Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Stan Dolan, 126A Harpenden Road, St Albans, Herts., AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by September 20th 2018. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award will be made. Please print your own name clearly! The names of all successful solvers will be published in the July 2018 edition.

The MA and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. The SPC permission form is available for download from

https://www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, the pupil ***may still participate and will be eligible for a prize in the same way as others****.*

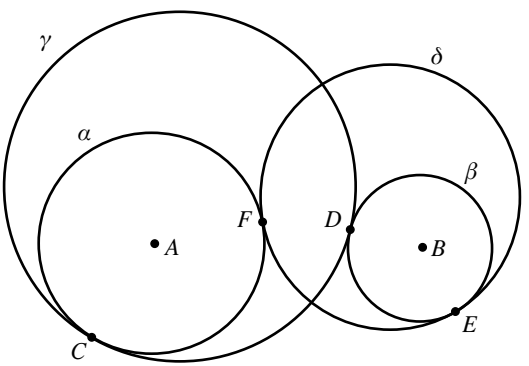
Problem 2018.3 **(Geoff Strickland)**

Two circles, α and β, have centres at A and B respectively.

Circle γ contains circle α and touches it at C. Circle γ also touches circle β (externally) at D.

Circle δ contains circle β and touches it at E. Circle δ also touches circle α (externally) at F.

Prove that AB, CD and EF are concurrent.



Problem 2018.4 **(Nick Lord)**

For positive real numbers x and y find the minimum value of

$$\frac{(1+x)(x+y)(y+8)}{xy}.$$

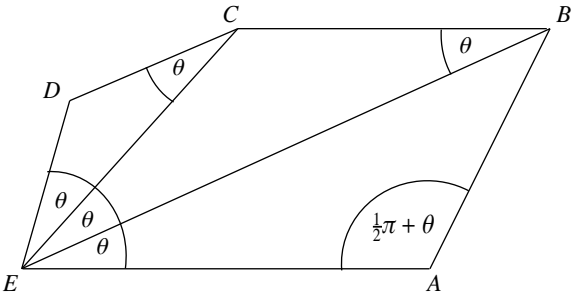
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Solutions to 2018.1 and 2018.2

Problem 2018.1 was solved by Cartesian Gains Student Problem Solving Group (Mountain Lakes High School, USA). Problem 2018.2 was solved by Abigail Burn (South Wilts Grammar School), Thomas Jackson (Cedars Upper School, Leighton Buzzard), Sae Koyama (King's College London Mathematics School), Vincent Liu (Dulwich College) and Ayush Modi (The Sixth Form College Farnborough). Both problems were solved by Donald Flynn (Nottingham High School), Elliot Gathercole (Cedars Upper School, Leighton Buzzard), Andy Li (Dulwich College) and Andrew Ng (Tonbridge School).

Problem 2018.1 **(Stan Dolan)**

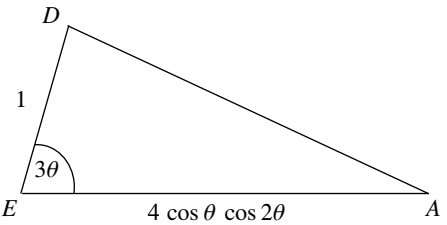
The diagram shows some of the angles in pentagon $ABCDE$.



Find, in terms of θ, the angle between lines AD and BE.

Solution

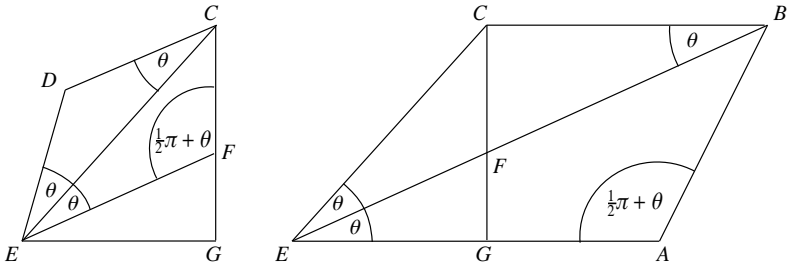
Mountain High, Elliot, Andy and Andrew all used the Sine Rule to connect various lengths. This straightforward approach leads to triangle ADE being similar to the triangle shown.



Noting that $4\cos\theta\cos 2\theta = \dfrac{\sin(\pi - 4\theta)}{\sin\theta}$ now makes it easy to show that $\angle EAD = \theta$.

Donald used a very different approach. He started by letting the perpendicular from C to AE intersect BE and AE at F and G, respectively. Then quadrilaterals $FCDE$ and $ABCE$ are similar.

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Donald used complex numbers to show that G is the midpoint of AE (although trigonometry works just as well). His argument was then extremely elegant. By similarity, $\angle EFD = \angle EAC = 2\theta$, whereas $\angle EFA = \pi - 2\theta$. Therefore AD and BE intersect (at F) at an angle 2θ.

Problem 2018.2 **(Geoff Strickland)**

(a) Find three rational numbers such that their sum is 6, the sum of their squares is $\frac{25}{2}$ and the sum of their cubes is 27.

(b) Is your solution unique?

Solution

Abigail, Thomas, Sae, Vincent, Elliot, Andy and Andrew all gave very similar solutions. The most succinct of these were Andrew's and Ayush's and were as follows.

Let the numbers be x, y and z,

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx) \Rightarrow xy + yz + zx = \tfrac{47}{4},$$

$$x^3 + y^3 + z^3 - 3xyz = \big(x + y + z\big)\big(x^2 + y^2 + z^2 - xy - yz - zx\big) \Rightarrow xyz = \tfrac{15}{2}.$$

Therefore x, y and z satisfy the equation $t^3 - 6t^2 + \frac{47}{4}t - \frac{15}{2} = 0$.

Then $\big(t - \frac{3}{2}\big)\big(t - 2\big)\big(t - \frac{5}{2}\big) = 0$ and so x, y and z are $\frac{3}{2}$, 2 and $\frac{5}{2}$ in some order.

Since the conditions uniquely determine a cubic which has a unique set of roots, the solution is unique.

Donald developed his own method for solving cubic equations, involving an imaginative use of 3D-geometry. It would be great if he wrote this up as an article in a future *Gazette*.

The first prize of £25 is awarded to Donald Flynn. The second prize of £20 is awarded to Andrew Ng.

10.1017/mag.2018.82 STAN DOLAN

Note from the Editor

See *www.m-a.org.uk/the-mathematical-gazette* for the SPC questions.

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