PARAMETRIC SURFACES

II. LOWER SEMI-CONTINUITY OF THE AREA

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In 1914 Carathéodory defined m-dimensional measure in n-dimensional space†. He considered one-dimensional measure as a generalization of length and he proved that the length of a rectifiable curve coincides with its one-dimensional measure.

In 1919, Hausdorff‡, developing Carathéodory's ideas, constructed a continuous scale of measures. After the work of Carathéodory and Hausdorff, the obvious problem was to treat area as two-dimensional measure and to establish the well-known integral formulae. But a considerable amount of work on establishing the integral formulae for the area had been done before that, with a variety of definitions of the area, due no doubt to the difficulty of the problem. Later the Lebesgue definition, somewhat modified by Frechet, became the dominant one. A particularly valuable feature of the Lebesgue-Frechet (L.-F.) definition is that it provides the area with the property of lower semi-continuity. When applied to plane figures bounded by a Jordan curve the L.-F. definition gives in fact the value of the interior area of the figure, and thus in cases when the boundary is of positive L_2-measure, the L.-F. area differs from the value of the closed area in the usual Lebesgue sense. This, of course, need not be considered as in any sense a serious fault with the definition, and, of course, there is nothing astonishing then if the area of the projection measured as a *closed area* happens to be greater than the area of the projected figure measured as an *open area*. This is the so-called projection conflict§. Being entirely based on an obvious confusion of two measures, it could not have shaken any confidence in the L.-F. definition. As a matter of fact, there is no conflict whatsoever, and the L.-F. area of the orthogonal projection of a parametric surface, which (the projection) itself is a parametric surface, is always less than or equal to the L.-F. area of the projected surface. This is obvious for polyhedral surfaces, and then it follows at once for the general case.

In 1913, Geöcze‖ gave an example of a parametric surface that fills up the cube and has at the same time L.-F. area equal to zero. But the Geöcze parametric surface is no surface in any geometric sense, and the notion of its area is entirely artificial. Whatever value might have been obtained for its area, it could not have been a test for the definition. Thus no fault was suspected in the L.-F. definition until recently, and a large amount of work was done on the basis of the L.-F. definition, culminating in the proof of the integral formula for parametric surfaces under the most general

† 'Über das lineare Mass von Punktmengen—eine Verallgemeinerung des Längenbegriffs', *Nachr. Ges. Wiss. Göttingen* (1914), 404–26.

‡ 'Dimension und äusseres Mass', *Math. Ann.* 79 (1919), 157–79.

§ T. Rado, 'On the problem of plateau', *Ergebn. Math. Grenzgeb.* 2 (1933).

‖ Z. Geöcze, 'Sur l'exemple d'une surface dont l'aire est égale à zéro et qui remplit un cube', *Bull. Soc. Math. France*, 41 (1913), 29–31.

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conditions. But the study of area cannot be restricted to the very narrow class of the parametric surfaces, that is, to the class of continuous images of a disk. A notion of area is attached to a much wider class of surfaces, such as continuous images of other figures than a disk, many cases of discontinuous images and to surfaces given as point sets. With the experience of more than fifty years of the modern theory of functions of a real variable, any attempt at a general study of the problem of area would inevitably lead to treating the area as a measure. In 1944 I set myself the problem of treating the area as Λ^2-measure and of proving first the identity of the L.-F. area of parametric surfaces and of their Λ^2-measure, excluding the boundaries. However, I came to results very different from the ones I was hoping for. In my paper 'On the definition and the value of the area of a surface'†, I constructed a *topological sphere* whose surface, on the one hand, has L.-F. area as small as we please, and, on the other hand, has a positive Lebesgue three-dimensional measure. This conflicts with all geometrical ideas of the area:

(i) It shows that the L.-F. area cannot be fitted into any scale of measures.

(ii) It has no property of additivity, even in the most clear cases. A construction similar to one employed in my paper leads to a parametric surface that includes all points of a unit square, and whose area is as small as we please. If we take the case of the ideal material surfaces whose plane density is 1, it would be very difficult to attribute to the above surface the value of mass as small as we please.

(iii) It has not been, and it cannot be, extended to surfaces given as point-sets. For these reasons, clinging to the L.-F. definition of area would imply treating it as a functional defined in an artificially restricted class of surfaces, and completely detached from the general study of the area‡.

The property of the semi-continuity attached to the L.-F. area is very important, but the conflicts with the idea of an area are too deep. The L.-F. definition is to be abandoned and the problems that have been solved with it are to be done anew.

I define the area of a surface as the Hausdorff two-dimensional measure. The area thus defined has no property of semi-continuity, as follows immediately from my paper quoted above. In the present paper, however, I show that in the most important and very general cases the property of semi-continuity holds.

We shall first specify the notation and the definition concerning τ-dimensional measure. All that we consider is always in three-dimensional Euclidean space. For any $\delta > 0$, $U(\delta)$ denotes a finite or an enumerable set of spheres all of diameters less than or equal to δ. If E is a set of points, $U(\delta, E)$ denotes a $U(\delta)$, such that $E \subset \{U(\delta)\}^0$. Denoting by d the diameter of the general sphere in $U(\delta)$, we write, for $0 < \tau \leqslant 3$,

$$A_\tau \text{ l.bd.} \sum_{U(\delta, E)} d^\tau = \Lambda^\tau_\delta E, \quad \lim_{\delta \to 0} \Lambda^\tau_\delta E = \Lambda^\tau E,$$

where A_τ is a constant for any fixed τ, and for $\tau = 1, 2, 3$ is equal to $1, \frac{1}{4}\pi, \frac{1}{6}\pi$ respectively.

† *Quart. J. Math.* (Oxford Series), 16 (1945), 86–102.

‡ The surface quoted above has an important bearing on the classical problem of the minimum value of the area of the surface of a solid, whose volume has a fixed value. It gives the answer that if the L.-F. definition is adopted, the area may be as small as we please. Since my paper was published it has been pointed out more than once that if the interior volume is constant then the old answer remains valid. Of course, I have never said that any application, past or future, of the L.-F. definition has led or would lead necessarily to unexpected results.

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LEMMA 1. *There exists a positive function* $\delta_0 = \delta_0(\epsilon, E)$ *of* $\epsilon > 0$ *and of a set* E *of finite* Λ^τ*-measure, such that for any* $U(\delta_0)$

$$A_\tau \sum_{U(\delta_0)} d^\tau > \Lambda^\tau\{EU(\delta_0)\} - \epsilon.$$

By the definition of measure there exists a positive function $\delta_0 = \delta_0(\epsilon, E)$ such that for any $U(\delta_0, E)$

$$A_\tau \sum_{U(\delta_0, E)} d^\tau > \Lambda^\tau E - \tfrac{1}{2}\epsilon. \tag{1}$$

Suppose that there exists a $U(\delta_0)$ such that

$$A_\tau \sum_{U(\delta_0)} d^\tau \leqslant \Lambda^\tau\{EU(\delta_0)\} - \epsilon. \tag{2}$$

Define a $U_1(\delta_0) = U\{\delta_0, E - EU(\delta_0)\}$ such that

$$A_\tau \sum_{U_1(\delta_0)} d^\tau < \Lambda^\tau\{E - EU(\delta_0)\} + \tfrac{1}{2}\epsilon. \tag{3}$$

Writing $U(\delta_0) + U_1(\delta_0) = U(\delta_0, E)$ we conclude, from (2) and (3), that

$$\sum_{U(\delta_0, E)} d^\tau < \Lambda^\tau E - \tfrac{1}{2}\epsilon, \tag{4}$$

which contradicts (1), and thus (2) is impossible, which proves the lemma.

LEMMA 2. *If* $\Lambda^\tau E < \infty$, E_1 *is a* Λ^τ*-measurable subset of* E *and* $\epsilon > 0$, *then there exists a* $U(\delta, E_1)$ *such that*

$$\Lambda^\tau\{(E - E_1)\, U(\delta, E_1)\} < \epsilon.$$

For let δ_0 be the value of the function $\delta_0(\frac{1}{2}\epsilon, E)$ of Lemma 1. For any $U(\delta_0, E_1)$ we have

$$A_\tau \sum_{U(\delta_0, E_1)} d^\tau > \Lambda^\tau\{EU(\delta_0, E_1)\} - \tfrac{1}{2}\epsilon. \tag{5}$$

On the other hand, there exists a $U(\delta_0, E_1)$ such that

$$A_\tau \sum_{U(\delta_0, E_1)} d^\tau < \Lambda^\tau E_1 + \tfrac{1}{2}\epsilon. \tag{6}$$

From (5) and (6) the lemma follows.

Let E be a Λ^2-measurable set lying between the planes $z = \zeta_1$ and $z = \zeta_2$ and $E(\zeta)$ be the intersection of E with the plane $z = \zeta$.

LEMMA 3.
$$\Lambda^2 E \geqslant \int_{\zeta_1}^{\zeta_2} \Lambda^* E(\zeta)\, d\zeta,$$

where Λ^* *is the outer linear measure.*

If $\Lambda^2 E = \infty$ there is nothing to prove. Let then $\Lambda^2 E < \infty$. Take a $U(\delta, E) = \Sigma S_i$ and let $S_i(\zeta)$ denote the intersection of the sphere S_i with the plane $z = \zeta$. We may assume that $U(\delta, E)$ also lies between the planes $z = \zeta_1$, and $z = \zeta_2$. We have

$$\tfrac{1}{4}\pi(dS_i)^2 = \int_{\zeta_1}^{\zeta_2} \{dS_i(\zeta)\}\, d\zeta,$$

and

$$\tfrac{1}{4}\pi\Sigma(dS_i)^2 = \int_{\zeta_1}^{\zeta_2} \{\Sigma dS_i(\zeta)\}\, d\zeta,$$

from which the lemma easily follows.

We shall denote by $S(P, r)$ the set of points inside and on the surface of the sphere of centre P and radius r. The set of the interior points will be denoted by $S^0(P, r)$ and the surface itself by $s(P, r)$.

Given a Λ^2-measurable set $E \subset S(P, R)$, denote by $E(r)$ the set $Es(P, r)$.

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LEMMA 3·1. $$\Lambda^2 E \geqslant \int_0^R \Lambda^* E(r)\, dr.$$

The proof is very similar to the above one and follows from the asymptotic equalities

$$\tfrac{1}{4}\pi\{dS(P_1,\rho)\}^2 \sim \int_{r_1-\rho}^{r_1+\rho} \{dS(r)\}\, dr,$$

where $d(PP_1) = r_1$, $S(r) = S(P_1,\rho)\, s(P,r)$, and ρ is small.

LEMMA 4. *If γ is a simple closed curve on a sphere of radius r, such that for every great circle of the sphere there is a point M on γ whose distance from the plane of the circle does not exceed ϵ, then $\Lambda\gamma > 2\pi r - 8\epsilon$.*

Take a pair of points A, B on γ dividing it into two arcs AMB and BNA of equal length, and let c be the equator of the sphere with respect to the mid-point of an arc AB of the great circle. Let M, M' be the points on γ and c respectively nearest to each other. Denote by $\cup AM'$, $\cup M'B$, $\cup MM'$ the shorter arcs, and their lengths, of great circles joining A to M', and so on, and by AM, MB the arcs, and their lengths, of γ. We have

$$\cup MM' < 2\epsilon, \quad AM + \cup MM' + \cup M'M + MB > \cup AM' + \cup M'B = \pi r.$$

Thus $AMB > \pi r - 4\epsilon$ and hence $\gamma > 2\pi r - 8\epsilon$.

A parametric surface $P = \Phi(M)$, $M \in H$, is said to be *ϵ-smooth* ($\epsilon \geqslant 0$) *with respect to a plane p*, if for any simply connected domain $U \subset H$, the Φ-image of the boundary of which is in p, the distance of the Φ-image of any point of U from p does not exceed ϵ. If $\epsilon = 0$ we say that the surface is *smooth with respect to p*.

If the parametric surface is ϵ-smooth with respect to any plane we say that it is *ϵ-smooth* (*smooth*, if $\epsilon = 0$).

If a Φ-element Q is not simply connected then the image of each of the inner complementary domains is called a *bulge*. $\Phi(Q)$ is the *base* of the bulge.

LEMMA 5. *An ϵ-smooth parametric surface cannot have a bulge of diameter greater than 2ϵ.*

For let U be an inner complementary domain of a Φ-element Q. If $d\Phi(U) = \delta$, we can find a point $M \in U$ such that the segment joining $\Phi(M)$ to $\Phi(Q)$ is of length not less than $\frac{1}{2}\delta$. Let p be the plane through $\Phi(Q)$ perpendicular to the segment. The image of the boundary of U, being the point $\Phi(Q)$, is in p and $\Phi(M)$ is at a distance not less than $\frac{1}{2}\delta$ from p. By ϵ-smoothness $\frac{1}{2}\delta \leqslant \epsilon$, which proves the lemma.

Let a parametric surface Π, defined by the equation $P = \Phi(M)$, be ϵ-smooth and $\Phi(M_0)$ its point at Π-distance $\Delta > 2\epsilon$ from the boundary. Write $\Pi = \Sigma(P, Q)$, where the summation is extended over all Φ-elements of H. Take the sphere $S\{\Phi(M_0), r\}$, $2\epsilon < r < \Delta$, and denote by $K(M_0, r)$ the saturated domain in H which contains M_0 and whose Φ-image is inside the sphere. The same domain will also be denoted by $K(Q_0, r)$, where Q_0 is the Φ-element containing M_0.

We shall establish a lower bound for $\Lambda^2\Phi\{K(M_0, r)\}$.

Denote by $Q(M_0)$ the Φ-element satisfying the following conditions: (i) M_0 belongs either to $Q(M_0)$ or to an interior complementary domain of $Q(M_0)$. (ii) $Q(M_0)$ is not contained in an interior complementary domain of any other Φ-element.

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It is easy to see that $Q(M_0)$ exists and is defined uniquely. We have

$$D_\Pi\{\Phi(M_0), \Phi(Q(M_0))\} \leqslant 2\epsilon,$$

for otherwise we would have a bulge of diameter greater than 2ϵ, which is impossible. Thus $Q(M_0) \subset K(M_0, r)$. Define now the subset $\Pi(M_0)$ of Π by the equation

$$\Pi(M_0) = \Sigma'(P, Q),$$

where the summation Σ' is extended over those Φ-elements Q, that are not separated from $Q(M_0)$ by any other Φ-element and do not belong to an interior complementary domain of $Q(M_0)$ (thus Σ' contains also the point $(P, Q(M_0))$).

LEMMA 6. *$\Pi(M_0)$ is homoeomorphic with a disk.*

Let $(P, Q) \in \Pi(M_0)$ and denote by $\bar{Q}$ the continuum consisting of Q and of all Φ-elements (if any) that are separated from $Q(M_0)$ by Q. Define the function $\Phi'(M)$ by the equation

$$\Phi'(M) = \Phi(Q) \quad \text{for any} \quad M \in \bar{Q}.$$

$\Phi'(M)$ is continuous and defines a parametric surface Π'. Φ'-elements of H are $\bar{Q}$. No two Φ'-elements are separated by a third one and the boundary of Π' is different from a point, for if it were a point, it would imply that $Q(M_0)$ is a boundary Φ-element, and as $\Phi(M_0)$ is Π-distant $\Delta > 2\epsilon$ from the boundary $\Pi(M_0)$ would be a bulge of diameter greater than or equal to Δ which is impossible. Hence Π is homoeomorphic with a disk. The correspondence

$$(P, Q) \sim (P, \bar{Q})$$

between points of $\Pi(M_0)$ and Π' being topological, the lemma is proved.

The saturated domain in $\Pi(M_0)$ containing the point $\Phi\{Q(M_0)\}$, and included in the sphere $S\{\Phi(M_0), r\} = S$, is homoeomorphic with a plane domain and its total boundary is on S. Call this domain $\Pi(M_0, r)$, and the outer frontier of its Π-closure $\gamma(r)$. $\gamma(r)$ is on S and it divides $\Pi(M_0)$ into two domains, having $\gamma(r)$ as a common boundary. Obviously $\gamma(r) \subset \Phi\{K(M_0, r')\}$ for any $r < r' < \Delta$. If $\gamma(r)$ has a continuum of convergence then $\Lambda\gamma(r) = \infty$; if not then $\gamma(r)$ is a Jordan curve in the sense of Π-distances. (It need not be a Jordan curve in the sense of the Euclidean distance.) Denote by H_1 the complementary domain of $\Phi^{-1}\{\gamma(r)\}$ in H containing the point M_0.

The total boundary of $\Phi(H_1)$ is $\gamma(r)$. The distance of $\gamma(r)$ from the plane of any great circle on S does not exceed ϵ. For if the distance of $\gamma(r)$ from a plane p through $\Phi(M_0)$ were greater than ϵ there would exist a plane p' parallel to p distant more than ϵ from $\Phi(M_0)$ and separating $\Phi(M_0)$ from $\gamma(r)$. Then there would exist a neighbourhood of $\Phi(M_0)$ with its total boundary in p', which is impossible by ϵ-smoothness.

By Lemma 4,

$$\Lambda\gamma(r) > 2\pi r - 8\epsilon,$$

and by Lemma 3 we deduce

LEMMA 7.
$$\Lambda^2\Phi\{K(M_0, r)\} > \pi r^2 - 8\epsilon r.$$

We shall need the following two lemmas proved elsewhere:

LEMMA 8†. *If Γ is a set of spheres in a bounded domain such that the centre of no one of them is inside another one, then Γ can be divided into* 104 *subsets Γ_j ($j = 1, \ldots, 104$), such that no pair of spheres belonging to the same subset have points in common.*

† A. S. Besicovitch, 'On the definition and value of the area of a surface', *Quart. J. Math.* (Oxford Series), 16 (1945), 92.

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LEMMA 9†. *Vitali Covering (in the narrow sense) Principle. If $F(X)$ is a non-negative additive function of a measurable Ξ set X (i.e. belonging to the additive class Ξ of sets), and if a set Γ of spheres covers the set E, which need not be measurable in the narrow Vitali sense (which means that, for any $x \in E$, Γ contains a sphere $S(x,r)$ with arbitrarily small r), then Γ contains a subset Γ' of disjoint spheres such that $F^*(E - \Gamma') = 0$.*

Given a plane t, a point P_0 in t, and two positive numbers α, r, then (t, P_0, α, r) denotes the set of points P', satisfying the conditions that the length of the segment P_0P' is not greater than r and the segment P_0P' makes an angle not exceeding α with the plane t. We say that the plane t is the tangential plane to Π at the point (P_0, Q_0) if (i) $\Phi(M) \neq P_0$ in a neighbourhood of Q_0, (ii) there exists $r(\alpha) > 0$ for $\alpha > 0$ such that for $r < r(\alpha)$

$$\Phi\{K(Q_0, r)\} \subset (t, P_0, \alpha, r),$$

(iii) the boundary of $\Phi\{K(Q_0, r)\}$, which obviously lies on the spherical part of the surface of (t, P_0, α, r), cannot be deformed continuously into a point without leaving that part of the surface.

Denote by E_k $(k = 1, 2, \ldots, \infty)$ the set of points of Π of multiplicity k. The measure $\Lambda^2\Pi$ is defined in the following way: If $\Lambda^2 E_\infty > 0$ then $\Lambda^2\Pi = \infty$, otherwise

$$\Lambda^2\Pi = \Lambda^2 E_1 + 2\Lambda^2 E_2 + 3\Lambda^2 E_3 + \ldots.$$

Similarly Λ^1- and Λ^2-measure of a part of parametric surface are defined. Lemmas 3 and 3·1 have been proved on the assumption that all the points of the sets are simple. With the new definition of measure they remain true when the multiplicity of points is counted, and Lemma 4 holds when γ is a Jordan curve only in a parametric sense.

Let $\{\Pi_n\}$ be a convergent sequence of parametric surfaces defined by a uniformly convergent sequence $\{\Phi_n(M)\}$ of continuous functions, and Π the limit surface defined by the function $\Phi(M) = \lim \Phi_n(M)$. Denote by $K_n(M_0, r)$ the domain defined with respect to Π_n as $K(M_0, r)$ is defined with respect to Π. We shall consider two cases, (i) Π is smooth, (ii) at almost all points of Π^0 (in the sense of Λ^2-measure) a tangential plane exists.

Let $\Phi(M_0)$ be an interior point of Π and, in addition, in the case (ii), a point at which a tangential plane exists.

LEMMA 10. *There exists a positive function $\rho(\eta, M_0)$ of a positive variable η such that $0 < r < \rho(\eta, M_0)$ implies*

$$\underline{\lim}\, \Lambda^2\Phi_n\{K(M_0, r)\} > (1-\eta)^2 \pi r^2.$$

(i) If Π is smooth then Π_n is ϵ_n-smooth, where $\epsilon_n \to 0$. For any $r' < r$,

$$K_n(M_0, r') \subset K(M_0, r)$$

for large n. By Lemma 7, $\underline{\lim}\, \Lambda^2\Phi_n\{K_n(M_0, r)\} \geqslant \pi r'^2$.

Hence $\underline{\lim}\, \Lambda^2\Phi_n\{K(M_0, r)\} \geqslant \pi r^2$.

(ii) If t is the tangential plane at (P_0, Q_0), where Q_0 is the Φ-element containing M_0 and $P_0 = \Phi(M_0)$, then for sufficiently small r

$$\Phi\{K(M_0, r)\} \subset (t, P_0, \alpha, r),$$

† A. S. Besicovitch, 'A general form of the covering principle', *Proc. Cambridge Phil. Soc.* 42 (1946), 2.

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where $\cos\alpha > 1-\frac{1}{2}\eta$. Hence the orthogonal projection of $\Phi\{K(M_0, r)$ on t covers the whole of the circle $c(P_0, r\cos\alpha)$ in the plane t, and thus the whole of the circle $c(P_0, (1-\frac{1}{2}\eta)r)$.

By the uniform convergence of the sequence $\{\Phi_n(M)\}$ the orthogonal projection of $\Phi_n\{K(M_0, r)\}$ covers the whole of $c(P_0, (1-\eta)r)$ for large n, from which the lemma follows.

Suppose that $\Lambda^2\Pi_n$ is bounded, say less than b for all n. Given $\eta > 0$, $\delta > 0$, denote by $\Pi(\delta)$ the set of those points $\Phi(M)$ of Π whose Π-distance from the boundary is greater than δ and for which $\rho(\eta, M)$ of Lemma 10 is greater than δ. We obviously have $\Pi^0 = \lim_{\delta\to 0} \Pi(\delta)$. Define the sets $|\Pi(\delta)|, |\Pi^0|$ by the equations

$$P \in |\Pi(\delta)| \sim (P, Q) \in \Pi(\delta), \quad P \in |\Pi^0| \sim (P, Q) \in \Pi^0.$$

All the points of $|\Pi(\delta)|, |\Pi^0|$ are considered as simple.

LEMMA 11. $$\Lambda^2|\Pi^0| < \infty.$$

The whole set $|\Pi(\delta)|$ can be included in a finite set Γ of spheres of radius δ satisfying the following conditions: (i) The centres of the spheres are at points of $|\Pi(\delta)|$. (ii) The centre of no sphere is inside any other one. By Lemma 8

$$\Gamma = \sum_{i=1}^{104} \Gamma_i,$$

where for every i, Γ_i is a set of disjoint spheres of Γ. Let $\Phi(M_{i,k})$ $(k = 1, 2, \ldots, n_i)$ be the centres of the spheres of Γ_i. The domains $K(M_{i,k}, \delta)$ are non-overlapping, whence

$$\Lambda^2\Pi_n \geqslant \sum_{k=1}^{n_i} \Lambda^2\Phi_n\{K(M_{i,k}, \delta)\}.$$

By Lemma 10

$$b \geqslant \varliminf \Lambda^2\Pi_n > n_i(1-\eta)^2\pi\delta^2 = (1-\eta)^2\pi \sum_{\Gamma_i} r^2.$$

Hence

$$\sum_{\Gamma} \pi r^2 < \frac{104b}{(1-\eta)^2},$$

from which it easily follows that

$$\Lambda^2|\Pi^0| \leqslant 104b.$$

Denoting by E_k^0 the set of points of Π^0 of multiplicity k we have $|\Pi^0| = E_1^0 + E_2^0 + \ldots + E_\infty^0$. Thus $\Lambda^2E_1^0 + \ldots + \Lambda^2E_\infty^0 < \infty$.

LEMMA 12. *If for two parametric surfaces* Π, Π' *defined by the functions* $\Phi(M), \Phi'(M)$

$$D_{\text{Euc.}}\{\Phi(M), \Phi'(M)\} < \epsilon$$

for all $M \in H$, *then*

$$-2\epsilon < D_\Pi\{\Phi(M_1), \Phi(M_2)\} - D_{\Pi'}\{\Phi'(M_1), \Phi'(M_2)\} < 2\epsilon$$

for any pair M_1, M_2 *in* H.

The proof follows at once from the definition of parametric distances.

Given a parametric surface Π and a point P of multiplicity k $(1 \leqslant k < \infty)$ on Π^0, let $(P, Q_1), \ldots, (P, Q_k)$ be the points of Π^0 forming the multiple point P. Denote by $\Delta(P, \Pi)$ the smallest of the Π-distances of these points from one another and from the boundary of Π.

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If a point P is of infinite multiplicity on Π^0 and N a positive integer we denote by $\Delta(P, \Pi, N)$ the largest number d satisfying the following condition. There are N points of Π^0 $(P, Q_1), \ldots, (P, Q_N)$ whose Π-distances from one another and from the boundary are not less than d.

SEMI-CONTINUITY THEOREM. $\Lambda^2\Pi^0 \leqslant \underline{\lim}\, \Lambda^2\Pi^0_n$ *in two cases,* (i) *when* Π *is smooth,* (ii) *when* Π *has a tangential plane at almost all interior points.*

Obviously we may assume without any loss of generality the convergence of $\Lambda^2\Pi^0_n$ and thus we have to prove that

$$\Lambda^2\Pi^0 \leqslant \lim \Lambda^2\Pi^0_n.$$

Denote by Π', Π^0 in the case (i) and Π^0 reduced by those points at which no tangential plane exists in the case (ii). In either case $\Lambda^2\Pi' = \Lambda^2\Pi^0$. Denote by E_k $(k = 1, 2, \ldots, \infty)$ the set of points of Π' of multiplicity k. By Lemma 11

$$\Lambda^2 E_1 + \Lambda^2 E_2 + \ldots + \Lambda^2 E_\infty < \infty.$$

Take positive integers k, N and positive numbers ϵ, δ and write $E = E_1 + \ldots + E_k + E_\infty$. For each $i = 1, 2, \ldots, k, \infty$ define a finite or an enumerable set $U_i(E_i, \delta)$ of closed spheres each of diameter less than δ, including E_i in their interior and such that

$$\Lambda^2\{(E - E_i)\, U_i\} < \epsilon.$$

Denote by Γ_1 the set of spheres $S(P, r)$ satisfying the following conditions:

$$P = \Phi(M) \in E_1, \quad S(P, r) \subset U_1, \quad r < \min\{\delta, \tfrac{1}{2}\Delta(P, \Pi), \rho(\eta, M)\}.$$

Γ_1 covers the set E_1 in the Vitali narrow sense. Therefore, by Lemma 9, it contains a subset Γ_1' of disjoint spheres such that

$$\Lambda^2(E_1 \Gamma_1') = \Lambda^2 E_1.$$

If δ is sufficiently small

$$\tfrac{1}{4}\pi \sum_{\Gamma_1'} d^2 > \Lambda^2 E_1 - \epsilon.$$

Denote by Γ_1'' a finite subset of Γ_1' satisfying a similar inequality

$$\tfrac{1}{4}\pi \sum_{\Gamma_1''} d^2 > \Lambda^2 E_1 - \epsilon.$$

Denote now by Γ_2 the set of spheres $S(P, r)$ satisfying the following conditions: $P \in E_2 - \Gamma_1''$; $S(P, r)$ is inside U_2 and outside Γ_1'', and if (P, Q_1), (P, Q_2) are the pair of points of Π forming the double points P and $M_1 \in Q_1$, $M_2 \in Q_2$, then $r < \min\{\delta, \tfrac{1}{2}\Delta(P, \Pi), \rho(\eta, M_1), \rho(\eta, M_2)\}$. As before Γ_2 has a disjoint subset Γ_2', such that

$$\Lambda^2\{(E_2 - \Gamma_1'')\, \Gamma_2'\} = \Lambda^2\{E_2 - \Gamma_1''\} \geqslant \Lambda^2 E_2 - \epsilon.$$

If δ is sufficiently small a finite subset Γ_2'' of Γ_2' can be found such that

$$\tfrac{1}{4}\pi \sum_{\Gamma_2''} d^2 > \Lambda^2 E_2 - 2\epsilon.$$

Similarly, Γ_3 is the set of spheres $S(P, r)$ satisfying the following conditions: $P \in E_3 - \Gamma_1'' - \Gamma_2''$; $S(P, r)$ is inside U_3 and outside $\Gamma_1'' + \Gamma_2''$, and if (P, Q_j) $(j = 1, 2, 3)$ are the three points of Π forming the triple point P and $M_j \in Q_j$, then

$$r < \min\{\delta, \tfrac{1}{2}\Delta(P, \Pi), \rho(\eta, M_1), \rho(\eta, M_2), \rho(\eta, M_3)\}.$$

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Then a disjoint subset Γ_3' of Γ_3 and a finite subset Γ_3'' of Γ_3' are defined so that

$$\Lambda^2\{(E_3-\Gamma_1''-\Gamma_2'')\,\Gamma_3'\} = \Lambda^2(E_3-\Gamma_1''-\Gamma_2'') > \Lambda^2 E_3 - 2\epsilon, \quad \tfrac{1}{4}\pi\sum_{\Gamma_3''} d^2 > \Lambda^2 E_3 - 3\epsilon,$$

and so on up to Γ_k''.

Finally, the set Γ_∞ of spheres $S(P,r)$ is defined to satisfy the conditions:

$$P \in E_\infty - \Gamma_1'' - \ldots - \Gamma_k'';$$

$S(P,r)$ is inside U_∞ and outside $\Gamma_1'' + \ldots + \Gamma_k''$, and if (P, Q_j) $(j = 1, \ldots, N)$ are the points defining $\Delta(P, \Pi, N)$ then

$$r < \min\{\delta, \tfrac{1}{2}\Delta(P,\Pi,N); \rho(\eta, M_j) \quad (j = 1, \ldots, N)\},$$

where $M_j \in Q_j$. Then a disjoint subset Γ_∞' of Γ_∞ and a finite subset Γ_∞'' of Γ_∞' are defined so that

$$\tfrac{1}{4}\pi\sum_{\Gamma_\infty''} d^2 > \Lambda^2 E_\infty - (k+1)\,\epsilon.$$

We have

$$\Lambda^2 E_1 + 2\Lambda^2 E_2 + \ldots + k\Lambda^2 E_k + N\Lambda^2 E_\infty < \frac{\pi}{4}\sum_{\Gamma_1''} d^2$$

$$+ 2\frac{\pi}{4}\sum_{\Gamma_2''} d^2 + \ldots + k\frac{\pi}{4}\sum_{\Gamma_k''} d^2 + N\frac{\pi}{4}\sum_{\Gamma_\infty''} d^2 + \{1^2 + 2^2 + \ldots + k^2 + N(k+1)\}\epsilon. \qquad (7)$$

Let $S(P,r)$ be a sphere of Γ_2'' and $\Phi(M_1)$, $\Phi(M_2)$ the two points of Π forming the double point P. As $\Phi(M_1) = \Phi(M_2)$ the two domains $K(M_1, r)$, $K(M_2, r)$ are either identical or disjoint. They cannot be identical because if so they would contain an arc c joining M_1 and M_2. As $\Phi(c) \subset S(P,r)$, $d\Phi(c) < 2r$ and $D_\Pi\{\Phi(M_1), \Phi(M_2)\} < 2r$, which is in contradiction with the inequality

$$2r < \Delta(P,\Pi) < D_\Pi\{\Phi(M_1), \Phi(M_2)\}.$$

By Lemma 10 $\quad \underline{\lim}\,\Lambda^2\Phi_n\{K(M_j,r)\} \geqslant (1-\eta)^2\pi r^2 \quad (j = 1, 2).$

Similarly, any sphere of Γ_i'' defines i domains for $i \leqslant k$ and N domains for $i = \infty$ of the form $K(M_j, r)$.

All these domains corresponding to the same sphere or to different ones are disjoint, and for each of them $\quad \underline{\lim}\,\Lambda^2\Phi_n\{K(M_j,r)\} \geqslant (1-\eta)^2\pi r^2.$

We have $\Sigma'\Phi_n\{K(M,r)\} \subset \Pi^0$, where the summation Σ' is extended over all the domains $K(M,r)$ corresponding to the spheres of $\Gamma_1'' + \ldots + \Gamma_k'' + \Gamma_\infty''$. Hence

$$a = \lim \Lambda^2\Pi_n^0 \geqslant \underline{\lim}\,\Sigma'\Lambda^2\Phi_n\{K(M,r)\} \geqslant (1-\eta)^2\pi\Big(\sum_{\Gamma_1''} r^2 + \ldots + k\sum_{\Gamma_k''} r^2 + N\sum_{\Gamma_\infty''} r^2\Big). \qquad (8)$$

By (7) and (8)

$$\Lambda^2 E_1 + 2\Lambda^2 E_2 + \ldots + k\Lambda^2 E_k + N\Lambda^2 E_\infty < \frac{a}{(1-\eta)^2} + \{1^2 + 2^2 + \ldots + k^2 + N(k+1)\}\epsilon.$$

This being true for any k, N, ϵ, η we conclude that $\Lambda^2 E_\infty = 0$, and that

$$\Lambda^2\Pi^0 = \Lambda^2 E_1 + 2\Lambda^2 E_2 + \ldots \leqslant a,$$

which was to be proved.

The theorem is not true if we take parametric surfaces with their boundaries, for Λ^2-measure of a parametric contour has no property of semi-continuity as the following example shows. Let the first contour C_0 be the boundary of a unit square. Take a

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series of positive terms $\eta_1+\eta_2+\ldots<\frac{1}{4}$. Divide each side of C_0 into n_1 equal segments of length $\delta_1 = 1/n_1$, and from the centre of each segment draw the perpendicular h_1 of length $h_1 = \frac{1}{2}$ outwards from the square. The contour C_0 together with each perpendicular h_1 taken twice form a parametric contour C_1. Divide each h_1 into n_2 equal segments of length $\delta_2 = h_1/n_2$, and from the centre of each segment draw the perpendicular h_2 of length h_2 from each side of h_1 assuming that $0<(\delta_1-2h_2)/\delta_1<\eta_1$. The contour C_1 together with all perpendiculars h_2 taken twice forms the contour C_2. Similarly, perpendiculars h_3 are drawn, $0<(\delta_2-2h_3)/\delta_2<\eta_2$, and the contour C_3 is defined, and so on. The sequence $\{C_n\}$ of parametric contours converges to a parametric contour C and $\Lambda^2 C_n = 0$ for any n, while $\Lambda^2 C > 1$.

With very slight and obvious modifications C_n and C can be made Jordan curves.

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