \mathrm{GU}(n-1, 1)$, $\mathbf{G}_1(\mathbb{R}) = \mathrm{U}(n-1, 1)$ and one can see that $H(\mathbb{R}) = G(U(n-1) \times U(1))$, $H_1(\mathbb{R}) = U(n-1) \times U(1)$.

We can restrict h to a morphism of algebraic groups $h \colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$, denoted again by h. Then, $(\mathbf{G}_{\mathbb{Q}}, h)$ is a Shimura datum, as defined in [10, Section 1.5] and [11, Section 2.1.1].

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2.3 The moduli problem

We formulate the PEL moduli problem we are interested in. We work with a neat, p-hyperspecial level $K \subseteq \mathbf{G}(\mathbb{A}^\infty)$ (see [34, Definition 1.4.1.8, Theorem 1.4.1.12]).

Let S be a scheme defined over $\mathcal{O}_{E,(p)} \coloneqq \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. We denote the category of such schemes by $\underline{\mathrm{Sch}}_{\mathcal{O}_{E,(p)}}$. Recall that we assume all schemes to be locally Noetherian. To S, we associate a quadruple $\underline{A} = (A, \lambda, \iota, \eta_K)$:

(1) $A \to S$ is an abelian scheme.
(2) $\lambda\colon A \to A^\vee$ is a prime-to-p polarization.
(3) $\iota\colon \mathcal{O}_E \to \mathrm{End}_S(A)$ is a ring homomorphism such that:
   (a) The *Rosati relation* $\lambda\iota(\overline{\alpha}) = \iota(\alpha)^\vee\lambda$, $\alpha \in \mathcal{O}_E$ holds.
   (b) For any $\alpha \in \mathcal{O}_E$, we have $\det(X - \iota(\alpha)|_{\mathrm{Lie}(A/S)}) = p_\alpha(X) \in \mathcal{O}_E[X] \subset \mathcal{O}_S[X]$.
(4) η_K is an $\mathcal{O}_E$-linear integral level K-structure (see [34, Section 1.3.7, Definition 1.3.7.8]).

An isomorphism of two such quadruples is an isomorphism of the underlying abelian schemes compatible with the remaining data in the natural way. Under our assumptions on K, the functor

$$\begin{aligned} \underline{\mathrm{Sch}}_{\mathcal{O}_{E,(p)}} &\longrightarrow \underline{\mathrm{Set}}, \\ S &\longmapsto \{(A, \lambda, \iota, \eta_K)\}/_{\cong}, \end{aligned}$$

is represented by a smooth, quasi-projective scheme $\mathcal{S}_K \in \underline{\mathrm{Sch}}_{\mathcal{O}_{E,(p)}}$ of relative dimension $n-1$ (see [34, Chapter 2]).

2.4 Hodge and determinant sheaves

For $f\colon G \to S$ a group scheme, we write

$$\underline{\omega}_{G/S} \coloneqq e^*(\Omega^1_{G/S}),$$

where e is the identity of G. We call the sheaf $\underline{\omega}_{G/S}$ the *Hodge sheaf* of G/S. If G and S are clear from the context, we simply write $\underline{\omega}$. If $G = A$ is an abelian scheme, then $\underline{\omega}_{A/S} \cong f_*(\Omega^1_{A/S})$.

Let $\underline{A} = (A, \lambda, \iota, \eta_N) \in \mathcal{S}_K(S)$, for $S \in \underline{\mathrm{Sch}}_{\mathcal{O}_{E,(p)}}$. Then, $\underline{\omega}_{A/S}$ is locally free of rank n with an action of $\mathcal{O}_E$ of signature $(n-1, 1)$, by which we mean that

$$\underline{\omega}_{A/S,\sigma} \coloneqq e_\sigma \cdot \underline{\omega}_{A/S}, \quad \underline{\omega}_{A/S,\overline{\sigma}} \coloneqq e_{\overline{\sigma}} \cdot \underline{\omega}_{A/S},$$

are locally free $\mathcal{O}_{\mathcal{S}_K}$-sheaves of ranks $n-1$ and 1, respectively. This is the CM decomposition of $\underline{\omega}_{A/S}$ (see Remark 2.1). We call $\underline{\omega}_{A/S,\sigma}$, $\underline{\omega}_{A/S,\overline{\sigma}}$ the σ, $\overline{\sigma}$-components of the Hodge sheaf, respectively.

We recall some basic facts concerning the de Rham cohomology of abelian schemes, following [4, 2.5.1]. For $f\colon X \to S$ a smooth morphism of schemes of relative dimension g, we have the de Rham complex $(\Omega^\bullet_{X/S}, d)$. The i-th de Rham cohomology of X/S is defined as $H^i_{\mathrm{dR}}(X/S) \coloneqq R^i f_* \Omega^\bullet_{X/S}$. We have the following.

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Proposition 2.3 [4, Proposition 2.5.2] *Let A/S be as above. Then:*

(1) *For $i \geq 0$, the sheaves $H^i_{\mathrm{dR}}(A/S)$ are locally free and their formation commutes with base change.*
(2) *The natural morphism $\bigwedge^i H^1_{\mathrm{dR}}(A/S) \to H^i_{\mathrm{dR}}(A/S)$ is an isomorphism for all $i \geq 0$.*
(3) *The Hodge–de Rham spectral sequence of A/S degenerates on the first page.*

We are interested in the case $p + q = 1$ of Proposition 2.3(2) (see [4, Lemma 2.5.3]). By [4, Section 5.1.1], $R^1 f_* \mathcal{O}_A \cong \underline{\omega}^{\vee}_{A^\vee/S} \coloneqq \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\omega}_{A^\vee/S}, \mathcal{O}_S)$. In particular, we have the short exact sequence

$$0 \longrightarrow \underline{\omega}_{A/S} \longrightarrow H^1_{\mathrm{dR}}(A/S) \longrightarrow \underline{\omega}^{\vee}_{A^\vee/S} \longrightarrow 0. \tag{2.1}$$

From this and Proposition 2.3(1), we deduce that $H^1_{\mathrm{dR}}(A/S)$ has rank $2n$. Using the prime-to-p polarization λ, we can identify $\omega^{\vee}_{A^\vee/S}$ with $\omega^{\vee}_{A/S}$ to get

$$0 \longrightarrow \underline{\omega}_{A/S} \longrightarrow H^1_{\mathrm{dR}}(A/S) \longrightarrow \underline{\omega}^{\vee}_{A/S} \longrightarrow 0. \tag{2.2}$$

We call (2.1) the *natural Hodge filtration*, as opposed to (2.2), which we call the *polarized Hodge filtration*. We simply call the short exact sequences (2.1) and (2.2) the *Hodge filtration* when it is clear from the context to which one we are referring. The action of $\mathcal{O}_E$ splits the Hodge filtration according to the CM decomposition of its terms, giving

$$0 \longrightarrow \underline{\omega}_{A/S,\sigma} \longrightarrow H^1_{\mathrm{dR}}(A/S)_\sigma \longrightarrow \underline{\omega}^{\vee}_{A^\vee/S,\sigma} \longrightarrow 0, \tag{2.3}$$

$$0 \longrightarrow \underline{\omega}_{A/S,\overline{\sigma}} \longrightarrow H^1_{\mathrm{dR}}(A/S)_{\overline{\sigma}} \longrightarrow \underline{\omega}^{\vee}_{A^\vee/S,\overline{\sigma}} \longrightarrow 0, \tag{2.4}$$

where the action of $\mathcal{O}_E$ on $\underline{\omega}_{A^\vee/S}$ is induced by the action of $\mathcal{O}_E$ on $A^\vee$ given by $a \mapsto \iota(a)^\vee$. By the Rosati relation, λ gives isomorphisms $\underline{\omega}_{A^\vee/S,\sigma} \cong \underline{\omega}_{A/S,\overline{\sigma}}$, $\underline{\omega}_{A^\vee/S,\overline{\sigma}} \cong \underline{\omega}_{A/S,\sigma}$. In particular, splitting (2.2) according to the $\mathcal{O}_E$-action, we obtain

$$0 \longrightarrow \underline{\omega}_{A/S,\sigma} \longrightarrow H^1_{\mathrm{dR}}(A/S)_\sigma \longrightarrow \underline{\omega}^{\vee}_{A/S,\overline{\sigma}} \longrightarrow 0, \tag{2.5}$$

$$0 \longrightarrow \underline{\omega}_{A/S,\overline{\sigma}} \longrightarrow H^1_{\mathrm{dR}}(A/S)_{\overline{\sigma}} \longrightarrow \underline{\omega}^{\vee}_{A/S,\sigma} \longrightarrow 0. \tag{2.6}$$

We call (2.3) and (2.4) the σ and $\overline{\sigma}$-components, respectively, of the natural Hodge filtration. Similarly, we call (2.5) and (2.6) the σ and $\overline{\sigma}$-components, respectively, of the polarized Hodge filtration. From (2.5) and (2.6), given that $\underline{\omega}_{A/S,\sigma}$ and $\underline{\omega}_{A/S,\overline{\sigma}}$ have ranks $n-1$ and 1, respectively, we can deduce that $H^1_{\mathrm{dR}}(A/S)_\sigma$ and $H^1_{\mathrm{dR}}(A/S)_{\overline{\sigma}}$ both have rank n. From the Hodge filtration (2.2), we deduce the isomorphism $\det H^1_{\mathrm{dR}}(A/S) \cong \mathcal{O}_S$, which depends on the choice of the polarization λ. We call $\delta_{A/S} \coloneqq \det H^1_{\mathrm{dR}}(A/S)$ the *determinant sheaf*. Write $\delta_{A/S,\sigma} \coloneqq \det(H^1_{\mathrm{dR}}(A/S)_\sigma)$, $\delta_{A/S,\overline{\sigma}} \coloneqq \det(H^1_{\mathrm{dR}}(A/S)_{\overline{\sigma}})$. Notice that $\delta_{A/S,\sigma} \otimes \delta_{A/S,\overline{\sigma}} \cong \delta_{A/S}$. In particular, the choice of a polarization will induce an isomorphism $\delta^{-1}_{A/S,\sigma} \cong \delta_{A/S,\overline{\sigma}}$. Taking determinants of (2.3) and (2.4), we deduce

$$\det \underline{\omega}_{A/S,\sigma} \cong \delta_{A/S,\sigma} \otimes \underline{\omega}_{A^\vee/S,\sigma}, \quad \underline{\omega}_{A/S,\overline{\sigma}} \cong \delta_{A/S,\overline{\sigma}} \otimes \det \underline{\omega}_{A^\vee/S,\overline{\sigma}}, \tag{2.7}$$

identifications which do not involve the polarization. By making use of λ instead, one can deduce similar isomorphisms from (2.5) and (2.6). We call $\delta_{A/S,\sigma}$ and $\delta_{A/S,\overline{\sigma}}$ the σ and $\overline{\sigma}$-*determinant sheaf*, respectively.

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Remark 2.4 The non-canonicity of the isomorphism $\det H^1_{\mathrm{dR}}(A/S) \cong \mathcal{O}_S$ is relevant when one considers the action of Hecke operators. In fact, the isomorphism in question is not Hecke-equivariant and the invertible sheaf $\delta_{A/S}$ is not trivial as a Hecke module.

2.5 The Weyl modules of GL_m

We recall the description of a natural class of algebraic representations $\underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$ of $\mathrm{GL}_{m,R}$, $m \geq 2$, into locally free finite rank R-modules, for R a Dedekind domain. Our main reference will be [25]. Until the end of this discussion, we write $G = \mathrm{GL}_{m,R}$, to ease notations.

Let $B \subset G$ be the Borel subgroup of upper-triangular matrices, $T \subset B$ the diagonal torus, and $U \subset B$ the unipotent radical of B. We denote by $B^- \supset U^-$ the Borel of G opposite to B with respect to T, with its unipotent radical. Write $X(T) \coloneqq \mathrm{Hom}(T, \mathbb{G}_m)$ for the group of characters of T. We are interested in representations associated with dominant weights of G with respect to $B \supset T$, which form the submonoid $X(T)_+ \subset X(T)$. Through the usual isomorphism $X(T) \cong \mathbb{Z}^m$, dominant weights correspond to tuples $(k_1, k_2, \ldots, k_m) \in \mathbb{Z}^m$ such that $k_i \geq k_{i+1}$. We write $\underline{k} = (k_1, k_2, \ldots, k_m) \in X(T)$ for short. The projection $B^- \to T$ allows us to consider $\underline{k}$ a degree one representation of B^-.

By [25, Section I.5.8], we can associate to any B^--module M a sheaf $\mathcal{L}(M)$ on the quotient G/B^-. In particular, we can apply this construction to any $\underline{k} \in X(T)$. By [25, Proposition I.5.12], $H^0(G/B^-, \mathcal{L}_{\underline{k}}) \cong \mathrm{ind}_{B^-}^G(\underline{k})$. To ease notations, like [25], we write $H^i(\underline{k}) \coloneqq H^i(G/B^-, \mathcal{L}_{\underline{k}})$.
For $\underline{k}$ dominant with $k_m \geq 0$, we define a *Schur functor*

$$\mathbb{S}^{\underline{k}} \colon \underline{\mathrm{Mod}}_R \to \underline{\mathrm{Mod}}_R$$

from the category of R-modules to itself (or, more generally, the category of quasi-coherent sheaves over a scheme S). See, for instance, [1] or [20, Chapter 8]. Let $\underline{l}$ denote the partition of $|\underline{k}| = \sum_i k_i$ conjugate to $\underline{k}$, that is, $l_j \coloneqq \#\{i \mid k_i \geq j\}$, for $j \geq 1$. For $M \in \underline{\mathrm{Mod}}_R$, we have

$$\mathbb{S}^{\underline{k}}(M) = \mathrm{im}(\otimes_j \wedge^{l_j}(M) \to \otimes_j M^{\otimes l_j} \to M^{\otimes |k|} \to \otimes_i M^{\otimes k_i} \to \otimes_i \mathrm{Sym}^{k_i}(M)),$$

where the first map is induced by $\wedge^l M \to M^{\otimes l}$, $m_1 \wedge m_2 \wedge \cdots \wedge m_l \mapsto \sum_{v \in \mathrm{S}_l} \mathrm{sgn}(v) m_{v(1)} \otimes m_{v(2)} \otimes \ldots m_{v(l)}$, the maps in the middle are identifications of $M^{\otimes |k|}$, grouping its factors first in terms of the columns and then in terms of the rows of the Young diagram for $\underline{k}$, and the last map is the natural quotient. In particular, if M is free of rank r with ordered basis $m_1, m_2, \ldots, m_r \in M$, so is $\mathbb{S}^{\underline{k}}(M)$, with basis elements corresponding with semi-standard Young tableaux of diagram $\underline{k}$ and entries in $\{1, 2, \ldots, r\}$.

Example 2.5 Let $M \in \underline{\mathrm{Mod}}_R$ be locally free of constant rank r. Then, for $\underline{k} = (k, 0, \ldots, 0)$, we get $\mathbb{S}^{\underline{k}}(M) = \mathrm{Sym}^k(M)$. For $\underline{k} = (1, 1, \ldots, 1)$, instead, we have $\mathbb{S}^{\underline{k}}(M) = \wedge^m(M)$ (which is 0, if $r < m$).

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More generally, for $n \geq 1$, consider $\underline{k} = (k_1, k_2, \ldots, k_{n-1}, 0)$. The morphism defining $\mathbb{S}^{\underline{k}}(M)$ factors through a natural surjection $\otimes_{i=1}^{n-1} \mathrm{Sym}^{k_i - k_{i+1}}(\wedge^i M) \to \mathbb{S}^{\underline{k}}(M)$, which is not injective in general, but gives an isomorphism when $n = 2, 3$ and $r = n - 1$.

Proposition 2.6 *We have the following:*

(1) *The G-module $H^0(\underline{k})$ is finite locally free over R and, for a dominant weight $\underline{k}$, nonzero. Moreover, $H^i(\underline{k}) = 0$, for $i > 0$, for $\underline{k}$ dominant.*
(2) *If $k_m \geq 0$, we have a natural isomorphism $\mathbb{S}^{\underline{k}}(\mathrm{st}) \cong H^0(\underline{k})$ in $\underline{\mathrm{Rep}}_R(G)$, where $\mathrm{st} = R^m$ denotes the standard representation of G.*

Proof The first statement follows, by standard base change arguments, from the case in which R is a field. See [25, Section II.1.10(2), Section I.5.16(2), Proposition II.4.4] and Kempf's vanishing theorem, [25, Proposition II.4.5].

We recall the definition of the map $\phi: \mathbb{S}^{\underline{k}}(\mathrm{st}) \to H^0(\underline{k})$ from the last point. Let $\underline{k}$ be a dominant weight such that $k_m \geq 0$ and let I be a semi-standard Young tableaux of diagram $\underline{k}$. Let $e_I \in \mathbb{S}^{\underline{k}}(\mathrm{st})$ denote the corresponding basis element, obtained from the standard basis of st. Given I, we can also construct an element f_I of $H^0(\underline{k}) \subseteq R[X_{i,j}, (\det X)^{-1}]$. For each column of $\underline{k}$, say the j-th one, consider the minor determinant $M_{I,j}$ of the sub-matrix of $X = (X_{i,j})$ obtained removing the first l_j columns of X and the rows corresponding to the entries in the j-th column of the tableau I; if $l_1 = m$, we set $M_{I,j} = 1$. We take $f_I = \det(X)^{-k_1} \prod_j M_{I,j}$. We define the map $\phi: \mathbb{S}^{\underline{k}}(\mathrm{st}) \to H^0(\underline{k})$ sending e_I to f_I. That ϕ is an isomorphism follows from base change and Nakayama's lemma. We omit the details. ∎

We call the G-module $H^0(\underline{k})$ the *(dual) Weyl module* associated with the dominant weight $\underline{k}$. Let $\underline{k}, \underline{k}' \in X(T)_+$. Then $\underline{k} + \underline{k}' \in X(T)_+$ and we have a morphism of G-representations

$$H^0(\underline{k}) \otimes_R H^0(\underline{k}') \longrightarrow H^0(\underline{k} + \underline{k}') \tag{2.8}$$

defined by multiplication of global sections. This lets us define corresponding morphisms of twisted modules.

2.6 Twist of a sheaf by a representation

Let R be a Dedekind domain, $S \in \underline{\mathrm{Sch}}_R$, $\rho \in \underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$ and take $\mathcal{F}$ a locally free $\mathcal{O}_S$-sheaf of rank m over S. Consider the contracted product

$$F^\rho = \underline{\mathrm{Isom}}(\mathcal{O}_S^m, \mathcal{F}) \times^{\mathrm{GL}_m} \rho.$$

This is a vector bundle $F^\rho \to S$ (see [23, Exposé XI, Section 4]). The sheaf of sections $\mathcal{F}^\rho$ of F^ρ is the twist of $\mathcal{F}$ by ρ.

Example 2.7 Take $\rho = \mathbb{S}^{\underline{k}}(\mathrm{st})$, for $\underline{k}$ dominant, $k_m \geq 0$. Then $\mathcal{F}^\rho = \mathbb{S}^{\underline{k}}(\mathcal{F})$. We denote this sheaf by $\mathcal{F}^{\underline{k}}$.

The construction $\mathcal{F} \mapsto \mathcal{F}^\rho$, for a fixed ρ, is compatible with isomorphisms $\mathcal{F} \cong \mathcal{F}'$: from every isomorphism $\phi: \mathcal{F} \to \mathcal{F}'$ of locally free sheaves of rank m over S, we obtain an isomorphism $\phi^\rho: \mathcal{F}^\rho \to \mathcal{F}'^\rho$. Similarly, $\mathcal{F} \mapsto \mathcal{F}^\rho$ is compatible with twists by Frobenius, when S has positive characteristic.

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The twist $\mathcal{F}^{\rho}$ is functorial in ρ and gives rise to a tensor functor $\rho \otimes \rho' \rightsquigarrow \mathcal{F}^{\rho} \otimes \mathcal{F}^{\rho'}$. In particular, from Equation (2.8), for $\underline{k}, \underline{k}' \in X(T)_+$, we obtain a morphism of sheaves $\mathcal{F}^{\underline{k}} \otimes \mathcal{F}^{\underline{k}'} \longrightarrow \mathcal{F}^{\underline{k}+\underline{k}'}$. We often use this morphism implicitly in our constructions.

2.7 The automorphic sheaves

Take $R = \mathcal{O}_{E,(p)}$. By Lemma 2.2, we have $\mathbf{G}_{1,R} \cong \mathrm{GL}_{n,R}$ and $H_{1,R} \cong \mathrm{GL}_{n-1} \times \mathrm{GL}_1$. Let $S \in \underline{\mathrm{Sch}}_R$ and consider $\underline{A} = (A, \lambda, \iota, \eta_K) \in \mathcal{S}_K(S)$. For $\underline{k}$ a dominant weight for GL_{n-1} and $w \in \mathbb{Z}$, we define the sheaves

$$\underline{\omega}^{\underline{k},w}_{\underline{A}/S} \coloneqq \underline{\omega}^{\underline{k}}_{\underline{A}/S,\sigma} \otimes \delta^{w}_{A/S,\sigma}.$$

Definition 2.1 (Automorphic sheaves) Let $\underline{k}$ be a dominant weight for GL_{n-1} and $w \in \mathbb{Z}$. We call $\underline{\omega}^{\underline{k},w}_{\underline{A}/S}$ the *automorphic sheaf of weight* $(\underline{k}, w)$ *over S associated with* $\underline{A}$.

This construction is compatible with base change, thus we work mainly with the automorphic sheaves of weights $(\underline{k}, w)$ associated with $\underline{A}$ in the universal class of $\mathcal{S}_K$, or some base change of it. We denote these sheaves by $\underline{\omega}^{\underline{k},w}$. Our convention for $\underline{\omega}^{\underline{k},w}$ has the advantage of allowing us to define Hecke-equivariant theta operators. For R any $\mathcal{O}_{E,(p)}$-algebra, let $B = \mathrm{Spec}\, R$ and consider $\mathcal{S}_{K,B}$ the base change of $\mathcal{S}_K$.

Definition 2.2 We call $H^0(\mathcal{S}_{K,B}, \underline{\omega}^{\underline{k},w}_B)$ the space of *modular forms of weight* $(\underline{k}, w)$ *and level K with coefficients in R*.

Remark 2.8 (Algebraic Koecher principle) In our setting, the results of [35] apply, since $\dim(\mathcal{S}^{\min}_K \backslash \mathcal{S}_K) = n - 1 \geq 2$, and we have, for a smooth, projective, toroidal compactification $\mathcal{S}^{\mathrm{tor}}_K$ and the minimal compactification $\mathcal{S}^{\min}_K$, that

$$H^0(\mathcal{S}_{K,B}, \underline{\omega}^{\underline{k},w}_B) = H^0(\mathcal{S}^{\mathrm{tor}}_{K,B}, \underline{\omega}^{\mathrm{can},\underline{k},w}_B) = H^0(\mathcal{S}^{\min}_{K,B}, \underline{\omega}^{\min,\underline{k},w}_B).$$

This is called the *Koecher principle*. This allows us to carry out most of our constructions and computations on $\mathcal{S}_{K,B}$, while still being able to deduce interesting consequences that make use of the geometry of $\mathcal{S}^{\mathrm{tor}}_{K,B}$ and $\mathcal{S}^{\min}_{K,B}$ and the extensions of the bundles $\underline{\omega}^{\underline{k},w}_B$.

3 Hecke operators

Here, we recall the definition of tame Hecke operators via cohomological correspondences.

Let $f\colon X \to Y$ be a finite flat morphism of schemes and $\mathcal{G}$ a quasi-coherent sheaf over Y. Since f is affine, quasi-coherent $f_*(\mathcal{O}_X)$-modules are equivalent to quasi-coherent $\mathcal{O}_X$-modules, via f_*. Write $f^!\mathcal{G}$ for the $\mathcal{O}_X$-sheaf corresponding to $f_*(\mathcal{O}_X)^\vee \otimes_{\mathcal{O}_Y} \mathcal{G}$. The trace map $\mathrm{Tr}_{\mathcal{G}}$ is a natural morphism $\mathrm{Tr}_{\mathcal{G}}\colon f^*\mathcal{G} \longrightarrow f^!\mathcal{G}$. This map is an isomorphism when f is finite étale. Since the functor $f^!$ is right adjoint to f_*, we have a counit morphism $f_*f^!\mathcal{G} \to \mathcal{G}$. The composition

$$\mathrm{tr}_{\mathcal{G}}\colon f_*f^*\mathcal{G} \xrightarrow{\mathrm{Tr}_{\mathcal{G}}} f_*f^!\mathcal{G} \longrightarrow \mathcal{G}$$

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is also sometimes called the trace morphism. Let X, Y be two schemes over a base scheme S and $\mathcal{F}, \mathcal{G}$, two quasi-coherent modules, over X and Y, respectively.

Definition 3.1 A *finite flat correspondence* between $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ is the datum (Z, p_1, p_2, ϕ) of an S-scheme Z with $p_1 \colon Z \to X$, $p_2 \colon Z \to Y$, finite flat morphisms of S-schemes, and $\phi \colon p_2^* \mathcal{G} \longrightarrow p_1^* \mathcal{F}$ a morphism of quasi-coherent sheaves over Z.

From (Z, p_1, p_2, ϕ), we obtain

$$T_\phi = T_\phi^i \colon H^i(Y, \mathcal{G}) \xrightarrow{p_2^*} H^i(Z, p_2^* \mathcal{G}) \xrightarrow{H^i(\phi)} H^i(Z, p_1^* \mathcal{F}) \xrightarrow{\mathrm{tr}_{\mathcal{F}}} H^i(X, \mathcal{F}), \quad i \geq 0,$$

a family of morphisms T_ϕ^i in cohomology. If $S = \mathrm{Spec}\, R$, then the T_ϕ^i are R-linear.

Write $\mathbf{G}^{p,\infty} = \mathbf{G}(\mathbb{A}^{p,\infty})$, consider $g \in \mathbf{G}^{p,\infty}$ and take $K_g = K \cap gKg^{-1}$. In particular, $K_{g,p} = K_p = \mathbf{G}(\mathbb{Z}_p)$ and since we assumed that K is neat, so is K_g. Consider also $g^{-1}K_g g = K \cap g^{-1}Kg$. Then we have two finite étale maps

$$p_1 \colon \mathcal{S}_{K_g} \longrightarrow \mathcal{S}_K, \quad p_2 \colon \mathcal{S}_{K_g} \longrightarrow \mathcal{S}_K.$$

We sketch their definitions. The first one is defined on points of $\mathcal{S}_{K_g}$ by sending

$$\underline{A} = (A, \iota, \lambda, \eta_{K_g}) \longmapsto (A, \iota, \lambda, \eta_K),$$

where η_K is the level K-structure obtained from η_{K_g} taking the K-orbit of any η in η_{K_g}. In particular, we see that p_1 has degree given by the cardinality of the finite set K^p / K_g^p. On the other hand, p_2 is obtained as the composition of the finite, étale, surjective morphism $\mathcal{S}_{g^{-1}K_g g} \longrightarrow \mathcal{S}_K$ defined like p_1, with the map

$$\begin{aligned} [g] \colon \mathcal{S}_{K_g} &\longrightarrow \mathcal{S}_{g^{-1}K_g g}, \\ (A, \iota, \lambda, \eta_{K_g}) &\longmapsto (A', \iota', \lambda', \eta'_{g^{-1}K_g g}), \end{aligned}$$

where A' is the unique (up to isomorphism) abelian scheme prime-to-p quasi-isogenous to A via $f \colon A \to A'$ such that $V^p(f_s)^{-1}(T^p A'_s) = \eta(g(\Lambda \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p))$, for s any geometric point in S, and $\iota', \lambda', \eta'_{g^{-1}K_g g}$ defined as in [34, Proposition 1.4.3.4]. See also [34, Corollary 1.3.5.4]. Then, $[g]$ is clearly an isomorphism with inverse $[g^{-1}]$. For any $g' \in K^p g K^p$ the morphisms $p_{1,g'}, p_{2,g'}$ are the same as $p_{1,g}, p_{2,g}$, respectively. In particular, both p_1 and p_2 only depend on the double coset $K^p g K^p$.

For every automorphic weight $(\underline{k}, w)$, we have the natural identifications given by base change

$$p_1^* \underline{\omega}^{\underline{k},w}_{A/\mathcal{S}_K} \cong \underline{\omega}^{\underline{k},w}_{A/\mathcal{S}_{K_g}} \cong p_2^* \underline{\omega}^{\underline{k},w}_{A/\mathcal{S}_K}.$$

More generally, suppose that we have a family $\{\mathcal{F}_{K'}\}_{K'}$, where the $K' \subseteq K$ are neat p-hyperspecial compact opens, and the $\mathcal{F}_{K'}$ are quasi-coherent sheaves on $\mathcal{S}_{K'}$

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compatible with pullback along the morphisms

$$\mathcal{S}_{K'} \longrightarrow \mathcal{S}_{K''}, \quad K' \subseteq K'' \subseteq K,$$
$$[g]\colon \mathcal{S}_{K'} \longrightarrow \mathcal{S}_{g^{-1}K'g}, \quad g \in \mathbf{G}^{p,\infty}.$$

We obtain a natural cohomological correspondence $(\mathcal{S}_{K_g}, p_1, p_2, \phi)$ and $\mathcal{O}_{E,(p)}$-linear operators

$$T_g = T_g^i \colon H^i(\mathcal{S}_K, \mathcal{F}_K) \longrightarrow H^i(\mathcal{S}_K, \mathcal{F}_K).$$

We call T_g the (tame) Hecke operator given by $g \in \mathbf{G}^{p,\infty}$. Let R be an $\mathcal{O}_{E,(p)}$-algebra. We can pull the Hecke correspondence back to $B = \operatorname{Spec} R$ and obtain R-linear operators

$$T_g \colon H^i(\mathcal{S}_{K,B}, \mathcal{F}_K) \longrightarrow H^i(\mathcal{S}_{K,B}, \mathcal{F}_K).$$

Furthermore, suppose that we have a family of locally closed (resp. closed, resp. open) subschemes $\{\tilde{\mathcal{S}}_{K',B} \to \mathcal{S}_{K',B}\}_{K'}$, compatible in the sense that the diagrams

$$\begin{array}{ccc} \tilde{\mathcal{S}}_{K',B} & \longrightarrow & \mathcal{S}_{K',B} \\ \downarrow & \ulcorner & \downarrow p_{K'/K''} \\ \tilde{\mathcal{S}}_{K'',B} & \longrightarrow & \mathcal{S}_{K'',B}, \end{array}$$

for $K' \subseteq K'' \subseteq K$ are all Cartesian. Again, we can pull the Hecke correspondence back to $\tilde{\mathcal{S}}_{\cdot,B}$ and obtain

$$T_g \colon H^i(\tilde{\mathcal{S}}_{K,B}, \mathcal{F}_K) \longrightarrow H^i(\tilde{\mathcal{S}}_{K,B}, \mathcal{F}_K).$$

This allows us to talk about Hecke operators on non-maximal Ekedahl–Oort strata. The action of T_g makes sense for compatible families of quasi-coherent sheaves $\{\mathcal{F}_{K'}\}_{K'}$ defined on the subschemes $\{\tilde{\mathcal{S}}_{K',B}\}_{K'}$, even when these are not defined as restrictions of sheaves on the tower $\{\mathcal{S}_{K',B}\}_{K'}$.

Remark 3.1 (Unramified Hecke algebra) The construction given above defines an R-linear action of the *unramified Hecke algebra* on $H^\bullet(\tilde{\mathcal{S}}_{K,B}, \mathcal{F}_K)$. This algebra is the restricted tensor product of the local Hecke algebras at unramified primes $\mathcal{H}_{\mathrm{ur}}(\mathbf{G}, K, R) = \otimes'_{\ell \nmid pD_{E/\mathbb{Q}}} \mathcal{H}(\mathbf{G}_\ell, K_\ell, R)$, where $\mathbf{G}_\ell = \mathbf{G}(\mathbb{Q}_\ell)$. The local Hecke algebras $\mathcal{H}(\mathbf{G}_\ell, K_\ell, R)$ are the convolution algebras of locally constant and compactly supported functions $f\colon K_\ell \backslash \mathbf{G}_\ell / K_\ell \to R$. The action is given by

$$f \longmapsto \sum_{K_\ell g K_\ell} f(g) T_g \in \operatorname{End}_R(H^\bullet(\tilde{\mathcal{S}}_{K,B}, \mathcal{F}_K)),$$

which is well defined because of the properties of f.

Remark 3.2 (Hecke operators at p) As shown in [18, Chapter 7], one can define an action of Hecke operators *at* p, for some $g \in \mathbf{G}(\mathbb{Q}_p)$. We will not discuss these operators, but we remark that they are still constructed from cohomological correspondences, even though the maps p_1, p_2, notation as above, involved are finite flat and *not étale*, so one has to take the ramification into account and normalize the action accordingly.

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4 Gauss–Manin connection and Kodaira–Spencer map

In this section, we recall how the Gauss–Manin connection and the Kodaira–Spencer map are defined and how they are related. We then proceed to describe some of their basic properties. The classical reference for the definition of the GM connection is [32]. We follow *loc. cit.* closely, but also benefit from the exposition given in the first chapter of [28] and in [27]. For the more context-specific, or arithmetic, properties of the KS morphism we will follow [34, Section 2.3.4] and provide other references where we deemed it necessary.

4.1 Generalities on filtrations

We record here general, basic facts concerning filtrations.

4.1.1 Tensor product of filtrations

Let S be a scheme and consider $(\mathcal{F}, F^\bullet(\mathcal{F}))$, $(\mathcal{G}, F^\bullet(\mathcal{G}))$ finite locally free sheaves over S endowed with finite, separated, exhaustive, descending filtrations

$$\mathcal{F} = F^0(\mathcal{F}) \supseteq F^1(\mathcal{F}) \supseteq \cdots \supseteq F^r(\mathcal{F}) \supseteq F^{r+1}(\mathcal{F}) = 0,$$
$$\mathcal{G} = F^0(\mathcal{G}) \supseteq F^1(\mathcal{G}) \supseteq \cdots \supseteq F^s(\mathcal{G}) \supseteq F^{s+1}(\mathcal{G}) = 0,$$

by finite locally free subsheaves $F^i(\mathcal{F}), F^j(\mathcal{G})$, such that the graded pieces $\mathrm{gr}^i\mathcal{F}, \mathrm{gr}^j\mathcal{G}$, are themselves finite locally free. Then we can defined a tensor product filtration on $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ with the same properties by setting

$$F^k(\mathcal{F} \otimes \mathcal{G}) \coloneqq \sum_{i+j=k} F^i(\mathcal{F}) \otimes F^j(\mathcal{G}).$$

We write $P(\mathcal{F} \otimes \mathcal{G}) = F^{r+s-1}(\mathcal{F} \otimes \mathcal{G})$ to denote the penultimate step of the filtration. The graded pieces of $F^\bullet(\mathcal{F} \otimes \mathcal{G})$ are $\mathrm{gr}^k(\mathcal{F} \otimes \mathcal{G}) \cong \oplus_{i+j=k}\mathrm{gr}^i(\mathcal{F}) \otimes \mathrm{gr}^j(\mathcal{G})$. If we have a finite family of finite locally free sheaves $(\mathcal{F}_i, F^\bullet(\mathcal{F}_i))_{1\leq i\leq m}$ with filtrations subject to the same conditions as above, we can define by induction a filtration on their tensor product. Again, we denote by $P(\otimes_{i=1}^m \mathcal{F}_i)$ the penultimate step of the filtration.

4.1.2 Dual filtration

Let S and $(\mathcal{F}, F^\bullet(\mathcal{F}))$ be as above. We can define a decreasing filtration on $\mathcal{F}^\vee$ by setting $F^i(\mathcal{F}^\vee) \coloneqq \left(\mathcal{F}/F^{r-i+1}(\mathcal{F})\right)^\vee$. Its graded pieces satisfy $\mathrm{gr}^i(\mathcal{F}^\vee) \cong \mathrm{gr}^{r-i}(\mathcal{F})^\vee$. Its penultimate step is $P(\mathcal{F}^\vee) = F^{r-1}(\mathcal{F}^\vee)$.

4.1.3 Koszul filtration

Let S be as above and consider a short exact sequence of finite locally free sheaves over S of constant rank

$$0 \to \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \to 0.$$

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The *Koszul filtration* of $\wedge^j\mathcal{F}$, for $0 \le j \le \mathrm{rk}(\mathcal{F})$, is defined by

$$K^i(\wedge^j\mathcal{F}) \coloneqq \mathrm{im}(\wedge^i\mathcal{F}' \otimes_{\mathcal{O}_S} \wedge^{j-i}\mathcal{F} \longrightarrow \wedge^j\mathcal{F}), \quad 0 \le i \le j.$$

The penultimate step of this filtration is $P(\wedge^j\mathcal{F}) = K^{j-1}(\wedge^j\mathcal{F})$. This filtration can also obtained by taking quotients from the tensor product filtration on $\mathcal{F}^{\otimes j}$. We extend this filtration by setting $K^i(\wedge^j\mathcal{F}) = \wedge^j\mathcal{F}$ for $i < 0$ and $K^i(\wedge^j\mathcal{F}) = 0$ for $i > j$. We have $K^i \wedge K^{i'} \subseteq K^{i+i'}(\wedge^\bullet\mathcal{F})$. The graded terms of the filtration are

$$\mathrm{gr}^i(\wedge^j\mathcal{F}) = K^i/K^{i+1} \cong \wedge^i\mathcal{F}' \otimes_{\mathcal{O}_S} \wedge^{j-i}\mathcal{F}'', \quad 0 \le i \le j \le \mathrm{rk}(\mathcal{F}).$$

4.1.4 Filtration of the symmetric algebra

Let S be as above. Let $\mathcal{G}$ be a finite locally free sheaf. For $j < 0$, we set $\mathrm{Sym}^j(\mathcal{G}) \coloneqq \left(\mathrm{Sym}^{-j}(\mathcal{G})\right)^\vee$ and whenever $\mathrm{Sym}^{-j}(\mathcal{G})$ is endowed with a filtration, $\mathrm{Sym}^j(\mathcal{G})$ is endowed with the dual filtration. One can consider $(\mathcal{F}, F^\bullet(\mathcal{F}))$ as above. By taking the quotient of the filtration on $\mathcal{F}^{\otimes j}$, we get a filtration on $\mathrm{Sym}^j\mathcal{F}$.

Remark 4.1 We will consider a filtration on $\mathbb{S}^{\underline{k}}(H_\sigma)$, where $\underline{k} = (k_1 \ge k_2 \ge \cdots \ge k_{n-1}) \in \mathbb{Z}^{n-1}_{\ge 0}$ and $H = H^1_{\mathrm{dR}}(A/S)$, for A/S an abelian scheme with $\mathcal{O}_E$-action. To construct it, first, we consider the Koszul filtration $K^\bullet(\wedge^j H_\sigma)$ induced by the Hodge filtration (2.5). Taking tensor products, we obtain a filtration on $\otimes_j \wedge^{l_j}(H_\sigma)$, $\underline{l}$ conjugate to $\underline{k}$. Similarly, we construct filtrations on $H_\sigma^{|\underline{k}|}$ and $\otimes_i \mathrm{Sym}^{k_i}(H_\sigma)$. Finally, we notice that the construction of $\mathbb{S}^{\underline{k}}(H_\sigma)$ is compatible with these filtrations. In particular, the penultimate step of this filtration is given by the image of

$$\bigoplus_j \left(\left(\bigotimes_{j' \ne j} \wedge^{l_{j'}} \underline{\omega}_\sigma \right) \bigotimes (\wedge^{l_j - 1} \underline{\omega}_\sigma \otimes H_\sigma) \right)$$

in $\mathbb{S}^{\underline{k}}(H_\sigma)$. When constructing generalized theta operators, we will consider more complicated filtrations, but they will be built following the same ideas.

4.2 Generalities on connections

We fix for the rest of this section the following data, a special case of [28, Section 1]: $X \xrightarrow{f} S \xrightarrow{g} B$, where f is smooth and proper and g is smooth. In particular, the sheaves $H^i_{\mathrm{dR}}(X/S)$ are finite locally free. We are mainly interested in the case $X = A$, where $\underline{A}$ is the universal object over $S = \mathcal{S}_{K,B}$, for some $B \in \underline{\mathrm{Sch}}_{\mathcal{O}_{E,(p)}}$.

Definition 4.1 (Connection) Let $\mathcal{F}$ be a quasi-coherent sheaf on S. A *B-linear connection* on $\mathcal{F}$ is a morphism $\nabla\colon \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \Omega^1_{S/B}$ of abelian sheaves satisfying the *Leibniz rule*

$$\nabla(hs) = h\nabla(s) + s \otimes d_{S/B}(h),$$

where h and s are sections of $\mathcal{O}_S$ and $\mathcal{F}$, respectively, over some open $U \subset S$ and $d_{S/B}\colon \mathcal{O}_S \to \Omega^1_{S/B}$ is the exterior differential.

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We can derive from ∇, for $(\mathcal{F}, \nabla)$ a sheaf with connection, a family of morphisms of abelian sheaves

$$\nabla_i : \mathcal{F} \otimes \Omega^i_{S/B} \longrightarrow \mathcal{F} \otimes \Omega^{i+1}_{S/B}$$

defining it on sections s and ω of $\mathcal{F}$ and $\Omega^i_{S/B}$ via a generalized Leibniz rule

$$s \otimes \omega \longmapsto \nabla(s) \wedge \omega + s \otimes d\omega$$

and then extending by linearity. The *curvature* of the connection ∇ is defined as $K(\nabla) \coloneqq \nabla_1 \circ \nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^2_{S/B}$, which is an $\mathcal{O}_S$-linear morphism. A connection is said to be *integrable* if $K(\nabla) = 0$.

Given two quasi-coherent sheaves with connections $(\mathcal{F}, \nabla_{\mathcal{F}}), (\mathcal{G}, \nabla_{\mathcal{G}})$, we can use the Leibniz rule to define the tensor product connection $(\mathcal{F} \otimes \mathcal{G}, \nabla_{\mathcal{F}\otimes\mathcal{G}})$, obtained on local sections as

$$\begin{aligned} \nabla_{\mathcal{F}\otimes\mathcal{G}} : \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G} &\longrightarrow \mathcal{F} \otimes \mathcal{G} \otimes \Omega^1_{S/B}, \\ f \otimes g &\longmapsto \nabla(f) \otimes g + f \otimes \nabla(g) \end{aligned}$$

and then extending by linearity, where we identify $\mathcal{F} \otimes \Omega^1 \otimes \mathcal{G} \cong \mathcal{F} \otimes \mathcal{G} \otimes \Omega^1$. Thus, from $(\mathcal{F}, \nabla)$, we can define a natural connection $\nabla : \mathcal{F}^{\otimes j} \to \mathcal{F}^{\otimes j} \otimes \Omega^1_{S/B}$, for $j \geq 0$, still denoted by ∇. From this, taking quotients, we get well-defined connections on $\wedge^j\mathcal{F}$ and $\mathrm{Sym}^j\mathcal{F}$.

Lemma 4.2 (Koszul transversality) *Consider $(\mathcal{F}, F^\bullet(\mathcal{F}))$ a finite locally free sheaf finitely filtered by finite locally free sheaves with finite locally free quotients. Suppose that $\mathcal{F}$ is endowed with a connection ∇ such that $\nabla(F^i(\mathcal{F})) \subseteq F^{i-1}(\mathcal{F})$. Then, for any $j \geq 1$, the induced filtration $(\mathcal{F}^{\otimes j}, F^\bullet)$ and connection $(\mathcal{F}^{\otimes j}, \nabla)$ satisfy*

$$\nabla(F^i(\mathcal{F}^{\otimes j})) \subseteq F^{i-1}(\mathcal{F}^{\otimes j}) \otimes \Omega^1_{S/B},$$

for $i \geq 1$. Similarly,

$$\nabla(K^i(\wedge^j\mathcal{F})) \subseteq K^{i-1}(\wedge^j\mathcal{F}) \otimes \Omega^1_{S/B}, \nabla(F^i(\mathrm{Sym}^j\mathcal{F})) \subseteq F^{i-1}(\mathrm{Sym}^j(\mathcal{F})) \otimes \Omega^1_{S/B}.$$

Proof This follows immediately from the Leibniz rule. ∎

4.3 The Gauss–Manin connection

For more details, see [32, Section 2]. The first exact sequence of $X/S/B$ is

$$0 \longrightarrow f^*\Omega^1_{S/B} \longrightarrow \Omega^1_{X/B} \longrightarrow \Omega^1_{X/S} \longrightarrow 0. \tag{4.1}$$

The corresponding Koszul filtration of $\Omega^j_{X/B}$ has graded pieces $\mathrm{gr}^i(\Omega^j_{X/B}) = \Omega^{j-i}_{X/S} \otimes_{\mathcal{O}_X} f^*\Omega^i_{S/B}$. Since $K^i(\Omega^\bullet_{X/B})$ is a sub-complex of $\Omega^\bullet_{X/B}$, the finite filtration $K^\bullet(\Omega^\bullet_{X/B})$ gives a convergent spectral sequence whose first page is

$$E_1^{i,j} \coloneqq R^{i+j}f_*(\mathrm{gr}^i(\Omega^\bullet_{X/B})) \cong H^j_{\mathrm{dR}}(X/S) \otimes_{\mathcal{O}_S} \Omega^i_{S/B}.$$

Consider $\nabla = \nabla^j = d_1^{0,j}$, the differentials of this spectral sequence. We have the following.

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Proposition 4.3 *The map ∇^j is an integrable connection on $H^j_{\mathrm{dR}}(X/S)$.*

Definition 4.2 (Gauss–Manin connection) We call ∇ the *Gauss–Manin connection on $H^i_{\mathrm{dR}}(X/S)$ (relative to $X/S/B$).*

Since the GM connection and the ones derived from it are essentially the only connections we will discuss, we reserve the notation ∇ for them, from now on.

Proposition 4.4 (Griffiths transversality) *Let $F^\bullet = F^\bullet(H^\bullet_{\mathrm{dR}}(X/S))$ denote the Hodge filtration. The GM connection satisfies*

$$\nabla(F^i(H^j_{\mathrm{dR}}(X/S))) \subseteq F^{i-1}(H^j_{\mathrm{dR}}(X/S)) \otimes \Omega^1_{X/S}.$$

In particular, the GM connection induces a mapping between graded terms.

For more details, see [28, Section 1.4.1, Proposition 1.4.1.6].

4.3.1 Base change

Let $f' : X' \to S$ be a proper, smooth morphism and let $\phi : X \to X'$ be a morphism of S-schemes. The construction of the GM connection is functorial in the following sense.

Lemma 4.5 *We have the natural commutative diagram*

$$\begin{array}{ccc} H^i_{\mathrm{dR}}(X'/S) & \xrightarrow{\nabla} & H^i_{\mathrm{dR}}(X'/S) \otimes \Omega^1_{S/B} \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^* \otimes \mathrm{id}} \\ H^i_{\mathrm{dR}}(X/S) & \xrightarrow{\nabla} & H^i_{\mathrm{dR}}(X/S) \otimes \Omega^1_{S/B}. \end{array}$$

Let T be a smooth scheme over B and $u : T \to S$ a morphism of B-schemes. The base change map $u^* H^i_{\mathrm{dR}}(X/S) \longrightarrow H^i_{\mathrm{dR}}(X_T/T)$ is an isomorphism for every u. The construction of ∇ immediately implies the following.

Lemma 4.6 *With X, S, B, and $u : T \to S$ as above, we have the commutative diagram*

$$\begin{array}{ccc} u^* H^i_{\mathrm{dR}}(X/S) & & \\ \| & \searrow^{u^*(\nabla)} & \\ H^i_{\mathrm{dR}}(X_T/T) & \xrightarrow{\nabla} & H^i_{\mathrm{dR}}(X_T/T) \otimes \Omega^1_{T/B}, \end{array}$$

where the diagonal map is the pullback connection induced by base change $u^* H^i_{\mathrm{dR}}(X/S) \cong H^i(X_T/T)$ and the natural map $du : u^* \Omega^1_{S/B} \to \Omega^1_{T/B}$.

Remark 4.7 Consider the case $X = A$, for $\underline{A} \in \mathcal{S}_{K,B}(S)$, $B \in \underline{\mathrm{Sch}}_{\mathcal{O}_{E,(p)}}$. Lemma 4.5 implies that the GM connection respects the $\mathcal{O}_E$-action on $H^i_{\mathrm{dR}}(A/S)$, that is, for $H = H^1_{\mathrm{dR}}(A/S)$, we have restrictions

$$\nabla : H_\sigma \longrightarrow H_\sigma \otimes_{\mathcal{O}_S} \Omega^1_{S/B}, \nabla : H_{\overline{\sigma}} \longrightarrow H_{\overline{\sigma}} \otimes_{\mathcal{O}_S} \Omega^1_{S/B}.$$

Similarly, if B is of characteristic p, we can consider the isogenies Frobenius and Verschiebung and take their pullbacks $F : H^{(p)} \to H$ and $V : H \to H^{(p)}$. Lemma 4.5

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tells us that ∇ commutes with F, V, and therefore it respects kernels and images of both. This will be relevant when defining generalized theta operators.

4.4 The Kodaira–Spencer map

The short exact sequence (4.1) gives a natural section

$$\underline{\mathrm{KS}} \in H^0(S, R^1 f_*(\underline{\mathrm{Der}}(X/S) \otimes_{\mathcal{O}_X} f^* \Omega^1_{S/B})) \cong \mathrm{Hom}_S(\underline{\mathrm{Der}}(S/B), R^1 f_*(\underline{\mathrm{Der}}(X/S))),$$

called the *Kodaira–Spencer section*.

Definition 4.3 The morphism $\underline{\mathrm{KS}}\colon \underline{\mathrm{Der}}(S/B) \to R^1 f_*(\underline{\mathrm{Der}}(X/S))$ is called the *Kodaira–Spencer mapping*.

Proposition 4.8 [28, Proposition 1.4.1.7] *Suppose that the Hodge–de Rham spectral sequence of X/S degenerates on the first page. Then, the morphism induced on the graded pieces by the GM connection, as in Proposition 4.4, is given by the cup product with the KS section.*

Let $X = A$ be an abelian scheme over S. We are interested in the special case of Proposition 4.8 for $A/S/B$. Let $H = H^1_{\mathrm{dR}}(A/S)$. We have, on the one hand, a morphism:

$$\underline{\omega}_{A/S} \subset H \xrightarrow{\nabla} H \otimes \Omega^1_{S/B} \twoheadrightarrow \underline{\omega}^{\vee}_{A^\vee/S} \otimes_{\mathcal{O}_S} \Omega^1_{S/B}$$

whence, equivalently, a morphism

$$\underline{\omega}_{A/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{A^\vee/S} \longrightarrow \Omega^1_{S/B}.$$

On the other hand, $R^1 f_* \underline{\mathrm{Der}}(A/S) \cong \underline{\omega}^{\vee}_{A^\vee/S} \otimes_{\mathcal{O}_S} \underline{\omega}^{\vee}_{A/S}$. By duality, the KS morphism is equivalent to

$$\underline{\mathrm{KS}}\colon \underline{\omega}_{A/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{A^\vee/S} \longrightarrow \Omega^1_{S/B}.$$

Then, Proposition 4.8 tells us that these two morphisms $\underline{\omega}_{A/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{A^\vee/S} \to \Omega^1_{S/B}$ are the same.

Assume now that $A \to S = \mathcal{S}_{K,B}$ is the universal object, $B \to \mathrm{Spec}\, \mathcal{O}_{E,(p)}$. Then, using the identification,

$$\underline{\omega}_{A/S,\sigma} \otimes_{\mathcal{O}_S} \underline{\omega}_{A^\vee/S,\sigma} \cong \underline{\omega}_{A/S,\sigma} \otimes_{\mathcal{O}_S} \det \underline{\omega}_{A/S,\sigma} \otimes \delta^{-1}_{A/S,\sigma},$$

which follows from Equation (2.7), we see that $\underline{\mathrm{KS}}$ induces a morphism

$$\underline{\mathrm{ks}} = \underline{\mathrm{ks}}_{A/S/B} \coloneqq \underline{\mathrm{KS}}_\sigma\colon \underline{\omega}_{A/S,\sigma} \otimes_{\mathcal{O}_S} \det \underline{\omega}_{A/S,\sigma} \otimes \delta^{-1}_{A/S,\sigma} \longrightarrow \Omega^1_{S/B}.$$

Then we have the following.

Proposition 4.9 [34, Proposition 2.3.4.2] *The map $\underline{\mathrm{ks}}$ is an isomorphism.*

Proof The proof in [34, Proposition 2.3.4.2] is given for $B = \mathrm{Spec}\, \mathcal{O}_{E,(p)}$ but the construction of $\underline{\mathrm{ks}}$ is compatible with base change, so the statement over a general base $B \to \mathrm{Spec}\, \mathcal{O}_{E,(p)}$ follows. Notice that, in the notation of [34, Definition 2.3.4.1], the injection $\underline{\omega}_{A/S,\sigma} \otimes_{\mathcal{O}_S} \underline{\omega}_{A^\vee/S,\sigma} \to \underline{\omega}_{A/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{A^\vee/S}$ composed with the projection

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$$\underline{\omega}_{A/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{A^\vee/S} \to \underline{\omega}_{A/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{A^\vee/S} \Big/ \big(\lambda^*(y) \otimes z - \lambda^*(z) \otimes y, (\alpha \cdot x) \otimes y - x \otimes (\alpha \cdot y)\big)$$

is an isomorphism, where $x \in \underline{\omega}_{A/S}(U), y, z \in \underline{\omega}_{A^\vee/S}(U), \alpha \in \mathcal{O}_E$ are sections over $U \subset S$ some open.

Equivalently, from [19, Section III.9], one can deduce a description of the KS map in terms of the universal vector extension. This and an argument using Grothendieck–Messing theory, as explained, for instance, in [22, pp. 116–118], also gives a proof of the proposition: see [3, Proposition 2.1.5] for the case $n = 3$, which can be easily generalized to our setting. See also the proof of Proposition 5.5. ∎

5 Ekedahl–Oort stratification

We refer to [5, 37] for the general theory of EO stratifications on PEL Shimura varieties and to [39] for the unitary case in particular.

We write S_K for the geometric special fiber $\mathcal{S}_K \times \mathrm{Spec}(\mathbb{F})$. By Lemma 2.2, $\mathbf{G}_{\mathbb{F}} \cong \mathrm{GL}_n \times \mathbb{G}_m$ and we can take $T \subset B \subset \mathbf{G}_{\mathbb{F}}$ to be the diagonal torus and Borel of upper-triangular matrices, respectively, determined by this isomorphism. Let W be the Weyl group of $\mathbf{G}_{\mathbb{F}}$ with respect to T, with set of simple reflections Δ determined by B. We can identify the Coxeter system (W, Δ) with S_n, the symmetric group of order n, together with the simple reflections $\{s_i = (i, i+1), 1 \le i \le n-1\}$. Write $[s(1), s(2), \ldots, s(n)]$ for the permutation of n elements acting as $i \mapsto s(i)$.

Denote by $\underline{\mathrm{BT}}_{1,\mathbb{F}}^{(n-1,1)}$ the set of isomorphism classes of principally quasi-polarized BT_1's over $\mathbb{F}$ with $\mathcal{O}_E$-action, height $2n$, and CM dimensions $d_\sigma = n-1, d_{\overline{\sigma}} = 1$. Let $G \in \underline{\mathrm{BT}}_{1,\mathbb{F}}^{(n-1,1)}$ and consider $D = \mathbb{D}(G)$ the (contravariant) Dieudonné module of G. By [5, Proposition 4.1.17–4.1.19], we can choose an isomorphism $D \cong \Lambda_{\mathbb{F}}$, with Λ as in Section 2.1, compatible with the $\mathcal{O}_E$-actions on both sides, as well as the symplectic pairings, the one on D being induced by $\mathbb{D}(\lambda)$, for λ the quasi-polarization of G, and the one on $\Lambda_{\mathbb{F}}$ being part of the datum. This choice endows D with a natural action of the reductive group $\mathbf{G}_{\mathbb{F}}$. We consider the parabolic subgroup $P \subset \mathbf{G}_{\mathbb{F}}$, of type $I \subseteq \Delta$, defined as the stabilizer of the flag $\ker F \subset D$. Write ${}^I W$ for the set of minimal length representatives of the quotient $W_I \backslash W$, where W_I is the Weyl group of P. Notice that the type I does not depend on the choice of the isomorphism $D \cong \Lambda_{\mathbb{F}}$, or of G, by [5, Proposition 4.19].

Lemma 5.1 [39, Example 3.1.2, Lemma 3.4.1] *The type I of the parabolic P is the same as that of the parabolic fixing the filtration $\Lambda_{1,\mathbb{F}} \subseteq \Lambda_{\mathbb{F}} \cong \Lambda_{1,\mathbb{F}} \oplus \Lambda_{2,\mathbb{F}}$ from Lemma 2.2. We have $I = \Delta \backslash \{s_{n-1}\}$. Moreover, the elements of ${}^I W$ are of the form $w_r = [1, 2, \ldots, r-1, n, r, \ldots, n-1]$, with $l(w_r) = n - r$. The order on these elements is $w_1 \ge w_2 \ge \cdots \ge w_n = \mathrm{id}$.*

For each $G \in \underline{\mathrm{BT}}_{1,\mathbb{F}}^{(n-1,1)}$, we also have a flag $D_\bullet$ in $\mathbb{D}(G)$ coming from the *canonical filtration* of G (see, for instance, [5, Section 4.2]). Denote by $P_G \subset \mathbf{G}_{\mathbb{F}}$ the parabolic fixing this flag and write $w(G) = w(P, P_G) \in {}^I W$ for the relative position of the two parabolic subgroups P and P_G.

Theorem 5.2 [5, Proposition 4.3.2, Theorem 4.2.8, Proposition 4.2.10, Theorem 4.2.18] and [36, Theorem 6.7] *We have the following:*

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(1) *There is a bijection*

$$\begin{aligned} \underline{\mathrm{BT}}_{1,\mathbb{F}}^{(n-1,1)} &\longrightarrow {}^{I}W, \\ G &\longmapsto w(G). \end{aligned}$$

(2) *There is a decomposition of S_K into a disjoint union of reduced locally closed subschemes $S_{K,w}$, $w \in {}^{I}W$, such that:*
   (a) *For $k/\mathbb{F}$ algebraically closed, and $s \in S_K(k)$, then $s \in S_{K,w}(k)$ if and only if $w = w(A_s[p])$.*
   (b) *For each $w \in {}^{I}W$, the subscheme $S_{K,w}$ is non-empty, equidimensional of dimension $l(w)$, smooth and quasi-affine.*
   (c) *We have the closure relation $\overline{S}_{K,w_r} = \sqcup_{r \le i \le n} S_{K,w_i}$, for all $1 \le r \le n-1$.*

In particular, we have $S_K = \overline{S}_{K,w_1} = S_{K,w_1} \sqcup \overline{S}_{K,w_2}$, so that S_{K,w_1} is a dense open, called the *ordinary locus*. We sometimes denote the ordinary locus S_{K,w_1} by S_K^{μ}. At the other end, we have the zero-dimensional closed stratum S_{K,w_n}, which we call the *core locus*. We also call S_{K,w_2} the *almost ordinary locus* and $S_{K,w_{n-1}}$ the *almost core locus*.

5.1 The standard Dieudonné modules

Following [39, Section 3.5], for a given point $s \in S_{K,w_r}(k)$, where $k/\mathbb{F}$ is an algebraically closed extension and $1 \le r \le n$, we can describe the structure of the Dieudonné module $D = \mathbb{D}(A_s[p])$ using standard objects, defined in [36, Section 4.9]. In fact, from [36, Theorem 4.7], we deduce the following.

Proposition 5.3 *Let $D = D_\sigma \oplus D_{\overline{\sigma}}$ be the CM decomposition of D. There are k-bases $\{e_i\}_{1\le i\le n}$, $\{f_i\}_{1\le i\le n}$ of D_σ, $D_{\overline{\sigma}}$, respectively, such that $\langle e_i, f_j \rangle = \delta_{ij}$ and the semilinear Frobenius $F\colon D \to D$ and Verschiebung $V\colon D \to D$ are described as follows:*

$$F(e_i) = \begin{cases} 0 & \text{if } i \ne r, \\ e_1 & \text{if } i = r, \end{cases} \quad F(f_i) = \begin{cases} f_{i+1} & \text{if } 1 \le i \le r-1, \\ 0 & \text{if } i = r, \\ f_i & \text{if } r+1 \le i \le n, \end{cases} \tag{5.1}$$

$$V(e_i) = \begin{cases} 0 & \text{if } i = 1, \\ e_{i-1} & \text{if } 2 \le i \le r, \\ e_i & \text{if } r+1 \le i \le n, \end{cases} \quad V(f_i) = \begin{cases} f_r & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

In particular, we have

$$\begin{aligned} D[F] &= \ker F = \mathrm{Span}_k \langle e_1, \dots, e_{r-1}, f_r, e_{r+1}, \dots, e_n \rangle, \\ D[V] &= \ker V = \mathrm{Span}_k \langle e_1, f_2, \dots, f_n \rangle, \end{aligned}$$

and:

(1) *if $r = 1$, the p-rank of $A_s[p]$ is n and that of $A_s[p]_{\overline{\sigma}}$ is $n-1$;*
(2) *if $r > 1$, the p-rank of $A_s[p]$ and $A_s[p]_{\overline{\sigma}}$ is $n-r$.*

Thus, the EO stratification coincides with the p-rank stratification for the $\overline{\sigma}$-component of the p-torsion.

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5.2 Partial Hasse invariants

We are interested in the action of V on $\underline{\omega}_{A_s/k} \cong D[F]$ for $s \in S_{K,w_r}(k)$, $D = \mathbb{D}(A_s[p])$, $k/\mathbb{F}$ algebraically closed, and $1 \le r \le n-1$. Consider the sheaf

$$\underline{\omega}_{0,r} \coloneqq \ker\left(V^{r-1}\colon \underline{\omega}_{A/\overline{S}_{K,w_r},\sigma} \to \underline{\omega}^{(p^{r-1})}_{A/\overline{S}_{K,w_r},\sigma}\right)$$

on $\overline{S}_{K,w_r}$. By Proposition 5.3, the rank of $\underline{\omega}_{0,r}$ is constant and equal to $r-1$. Therefore, $\underline{\omega}_{0,r}$ locally free of rank $r-1$. The quotient sheaf $\underline{\omega}_{\mu,r} \coloneqq \underline{\omega}_\sigma/\underline{\omega}_{0,r}$ is also locally free, of rank $n-r$, over $\overline{S}_{K,w_r}$. Over S_{K,w_r}, $\underline{\omega}_{0,r}$ and $\underline{\omega}_{\mu,r}$ correspond to the local–local and multiplicative part of the canonical filtration of the p-torsion of A, hence the names. In particular, the quotient morphism $V\colon \underline{\omega}_{\mu,r} \to \underline{\omega}^{(p)}_{\mu,r}$ is an isomorphism over the dense open $S_{K,w_r} \subseteq \overline{S}_{K,w_r}$ and has a kernel of rank 1 over $\overline{S}_{K,w_{r+1}} = \overline{S}_{K,w_r} \backslash S_{K,w_r}$. This is because over $\overline{S}_{K,w_{r+1}}$, $\ker(V^r\colon \underline{\omega}_\sigma \to \underline{\omega}^{(p^r)}_\sigma)$ has constant rank r, by Proposition 5.3, and $\ker(V\colon \underline{\omega}_{\mu,r} \to \underline{\omega}^{(p)}_{\mu,r}) \cong \underline{\omega}_{0,r+1}/\underline{\omega}_{0,r}$ has constant rank 1. Consider the corresponding section $A_r = \det V \in H^0(\overline{S}_{K,w_r}, (\det \underline{\omega}_{\mu,r})^{p-1})$.

Lemma 5.4 *The section A_r is nowhere vanishing on S_{K,w_r} and zero everywhere on the boundary $\partial\overline{S}_{K,w_r}$. Moreover, the construction of A_r is independent of the level K.*

We call the A_r's *(generalized) partial Hasse invariants*. They are factors of the generalized Hasse invariants of [5].

5.3 Smoothness of closures in the EO stratification

We establish a geometric property of the KS morphism on lower EO strata. Consider $2 \le r \le n-1$. We show in particular that $\overline{S}_{K,w_r}$ is smooth. We write

$$\nabla_r\colon H^1_{\mathrm{dR}}(A/\overline{S}_{K,w_r}) \longrightarrow H^1_{\mathrm{dR}}(A/\overline{S}_{K,w_r}) \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}}$$

to denote the Gauss–Manin connection relative to $A/\overline{S}_{K,w_r}/\mathbb{F}$. We use the same notation for the restriction of ∇_r to $S_{K,w_r} \subseteq \overline{S}_{K,w_r}$. Recall that ∇_r is related to $\nabla\colon H^1_{\mathrm{dR}}(A/S_K) \to H^1_{\mathrm{dR}}(A/S_K) \otimes \Omega^1_{S_K/\mathbb{F}}$ via base change (see Lemma 4.6). We write $u\colon S_{K,w_r} \to S_K$ and $\overline{u}\colon \overline{S}_{K,w_r} \to S_K$ for the natural immersions. To ease notations, we set $\tilde{\delta} = \det \underline{\omega}_{A/S,\sigma} \otimes \delta_\sigma^{-1}$, with S depending on the context. We also write $\mathcal{C}_{\overline{S}_{K,w_r}/S_K}$ to denote the conormal sheaf of $\overline{S}_{K,w_r}$ in S_K.

Proposition 5.5 *We have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\omega}_{0,r} \otimes \tilde{\delta} & \longrightarrow & \underline{\omega}_{A/\overline{S}_{K,w_r},\sigma} \otimes \tilde{\delta} & \longrightarrow & \underline{\omega}_{\mu,r} \otimes \tilde{\delta} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \overline{u}^*(\underline{\mathrm{ks}})} & & \downarrow{\scriptstyle \underline{\mathrm{ks}}_{\mu,r}} & & \\ 0 & \longrightarrow & \mathcal{C}_{\overline{S}_{K,w_r}/S_K} & \longrightarrow & \overline{u}^*(\Omega^1_{S_K/\mathbb{F}}) & \longrightarrow & \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}} & \longrightarrow & 0, \end{array} \tag{5.3}$$

whose vertical arrows are isomorphisms. In particular, $\overline{S}_{K,w_r}$ is smooth and $\nabla_r(\underline{\omega}_0) \subseteq \underline{\omega}_0 \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}}$.

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Proof We prove this via Grothendieck–Messing theory. For what we intend to prove the level K is irrelevant, so we drop it from the notation. We write S_r (resp. $\overline{S}_r$) instead of S_{K,w_r} (resp. $\overline{S}_{K,w_r}$).

Over S_r, we have $\nabla_r(\underline{\omega}_0) \subseteq \underline{\omega}_0 \otimes \Omega^1_{S_r/\mathbb{F}}$, by Lemma 4.5, Remark 4.7, and the fact that $\underline{\omega}_0 = \ker(V^{r-1}\colon H_{\mathrm{dR},\sigma} \to H_{\mathrm{dR},\sigma}^{(p^{r-1})})$, by Proposition 5.3. By Lemma 4.6 and Proposition 4.8, this means that, over S_r, the composition

$$\underline{\omega}_0 \otimes \tilde{\delta} \longrightarrow \underline{\omega}_{A/S_r,\sigma} \otimes \tilde{\delta} \xrightarrow{\mathrm{ks}} u^*(\Omega^1_{S/\mathbb{F}}) \longrightarrow \Omega^1_{S_r/\mathbb{F}}$$

is zero. By density of $S_r \subseteq \overline{S}_r$, we deduce the existence of the commutative diagram (5.3) and the morphism $\underline{\mathrm{ks}}_\mu$. We now show that its vertical arrows are isomorphism and that the second row is a short exact sequence.

Consider a point $s \in \overline{S}_r \subseteq S$ and write $k = k(s)$. We want to describe the Zariski tangent spaces

$$T_{\overline{S}_r,s} = T_{\overline{S}_r/\mathbb{F},s}, \qquad T_{S,s} = T_{S/\mathbb{F},s}.$$

These are k-vector spaces whose elements v correspond to lifts of $s\colon \operatorname{Spec} k \to \overline{S}_r$ to $v\colon \operatorname{Spec} k[\epsilon] \to \overline{S}_r$ (resp. S), where $k[\epsilon] \coloneqq k[T]/(T^2)$. By definition of S, resp. $\overline{S}_r$, any $v \in T_{S,s}$, resp. $v \in T_{\overline{S}_r,s}$, corresponds to an abelian scheme with PEL structure $\underline{A}_v \in S(k[\epsilon])$, resp. $\underline{A}_v \in \overline{S}_r(k[\epsilon])$, lifting the object $\underline{A} \in \overline{S}_r(k)$ corresponding to s. Grothendieck–Messing theory tells us that the category of such lifts $\underline{A}_v$ is equivalent to the category of lifts of the k-subspace $\underline{\omega} = \underline{\omega}_{A/k}$ of $H = H^1_{\mathrm{dR}}(A/k)$ to sub- $k[\epsilon]$-modules $\tilde{\underline{\omega}}$ of $H[\epsilon] = H \otimes_k k[\epsilon]$, the trivial lift of H, such that:

(1) $\tilde{\underline{\omega}}$ is a free direct summand of $H[\epsilon]$ of rank n.
(2) $\tilde{\underline{\omega}}$ is $\mathcal{O}_E$-stable (of type necessarily $(n-1,1)$).
(3) $\tilde{\underline{\omega}}$ is maximal isotropic for the perfect alternating pairing on $H[\epsilon]$ induced by the de Rham pairing $\langle\cdot,\cdot\rangle^\lambda_{\mathrm{dR}}$ on H.

Moreover, for $v \in T_{S,s}$, we have that:

(4) $v \in T_{\overline{S}_r,s}$ if and only if the lift $\underline{A}_v$ corresponds to $\tilde{\underline{\omega}}$ such that $\tilde{\underline{\omega}}_\sigma[V^{r-1}]$ is free of rank $r-1$.

Condition (4) follows from Proposition 5.3. By [6, Section 2], the BT_1 with extra structure $A_v[F]$ is determined by $(\tilde{\underline{\omega}}, V\colon \tilde{\underline{\omega}} \to \tilde{\underline{\omega}}^{(p)})$, with the extra structures detailed at the points (1–3) above. Therefore, for $r \neq 1$, $\tilde{\underline{\omega}}[V^{r-1}]$ is free of rank $r-1$ if and only if $V^{r-1}(A_v[p]_{\overline{\sigma}}) \subseteq A_v[F]_{\overline{\sigma}}$ exists and is finite locally free of rank $r-1$, which happens if and only if $v \in \overline{S}_r(k[\epsilon])$, by the structure of the canonical filtration on S_j, $j \geq r$ (see Proposition 5.3 and the beginning of Section 5.2). The case $r = 1$ is trivial.

Such a $\tilde{\underline{\omega}}$ corresponds to a k-linear morphism $h\colon \underline{\omega} \to H/\underline{\omega} \cong \underline{\omega}^\vee_{A^\vee/k} \underset{\lambda^*}{\cong} \underline{\omega}^\vee$ subject to the conditions:

(1) The pairing $\langle\cdot, h(\cdot)\rangle^\lambda_{\mathrm{dR}}$ on $\underline{\omega}$ is symmetric.
(2) h is $\mathcal{O}_E$-linear.

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The first condition corresponds to $\underline{\tilde{\omega}}$ being maximal isotropic, the second to the fact that $\underline{\tilde{\omega}}$ is $\mathcal{O}_E$-stable. Furthermore, the extra condition that $\underline{\tilde{\omega}}_\sigma[V^{r-1}]$ is free of rank $r-1$ corresponds to:

(3) h is identically 0 on $\underline{\omega}_\sigma[V^{r-1}] \subseteq \underline{\omega}_\sigma$.

Notice that the first two conditions together imply that h is determined by

$$h_\sigma\colon \underline{\omega}_\sigma \longrightarrow (H/\underline{\omega})_\sigma \cong \underline{\omega}^\vee_{A^\vee/k,\sigma} \underset{\lambda^*}{\cong} \underline{\omega}^\vee_{\overline{\sigma}}.$$

Therefore, the deformations $T_{S,s}$ of A are in correspondence with $\mathrm{Hom}_k(\underline{\omega}_\sigma, \underline{\omega}^\vee_{\overline{\sigma}}) \cong \underline{\omega}^\vee_\sigma \otimes_k \underline{\omega}^\vee_{\overline{\sigma}}$, and $T_{\overline{S}_r,s} \subset T_{S,s}$ corresponds to

$$\mathrm{Hom}_k(\underline{\omega}_\mu, \omega^\vee_{\overline{\sigma}}) = \{h\colon \underline{\omega}_\sigma \to \underline{\omega}^\vee_{\overline{\sigma}} \mid h|_{\underline{\omega}_\sigma[V^{r-1}]} = 0\} \subset \mathrm{Hom}_k(\underline{\omega}_\sigma, \underline{\omega}^\vee_{\overline{\sigma}}).$$

As discussed in [3, p. 40], this correspondence is given by isomorphisms of k-vector spaces and, as noted in Proposition 4.9, the isomorphism $T_{S,s} \to \underline{\omega}^\vee_\sigma \otimes_k \underline{\omega}^\vee_{\overline{\sigma}}$ is the dual of the fiber at s of the KS morphism. Hence, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\overline{S}_r,s} & \longrightarrow & T_{S,s} & \longrightarrow & \mathcal{N}_{\overline{S}_r/S,s} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\omega}^\vee_\mu \otimes \underline{\omega}^\vee_{\overline{\sigma}} & \longrightarrow & \underline{\omega}^\vee_\sigma \otimes \underline{\omega}^\vee_{\overline{\sigma}} & \longrightarrow & \underline{\omega}^\vee_0 \otimes \underline{\omega}^\vee_{\overline{\sigma}} & \longrightarrow & 0,
\end{array}$$

where the vertical arrows are isomorphisms and $\mathcal{N}_{\overline{S}_r/S,s}$ denotes the normal sheaf at s of $\overline{S}_r$ in S. This proves at once that $\overline{S}_r$ is smooth and by duality that the diagram of exact sequences (5.3) is exact and its vertical arrows are isomorphisms.

Finally, from Lemma 4.5 with $\phi = V^{r-1}$, we deduce that $\nabla_r(\underline{\omega}_0) \subseteq \underline{\omega}_0 \otimes \Omega^1_{\overline{S}_r/\mathbb{F}}$. ∎

Remark 5.6 The smoothness of closed EO strata in the p-split case stands in contrast with the p-inert case: for instance, when $n = 3, 4$, one can show that the one-dimensional closed strata are (unions of) chains of Fermat curves of degree $p+1$ intersecting at superspecial points, with every superspecial point being at the intersection of many irreducible components. See [8] or [38, Example G] for more details.

6 Generalized splittings and operators

6.1 Unit-root splitting

We present here the classic unit-root splitting of the Hodge filtration (2.3) over the ordinary locus S^μ_K and then we show how one can obtain a similar splitting over lower EO strata. In our discussion, we will make implicit use of Proposition 5.3 and of technical Lemma 6.1, inspired by [15, Section 3.3.1].

Let us introduce some notation. Let $\overline{S}$ be a scheme and consider $\mathcal{F} \subseteq \mathcal{G}$ finite locally free sheaves with constant rank over $\overline{S}$, such that $\mathcal{G}/\mathcal{F}$ is also finite locally free. Let r denote the rank of $\mathcal{F}$. Consider a tuple of integers $\underline{k} = (k_1 \geq k_2 \geq \cdots \geq k_r \geq 0)$. From the results of Section 4.1, we obtain a natural descending filtration on $\mathbb{S}^{\underline{k}}(\mathcal{G})$, whose last nonzero step is $\mathbb{S}^{\underline{k}}(\mathcal{F})$. Let us denote by $P(\mathbb{S}^{\underline{k}}(\mathcal{G}))$ the penultimate step of this filtration (see also Remark 4.1).

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Let $\mathcal{H}$ be another finite locally free sheaf on $\overline{S}$, with the same rank r as $\mathcal{F}$, and consider $\psi\colon \mathcal{F} \to \mathcal{H}$ a morphism. The *adjugate* morphism $\psi^{\mathrm{adj}}\colon \mathcal{H} \longrightarrow \mathcal{F} \otimes \det(\mathcal{H}) \otimes \det(\mathcal{F})^{-1}$ is defined as

$$\psi^{\mathrm{adj}}\colon \mathcal{H} \cong (\wedge^{r-1}\mathcal{H})^{\vee} \otimes \det(\mathcal{H}) \xrightarrow{(\wedge^{r-1}\psi)^{\vee}\otimes \mathrm{id}_{\det(\mathcal{H})}} (\wedge^{r-1}\mathcal{F})^{\vee} \otimes \det(\mathcal{H}) \cong \mathcal{F} \otimes \mathcal{H} \otimes \det(\mathcal{F})^{-1}.$$

Lemma 6.1 *Let $\overline{S}, \mathcal{F}, \mathcal{G}, \mathcal{H}$, and $\underline{k}$ be as above. Suppose that we have a morphism*

$$\phi\colon \mathcal{G} \longrightarrow \mathcal{H}$$

of $\mathcal{O}_{\overline{S}}$-sheaves such that the restriction $\phi|_{\mathcal{F}}$ of ϕ to $\mathcal{F}$ is an isomorphism on a dense open $S \subseteq \overline{S}$. Write $d = \det \phi|_{\mathcal{F}} \in H^0(\overline{S}, \det \mathcal{H} \otimes (\det \mathcal{F})^{-1})$ for the determinant of that restriction. Then, the morphism

$$\mathcal{G} \xrightarrow{\phi} \mathcal{H} \xrightarrow{\phi^{-1}} \mathcal{F} \xrightarrow{d} \mathcal{F} \otimes \det \mathcal{H} \otimes (\det \mathcal{F})^{-1}$$

extends naturally from S to $\overline{S}$. Similarly, the map

$$P(\mathbb{S}^{\underline{k}}(\mathcal{G})) \xrightarrow{\mathbb{S}^{\underline{k}}(\phi)} \mathbb{S}^{\underline{k}}(\mathcal{H}) \xrightarrow{\mathbb{S}^{\underline{k}}(\phi^{-1})} \mathbb{S}^{\underline{k}}(\mathcal{F}) \xrightarrow{d} \mathbb{S}^{\underline{k}}(\mathcal{F}) \otimes \det \mathcal{H} \otimes \det \mathcal{F}^{-1}$$

extends from S to $\overline{S}$.

Proof The key observation, due to [15, Section 3.3.1], is that the morphism $d \cdot \phi^{-1}\colon \mathcal{H} \longrightarrow \mathcal{F} \otimes \det \mathcal{H} \otimes (\det \mathcal{F})^{-1}$ is the adjugate of $\phi|_{\mathcal{F}}$ and thus it is naturally defined over all of $\overline{S}$. Notice that on $\mathcal{F}$ the morphism $\phi|_{\mathcal{F}}^{\mathrm{adj}} \circ \phi$ is simply multiplication by d.

The statement is trivial when $\underline{k} = \underline{0}$. We present the proof in the case $\underline{k} = (k, 0, \ldots, 0)$, where $\mathbb{S}^{\underline{k}}(\mathcal{G}) = \mathrm{Sym}^k(\mathcal{G})$. The general case is proved similarly. One can consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^{k-1}(\mathcal{F}) \otimes \mathcal{G} & \longrightarrow & F^{k-1}(\mathrm{Sym}^k(\mathcal{G})) \\ & \searrow & \downarrow \\ & & \mathrm{Sym}^k(\mathcal{F}) \otimes \det \mathcal{H} \otimes (\det \mathcal{F})^{-1}, \end{array}$$

with the vertical arrow being the morphism we want to extend. The diagonal arrow is simply $\mathrm{id} \otimes (\phi|_{\mathcal{F}}^{\mathrm{adj}} \circ \phi)$, so it extends from S to $\overline{S}$, and it clearly factors through $F^{k-1}(\mathrm{Sym}^k(\mathcal{G}))$, so we conclude. ∎

6.1.1 The ordinary splitting

On $S = S_K^{\mu}$, the morphism $F\colon H_{\sigma}^{(p)} \to H_{\sigma}$ induces via (2.3) an isomorphism $F\colon \underline{\omega}^{\vee,p}_{A^{\vee}/S,\sigma} \to \underline{\omega}^{\vee}_{A^{\vee}/S,\sigma}$. Moreover, F kills the subsheaf $\underline{\omega}^{(p)}_{A/S,\sigma} \subset H_{\sigma}^{(p)} = H^{1,(p)}_{\mathrm{dR}}(A/S_K^{\mu})$ and the composition

$$\underline{\omega}^{\vee}_{A^{\vee}/S,\sigma} \xrightarrow{F^{-1}} \underline{\omega}^{\vee,p}_{A^{\vee}/S,\sigma} \xrightarrow{F} H_{\sigma}$$

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gives a natural splitting of the Hodge filtration (2.3). We write $\mathcal{U}_\sigma = \mathrm{im}(F\colon H_\sigma^{(p)} \to H_\sigma)$, the locally free subsheaf giving the splitting. We have that $\mathcal{U}_\sigma = \ker(V\colon H_\sigma \to H_\sigma^{(p)})$. Moreover, $V\colon H_\sigma \to H_\sigma^{(p)}$ has image $\underline{\omega}_{A/S,\sigma}^{(p)}$ and, over S_K^μ, it induces the isomorphism $V\colon \underline{\omega}_{A/S,\sigma} \to \underline{\omega}_{A/S,\sigma}^{(p)}$. This implies that the composition

$$p_{\mathrm{ur},1}\colon H_\sigma \xrightarrow{V} \underline{\omega}_{A/S,\sigma}^{(p)} \xrightarrow{V^{-1}} \underline{\omega}_{A/S,\sigma}$$

is the projection $H_\sigma \to \underline{\omega}_{A/S,\sigma}$ parallel to $\mathcal{U}_\sigma$, or, in other words, the projection $H_\sigma \to \underline{\omega}_{A/S,\sigma}$ relative to the splitting of (2.3) given by $\mathcal{U}_\sigma$. This is usually called the *unit-root splitting*. We have proved the following.

Lemma 6.2 (Ordinary unit-root splitting) *Over the ordinary locus S_K^μ, the Hodge filtration (2.3) admits a natural splitting induced by the Frobenius $F\colon H_\sigma^{(p)} \to H_\sigma$. The corresponding projection $p_{\mathrm{ur},1}\colon H_\sigma \to \underline{\omega}_\sigma$ is induced by V as described above.*

The morphism $p_{\mathrm{ur},1}$ cannot be extended naturally to $S_K = \overline{S}_K^\mu$, because the factor $V^{-1}\colon \underline{\omega}_\sigma^{(p)} \to \underline{\omega}_\sigma$ does not make sense outside S_K^μ. If we consider instead the map $A_1 \cdot p_{\mathrm{ur},1}$, we see that it can be extended to the whole S_K: the linear map

$$A_1 \cdot V^{-1}\colon \underline{\omega}_\sigma^{(p)} \longrightarrow \underline{\omega}_\sigma \otimes (\det \underline{\omega}_\sigma)^{p-1}$$

is the adjugate to the morphism $V\colon \underline{\omega}_\sigma \to \underline{\omega}_\sigma^{(p)}$, which is defined over S_K. In fact, from Lemma 6.1, we deduce the following.

Lemma 6.3 *The morphism $A_1 \cdot p_{\mathrm{ur},1}\colon H_\sigma \to \underline{\omega}_\sigma \otimes (\det \underline{\omega}_\sigma)^{p-1}$ extends to S_K. Similarly, for $\underline{k} = (k_1, \ldots, k_{n-1})$, with $k_1 \geq k_2 \geq \cdots \geq k_{n-1} \geq 0$,*

$$A_1 \cdot \mathbb{S}^{\underline{k}}(p_{\mathrm{ur},1})|_{P(\mathbb{S}^{\underline{k}}(H_\sigma))}\colon P(\mathbb{S}^{\underline{k}}(H_\sigma)) \longrightarrow \mathbb{S}^{\underline{k}}(\underline{\omega}_\sigma) \otimes (\det \underline{\omega}_\sigma)^{p-1}$$

extends to S_K.

6.1.2 Partial splittings

Fix some $2 \leq r \leq n-1$. On $S = S_{K,w_r}$, the map $F\colon \underline{\omega}_{A^\vee/S,\sigma}^{\vee,p} \longrightarrow \underline{\omega}_{A^\vee/S,\sigma}^{\vee}$ is zero. Nevertheless, we can still look at the surjection $V\colon H_\sigma \to \underline{\omega}_\sigma^{(p)}$. On S, the map $V\colon \underline{\omega}_\sigma \to \underline{\omega}_\sigma^{(p)}$ is not surjective, but it induces an isomorphism between the multiplicative part $\underline{\omega}_\mu$ and its p-twist. Write $H_{\mu,r} \coloneqq H_\sigma/\underline{\omega}_{0,r}$. We have the composition

$$p_{\mathrm{ur},r}\colon H_{\mu,r} \xrightarrow{V} \underline{\omega}_\mu^{(p)} \xrightarrow{V^{-1}} \underline{\omega}_{\mu,r}.$$

This composition is a right inverse of the inclusion $\underline{\omega}_{\mu,r} \to H_{\mu,r}$. That is, the kernel of $p_{\mathrm{ur},r}$ gives a splitting of the short exact sequence

$$0 \longrightarrow \underline{\omega}_{\mu,r} \longrightarrow H_{\mu,r} \longrightarrow \underline{\omega}_{A^\vee/S,\sigma}^{\vee} \longrightarrow 0.$$

We call this the *(generalized) partial unit-root splitting*. Recall that we write $A_r = \det(V\colon \underline{\omega}_\mu \to \underline{\omega}_\mu^{(p)})$. As in the ordinary case, we can prove the following using Lemma 6.1.

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Lemma 6.4 *The morphism* $A_r \cdot p_{\mathrm{ur},r}\colon H_{\mu,r} \longrightarrow \underline{\omega}_{\mu,r} \otimes (\det \underline{\omega}_{\mu,r})^{p-1}$ *extends to* $\overline{S}_{K,w_r}$. *Similarly, for* $\underline{k} = (k_1, k_2, \ldots, k_{n-r})$, *with* $k_1 \geq k_2 \geq \cdots \geq k_{n-r} \geq 0$,

$$A_r \cdot \mathbb{S}^{\underline{k}}(p_{\mathrm{ur},r})|_{P(\mathbb{S}^{\underline{k}}(H_{\mu,r}))}\colon P(\mathbb{S}^{\underline{k}}(H_{\mu,r})) \longrightarrow \mathbb{S}^{\underline{k}}(\underline{\omega}_{\mu,r}) \otimes (\det \underline{\omega}_{\mu,r})^{p-1}$$

extends to $\overline{S}_{K,w_r}$.

In the process of constructing the generalized theta operators, we will give a mild generalization of Lemma 6.4, which again follows from Lemma 6.1.

6.2 Generalized theta operators

We prove here the main result of this article (Theorem 6.5). Let us start by recalling some notations. Pick $1 \leq r \leq n-1$ an integer and $(\underline{k}, w)$ an automorphic weight with $k_{n-1} \geq 0$. On $\overline{S}_{K,w_r}$, we have the short exact sequence

$$0 \longrightarrow \underline{\omega}_{0,r} \longrightarrow \underline{\omega}_{\sigma} \longrightarrow \underline{\omega}_{\mu,r} \longrightarrow 0,$$

where $\underline{\omega}_{0,r} = \ker(V^{r-1}\colon \underline{\omega}_{\sigma} \to \underline{\omega}_{\sigma}^{(p^{r-1})})$ is a locally free sheaf of rank $r-1$. This short exact sequence induces a filtration $F^{\bullet,r}$ on $\underline{\omega}^{\underline{k},w}$ over $\overline{S}_{K,w_r}$, which is trivial when $r = 1$ (see Section 4.1 for more details on the definition of $F^{\bullet,r}$). The filtration $F^{\bullet,r}$ gives rise to the graded sheaf $\mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w})$ over $\overline{S}_{K,w_r}$. Notice that the graded sheaf $\mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w})$ decomposes as a direct sum of sheaves

$$\mathrm{gr}^{\underline{a},r}(\underline{\omega}^{\underline{k},w}) \coloneqq \mathrm{im}(\otimes_j \wedge^{l_j - b_j}(\underline{\omega}_0) \otimes \wedge^{b_j}(\underline{\omega}_{\mu}) \to \otimes_i \mathrm{Sym}^{k_i - a_i}(\underline{\omega}_0) \otimes \mathrm{Sym}^{a_i}(\underline{\omega}_{\mu})),$$

where $\underline{a}, \underline{b} \in \mathbb{Z}_{\geq 0}^{n-1}$ are such that $a_i \leq k_i$, $a_i \geq a_{i+1}$ for all i, that is, $\underline{a}$ is a *subtableau* of $\underline{k}$, and $\underline{b}$ is the transpose of $\underline{a}$. In particular, given $f \in H^0(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w}))$, we can write it uniquely as $f = \sum_{\underline{a}} f_{\underline{a}}$, where $f_{\underline{a}} \in H^0(\overline{S}_{K,w_r}, \mathrm{gr}^{\underline{a},r}(\underline{\omega}^{\underline{k},w}))$.

Theorem 6.5 *Let* $1 \leq r < n$ *be an integer and* $(\underline{k}, w)$ *an automorphic weight with* $k_{n-1} \geq 0$. *There exists a differential operator*

$$\theta_r\colon \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w}) \longrightarrow \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k}+\underline{\Delta}_r,w-1}),$$

defined on the (closure of the) Ekedahl–Oort stratum $\overline{S}_{K,w_r}$, *with*

$$\underline{\Delta}_r = (p+1, p, \ldots, p, 1, \ldots, 1),$$

where exactly the last $r-1$ *entries are* 1. *The operator* θ_r *satisfies the following properties:*

(1) *The operator* θ_r *is* A_r*-linear, that is,* $\theta_r(A_r) = 0$, *where* A_r *is the partial Hasse invariant defined above.*
(2) *The operator* θ_r *commutes with the action of prime-to-p Hecke operators.*
(3) *Let* $f \in H^0(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w}))$ *and write it as* $f = \sum_{\underline{a}} f_{\underline{a}}$. *If* $r = n-1, n-2$, *then* $\theta_r(f)$ *is divisible by the Hasse invariant* A_r *if and only if for each component* $f_{\underline{a}}$ *either* $A_r \mid f_{\underline{a}}$ *or* $p \mid a_1$.

We prove this theorem throughout this section.

Remark 6.6 In what follows, we will consider several sheaves, as well as morphisms between them, which are naturally defined over S_K (or even $\mathcal{S}_K$), but will be restricted

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to subschemes $S \subseteq S_K$. To ease notations, for a sheaf $\mathcal{F}$ on S_K, we will not in general write $\mathcal{F}|_S$, and similarly, for a morphism $\phi: \mathcal{F} \to \mathcal{G}$, we will not write $\phi|_S: \mathcal{F}|_S \to \mathcal{G}|_S$, instead, we will simply keep writing $\mathcal{F}$ and $\phi: \mathcal{F} \to \mathcal{G}$, with the understanding that everything is restricted to a subscheme that will be clear from context.

6.2.1 Ordinary operator

Let $(\underline{k}, w)$ be an automorphic weight such that $k_{n-1} \geq 0$, with the corresponding automorphic sheaf $\underline{\omega}^{\underline{k},w}_{A/S_K}$, which we denote simply by $\underline{\omega}^{\underline{k},w}$. Consider $H = H^1_{\mathrm{dR}}(A/S_K)$ and write

$$H^{\underline{k},w} \coloneqq \mathbb{S}^{\underline{k}}(H_\sigma) \otimes_{\mathcal{O}_S} \delta_\sigma^w.$$

The GM connection induces a morphism

$$\underline{\omega}^{\underline{k},w} \xrightarrow{\nabla} H^{\underline{k},w} \otimes \Omega^1_{S_K/\mathbb{F}}.$$

By Lemma 4.2, the image of this map lies in the penultimate step of the natural filtration $F^\bullet$ of $H^{\underline{k},w}$ described in Remark 4.1. In particular, from Lemma 6.3, we deduce the following.

Lemma 6.7 [15, Section 3.3.2] *Consider the composition*

$$\psi: \underline{\omega}^{\underline{k},w} \xrightarrow{\nabla} H^{\underline{k},w} \otimes \Omega^1_{S/\mathbb{F}} \longrightarrow \underline{\omega}^{\underline{k},w} \otimes (\det \underline{\omega}_\sigma)^{p-1} \otimes \Omega^1_{S_K/\mathbb{F}} \longrightarrow \underline{\omega}^{\underline{k}+\underline{p-1},w} \otimes \Omega^1_{S_K/\mathbb{F}},$$

where the second arrow is $A_1 \cdot (\mathbb{S}^{\underline{k}}(p_{\mathrm{ur},1}) \otimes \mathrm{id}_{\delta_\sigma} \otimes \mathrm{id}_{\Omega^1})$ *and* $(\underline{p-1}, 0) = ((p-1, \ldots, p-1), 0)$ *is the weight of* A_1. *Then* ψ *extends from* S_K^μ *to* S_K.

We may now define θ_1, the *ordinary theta operator*, as follows. Composing ψ from Lemma 6.7 with $\mathrm{id} \otimes \underline{\mathrm{ks}}^{-1}$, we obtain

$$\underline{\omega}^{\underline{k},w} \longrightarrow \underline{\omega}^{\underline{k}+\underline{p-1},w} \otimes \underline{\omega}_\sigma \otimes \det \underline{\omega}_\sigma \otimes \delta_\sigma^{-1}. \tag{6.1}$$

To define θ_1, we further compose (6.1) with the natural morphism $\underline{\omega}^{\underline{k}+\underline{p-1},w} \otimes \underline{\omega}_\sigma \otimes \det \underline{\omega}_\sigma \to \underline{\omega}^{\underline{k}+\underline{\Delta}_1,w}$ and obtain

$$\theta_1: \underline{\omega}^{\underline{k},w} \longrightarrow \underline{\omega}^{\underline{k}+\underline{\Delta}_1,w-1},$$

where $\underline{\Delta}_1 = (p+1, p, \ldots, p)$. This proves the existence part of Theorem 6.5 in the case $r = 1$. Note that this theta operator is already defined in [15].

Proposition 6.8 *The operator* θ_1 *satisfies the following fundamental properties:*

(1) *For* $f \in H^0(S_K, \underline{\omega}^{\underline{k},w})$, $g \in H^0(S_K, \underline{\omega}^{\underline{k}',w'})$, *we have* $\theta_1(fg) = f\theta_1(g) + \theta_1(f)g$, *that is, the operator* θ_1 *is a derivation of the algebra of modular forms of level* K *with coefficients in* $\mathbb{F}$.
(2) *The operator* θ_1 *is* A_1*-linear, that is,* $\theta_1(A_1) = 0$.
(3) *The operator* θ_1 *is Hecke-equivariant.*

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Proof

(1) This is clear from the definition of ∇ and thus of θ_1.

(2) We can choose a small enough dense open $U \subseteq S_K^\mu$ such that $\underline{\omega}_\sigma$ and H_σ are both free on U and the Hodge filtration (2.3) splits. In particular, we can pick a local basis $e_1, \dots, e_{n-1}$ of $\underline{\omega}_\sigma$ and complete it to a local basis of H_σ with the addition of some e_n. Then, in these local coordinates, the ordinary Hasse invariant is $A_1|_U = a \cdot (e_1 \wedge \cdots \wedge e_{n-1})^{p-1}$, for some $a \in \mathcal{O}_{S_K}(U)$. With these notations, from the definition of A_1, one can compute that

$$\begin{aligned}(\wedge^{p-1}V)(A_1)|_U &= a(Ve_1 \wedge \cdots \wedge Ve_{n-1})^{p-1} = a(a(e_1 \wedge \cdots \wedge e_{n-1})^p)^{p-1} \\ &= a^p(e_1 \wedge \cdots \wedge e_{n-1})^{p(p-1)} = A_1^p.\end{aligned}$$

Therefore, from the definition of $p_{\mathrm{ur},1} = V|^{-1}_{\underline{\omega}_\sigma} \circ V_\sigma$ and Lemma 4.5, we have on U that

$$\begin{aligned}(\wedge^{p-1}p_{\mathrm{ur},1}) \circ \nabla(A_1) &= (\wedge^{p-1}V^{-1})(\wedge^{p-1}V)\nabla(A_1) \\ &= (\wedge^{p-1}V^{-1})\nabla((\wedge^{p-1}V)A_1) = (\wedge^{p-1}V^{-1})\nabla(A_1^p) = 0.\end{aligned}$$

Hence, $\theta_1(A_1)|_U = 0$. By density of $U \subset S_K$, we deduce that $\theta_1(A_1) = 0$.

(3) The construction of θ_1 does not depend on the level structure and it is compatible with changes of base involved in the definition of the Hecke operators. Moreover, any prime-to-p quasi-isogeny $f\colon A \to A'$, for $A, A' \in S_K(T)$, $T \to S_K$ finite étale, induces an isomorphism $f^*\colon \underline{\omega}_{A'/T} \to \underline{\omega}_{A/T}$ such that

$$\begin{array}{ccc} \underline{\omega}_{A'/T,\sigma} \otimes_{\mathcal{O}_S} \det \underline{\omega}_{A'/T,\sigma} \otimes \delta^{-1}_{A'/T,\sigma} & \xrightarrow{\underline{\mathrm{ks}}_{A'/T}} & \Omega^1_{T/\mathbb{F}} \\ \downarrow f^* & & \| \\ \underline{\omega}_{A/T,\sigma} \otimes_{\mathcal{O}_S} \det \underline{\omega}_{A/T,\sigma} \otimes \delta^{-1}_{A/T,\sigma} & \xrightarrow{\underline{\mathrm{ks}}_{A/T}} & \Omega^1_{T/\mathbb{F}} \end{array}$$

is commutative. All of this, combined with Lemma 4.5, implies that θ_1 is Hecke-equivariant. Notice that this relies on our convention for the definition of $\underline{\mathrm{ks}}$: had the polarization been involved, the morphism $\underline{\mathrm{ks}}$ would have introduced a nontrivial Hecke-action (see [15, Lemma 4.1.2]).

■

Remark 6.9 We can provide an alternative proof of point (2) of Proposition 6.8. From, for instance, the proof of [29, Proposition 4.1.1], we see that we can cover S_K^μ with maps $U \to S_K^\mu$, finite étale of degree $\#\mathrm{GL}_{n-1}(\mathbb{F}_p)$, on which we can find a local basis $e_1, \dots, e_{n-1} \in \underline{\omega}_\sigma(U)$, which extends to a basis $e_1, \dots, e_n \in H_\sigma(U)$, such that $V(e_i) = e_i^{(p)} = e_i \otimes 1$, for $i = 1, \dots, n-1$, and $V(e_n) = 0$. This follows from applying the proposition in *loc. cit.* with $H = \underline{\omega}_\sigma^\vee|_O$, $S_1 = O$, for some non-empty affine open $O \subseteq S_K^\mu$, with notations H and S_1 as in the reference, and $F = V^\vee$. In that case, we have

$$A_1|_U = (e_1 \wedge \cdots \wedge e_{n-1})^{p-1}.$$

Moreover, by Lemmas 4.5 and 4.6, we have that on U

$$0 = \nabla(e_i^{(p)}) = \nabla(V(e_i)) = V(\nabla(e_i)), \quad i = 1, 2, \dots, n-1,$$

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since $dF_{U,\mathrm{abs}} = 0$. This implies that $\nabla(e_i) = e_n \otimes \omega_i$, for some sections $\omega_i \in \Omega^1_{S_K/\mathbb{F}}(U)$, which form a local basis of $\Omega^1_{S_K/\mathbb{F}}$, by Proposition 4.9. With this description of the $\nabla(e_i)$'s, one can see that, over U, we have $\theta_1(A_1)|_U = 0$, thanks to a simple computation, using the definitions of ∇, $p_{\mathrm{ur},1}$ and thus θ_1. Therefore, since we can cover S_K^μ with such finite étale U's, $\theta_1(A_1)$ vanishes on the ordinary locus S_K^μ, and by density, we have $\theta_1(A_1) = 0$ on all of S_K.

This shows that θ_1 admits a relatively simple and explicit description étale-locally on the ordinary locus. In fact, one can prove that it is enough to take $U = I$, with I being the (small) Igusa variety $I \to S_K^\mu$ of level 1 and achieve the same results working globally. This is related to the approach taken in [9] to construct theta operators when p is inert. See, in particular, [9, Section 2.1]. See also [24].

One could use this approach more generally on S_{K,w_r}, $2 \le r \le n-1$, instead of S_K^μ, to obtain étale-local bases of $\underline{\omega}_{\mu,r}$ such that $V\colon \underline{\omega}_{\mu,r} \to \underline{\omega}^{(p)}_{\mu,r}$ admits a description relative to them analogous to the one given above. We do not pursue this direction here.

6.2.2 Generalized theta operators

Fix $2 \le r \le n-1$. Let $(\underline{k}, w)$ be an automorphic weight such that $k_{n-1} \ge 0$ and consider the sheaf $\underline{\omega}^{\underline{k},w}$. We look in particular at the restriction of $\underline{\omega}^{\underline{k},w}$ to $\overline{S}_{K,w_r}$, where we have, as above, the short exact sequence

$$0 \longrightarrow \underline{\omega}_{0,r} \longrightarrow \underline{\omega}_\sigma \longrightarrow \underline{\omega}_{\mu,r} \longrightarrow 0. \tag{6.2}$$

From this sequence, we get a Koszul filtration on $\wedge^j \underline{\omega}_\sigma$, for all $j = 1, \dots, n-1$, following the notations of Section 4.1.3. Similarly, we can define a Koszul filtration on $\wedge^j H_\sigma$, $j = 1, \dots, n-1$, from

$$0 \longrightarrow \underline{\omega}_{0,r} \longrightarrow H_\sigma \longrightarrow H_{\mu,r} \longrightarrow 0. \tag{6.3}$$

From Lemma 4.5 and Proposition 5.5, we deduce that the GM connection on $\overline{S}_{K,w_r}$ induces morphisms between the graded pieces of these two Koszul filtrations as follows:

$$\begin{array}{ccc} \mathrm{gr}^i(\wedge^j \underline{\omega}_\sigma) & \xrightarrow{\nabla_r} & \mathrm{gr}^i(\wedge^j H_\sigma) \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}} \\ \| & & \| \\ \wedge^i \underline{\omega}_{0,r} \otimes \wedge^{j-i} \underline{\omega}_{\mu,r} & \xrightarrow{\nabla_r} & \wedge^i \underline{\omega}_{0,r} \otimes \wedge^{j-i} H_{\mu,r} \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}}. \end{array}$$

In fact, we can be more precise. From the inclusion $\underline{\omega}_{\mu,r} \subset H_{\mu,r}$, we obtain a third Koszul filtration on $\wedge^{j-i} H_{\mu,r}$, which we denote $K^\bullet(\wedge^{j-i} H_{\mu,r})$. Applying Lemma 4.2 to $K^\bullet$, we see that actually

$$\nabla_r(\mathrm{gr}^i(\wedge^j \underline{\omega}_\sigma)) \subseteq \wedge^i \underline{\omega}_{0,r} \otimes K^{j-i-1}(\wedge^{j-i} H_{\mu,r}) \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}} \subseteq \mathrm{gr}^i(\wedge^j H_\sigma) \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}}.$$

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Let ψ be the following composition:

$$\begin{aligned}\psi\colon \mathrm{gr}^i(\wedge^j \underline{\omega}_\sigma) &\xrightarrow{\nabla_r} \wedge^i \underline{\omega}_{0,r} \otimes K^{j-i-1}(\wedge^{j-i} H_{\mu,r}) \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}} \\ &\longrightarrow \mathrm{gr}^i(\wedge^j \underline{\omega}_\sigma) \otimes (\det \underline{\omega}_{\mu,r})^{p-1} \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}},\end{aligned}$$

considered over S_{K,w_r}, where the second map acts as $A_r \cdot \wedge^{j-i} p_{\mathrm{ur},r}$ on the factor $K^{j-i-1}(\wedge^{j-i} H_{\mu,r})$ and is the identity on the others. From Lemma 6.4, we deduce that ψ extends to $\overline{S}_{K,w_r}$.

This construction can be generalized. From the Koszul filtrations on $\wedge^j \underline{\omega}_\sigma$, $\wedge^j H_\sigma$ defined by the subsheaf $\underline{\omega}_{0,r}$, we obtain filtrations on

$$\otimes_j \wedge^{l_j}(\underline{\omega}_\sigma), \otimes_j \wedge^{l_j}(H_\sigma), \quad j = 1, 2, \ldots, n-1,$$

as explained in Section 4.1.4. From these, in turn, we obtain filtrations on $\underline{\omega}^{\underline{k},w}$ and $H^{\underline{k},w} = \mathbb{S}^{\underline{k}}(H_\sigma) \otimes \delta_\sigma^w$, the filtration on δ_σ^w being trivial. The morphism $p_{\mathrm{ur},r}\colon H_{\mu,r} \to \underline{\omega}_{\mu,r}$ from Section 6.1.2, along with $\mathrm{id}_{\underline{\omega}_{0,r}}$, $\mathrm{id}_{\delta_\sigma}$, lends naturally a morphism of sheaves

$$\mathrm{gr}^\bullet(p_{\mathrm{ur},r})\colon \mathrm{gr}^\bullet(H^{\underline{k},w}) \longrightarrow \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w})$$

on S_{K,w_r}. In fact, under the natural identification $\mathrm{gr}^\bullet(\mathbb{S}^{\underline{k}}(H_\sigma)) = \mathbb{S}^{\underline{k}}(\mathrm{gr}^\bullet(H_\sigma)) = \mathbb{S}^{\underline{k}}(\underline{\omega}_{0,r} \oplus H_{\mu,r})$, see (6.2) and (6.3), we have $\mathrm{gr}^\bullet(p_{\mathrm{ur},r}) = \mathbb{S}^{\underline{k}}(\mathrm{id}_{\underline{\omega}_{0,r}} \oplus p_{\mathrm{ur},r}) \otimes \mathrm{id}^w_{\delta_\sigma}$. From Lemmas 6.1 and 6.4, we deduce the following, which is analogous to Lemma 6.7.

Lemma 6.10 *Consider the composition*

$$\psi\colon \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}) \xrightarrow{\nabla_r} \mathrm{gr}^\bullet(H^{\underline{k},w}) \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}} \longrightarrow \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}) \otimes (\det \underline{\omega}_\mu)^{p-1} \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}},$$

where the second arrow is $A_r \cdot \mathrm{gr}^\bullet(p_{\mathrm{ur},r}) \otimes \mathrm{id}_{\Omega^1}$. Then, ψ, which a priori is only defined on S_{K,w_r}, extends to $\overline{S}_{K,w_r}$.

We can further compose ψ with $\mathrm{id} \otimes \underline{\mathrm{ks}}^{-1}_{\mu,r}$ defined in Proposition 5.5 to get

$$\begin{aligned}\mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}) &\xrightarrow{\psi} \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}) \otimes (\det \underline{\omega}_{\mu,r})^{p-1} \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}} \\ &\xrightarrow{\mathrm{id}\otimes \underline{\mathrm{ks}}^{-1}_\mu} \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}) \otimes (\det \underline{\omega}_{\mu,r})^{p-1} \otimes \underline{\omega}_\mu \otimes \det \underline{\omega}_\sigma \otimes \delta_\sigma^{-1} \\ &\longrightarrow \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k}+\underline{\Delta}_r, w-1}),\end{aligned}$$

where the last map is given by the natural multiplication morphisms. Therefore, we obtain a morphism on $\mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w})$, shifting the weight by $\underline{\Delta}_r = (p+1, p, \ldots, p, 1, \ldots, 1)$. We call this composition

$$\theta_r\colon \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}) \longrightarrow \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k}+\underline{\Delta}_r, w-1})$$

the *generalized (partial) theta operator* relative to the stratum $\overline{S}_{K,w_r}$. This proves the existence part of Theorem 6.5 in the case $r \geq 2$.

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Proposition 6.11 (Basic properties of θ_r) *The operator θ_r satisfies the following fundamental properties:*

(1) *For $f \in H^0(\overline{S}_{K,w_r}, \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k},w}))$, $g \in H^0(\overline{S}_{K,w_r}, \mathrm{gr}^\bullet(\underline{\omega}^{\underline{k}',w'}))$, we have $\theta_r(fg) = f\theta_r(g) + \theta_r(f)g$.*
(2) *The operator θ_r is A_r-linear, that is, $\theta_r(A_r) = 0$.*
(3) *The operator θ_r is Hecke-equivariant.*

Proof

(1) This follows from the construction of θ_r and the properties of ∇_r.
(2) Consider $U \subseteq S_{K,w_r}$ a small enough dense open, so that we can choose a local basis $e_1, \dots, e_n$ of H_σ over U such that $e_1, \dots, e_{r-1}$ is a basis of $\underline{\omega}_0$, $e_1, \dots, e_{n-1}$ a basis of $\underline{\omega}_\sigma$, and e_n reduces to a basis of $H_\sigma/\underline{\omega}_\sigma$. In particular, we see that
$$A_r|_U = a(\overline{e}_r \wedge \cdots \wedge \overline{e}_{n-1})^{p-1},$$
for some $a \in \mathcal{O}_{S_{K,w_r}}(U)$, where $\bar{\cdot}$ denotes the reduction through $H_\sigma \to H_{\mu,r}$. As in Proposition 6.8, one can see that $\theta_r(A_r)|_U = 0$, so that by density $\theta_r(A_r) = 0$.
(3) The same arguments that we used for θ_1 work. ∎

Remark 6.12 Notice that multiplication by A_r, for $1 \le r \le n-1$, is also Hecke-equivariant. By this, we mean that it induces the commutative diagram
$$\begin{array}{ccc} H^i(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w})) & \xrightarrow{T_g} & H^i(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w})) \\ \downarrow{\scriptstyle A_r\cdot} & & \downarrow{\scriptstyle A_r\cdot} \\ H^i(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k}+\underline{w}_r,w})) & \xrightarrow{T_g} & H^i(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k}+\underline{w}_r,w})), \end{array}$$
for any $i \ge 0$, $g \in \mathbf{G}(\mathbb{A}^{p,\infty})$ and automorphic weight $(\underline{k}, w)$, where $\underline{w}_r = (p-1, \dots, p-1, \dots, 0)$ is the weight shift produced by A_r. This follows from the fact that any prime-to-p quasi-isogeny $f\colon A \to A'$, for $A, A' \in \overline{S}_{K,w_r}(T)$, $T \to \overline{S}_{K,w_r}$ finite étale, induces an isomorphism $f^*\colon \underline{\omega}_{A'/T} \to \underline{\omega}_{A/T}$ that respects the filtration $\underline{\omega}_{0,r} \subseteq \underline{\omega}_\sigma$ and sends $A_r(A'/T) \in H^0(T, (\det \underline{\omega}_{A'/T,\mu})^{p-1})$ to $A_r(A/T) \in H^0(T, (\det \underline{\omega}_{A/T,\mu})^{p-1})$.

Remark 6.13 Notice that even though θ_r is not defined on $\underline{\omega}^{\underline{k},w}$, it is defined on the graded parts of a filtration on such sheaves. Since this filtration is Hecke-equivariant, so is the one it induces on $H^0(\overline{S}^{\min}_{K,w_r}, \underline{\omega}^{\underline{k},w})$ and, therefore, every Hecke-eigensystem appearing in $H^0(\overline{S}_{K,w_r}, \underline{\omega}^{\underline{k},w})$ will also appear in
$$H^0(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w})).$$
Furthermore, thanks to the work of [21], the Hecke-eigensystems found in $\{H^0(\overline{S}_{K,w_r}, \underline{\omega}^{\underline{k},w})\}_{\underline{k},w}$ are the same as those found in $\{H^0(\overline{S}_K, \underline{\omega}^{\underline{k},w})\}_{\underline{k},w}$.

6.2.3 Operators B_r

Here, we conclude the proof of Theorem 6.5. In particular, we prove point (3). We achieve this by describing the action of the restriction of θ_r to $\overline{S}_{K,w_{r+1}}$. Let us consider

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$V\colon \underline{\omega}_{\mu,r} \to \underline{\omega}^{(p)}_{\mu,r}$ over the substratum $S_{K,w_{r+1}} \subset \overline{S}_{K,w_r}$. From the canonical filtration of $A[\mathfrak{p}_\sigma]$, we deduce, over $\overline{S}_{K,w_{r+1}}$, the existence of a short exact sequence

$$0 \longrightarrow \underline{\omega}_{\mu,r,0} \longrightarrow \underline{\omega}_{\mu,r} \longrightarrow \underline{\omega}_{\mu,r+1} \longrightarrow 0, \tag{6.4}$$

where $\underline{\omega}_{\mu,r,0} = \ker(V\colon \underline{\omega}_{\mu,r} \to \omega^{(p)}_{\mu,r}) = \underline{\omega}_{0,r+1}/\underline{\omega}_{0,r}$ is an invertible sheaf. Notice that $V\colon \underline{\omega}_{\mu,r+1} \to \underline{\omega}^{(p)}_{\mu,r+1}$ gives a splitting of the p-twist of (6.4) over $S_{K,w_{r+1}}$, since that map is an isomorphism on $S_{K,w_{r+1}}$, so that we get in particular a projection $\pi_{r,0}\colon \underline{\omega}^{(p)}_{\mu,r} \to \underline{\omega}^{p}_{\mu,r,0}$. Considering now $V\colon H_{\mu,r} \to \underline{\omega}^{(p)}_{\mu,r} \subseteq H^{(p)}_{\mu,r}$ over $S_{K,w_{r+1}}$, we obtain an induced isomorphism $\pi_{r,0}V\colon H_{\mu,r}/\underline{\omega}_{\mu,r} \cong (\det \underline{\omega}_\sigma)^{-1} \otimes \delta_\sigma \to \underline{\omega}^{p}_{\mu,r,0}$ and thus a nowhere vanishing section $B_{r,0} \in H^0(S_{K,w_{r+1}}, \underline{\omega}^{p}_{\mu,r,0} \otimes \det \underline{\omega}_\sigma \otimes \delta_\sigma^{-1})$. One can also consider the morphism

$$\underline{\omega}^{p}_{\mu,r,0} \cong \underline{\omega}_{\mu,r,0} \otimes \underline{\omega}^{p-1}_{\mu,r,0} \hookrightarrow \underline{\omega}_{\mu,r} \otimes \underline{\omega}^{p-1}_{\mu,r,0} \xrightarrow{A_{r+1}\cdot} \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1},$$

which, by slight abuse of notation, we denote again $(A_{r+1}\cdot)$. We set $B_r \coloneqq A_{r+1} \cdot B_{r,0}$, which we identify with a section of $\underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1} \otimes \det \underline{\omega}_\sigma \otimes \delta_\sigma^{-1}$ over $S_{K,w_{r+1}}$. Notice that B_r is nowhere vanishing on $S_{K,w_{r+1}}$. We now extend B_r to a section over the closure $\overline{S}_{K,w_{r+1}}$, which vanishes on the complement of $S_{K,w_{r+1}}$.

First, consider the restriction of the partial unit-root projection $A_r \cdot p_{\mathrm{ur},r}\colon H_{\mu,r} \to \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1}$ to $\overline{S}_{K,w_{r+1}}$. Since it coincides with multiplication by $A_r \equiv 0$ when restricted to $\underline{\omega}_{\mu,r}$, we see that it factors through $H_{\mu,r}/\underline{\omega}_{\mu,r}$. On that quotient, when further restricted to $S_{K,w_{r+1}}$, one can see that $A_r \cdot p_{\mathrm{ur},r}$ coincides with multiplication by B_r. In fact, one can write $A_r \cdot p_{\mathrm{ur},r}\colon H_{\mu,r}/\underline{\omega}_{\mu,r} \to \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1}$ as $B_{r,0} = \pi_{r,0}V\colon H_{\mu,r}/\underline{\omega}_{\mu,r} \to \underline{\omega}^{p}_{\mu,r,0}$ composed with $V|^{\mathrm{adj}}_{\underline{\omega}_{\mu,r}}\colon \underline{\omega}^{p}_{\mu,r,0} \hookrightarrow \underline{\omega}^{(p)}_{\mu,r} \to \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1}$. The map $V|^{\mathrm{adj}}_{\underline{\omega}_{\mu,r}}$ coincides with multiplication by A_{r+1} when restricted to $\underline{\omega}^{p}_{0}$. This follows from the commutative diagram

$$\begin{array}{ccc}
\underline{\omega}^{(p)}_{\mu,r} \cong \mathrm{Hom}(\overset{n-r-1}{\wedge}\underline{\omega}^{(p)}_{\mu,r}, \det(\underline{\omega}_{\mu,r})^p) & \xrightarrow{(\wedge V)^*} & \mathrm{Hom}(\overset{n-r-1}{\wedge}\underline{\omega}_{\mu,r}, \det(\underline{\omega}_{\mu,r})^p) \cong \underline{\omega}_{\mu,r} \otimes \det(\omega_{\mu,r})^{p-1} \\
\uparrow & & \uparrow \\
\underline{\omega}^{(p)}_{\mu,r,0} \cong \mathrm{Hom}(\det(\underline{\omega}_{\mu,r+1})^p, \det(\underline{\omega}_{\mu,r})^p) & \xrightarrow{A_{r+1}\cdot} & \mathrm{Hom}(\det \underline{\omega}_{\mu,r+1}, \det(\underline{\omega}_{\mu,r})^p) \cong \underline{\omega}_{\mu,r,0} \otimes \det(\underline{\omega}_{\mu,r})^{p-1},
\end{array}$$

where the first row is the definition of $V|^{\mathrm{adj}}_{\underline{\omega}_{\mu,r}}$, and in the second row, we are using the surjection $\wedge^{n-r-1}\underline{\omega}_{\mu,r} \to \det(\underline{\omega}_{\mu,r+1})$ and the isomorphism $\underline{\omega}_{\mu,r,0} \cong \det(\underline{\omega}_{\mu,r+1})^{-1} \otimes \det(\underline{\omega}_{\mu,r})$ deduced from (6.4). This, in particular, shows what we have claimed above, that B_r extends to a section defined over $\overline{S}_{K,w_{r+1}}$. This is the section corresponding to the morphism $A_r \cdot p_{\mathrm{ur},r}\colon H_{\mu,r}/\underline{\omega}_{\mu,r} \to \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1}$. Moreover, over deeper strata S_{K,w_s}, $s > r+1$ (when there are any), we have that the image of $V\colon H_{\mu,r}/\underline{\omega}_{\mu,r} \to \underline{\omega}^{(p)}_{\mu,r}$ intersects $\underline{\omega}^{p}_{\mu,r,0}$ trivially, by Proposition 5.3. Therefore, $V|^{\mathrm{adj}}_{\underline{\omega}_{\mu,r}}$ vanishes on the image of $V\colon H_{\mu,r}/\underline{\omega}_{\mu,r} \to \underline{\omega}^{(p)}_{\mu,r}$, as one can see working through the definition of the adjugate morphism again. Let us summarize what we have proved so far.

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Lemma 6.14 *Over the closed stratum $\overline{S}_{K,w_{r+1}}$, the morphism $A_r \cdot p_{\mathrm{ur},r}\colon H_{\mu,r} \to \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1}$ factors through $H_{\mu,r}/\underline{\omega}_{\mu,r} \cong (\det \underline{\omega}_\sigma)^{-1} \otimes \delta_\sigma$ and thus gives a section*

$$B_r \in H^0\big(\overline{S}_{K,w_{r+1}}, \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1} \otimes \det(\underline{\omega}_\sigma) \otimes \delta_\sigma^{-1}\big),$$

which is nowhere vanishing over $S_{K,w_{r+1}}$ and identically zero on $\overline{S}_{K,w_{r+1}} \backslash S_{K,w_{r+1}}$.

We record the following lemma.

Lemma 6.15 *The generalized Hasse invariant A_r vanishes with simple zeros along $\overline{S}_{K,w_{r+1}}$. In particular, $f \in H^0\big(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w})\big)$ vanishes along $\overline{S}_{K,w_{r+1}}$ if and only if there is some $g \in H^0\big(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k}-\underline{w}_r,w})\big)$ such that $f = A_r \cdot g$.*

Proof This follows from the Grothendieck–Messing theory. We use the same notations from the proof of Proposition 5.5. Let $s \in S_{r+1} \subseteq \overline{S}_r$ be a point and take $k = k(s)$. Consider $\underline{A}_v \in \overline{S}_r(k[\epsilon]/(\epsilon^2))$ any lift of $\underline{A}_s$ corresponding to s. This will correspond to a linear morphism of k-vector spaces $h_{\mu,r}\colon \underline{\omega}_{\mu,r} \to H/\underline{\omega} = H_{\mu,r}/\underline{\omega}_{\mu,r}$. Furthermore, $v \in \overline{S}_{r+1}(k[\epsilon]/(\epsilon^2)) \subseteq \overline{S}_r(k[\epsilon]/(\epsilon^2))$ if and only if $h_{\mu,r}$ is zero on the line $\underline{\omega}_{\mu,r,0} \subseteq \underline{\omega}_{\mu,r}$. We want to compute $A_r(v)$. We can choose a basis $e_1, \ldots, e_{n-r+1}$ of $H_{\mu,r}$ such that:

- $e_1, \ldots, e_{n-r}$ span $\underline{\omega}_{\mu,r}$,
- $V(e_1) = 0$ (so that e_1 spans $\underline{\omega}_{\mu,r,0}$), $V(e_i) = e_i^{(p)}, 2 \le i \le n-r$, $V(e_{n-r+1}) = e_1^{(p)}$, and
- $h_{\mu,r}(e_i) = a_i e_{n-r+1} + \underline{\omega}_{\mu,r}$, $a_i \in k$ for $1 \le i \le n-r$.

Notice that $v \in \overline{S}_{r+1}(k[\epsilon]/(\epsilon^2))$ if and only if $a_1 = 0$. We can lift each e_i to $\tilde{e}_i := e_i + \epsilon a_i e_{n-r+1}$ to obtain a basis of $\widetilde{\underline{\omega}}_{\mu,r}$ with respect to which $A_r(v) = \epsilon a_1(\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-r})^{p-1}$. This shows that A_r has simple zeros at any point of S_{r+1}. ∎

Finally, we prove the last desired property of θ_r.

Proposition 6.16 *Let $f \in H^0\big(\overline{S}_{K,w_r}, \mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w})\big)$ and write*

$$f = \sum_{\underline{a}} f_{\underline{a}}, \quad f_{\underline{a}} \in H^0\big(\overline{S}_{K,w_r}, \mathrm{gr}^{\underline{a},r}(\underline{\omega}^{\underline{k},w})\big).$$

If $r = n-1, n-2$, then $\theta_r(f)$ is divisible by the Hasse invariant A_r if and only if for each component $f_{\underline{a}}$ either $A_r \mid f_{\underline{a}}$ or $p \mid a_1$.

Proof The idea is to compare the action of θ_r over $\overline{S}_{K,w_{r+1}}$ and that of multiplication by B_r. We show that they are essentially the same, up to a scalar depending on the weight and the decomposition of $\mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w})$ in its graded terms. Indeed, one can consider the restriction to $\overline{S}_{K,w_{r+1}}$ of

$$\psi\colon \underline{\omega}_{\mu,r} \xrightarrow{\nabla_r} H_{\mu,r} \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}} \xrightarrow{A_r \cdot p_{\mathrm{ur},r} \otimes \mathrm{id}} \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1} \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}}.$$

This is an $\mathcal{O}_{\overline{S}_{K,w_{r+1}}}$-linear morphism that factors through the restriction to $\overline{S}_{K,w_{r+1}}$ of $\widetilde{\underline{\mathrm{ks}}}_{\mu,r}\colon \underline{\omega}_{\mu,r} \to H_{\mu,r}/\underline{\omega}_{\mu,r} \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}}$. By construction of the Kodaira–Spencer map, we

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have the following commutative diagram:

$$\begin{array}{ccc} \underline{\omega}_{\mu,r} \otimes (H_{\mu,r}/\underline{\omega}_{\mu,r})^{\vee} & \xrightarrow{\widetilde{\mathrm{ks}}_{\mu,r}\otimes \mathrm{id}} & (H_{\mu,r}/\underline{\omega}_{\mu,r}) \otimes (H_{\mu,r}/\underline{\omega}_{\mu,r})^{\vee} \otimes \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}} \\ \downarrow{\scriptstyle \mathrm{ks}_{\mu,r}} & & \downarrow \\ \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}} & = & \Omega^1_{\overline{S}_{K,w_r}/\mathbb{F}}. \end{array}$$

In particular, over $\overline{S}_{K,w_{r+1}}$, the operator $\theta_r = (\mathrm{id} \otimes \underline{\mathrm{ks}}_{\mu,r}^{-1})\psi \colon \underline{\omega}_{\mu,r} \to \mathrm{Sym}^2(\underline{\omega}_{\mu,r}) \otimes \det(\underline{\omega}_{\mu,r})^{p-1} \otimes \det(\underline{\omega}_{\sigma}) \otimes \delta_{\sigma}^{-1}$ coincides with multiplication by B_r. In fact, more precisely, before applying the symmetrization $\underline{\omega}_{\mu,r} \otimes \underline{\omega}_{\mu,r} \to \mathrm{Sym}^2(\underline{\omega}_{\mu,r})$, the operator acts on local sections of $\underline{\omega}_{\mu,r}$ as

$$s \longmapsto B_r \otimes s.$$

Therefore, for $k > 1$, by the Leibniz rule, the operator $\theta_r \colon \wedge^k \underline{\omega}_{\mu,r} \to \underline{\omega}_{\mu,r} \otimes \wedge^k \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1} \otimes \det(\underline{\omega}_{\sigma}) \otimes \delta_{\sigma}^{-1}$ acts on local sections as

$$s_1 \wedge s_2 \wedge \cdots \wedge s_k \longmapsto \sum_j (-1)^j s_j \otimes B_r \wedge s_1 \wedge \cdots \wedge \hat{s}_j \wedge \cdots \wedge s_k,$$

where, by abuse of notation, we write $B_r \wedge s$, s a section of $\underline{\omega}_{\mu,r}$, to denote the obvious section of $\wedge^2 \underline{\omega}_{\mu,r} \otimes \det(\underline{\omega}_{\mu,r})^{p-1} \otimes \det(\underline{\omega}_{\sigma}) \otimes \delta_{\sigma}^{-1}$. In particular, when $k = n - r = \mathrm{rk}(\underline{\omega}_{\mu,r})$, the formula on local sections simplifies to

$$s_1 \wedge s_2 \wedge \cdots \wedge s_{n-r} \longmapsto B_r \otimes s_1 \wedge \cdots \wedge s_{n-r},$$

as one can see, for instance, by working on a small enough open $U \subseteq \overline{S}_{K,w_{r+1}}$, where we can take, up to a coefficient, $s_1, \ldots, s_{n-r}$ to be a basis of $\underline{\omega}_{\mu,r}$. Recall that the graded sheaf $\mathrm{gr}^{\bullet,r}(\underline{\omega}^{\underline{k},w})$ decomposes as a direct sum of sheaves $\mathrm{gr}^{\underline{a},r}(\underline{\omega}^{\underline{k},w})$. We claim that, for $r = n-2, n-1$ and $s \in H^0(\overline{S}_{K,w_r}, \mathrm{gr}^{\underline{a},r}(\underline{\omega}^{\underline{k},w}))$, we have

$$\theta_r(s)|_{\overline{S}_{K,w_{r+1}}} = a_1 \cdot B_r \cdot s|_{\overline{S}_{K,w_{r+1}}}.$$

This is because, when $r = n-2, n-1$, $\underline{\omega}_{\mu,r}$ has rank 1 or 2. In particular, in both those cases, $\theta_r(s)|_{\overline{S}_{K,w_{r+1}}}$ can be described by considering Example 2.5 and the definition of $\mathrm{gr}^{\underline{a},r}(\underline{\omega}^{\underline{k},w})$. In fact, ignoring the tensor factors relative to $\underline{\omega}_{0,r}$ and δ_{σ}, which we can do because of the Leibniz rule and the fact that θ_r is zero on $\underline{\omega}_{0,r}$ over $\overline{S}_{K,w_{r+1}}$, s can be treated essentially as a section of $\mathbb{S}^{\underline{a}}(\underline{\omega}_{\mu,r})$. The sheaf $\mathbb{S}^{\underline{a}}(\underline{\omega}_{\mu,r})$ will be nonzero only if $b_1 \leq 2$ (recall that $\underline{b}$ is the Young tableau transpose of $\underline{a}$), in which case, by Example 2.5, we have the natural isomorphism

$$\mathbb{S}^{\underline{a}}(\underline{\omega}_{\mu,r}) \cong \mathrm{Sym}^{a_1-a_2}(\underline{\omega}_{\mu,r}) \otimes (\wedge^2 \underline{\omega}_{\mu,r})^{a_2}.$$

In this setting, we can apply the formula that describes the action of θ_r as multiplication by B_r, together with the Leibniz rule, and deduce our claim that $\theta_r(s)|_{\overline{S}_{K,w_{r+1}}} = a_1 \cdot B_r \cdot s|_{\overline{S}_{K,w_{r+1}}}$. This formula, together with Lemma 6.15, implies what we wanted to prove. ∎

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Remark 6.17 The restriction $r = n-1, n-2$ is unnecessary, but proving the same divisibility criterion without that assumption seems to require techniques beyond the scope of this article.

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