once we know that the limit is $\frac{e}{e-1}$, relatively crude bounds for e refine these bounds further. For example, $e < \frac{11}{4}$ leads to $\frac{11}{7} < \frac{e}{e-1}$, a nice companion to the upper bound of $\frac{13}{8}$ above.

References

1. M. Z. Spivey, The Euler-Maclaurin formula and sums of powers, *Math. Mag*. **79** (2006) pp. 61-65.
2. M. Z. Spivey, Letter to the Editor, *Math. Mag*. **83** (2010) pp. 54-55.
3. F. Holland, $\lim_{m\to\infty} \sum_{k=0}^{m} \left(\frac{k}{m}\right)^m = \frac{e}{e-1}$, *Math. Mag*. **83** (2010) pp. 51-54.

10.1017/mag.2016.72

NICK LORD
Tonbridge School, Kent TN9 1JP

100.20 Outside the box

Stephen Lee and Charlie Stripp recently wrote an article [1] about MEI (Mathematics in Education and Industry, a charitable body in the UK concerned with improving mathematical education). It included their *Maths Item of the Month* from September 2013, which was a delightful geometric problem. I stopped to admire the scenery, as it were, and wander around the mathematical neighbourhood of the problem. It is the connections between different parts of mathematics that I find so fascinating, and here, what started out as school trigonometry turned into elementary number theory, which sadly is not in the school syllabus, at least not in England, but is well within the grasp of the interested student.

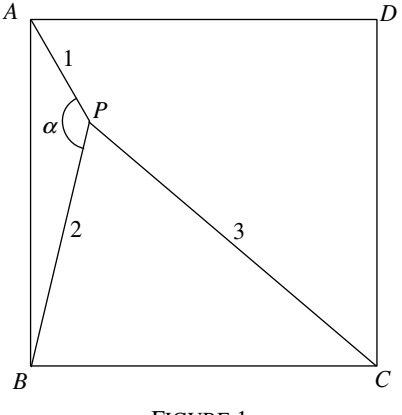


FIGURE 1

The original problem asked 'What is angle α?' in Figure 1, in which *ABCD* is a square. The solution on the MEI website, [2], is pretty and purely geometric, but I dived in and did the general case in Figure 2, which is a great little exercise in school trigonometry.

https://doi.org/10.1017/mag.2016.73 Published online by Cambridge University Press

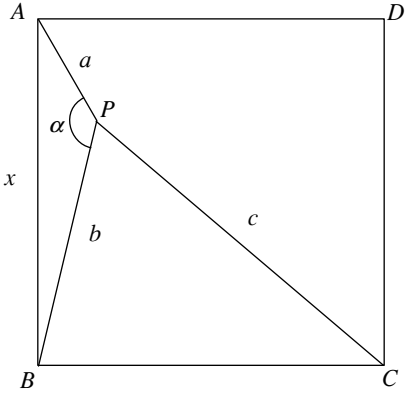


FIGURE 2

If we let the side of the square be x, the cosine rule in triangle APB tells us that

$$x^2 = a^2 + b^2 - 2ab\cos\alpha. \tag{1}$$

The sine rule in the same triangle yields $\sin\angle ABP = \dfrac{a}{x}\sin\alpha$, and this is equal to $\cos\angle PBC$, so by the cosine rule in triangle PBC

$$c^2 = b^2 + x^2 - 2bx\left(\frac{a}{x}\right)\sin\alpha. \tag{2}$$

Adding (1) and (2) yields $c^2 = a^2 + 2b^2 - 2ab(\sin\alpha + \cos\alpha)$, or

$$\sin\alpha + \cos\alpha = \frac{a^2 + 2b^2 - c^2}{2ab}.$$

In harmonic form, this becomes

$$\sin(\alpha + 45°) = \frac{a^2 + 2b^2 - c^2}{2\sqrt{2}ab}. \tag{3}$$

So we have a fairly simple equation for α. As Adam McBride said of a completely different surprise in his 2015 conference talk, you have no right to expect that. What you expect is two gruesomely non-linear simultaneous equations for x and α, offering no hope of an analytic solution. But we have got lucky, so let us go exploring. If P is inside the square, $45° < \alpha < 180°$, so $90° < \alpha + 45° < 225°$, and that means that for any triple (a, b, c), there is at most one α satisfying (3).

In the original problem, $a = 1$, $b = 2$ and $c = 3$, so $\sin(\alpha + 45°) = 0$, $\alpha + 45° = 180°$, and therefore $\alpha = 135°$. What other interesting cases are there? I guess that depends on what you find interesting, but I am going for a, b and c being integers, and α being rational (i.e. a rational number of degrees, or a rational multiple of π radians if you prefer), and we can restrict ourselves to primitive solutions, i.e. those where a, b and c have no common factor. We can start by looking for other cases where $\alpha = 135°$, which is to

https://doi.org/10.1017/mag.2016.73 Published online by Cambridge University Press

say other integral solutions to

$$a^2 + 2b^2 = c^2. \tag{4}$$

I can find nothing on the internet about this equation, but it is very like Pythagoras's equation, $p^2 + q^2 = r^2$, and you may know that you can generate all the primitive solutions to that by running through coprime m and n not both odd, and setting

$$(p, q, r) = (m^2 - n^2,\ 2mn,\ m^2 + n^2).$$

A similar approach works for (4). Start by rearranging it as $(c - a)(c + a) = 2b^2$. Suppose we have a primitive solution. Set $\dfrac{c + a}{b} = \dfrac{m}{n}$, where m and n are coprime, so that $\dfrac{c - a}{b} = \dfrac{2n}{m}$. Adding and subtracting these two equations gives

$$\frac{c}{b} = \frac{m^2 + 2n^2}{2mn} \quad \text{and} \quad \frac{a}{b} = \frac{m^2 - 2n^2}{2mn}.$$

No odd prime can divide both the numerator and denominator of either of these, but one factor of 2 can, and it does so precisely when m is even (so that n is odd). Therefore

$$(a, b, c) = \begin{cases} m^2 - 2n^2,\ 2mn,\ m^2 + 2n^2), & \text{if } m \text{ is odd,} \\ \frac{1}{2}m^2 - n^2,\ mn,\ \frac{1}{2}m^2 + n^2, & \text{if } m \text{ is even.} \end{cases} \tag{5}$$

We can now generate all the primitive solutions to (4). Table 1 gives the first few, generated from coprime m and n with $m^2 > 2n^2$:

m	n	a	b	c
2	1	1	2	3
3	1	7	6	11
3	2	1	12	17
4	1	7	4	9
5	1	23	10	27
5	2	17	20	33
5	3	7	30	43

TABLE 1

Notice that there are two 1s and three 7s in the a column in this short list. That observation opens an interesting door. The 1s are solutions to $m^2 - 2n^2 = 1$ or $m^2 - 2n^2 = 2$. The first of these is a particular case of the famous Pell's equation, $m^2 - kn^2 = 1$, which has been studied since antiquity, because solutions provide good rational approximations to $\sqrt{k}$. (See, for example, [3].) Lagrange showed that so long as k is not a perfect square, Pell's equation has infinitely many solutions, and from that you can

https://doi.org/10.1017/mag.2016.73 Published online by Cambridge University Press

show that if the more general equation $m^2 - kn^2 = t$ has any solutions at all, then it has infinitely many. So, returning to our geometric problem, any a that occurs at all occurs infinitely often, but it is clear from (5) that any b or c that occurs only occurs finitely often. You would never have guessed that from the geometry; it only emerges from the number theory.

Digressing slightly, there is another way to view (4). In the theory of quadratic fields (see, for example, [4, chapters XIV and XV]), $a^2 + 2b^2$ is the norm of the algebraic integer $a + b\sqrt{-2}$ in the field $\mathbb{Q}(\sqrt{-2})$. So, because all the integers of this field are of this form, what we have done is to find a way to generate every integer of this field whose norm is the square of a rational integer or, equivalently whose modulus is a rational integer. For what it is worth, we can rephrase (4) as $a^2 = c^2 - 2b^2$, and so, since $c^2 - 2b^2$ is the norm of the algebraic integer $c + b\sqrt{2}$ in the field $\mathbb{Q}(\sqrt{2})$, we can also generate every integer of this field whose norm is the square of a rational integer.

Returning to the geometric problem, the solutions exemplified in Table 1 represent points on a quarter circular arc AB inside the square, but the squares are of different sizes. Could there be two different solutions (a, b, c) and (a', b', c') in the same-sized square? By (1) that would mean

$$a^2 + b^2 + ab\sqrt{2} = a'^2 + b'^2 + a'b'\sqrt{2}$$

but since $\sqrt{2}$ is irrational, that means

$$a^2 + b^2 = a'^2 + b'^2 \qquad \text{and} \qquad ab = a'b'.$$

So in $\mathbb{R}^2$, the points (a, b) and (a', b') are in the positive quadrant and on the same circle and the same rectangular hyperbola, both centred at the origin, so the only possibility is $a = b'$ and $b = a'$. But this cannot happen as b and b' are even, whilst a and a' are odd. Therefore, all the squares are of different sizes.

So much for $\alpha = 135°$. What other angles are possible? From (3), we need a rational angle whose sine is a rational multiple of $\sqrt{2}$. There is a pretty theorem in [4] to the effect that if a rational angle has a sine which is a rational multiple of $\sqrt{2}$, then that sine must be 0 or $\pm 1/\sqrt{2}$, so the only other case with P inside the square has $\sin(\alpha + 45°) = 1/\sqrt{2}$, and so $\alpha = 90°$. These points P all lie on a semicircle AB inside the square of course, and from (3) we have

$$c^2 = b^2 + (b - a)^2. \tag{6}$$

So apart from the special case $a = b = c = 1$, these are all derived from Pythagorean triples. If fact from $p < q < r$, with $p^2 + q^2 = r^2$, we can derive three solutions

$$(a, b, c) = (q - p, q, r), (p + q, p, r) \text{ or } (p + q, q, r).$$

I think there is less of interest going on in this case, but there is a theorem that there are infinitely many Pythagorean triples in which the two legs differ by 1. (Exercise: prove this using Pell's equation.) Therefore it

https://doi.org/10.1017/mag.2016.73 Published online by Cambridge University Press

follows that in this case too, there are infinitely many solutions with $a = 1$, but values for b and c occur only finitely often. Again we could ask whether any two solutions have the same-sized square, but I suspect this is a hard question, although certainly there are no very small solutions. The values of a, b and c are generated from the Pythagorean triple (p, q, r), and they in turn are generated from coprime m and n not both odd. So $a^2 + b^2$, which is the square of the length of the side of the square, is equal to one of three quartic expressions in m and n. That is hard to analyse.

Thinking outside the box, we can allow P outside the square. Before we step outside the box though, we can pause briefly on the threshold, and consider P on the line segment AB, where $\alpha = 180°$, so $\sin(\alpha + 45°) = -1/\sqrt{2}$, and (3) becomes

$$c^2 = b^2 + (b + a)^2. \tag{7}$$

Again solutions are derived from Pythagorean triples, with $(a, b, c) = (q - p, p, r)$, but this time the Pythagorean triangle is on the diagram: it is triangle PBC (see Figure 3).

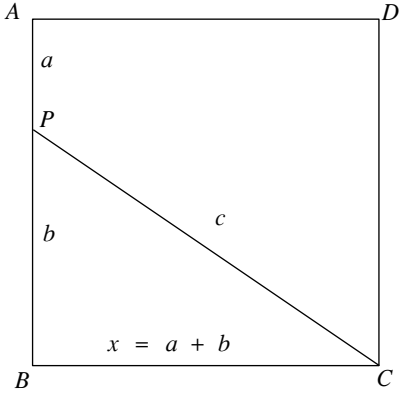


FIGURE 3

Remarkably, in this case too, there are infinitely many solutions with $a = 1$ (why?), and again values for b and c occur only finitely often. Here we certainly can have two different solutions in the same square: we just need to find two Pythagorean triangles with the same longer leg, and the first example of that is (29, 420, 421) and (341, 420, 541) which gives $(a, b, c) = (391, 29, 421)$ and (79, 341, 541) in a square of side 420.

Now we can step outside the box properly. Equation (3) still holds, but we have to be clear what we mean by the angle α. A good way to think about it is that it is the angle through which you have to rotate the vector $\overrightarrow{AP}$ anticlockwise to arrive at a vector parallel to $\overrightarrow{BP}$. We know $\sin(\alpha + 45°)$ takes one of the three values 0 or $\pm 1/\sqrt{2}$, and for each of these values there are two possible values of α. We have already met one of each, and the others all put P outside the box. Once we know $\sin(\alpha + 45°)$, equation (3)

https://doi.org/10.1017/mag.2016.73 Published online by Cambridge University Press

tells us which (a, b, c) triples are possible, so these new values of α do not introduce any new values of (a, b, c).

The case $\alpha = 0°$ puts P on AB extended, either above A or below B. Then $\sin(\alpha + 45°) = 1/\sqrt{2}$, just as when $\alpha = 90°$, so the same (a, b, c) triples appear as solutions: those that satisfy (6). However, we have a different-sized square: $x = |a - b|$ rather than $\sqrt{a^2 + b^2}$. Here we have infinitely many cases of two solutions with the same-sized square: from a Pythagorean triple (p, q, r) the solutions $(q - p, q, r)$ and $(p + q, q, r)$ both have squares of side p.

The others are similar. The case $\alpha = 270°$ puts P on a semicircle outside the square and $\sin(\alpha + 45°) = -1/\sqrt{2}$, just as when $\alpha = 180°$, so the same (a, b, c) triples appear as solutions: those that satisfy (7). Again we have a different-sized square: $x = \sqrt{a^2 + b^2}$ rather than $a + b$. As with $\alpha = 90°$, I suspect it will be a hard question whether or not there are two solutions with the same-sized square.

Finally, $\alpha = 315°$ puts P on a three-quarter circular arc outside the square and $\sin(\alpha + 45°) = 0$, just as when $\alpha = 135°$, so the same (a, b, c) triples appear as solutions: those that satisfy (4). Again we have a different-sized square: $x = \sqrt{a^2 + b^2 - \sqrt{2}ab}$ rather than $\sqrt{a^2 + b^2 + \sqrt{2}ab}$. As with $\alpha = 135°$, there definitely cannot be two solutions with the same-sized square.

So here we are, back at the start, with $(a, b, c) = (1, 2, 3)$. If you do not insist that P is inside the square (and the original problem did not, except pictorially), then there is another answer, $\alpha = 315°$, illustrated in Figure 4.

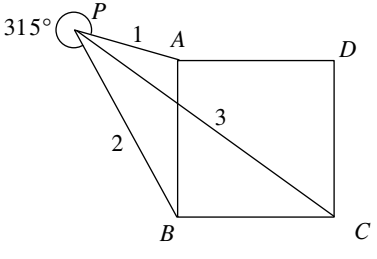


FIGURE 4

There is a lot more for the interested reader to explore. There are examples where the same-sized square has solutions with different values of α. The smallest size of square that allows 4 different placements of P is 12, disappointingly none of them strictly inside the square. What is special about 12 that allows this, and are there more numbers like that?

Alternatively, one can leave number theory aside, forget about a, b and c being integral and α being rational, and explore (3). For example, given a, b and c, in general there are two values of α, in two different-sized squares. As a final exercise, show that if x and x' are the sides of these squares, then $x^2 + x'^2 = a^2 + c^2$. There is no b in that formula! So says the algebra. What does that mean geometrically?

https://doi.org/10.1017/mag.2016.73 Published online by Cambridge University Press

Acknowledgement

I thank the referee for helpful comments that improved the article, any remaining faults being all my own work.

References

1. S. Lee and C. Stripp, MEI Insights I: About MEI, *Mathematics in School*, **44,** 3 (May 2015) pp. 17-19.
2. MEI, *Maths Item of the Month Archive* 2013, available at http://www.mei.org.uk/month_item_13
3. J. J. O'Connor and E. F. Robertson, *Pell's equation*, available at http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Pell.html
4. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (4th edn.), Oxford University Press (1960).
5. J. Jahnel. *When is the (co)sine of a rational angle equal to a rational number*?, available at http://www.uni-math.gwdg.de/jahnel/Preprints/cos.pdf

10.1017/mag.2016.73

JIM SIMONS
Foxcote, Noverton Lane, Prestbury, Cheltenham GL52 5BB
e-mail: *jimsimonsfoxcote@googlemail.com*

100.21 Generalising the arbelos

The shape shown in Figure 1 is known as the *arbelos*, the name given by Archimedes. The arbelos is bounded by three semicircles; the diameter of the larger semicircle is the sum of the diameters of the two smaller ones. As a result, the semicircles are mutually tangent.

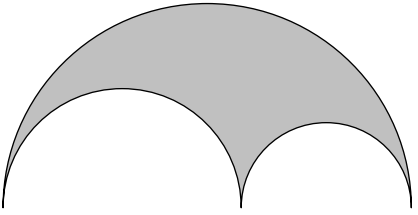


FIGURE 1

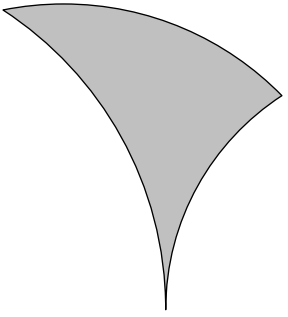


FIGURE 2

The arbelos has been extensively studied, originally by Archimedes (in his *Book of Lemmas*, see [1]), later by Pappus, and in a flurry of recent activity (see, for example, [2, 3]).

https://doi.org/10.1017/mag.2016.73 Published online by Cambridge University Press