Math. Proc. Camb. Phil. Soc. (1985), **97**, 127
Printed in Great Britain

Minimal projections in L_p-spaces

BY E. J. HALTON AND W. A. LIGHT
Mathematics Department, University of Lancaster

(*Received* 20 *November* 1983; *revised* 15 *June* 1984)

Introduction

Let X be a normed linear space and let W be a proper subspace of X. A *projection* is a surjective linear map $P: X \to W$ such that P is idempotent. It is immediately clear that P has norm at least unity. Thus the problem of calculating the number

$$\lambda(W, X) = \inf\{\|P\| : P \text{ is a projection from } X \text{ onto } W\}$$

has some interest. The number $\lambda(W, X)$ is often called the *relative projection constant* of W in X. If the infimum is attained, any attaining projection is called a *minimal projection*. The problems of calculating $\lambda(W, X)$ for a fixed X and W or finding a minimal projection turn out to be very difficult. For example, if $X = C[0, 1]$ with the usual supremum norm and W is the subspace of polynomials of degree at most two then $\lambda(W, X)$ remains unknown as does any example of a minimal projection. One of the few places where the problem shows much tractability is the case

$$X = L^p_{2\pi}, \quad W = M_n.$$

Here $L^p_{2\pi}$ is the space of pth power integrable functions periodic on $[0, 2\pi]$ with the usual norm and M_n is the subspace of trigonometric polynomials of degree at most n. Let

$$(S_n x)(t) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

where the a_k, b_k are the usual Fourier coefficients of x. Then for all p such that

$$1 \leqq p \leqq \infty,$$

S_n is a minimal projection from $L^p_{2\pi}$ onto M_n. At the heart of this result is an algebraic connection between S_n and any other projection from X onto W. This connection goes back essentially to Marcinkiewicz [5], and has been given in quite general form by Rudin [6].

In this paper we shall exhibit similar algebraic connections in an entirely different context. In our case X will be $L_p(S \times T)$ where S and T are either spaces of unit measure or finite sets. We will set $W = L_p(S) + L_p(T)$. Of course, some abuse of notation is involved here since $L_p(S)$ is not strictly speaking a subspace of $L_p(S \times T)$ but the meaning is self-evident. In [4], Jameson and Pinkus showed that if $X = C(S \times T)$, and $W = C(S) + C(T)$ then $\lambda(W, X) = 3$. They also exhibited a minimal projection. Following this, Halton and Light [3] showed that if

$$X = L_p(S \times T), \quad W = L_p(S) + L_p(T) \quad \text{and} \quad p = 1 \text{ or } \infty$$

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then $\lambda(W, X) = 3$ and a minimal projection is given by

$$(Px)(s,t) = \int_S x(a,t)\,d\mu(a) + \int_T x(s,b)\,d\nu(b) - \iint_{S\times T} x(a,b)\,d\mu(a)\,d\nu(b).$$

They also observed that P is the orthogonal projection from $L_2(S\times T)$ onto $L_2(S)+L_2(T)$ and hence is minimal with unit norm in this case. This is in fact an old result due to Golomb [2]. We shall show that this same projection remains minimal for all p, $1 \leqq p \leqq \infty$. As with the truncated Fourier series projection, we are unable to calculate $\lambda(W, X)$ for values of p other than $p = 1, 2, \infty$.

Part I: S and T are finite sets

We shall assume that S and T are finite sets which we identify immediately as $S = \{1, 2, \ldots, n\}$ and $T = \{1, 2, \ldots, m\}$. Initially, we consider the linear space X of all functions $x: S\times T \to \mathbb{R}$. Let G be the subspace of X consisting of functions which depend only on the s variable and let H be the subspace of X consisting of functions which depend only on the t variable. Let $W = G+H$. It will be important to remember throughout the discussion that a function $x \in X$ is in W if and only if x satisfies the 'four-point rule':

$$x(i,j) = x(k,j) + x(i,l) - x(k,l), \quad 1 \leqq i, k \leqq n,\ 1 \leqq j, l \leqq m.$$

Now let P be a projection from X onto W. It will be convenient to describe the action of P on the functions $x_{ab} \in X$, where

$$x_{ab}(i,j) = \delta_{ia}\,\delta_{bj}, \quad 1 \leqq a, i \leqq n,\ 1 \leqq b, j \leqq m.$$

We shall set $Px_{ab} = w_{ab}$. In order that P should be a projection we require that

(i) $\sum_{a=1}^{n} w_{ab}(i,j) = \delta_{bj}, \quad 1 \leqq i \leqq n,\ 1 \leqq b, j \leqq m,$

(ii) $\sum_{b=1}^{m} w_{ab}(i,j) = \delta_{ai}, \quad 1 \leqq a, i \leqq n,\ 1 \leqq j \leqq m,$

(iii) $w_{ab}(i,j) = w_{ab}(k,j) + w_{ab}(i,l) - w_{ab}(k,l), \quad 1 \leqq a, i, k \leqq n,\ 1 \leqq b, j, l \leqq m.$

The first two conditions ensure that P preserves functions in G and H while the final condition ensures the range of P is indeed W. These conditions first appeared in [4]. Before going on to our main result it will be convenient to present here some elementary facts. Firstly, the subspace W has dimension $n+m-1$. If we define

$$\langle x, y\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} x(i,j)\,y(i,j)$$

then this defines an inner product on X. The orthogonal complement $W^\perp$ has dimension $(m-1)(n-1)$, $W \cap W^\perp = \{0\}$, $W + W^\perp = X$. It is also clear that the elements

$$g_a = \sum_{b=1}^{m} x_{ab} \quad 1 \leqq a \leqq n; \quad h_b = \sum_{a=1}^{n} x_{ab} \quad 1 \leqq b \leqq m$$

span W. An immediate consequence of the relationship

$$W^\perp = (G+H)^\perp = G^\perp \cap H^\perp$$

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is that

$$W^{\perp} = \left\{x \in X: \sum_{i=1}^{n} x(i,j) = 0, \quad 1 \leqq j \leqq m, \quad \sum_{j=1}^{m} x(i,j) = 0, \quad 1 \leqq i \leqq n\right\}.$$

An elementary result from [3] was

LEMMA 1·1. *Let $J: X \to W$ be the projection defined by*

$$Jx_{ab} = w_{ab} = \begin{cases} -\dfrac{1}{nm}, & a \neq i, b \neq j \\ \dfrac{n-1}{nm}, & a = i, b \neq j \\ \dfrac{m-1}{nm}, & a \neq i, b = j \\ \dfrac{n+m-1}{nm}, & a = i, b = j. \end{cases}$$

Then $Jx = 0$ for all $x \in W^{\perp}$, i.e. J is the orthogonal projection.

Main Result

We define permutations η, τ on S and T respectively by

$$\eta(1) = n, \quad \eta(i) = i-1 \quad 2 \leqq i \leqq n,$$
$$\tau(1) = m, \quad \tau(j) = j-1 \quad 2 \leqq j \leqq m.$$

These induce linear operators A and B on X as follows: if $x \in X$ then

$$(Ax)(i,j) = x(\eta(i), j), \quad 1 \leqq i \leqq n,$$
$$(Bx)(i,j) = y(i, \tau(j)), \quad 1 \leqq j \leqq m.$$

THEOREM 1·2. *Let Q be any projection from X onto W. Let J be the projection defined in Lemma 1·1. Then*

$$J = \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} B^{-j} A^{-i} Q A^{i} B^{j}.$$

Proof. It is easy to check the veracity of the equation for the functions $\{g_a\}_{a=1}^{n}$ and $\{h_b\}_{b=1}^{m}$. For example, for $1 \leqq a \leqq n$ $Bg_a = g_a$ and $Ag_a = g_{\eta(a)}$. Hence $A^i B^j g_a \in W$ and so $QA^i B^j g_a = A^i B^j g_a$. Thus, for $1 \leqq a \leqq n$,

$$\frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} B^{-j} A^{-i} Q A^{i} B^{j} g_a = \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} B^{-j} A^{-i} A^{i} B^{j} g_a = g_a.$$

To investigate members of the orthogonal complement $W^{\perp}$, let us first consider the action of the right-hand side of our equation on x_{ab}, $1 \leqq a \leqq n$, $1 \leqq b \leqq m$. Denoting Qx_{ab} by w_{ab} we have

$$\begin{aligned} \sum_{i-0}^{n=1} \sum_{j=0}^{m-1} (B^{-j} A^{-i} Q A^{i} B^{j} x_{ab})(e,f) &= \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (B^{-j} A^{-i} w_{a+i,b+j})(e,f) \\ &= w_{ab}(e,f) + w_{a+1,b}(e+1,f) + \ldots + w_{a+n-1,b}(e+n-1,f) \\ &\quad + w_{a,b+1}(e,f+1) + \ldots + w_{a+n-1,b+1}(e+n-1,f+1) + \ldots \\ &\quad + w_{a,b+m-1}(e,f+m-1) + \ldots + w_{a+n-1,b+m-1}(e+n-1,f+m-1) \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} w_{a+l,b+k}(e+l,f+k). \end{aligned}$$

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In this computation we have attached the obvious meaning to subscripts and arguments outside the permitted range. They are to be read modulo n and m appropriately. Now let u_{abcd} denote the function $x_{ab}+x_{cd}-x_{ad}-x_{cb}$. It is clear that $u_{abcd}\in W^{\perp}$. Also

$$\sum_{k=0}^{n-1}\sum_{l=0}^{m-1} B^{l}A^{-k}QA^{k}B^{l}u_{abcd}(e,f)$$
$$=\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\{w_{a+l,b+k}(e+l,f+k)+w_{c+l,d+k}(e+l,f+k)$$
$$-w_{a+l,d+k}(e+l,f+k)-w_{c+l,b+k}(e+l,f+k)\}.$$

The contribution to this summation containing subscripts α and β is

$$w_{\alpha\beta}(e+\alpha-a,f+\alpha-b)+w_{\alpha\beta}(e+\alpha-c,f+\beta-d)$$
$$-w_{\alpha\beta}(e+\alpha-a,f+\beta-d)-w_{\alpha\beta}(e+\alpha-c,f+\beta-b)$$

which is zero by the four-point rule. Hence

$$\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} B^{-j}A^{-i}Q\,A^{i}B^{j}u_{abcd}=0,\quad \begin{cases}1\leqq a,c\leqq n\\ 1\leqq b,d\leqq m.\end{cases}$$

Clearly, there are enough functions of this type to span $W^{\perp}$ and this is sufficient to complete the proof. ∎

We now introduce a norm on X with certain invariance properties which allow us to draw conclusions about the minimality of the projection J. The following results can then be viewed as a consequence of a theorem of Rudin [6]. The setting for this theorem is as follows. We take a Banach space X and a compact topological group G whose elements can be thought of as a group of bounded linear transformations on X. W is a closed subspace of X which is invariant under G. If there exists a bounded projection Q from X onto W, then there exists a bounded projection P from X onto Y which commutes with every element of G. This is in a way the substance of Theorem 1·2, although we did not employ Rudin's theorem since the direct proof is brief.

THEOREM 1·3. *Suppose X is endowed with a norm under which the operators A and B have unit norm. Then $J:X\to W$ is a minimal projection.*

Proof. We have

$$\|J\|=\sup_{\|x\|=1}\|Jx\|\leqq\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\|B^{-j}\|\,\|A^{-i}\|\,\|Q\|\,\|A^{i}\|\,\|B^{j}\|\,\|x\|\leqq\|Q\|.\ \blacksquare$$

COROLLARY 1·4. *For $x\in X$, define*

$$\|x\|_{p}=\left\{\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}|x(i,j)|^{p}\right\}^{1/p}.$$

Then $J:X\to W$ is a minimal projection.

Proof. It is elementary to check that $\|A\|=\|B\|=1$ in this case. ∎

Corollary 1·4 can be rephrased by observing that S and T may be regarded as measure spaces (S,Σ,u) and (T,Θ,ν) by taking $\mu(i)=1/n$, $1\leqq i\leqq n$ and $\nu(j)=1/m$, $1\leqq j\leqq m$. Then J is the minimal projection from $L_{p}(S\times T)$ onto $L_{p}(S)+L_{p}(T)$ for $1\leqq p\leqq\infty$. In [3] and [4] the value of $\|J\|_{p}$ for p equals 1 or ∞ has been computed as $3-2(n+m-1)/nm$.

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Part II: Minimal projections in $L_p(S \times T)$

We now go on to consider the case $L_p(S \times T)$ where $1 < p < \infty$. We shall take (S, Σ, μ) and (T, θ, ν) to be finite, non-atomic measure spaces with $\mu(S) = \nu(T) = 1$. The measure space $(S \times T, \Phi, \sigma)$ is then constructed in the usual way. We shall consider $W = L_p(S) + L_p(T)$ as a subspace of $X = L_p(S \times T)$. If $p = 1$ or ∞ then it was shown in [3] that the projection

$$(Px)(s,t) = \int_S x(a,t)\,d\mu(a) + \int_T x(s,b)\,d\nu(b) - \iint_{S\times T} x(a,b)\,d\mu(a)\,d\nu(b)$$

is a minimal one from X onto W. We will establish that this projection is also minimal for $1 < p < \infty$.

Our approach will be similar to that of [3]. We will exploit the connections between X and the case when S and T are finite sets. Henceforward, we shall denote by X_n the space of real-valued functions on $\{1, 2, \ldots, n\}^2$ with the norm as given in Corollary 1·4. The subspace of X_n given by $W = G + H$ in part one will be denoted by W_n. Throughout this section there will be several meanings to the symbol $\|.\|$. Which norm is intended will always be obvious from the context.

It will be important in several instances to be able to subdivide a measure space. It is well-known (see [1], p. 308) that any finite non-atomic measure space (H, Δ, β) can be written as the union of a finite number of disjoint sets each with measure no more than some prescribed tolerance ϵ. Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < \beta(H)$. Then we can use the partitioning property to show that there is a set $A \in \Delta$ such that $\beta(A) = \alpha$. This is done as follows. Firstly partition H into m sets each of measure at most $\alpha/2$. By taking unions if necessary we can assume that at least one of these sets, A_1 say, has measure $\frac{1}{2}\alpha \leqq \beta(A_1) \leqq \alpha$. Partition $H \backslash A_1$ into sets of measure at most $\frac{1}{4}\alpha$. By taking the union of as many of these as necessary together with A_1 we obtain A_2 with $\frac{3}{4}\alpha \leqq \beta(A_2) \leqq \alpha$. Repeating we get $A_1, A_2, \ldots$ with $2^i - 1/2^i\alpha \leqq \beta(A_i) \leqq \alpha$. Now setting $A = \bigcup_{n=1}^{\infty} A_n$ gives the required set. This argument coupled with an elementary construction gives:

LEMMA 2·1. *Let (H, Δ, β) be a finite, non-atomic measure space. Suppose E belongs to Δ and $\beta(E) > 0$. Then*

(i) *for such an E and any α, $0 < \alpha < \beta(E)$, there exists $F \subset E$ such that $\beta(F) = \alpha$,*

(ii) *for such an E, there exists a partition of E, $E = \bigcup_{i=1}^{n} F_i$ such that $\beta(F_i) = \beta(E)/n$.*

Now for a fixed $n \in \mathbb{N}$ take $\{S_i\}_{i=1}^n$, $\{T_j\}_{j=1}^n$ forming partitions of S and T respectively such that $\mu(S_i) = \nu(T_j) = 1/n$, $1 \leqq i, j \leqq n$. Define a subspace $Y_n \subset X$ by

$$Y_n = \left\{ y \in X : y = \sum_{i,j=1}^{n} a_{ij} \chi_{S_i \times T_j} \right\}.$$

Here χ_E denotes the characteristic function of the set E. Now our link between X and X_n is established using the linear operators $U_n : X \to X_n$, $V_n : X_n \to X$, where

$$(U_n x)(i,j) = n^2 \iint_{S_i \times T_j} x\,d\sigma \quad \text{for} \quad x \in X,\ 1 \leqq i, j \leqq n$$

and

$$V_n z = \sum_{i,j=1}^{n} z(i,j)\,\chi_{S_i \times T_j} \quad \text{where} \quad z \in X_n.$$

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Clearly, the range of V_n is Y_n and $V_n U_n y = y$ for all y in Y_n.

LEMMA 2·2. *If Q is a projection from X onto W, then $U_n Q V_n$ is a projection from X_n onto W_n.*

Proof. We check that $U_n Q V_n$ has range W_n. Suppose $z \in X_n$. Since V_n has range Y_n and Q has range W we have $Q V_n z = g + h$ for some $g \in L_p(S)$, $h \in L_p(T)$. We show that $U_n Q V_n z$ satisfies the four-point rule and is therefore in W_n.

$$\begin{aligned}(U_n Q V_n z)(i,j) + (U_n Q V_n z)(l,k) &= n^2 \iint_{S_i \times T_j} (g+h)\,d\sigma + n^2 \iint_{S_l \times T_k} (g+h)\,d\sigma \\ &= n^2 \left\{ \frac{1}{n} \int_{S_i} g\,d\mu + \frac{1}{n} \int_{T_j} h\,d\nu + \frac{1}{n} \int_{S_l} g\,d\mu + \frac{1}{n} \int_{T_k} h\,d\nu \right\} \\ &= n^2 \left\{ \iint_{S_i \times T_k} g\,d\sigma + \iint_{S_l \times T_j} h\,d\sigma + \iint_{S_l \times T_j} g\,d\sigma + \iint_{S_i \times T_k} h\,d\sigma \right\} \\ &= (U_n Q V_n z)(i,k) + (U_n Q V_n z)(l,j).\end{aligned}$$

It now remains to show that if $z \in W_n$ then $U_n Q V_n z = z$. Suppose first that $z \in G$. Then $z(s_i, t_j) = k_i$, $1 \leqq j \leqq n$, and

$$V_n z = \sum_{i,j=1}^{n} k_i \chi_{S_i \times T_j}.$$

$V_n z$ is clearly an element of $L(S)$ and is hence preserved under Q.

Now

$$\begin{aligned}(U_n Q V_n z)(i,j) = (U_n V_n z)(i,j) &= n^2 \iint_{S_i \times T_j} (V_n z)\,d\sigma \\ &= n^2 \iint_{S_i \times T_j} \sum_{l,m} k_l \chi_{S_l \times T_m}\,d\sigma = n^2 \iint_{S_i \times T_j} k_i\,d\sigma = k_i, \quad 1 \leqq i,j \leqq n.\end{aligned}$$

A similar argument shows that if $z \in H$ then $U_n Q V_n z = z$.

The result then follows from the definition of W_n and the linearity of $U_n Q V_n$. ∎

Recall our definition of the projection $P: X \to W$,

$$(Px)(s,t) = \int_S x(a,t)\,d\mu(a) + \int_T x(s,b)\,d\nu(b) - \iint_{S \times T} x(a,b)\,d\mu(a)\,d\nu(b).$$

We let $N = \{x \in X : Px = 0\}$. Then $N + W = X$ and $W \cap N = \{0\}$. We now characterise the elements of N in the following lemma.

LEMMA 2·3. $N = \{x \in X : \int_S x\,d\mu = 0$ *for almost all* $t \in T$ *and* $\int_T x\,d\nu = 0$ *for almost all* $s \in S\}$.

Proof. If $x \in X$ and $Px = 0$, then

$$\int_S x(a,t)\,d\mu(a) - \iint_{S \times T} x(a,b)\,d\mu(a)\,d\nu(b) = -\int_T x(s,b)\,d\nu(b).$$

The left-hand side of this equation can be regarded as an element of $L_p(T)$ while the right-hand side is a member of $L_p(S)$. Hence both functions must be constant almost everywhere.

Set

$$\int_S x(a,t)\,d\mu(a) - \iint_{S \times T} x(a,b)\,d\mu(a)\,d\nu(b) = k, \quad \text{for almost all } t \in T.$$

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Then

$$\int_S x(a,t)\,d\mu(a) + \int_S k\,d\mu = k, \quad \text{for almost all } t \in T.$$

Thus

$$\int_S x(a,t)\,d\mu(a) + k = k, \quad \text{for almost all } t \in T.$$

or

$$\int_S x(a,t)\,d\mu(a) = 0, \quad \text{for almost all } t \in T.$$

By a similar argument we have $\int_T x(s,b)\,d\nu(b) = 0$ for almost all $s \in S$. ∎

We now show that the operator U_n is a means of relating the kernels and ranges of P and J where J is the orthogonal projection defined in the previous section and mapping X_n onto W_n.

LEMMA 2·4. (i) *If $y \in Y_n \cap N$ then $U_n y \in W_n^{\perp}$.* (ii) *If $y \in Y_n \cap W$ then $U_n y \in W_n$.*

Proof. (i) If $y \in Y_n \cap N$ then for $1 \leqq j \leqq n$ we have

$$\sum_{i=1}^{n} (U_n y)(i,j) = \sum_{i=1}^{n} n^2 \iint_{S_i \times T_j} y\,d\sigma = n^2 \iint_{S \times T_j} y\,d\sigma.$$

This last integral is zero due to the properties of N exhibited in Lemma 2·3. In a similar manner we have that

$$\sum_{j=1}^{n} (U_n y)(i,j) = 0 \quad \text{for} \quad 1 \leqq i \leqq n.$$

The representation of $W_n^{\perp}$ in Section I gives us that $U_n y \in W_n^{\perp}$.

(ii) If $y \in Y_n \cap W$ then we have already shown in Lemma 2·2 that $U_n y \in W_n$. ∎

If we restrict ourselves to Y_n we can use the results of Section I to establish a result about $P: X \to W$.

THEOREM 2·5. *If Q is any projection from X onto W and $y \in Y_n$, then*

$$Py = \frac{1}{n^2} \sum_{i,j=1}^{n} V_n B^{-j} A^{-i} U_n Q V_n A^i B^j U_n y.$$

Proof. We can express y uniquely as $y = w + w'$ where $w \in W$ and $w' \in N$. Since $w = Py$ we have

$$\begin{aligned} w &= \int_S y\,d\mu + \int_T y\,d\nu - \iint_{S \times T} y\,d\sigma \\ &= \sum_{k=1}^{n} \left\{ \int_{S_k} y\,d\mu + \int_{T_k} y\,d\nu \right\} - \iint_{S \times T} y\,d\sigma. \end{aligned}$$

For almost all $(s',t') \in S_i \times T_j$ we have

$$\begin{aligned} (Py)(s',t') &= \sum_{k=1}^{n} \left\{ \int_{S_k} y(a,t')\,d\mu(a) + \int_{T_k} y(s',b)\,d\nu(b) \right\} - \iint_{S \times T} y(a,b)\,d\mu(a)\,d\nu(b) \\ &= \sum_{k=1}^{n} (a_{kj} + a_{ik}) - \iint_{S \times T} y\,d\sigma. \end{aligned}$$

This expression is clearly constant for almost all $(s',t') \in S_i \times T_j$ and hence

$$w = Py \in Y_n \cap W.$$

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Since $y - Py = w'$ then $w' \in Y_n \cap N$. Lemma 2·4 now gives $U_n w \in W_n$ and $U_n w' \in W_n^{\perp}$. Recalling from Section I that the projection

$$J = \frac{1}{n^2} \sum_{i,j=1}^{n} B^{-j} A^{-i} U_n Q V_n A^i B^j,$$

where $U_n Q V_n$ is a projection from X_n onto W_n by Lemma 2·2, we have

$$\frac{1}{n^2} \sum_{i,j=1}^{n} B^{-j} A^{-i} U_n Q V_n A^i B^j U_n w' = 0$$

and

$$\frac{1}{n^2} \sum_{i,j=1}^{n} B^{-j} A^{-i} U_n Q V_n A^i B^j U_n w = U_n w.$$

This is sufficient to establish the required result. ∎

Before arriving at our objective we need the following Lemmas.

LEMMA 2·6. *Let $f \in L_p(H, \Delta, \beta)$, $\beta(H) = 1$. Given $\epsilon > 0$ there exists a partition of H, $\{H_i\}_1^n$ and $a_i \in \mathbb{R}$, $1 \leqq i \leqq n$, such that $\|f - \Sigma_{i=1}^n a_i \chi_{H_i}\| < \epsilon$, and $\beta(H_i) = 1/n$, $1 \leqq i \leqq n$.*

Proof. Firstly, it is possible to approximate f to any required accuracy by simple functions. By an easy restructuring these can be thought of as functions of the form $\Sigma_{i=1}^m a_i \chi_{E_i}$ where the E_i are a partition of H. Suppose the function $g = \Sigma_{i=1}^m a_i \chi_{E_i}$ satisfies $\|f - g\| < \epsilon/2$. Choose n large enough to ensure $2^p \max_i |a_i|^p \, m/n \leqq \epsilon^p$. Then use Lemma 2·1 (ii) to subdivide each E_i into as many sets as possible of measure $1/n$. Hence $E_i = (\bigcup_{k=1}^{r(i)} G_k^i) \cup F_i$ where $\beta(G_k^i) = 1/n$, $\beta(F_i) < 1/n$ and together the G_k^i and F_i form a partition of E_i. Set $\bigcup_{i=i}^m F_i = F$. Then $\beta(F) < m/n$. Partition F into sets of measure $1/n$. Then set

$$g_1 = \sum_{j=1}^{m} \sum_{k=1}^{r(j)} a_j \chi_{G_k^j}.$$

$$\begin{aligned}
\|g - g_1\|^p &= \sum_{j=1}^{m} \int_{E_j} |g - g_1|^p \, d\beta \\
&= \sum_{j=1}^{m} \left[\sum_{k=1}^{r(j)} \int_{G_k^j} |g - g_1|^p \, d\beta + \int_{F_j} |g - g_1|^p \, d\beta \right] \\
&= \sum_{j=1}^{m} \int_{F_j} |g|^p \, d\beta \\
&= \sum_{j=1}^{m} |a_j|^p \beta(F_j) \\
&\leqq \max_j |a_j|^p . m/n \\
&\leqq \left(\frac{\epsilon}{2}\right)^p .
\end{aligned}$$

Finally $\|f - g_1\| \leqq \|f - g\| + \|g - g_1\| \leqq \epsilon/2 + \epsilon/2 = \epsilon$. ∎

LEMMA 2·7. *The linear operators $U_n : X \to X_n$ and $V_n : X_n \to X$ are both of unit norm.*

Proof. Firstly

$$\begin{aligned}
\|V_n z\| &= \left\{ \iint |V_n z|^p \, d\sigma \right\}^{1/p} = \left\{ \iint \Big| \sum_{i,j} z(i,j) \chi_{S_i \times T_j} \Big|^p \, d\sigma \right\}^{1/p} \\
&= \left\{ \sum_{i,j} \iint_{S_i \times T_j} |z(i,j)|^p \, d\sigma \right\}^{1/p} = \left\{ \frac{1}{n^2} \sum_{i,j} |z(i,j)|^p \right\}^{1/p} = \|z\|.
\end{aligned}$$

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Secondly, we have

$$\begin{aligned}\|U_n x\| &= \left\{\frac{1}{n^2}\sum_{i,j}|(U_n x)(i,j)|^p\right\}^{1/p}\\ &= \left\{\frac{1}{n^2}\sum_{i,j}\left|n^2\iint_{S_i\times T_j} x\,d\sigma\right|^p\right\}^{1/p}\\ &\leqq \left\{\frac{1}{n^2}\sum_{i,j}\left[\iint_{S\times T}|x|\,n^2\chi^2_{S_i\times T_j}\,d\sigma\right]^p\right\}^{1/p}\\ &\leqq \left\{\frac{1}{n^2}\sum_{i,j}\iint_{S\times T}|x|^p\,\chi^p_{S_i\times T_j}\,d\sigma\cdot\left(\iint_{S\times T}n^{2q}\chi^q_{S_i\times T_j}\right)^{p/q}\right\}^{1/p}\end{aligned}$$

using Hölder's inequality and assuming $1/p+1/q=1$. Now

$$\left(\iint_{S\times T}n^{2q}\chi^q_{S_i\times T_j}\right)^{p/q} = n^{pq-1(2q-2)} = n^2,$$

and so

$$\begin{aligned}\|U_n x\| &\leqq \left\{\frac{1}{n^2}\sum_{i,j}n^2\iint_{S\times T}|x|^p\chi_{S_i\times T_j}\,d\sigma\right\}^{1/p}\\ &= \|x\|.\end{aligned}$$

Attainment is easily achieved. ∎

THEOREM 2·8. *The projection $P: L_p(S\times T)\to W$ given by*

$$(Px)(s,t) = \int_S x(a,t)\,d\mu(a)+\int_T y(s,b)\,d\nu(b)-\iint_{S\times T}x(a,b)\,d\mu(a)\,d\nu(b)$$

is minimal for any p such that $1\leqq p\leqq\infty$.

Proof. We shall again set $X = L_p(S\times T,\Phi,\sigma)$ where $1<p<\infty$ since the other cases are contained in [3]. Given $\epsilon>0$ we choose $x\in X$ with $\|x\|=1$ so that

$$\|Px\|\geqq\|P\|-\epsilon/2.$$

By Lemma 2·6 there exists a $y\in Y_n$ (for a suitable choice of n) such that

$$\|y\|=1 \quad\text{and}\quad \|x-y\|<\tfrac{1}{2}\epsilon\|P\|^{-1}.$$

Then

$$\begin{aligned}\|P\| &\leqq \|Px\|+\frac{\epsilon}{2}\leqq\|Py\|+\|P(x-y)\|+\frac{\epsilon}{2}\\ &\leqq\|Py\|+\epsilon\\ &\leqq\frac{1}{n^2}\sum_{i,j=1}^{n^2}\|V_nB^{-j}A^{-i}U_nQV_nA^iB^jU_n\|\,\|y\|+\epsilon,\end{aligned}$$

where Q is any projection from X onto W (Theorem 2·5). Hence

$$\|P\|\leqq\frac{1}{n^2}\sum_{i,j=1}^{n}\|V_n\|\,\|B^{-j}\|\,\|A^{-i}\|\,\|U_n\|\,\|Q\|\,\|V_n\|\,\|A^i\|\,\|B^j\|\,\|U_n\|+\epsilon\leqq\|Q\|+\epsilon.$$

Since ϵ was arbitrary this concludes the proof. ∎

The authors would like to thank the referee for his helpful comments on an earlier version of this paper.

The first author was supported by the Science and Engineering Research Council of Great Britain.

https://doi.org/10.1017/S0305004100062666 Published online by Cambridge University Press

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https://doi.org/10.1017/S0305004100062666 Published online by Cambridge University Press