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RESEARCH ARTICLE

Pathological MMP singularities as α_p-quotients

Quentin Posva

Institut de Mathématiques, Université de Neuchâtel, Rue Emile-Argand 11, 2000 Neuchâtel, Switzerland;
E-mail: quentin.posva@hhu.de, quentin.posva@unine.ch

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Abstract

We construct pathological examples of MMP singularities in every positive characteristic using quotients by α_p-actions. In particular, we obtain non-S_3 terminal singularities, as well as locally stable (respectively stable) families whose general fibers are smooth (respectively klt, Cohen–Macaulay, and F-injective) and whose special fibers are non-S_2. The dimensions of these examples are bounded below by a linear function of the characteristic.

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1. Introduction

Singularities of the Minimal Model Program (MMP) have poor cohomological properties in positive characteristic, unlike in characteristic 0. Striking examples are canonical and terminal singularities that are not Cohen–Macaulay [22, 7, 34, 32, 33]. In this article, we construct several new examples of non-S_3 singularities of the MMP in every positive characteristic $p > 0$.

The examples constructed in the aforementioned works are of two types: they stem either from a cone construction on a variety failing Kodaira vanishing or from a wild $\mathbb{Z}/p\mathbb{Z}$-quotient. Our examples are α_p-quotients: to understand their singularities we use the perspective of 1-foliations and the tools developed in [27].

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Our first series of examples gives the following:

Theorem 1.1 (Theorem 3.2). *Let k be an algebraically closed field of characteristic $p > 0$. Then there exists an isolated $\mathbb{Q}$-factorial singularity $(\mathbf{0} \in Y)$ over k that is canonical (resp. terminal) of dimension* $\max\{p, 3\}$ *(resp. $p + 1$) and non-S_3.*

In particular, we obtain a three-dimensional terminal singularity in characteristic $p = 2$ that is not S_3: the other known examples of this dimension were all constructed by Totaro in positive and mixed characteristics 2, 3, and 5 [32, 33]. For an arbitrary characteristic $p > 2$, Totaro also constructed terminal non-S_3 cone singularities of dimension $2p + 2$ [32]. In this regard, we achieve an improvement on the asymptotic dimensional lower bound.

Note that Yasuda exhibits in [34] a series of terminal and canonical singularities, starting in dimension 4, which have depth ≥ 3 but are not Cohen–Macaulay: their optimal asymptotic dimension, as a function of p, is roughly given by $\sqrt{2p}$.

We establish several other properties of the singularities $(\mathbf{0} \in Y)$ in Proposition 3.4, Proposition 3.5, Corollary 3.6, and Proposition 3.7. Notably, they are not F-injective, and they are rational up to Frobenius nilpotence.

We use the same method to produce another series of pathological examples. In characteristic 0, fibers of locally stable one-parameter families (in the sense of Definition 2.1) are S_2 [16, §7.3]. This is true for one-parameter families of surfaces with residue characteristics different from 2, 3, and 5 [3], but recent examples of Kollár [17] show that this property can fail dramatically for families of threefolds in every positive characteristic. Using α_p-quotients, we construct new examples of this kind. First we obtain:

Theorem 1.2 (Theorem 4.3). *Let k be an algebraically closed field of characteristic $p > 0$. Then there exists a locally stable family $Y \to \mathbb{A}^1$ over k of relative dimension* $\max\{p, 3\}$ *such that Y_t is smooth for $t \neq 0$ and Y_0 is reduced but non-S_2.*

Then we compactify the families of Theorem 1.2 to obtain pathological stable families (in the sense of Definition 2.1):

Theorem 1.3 (Theorem 4.7). *Let k be an algebraically closed field of characteristic $p > 0$. Then there exists a projective flat family of pairs $(\mathcal{Y}, \mathcal{B}) \to \mathbb{A}^1$ over k of relative dimension* $\max\{p, 3\}$ *such that:*

1. *$\mathcal{Y}$ is normal and $(\mathcal{Y}, \mathcal{B}) \to \mathbb{A}^1$ is stable,*
2. *$\mathcal{Y}_t$ has only μ_p-quotient singularities (or is regular if $p = 2$) for $t \neq 0$,*
3. *$\mathcal{Y}_0$ is reduced but non-S_2.*

Moreover, if C is a normal curve and if $C \to \mathbb{A}^1$ is a finite flat morphism, then $\mathcal{Y}_C$ is normal and $(\mathcal{Y}_C, \mathcal{B}_C) \to C$ is stable.

Our method produces boundary divisors $\mathcal{B} \neq 0$ whose coefficients are quite small, but the number of irreducible components of $\mathcal{B}$ and the volume of $K_{\mathcal{Y}_t} + \mathcal{B}_t$ are up to our liking: see Remark 4.10 and the proof of Theorem A.12.

The examples given by Theorem 1.3 differ from Kollár's ones [17] in two important ways: their dimension is constrained by the characteristic (while Kollár constructs pathological families of threefold pairs in every characteristic, and indicates how to generalize the construction to any higher dimension), but their general fibers are klt, Cohen–Macaulay, and F-injective by [27, Theorems 1 and 2] (while the general fibers in Kollár's examples are not klt nor Cohen–Macaulay: see Remark A.11). Boundary divisors appear in both situations. We discuss the implications of these examples for moduli theory in Section 1.1 below.

Let me sketch the construction of these examples. We start with a 1-foliation $\mathcal{F}$ of rank 1 on an affine space $\mathbb{A}^n$, and we wish to understand the singularities of $\mathbb{A}^n/\mathcal{F}$. There is a simple criterion guaranteeing that $\mathbb{A}^n/\mathcal{F}$ is only S_2 (see [27, Lemma 2.5.4]), so we only need to study its singularities in the sense of the MMP. If F is a divisor over $\mathbb{A}^n/\mathcal{F}$, then by [27, §4.2] we can compute the discrepancy $a(F; \mathbb{A}^n/\mathcal{F})$ in terms of $a(E; \mathbb{A}^n)$ and $a(E; \mathcal{F})$ where E is the closure of F in the function field of $\mathbb{A}^n$ (this is recalled

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in Theorem 2.10). The problem is that we cannot estimate $a(E;\mathcal{F})$ easily. So we blow-up $\mathbb{A}^n$ until the singularities of $\mathcal{F}$ become log canonical. This leads to a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Z = X/b^*\mathcal{F} \\ \downarrow b & & \downarrow \mathfrak{b} \\ \mathbb{A}^n & \longrightarrow & \mathbb{A}^n/\mathcal{F} \end{array}$$

where the horizontal arrows are the quotient morphisms, and both morphisms b and $\mathfrak{b}$ are birational. Writing

$$K_Z - \sum_i a_i F_i = \mathfrak{b}^* K_{\mathbb{A}^n/\mathcal{F}}$$

where the F_i's are the $\mathfrak{b}$-exceptional divisors, we reduce to computing the discrepancies over the sub-pair $(Z, -\sum_i a_i F_i)$. The advantage is now that $b^*\mathcal{F}$ is log canonical, so the tools given by Theorem 2.10 are much more efficient. The appearance of the characteristic in the computations is due to the adjunction formula of Proposition 2.9.

For Theorem 1.2 and Theorem 1.3 we use a relative version of this construction. Proving that the central fiber of the quotient is not S_2 is more involved (as the criterion used before does not apply): we use the tools developed in [27, §5] (see Theorem 2.12). To ensure that the fibers of the quotient family are regular in codimension one, we need to work in relative dimension d at least 3. When $d = 2 = p$ the output (Example 4.6) is quite different: we obtain a locally stable family of surfaces $\mathcal{S} \to \mathbb{A}^1$ where $\mathcal{S}$ is Cohen–Macaulay and $\mathcal{S}_0$ has an inseparable node in the sense of [28].

In our examples, the 1-foliation $\mathcal{F}$ is generated by (some variation of) the quadratic derivation $\partial = \sum_{i=1}^n x_i^2 \partial_{x_i}$. The latter satisfies $\partial^{[p]} = 0$ in every characteristic $p > 0$, and thus defines an α_p-action on $\mathbb{A}^n$ such that $\mathbb{A}^n/\alpha_p = \mathbb{A}^n/\partial$ [27, Proposition 2.3.7]. If $n = 2 = p$ then $\mathbb{A}^2/\partial$ is the RDP D_4^0. The quadratic derivation has the nice property that a single blow-up simplifies its singularities into lc ones, making the necessary computations straightforward.

In principle, other choices of 1-foliations $\mathcal{F}$ can lead to similar examples, possibly with better dimensional bounds. The difficult part is the computation of the blow-up sequence $b\colon X \to \mathbb{A}^n$ needed to simplify the singularities of $\mathcal{F}$. The results of [29] suggest that such a sequence usually exists, but the number of necessary blow-ups might be quite large. To illustrate this, in Section 4.2 we consider the derivation $y^3\partial_x + x\partial_y + t\partial_z$ on $\mathbb{A}^3_{x,y,z} \times \mathbb{A}^1_t$: this choice was motivated by the fact that the quotient of $\mathbb{A}^2$ by $y^3\partial_x + x\partial_y$ gives the RDP E_6^0 in characteristic 3. The quotient gives a pathological family of affine threefolds in characteristic 3 (Theorem 4.14). This family is more difficult to compactify, as explained in Remark 4.15: we shall not pursue this task here.

Remark 1.4. The underlying families $Y \to \mathbb{A}^1$ (or $\mathcal{Y} \to \mathbb{A}^1$) appearing in Theorem 1.2, Theorem 1.3, and Theorem 4.14 share the following properties: the central fibers Y_0 do not have liftable cohomology (in the sense of [20]), the pair (Y, Y_0) is not F-pure along Y_0, and Y is not F-injective at the non-S_2-points of Y_0. All the other fibers $Y_t (t \neq 0)$ have liftable cohomology, and $(Y \setminus Y_0, Y_t)$ is F-pure ([1]).

1.1. Consequences for KSBA moduli theory of stable pairs

The existence of the pathological stable families of Theorem 1.3 has the following consequence, which extends the main result of [17]:

[1]Here is a sketch of proof. The fiber Y_0 does not have liftable cohomology, as otherwise it would be S_2 (even CM) by [20, Theorem 8.5]. If (Y, Y_0) was F-pure, then $\operatorname{depth} \mathcal{O}_{Y,y} \geq \min\{3, \dim \mathcal{O}_{Y,y}\}$ for every $y \in Y_0$ by [26, Lemma 3.4], and so Y_0 would be S_2. The fact that Y is not F-injective is proved exactly as in Corollary 3.6. On the other hand, if $t \neq 0$ then Y_t has only μ_p-quotient singularities and thus it is F-pure [27, Theorem 1]; this implies that Y_t has liftable cohomology [20, Corollary 7.3] and that (Y, Y_t) is F-pure in a neighborhood of Y_t [26, Theorem A].

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Theorem 1.5 (Theorem A.12). *Fix an integer $n \geq 3$. Then in characteristics $p \leq n$ the stable families of n-dimensional Cohen–Macaulay pairs (resp. of pairs with F-injective underlying varieties) usually do not compactify to a proper KSBA moduli stack.*

The formulation of that theorem requires an explanation. In positive characteristic, no precise definition of the KSBA moduli stack ([2]) of stable pairs has been proposed so far in the literature: defining families of pairs over general bases is the delicate part. It is easier to define boundary-free families: to wit, the moduli stacks of stable surfaces are defined in [25, §1.3] and studied in [25, 3, 28]. That definition can be extended to any dimension d, but then only few properties of the resulting stacks are known: for example, when $d \geq 3$ we do not know if they are algebraic. In any case, the boundary-free case is not our setting.

There is however no ambiguity about what the values of the KSBA moduli stacks should be over regular curves: stable families of pairs with demi-normal underlying fibers (see Definition A.2). If the moduli stacks would satisfy the valuative criterion of properness, then a stable family with demi-normal fibers over a punctured curve could be extended with a demi-normal fiber, possibly after replacing the base with a finite base-change. The examples of Theorem 1.3 show that this is usually not possible when the characteristic is small compared to the dimension, even if the generic fiber has only klt, F-injective and Cohen–Macaulay singularities.

This discussion and the content of Theorem 1.5 are made precise in Appendix A.

Let us close this introduction with a few open questions related to KSBA moduli theory in positive characteristic:

Question 1.6. Let k be an algebraically closed field of characteristic $p > 0$.

1. *Are the poor properties of stable families of Cohen–Macaulay/F-injective pairs related to the ratio between the dimension and the characteristic?* More precisely, are there families of n-dimensional pairs such as those of Theorem 1.3 if $p \gg n$?
2. *Do such pathologies also appear for stable families of n-dimensional varieties?* More precisely, are there stable families of n-dimensional varieties (without boundaries) over regular curves with pathologies such as those of [17] and Theorem 1.3? What if we assume that $p \gg n$, or that the generic fiber is Cohen–Macaulay or F-injective?
3. *Are there pathological families of stable surface pairs in characteristics ≤ 5?*

2. Preliminaries

2.1. Conventions and notations

We work over an algebraically closed field k of positive characteristic $p > 0$.

1. A *variety* (over k) is an integral quasi-projective k-scheme of finite type. A *curve* (resp. *surface*, *threefold*) is k-variety of dimension one (resp. two, three).
2. The conditions S_i are the *Serre's conditions*, see [1, 0341].
3. Given a variety X, a divisor E over X is a prime divisor appearing on a (proper) birational model $\mu\colon Y \to X$. Its center $c_X(E) \subset X$ is the closure of $\mu(E)$.
4. Let us recall some standard MMP terminology from [21, 16]. Let X be a normal variety and Δ be a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. For a proper birational morphism $\mu\colon Y \to X$ with Y normal we write

$$K_Y = \mu^*(K_X + \Delta) + \sum_E a(E; X, \Delta) \cdot E.$$

[2]The distinction between moduli stacks and moduli functors is irrelevant for our discussion. While I use the language of stacks on a superficial level in this note, the reference for moduli theory of stable pairs in characteristic 0 [18] takes the viewpoint of moduli functors.

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We define

$$\operatorname{discrep}(X,\Delta) = \inf\{a(E;X,\Delta) \mid E \text{ exceptional over } X\}.$$

We say that (X,Δ) is *sub-lc* (resp. *sub-klt*) if for every μ and E we have $a(E;X,\Delta) \geq -1$ (resp. $a(E;X,\Delta) > -1$). If $\Delta \geq 0$ and (X,Δ) is sub-lc (resp. sub-klt), we say that (X,Δ) is *lc* (resp. *klt*). We say that (X,Δ) is *canonical* (resp. *terminal*) if Δ is effective, and for every μ and μ-exceptional E we have $a(E;X,\Delta) \geq 0$ (resp. $a(E;X,\Delta) > 0$). In case $(X, \operatorname{Supp}(\Delta))$ is log smooth, sufficient and necessary conditions on the coefficients of Δ for these conditions to hold are given in [16, Corollary 2.11]. In particular, a regular variety is terminal.

5. We will several times use tacitly the following fact [16, Lemma 2.5]: if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, if $\mu\colon Y \to X$ is birational with Y normal, and if

$$K_Y + \mu_*^{-1}\Delta + F = \mu^*(K_X + \Delta) \quad \text{where } F \text{ is } \mu\text{-exceptional},$$

then $a(E;X;\Delta) = a(E;Y,\mu_*^{-1}\Delta + F)$ for every divisor E over X.

We also use the following terminology (see Appendix A for more details):

Definition 2.1 [18, (2.3, 2.44)]. Let C be a normal k-variety of dimension ≤ 1, let $f\colon (X,\Delta) \to C$ be a family of pairs [18, Definition 2.2]–in particular, the fibers X_c are geometrically reduced–, and assume that X is normal. Then we say that f is *locally stable* if $(X, \Delta + X_\alpha)$ is lc for every $\alpha \in C(k)$. We say that f is *stable* if it is proper, locally stable and $K_X + \Delta$ is f-ample.

We record the following well-known result.

Lemma 2.2. *Let X be a regular k-variety and G be a divisor over X with nonempty center $\mathfrak{c} = c_X(G) \subset X$. Then $a(G;X) \geq \operatorname{codim}_X(\mathfrak{c}) - 1$.*

Proof. By a result of Zariski [21, Lemma 2.45], there is a finite sequence of birational proper morphisms

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} X_0 = X$$

such that each f_i is the blow-up of X_{i-1} along the center $\mathfrak{c}_{i-1}$ of G on X_{i-1}, and G appears on X_n as an f_n-exceptional divisor dominating $\mathfrak{c}_{n-1}$. To prove the statement we may localize each X_i in a neighbourhood of the generic point of $\mathfrak{c}_i$, and so we can assume that each $\mathfrak{c}_i$ is a regular subvariety and that each X_i is regular. Let $E_i \subset X_i$ be the unique f_i-exceptional divisor: since $E_i = f_i^{-1}(\mathfrak{c}_{i-1})$ it holds that $\mathfrak{c}_i \subseteq E_i$ for each i, with equality if and only if $i = n$. By abuse of notation, the strict transform of E_i on each $X_{i+1}, \dots, X_n$ will also be denoted by E_i.

Let $f\colon X_n \to X$ be the composition of all blow-ups. We write

$$f^*K_X = K_{X_n} - \sum_{i=1}^{n} a_i E_i.$$

We claim that $a_{i+1} > a_i$ for all i: since $a_1 = \operatorname{codim}(\mathfrak{c}) - 1$ and $E_n = G$, the result will follow. These inequalities are easily proved by induction: we have

$$f_i^*\left(K_{X_{i-1}} - \sum_{j=1}^{i-1} a_j E_j\right) = K_{X_i} - \sum_{j=1}^{i-1} a_j E_j - \left(\operatorname{codim}_{X_{i-1}}(\mathfrak{c}_{i-1}) - 1 + \operatorname{mult}_{\mathfrak{c}_{i-1}} \sum_j a_j E_j\right) \cdot E_i.$$

Since $\mathfrak{c}_{i-1} \subsetneq E_{i-1}$ we have

$$a_i = \operatorname{codim}_{X_{i-1}}(\mathfrak{c}_{i-1}) - 1 + \operatorname{mult}_{\mathfrak{c}_{i-1}} \sum_j a_j E_j \geq 1 + a_{i-1}.$$

This proves the lemma. □

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2.2. 1-Foliations

For basics on derivations and 1-foliations, we refer to [27]. We recall below the most important facts for us. For derivations we need:

Lemma 2.3. *Let ∂ be a p-closed derivation on an integral scheme* $\mathrm{Spec}(A)$. *If there exists $a \in A$ such that $\partial(a) \neq 0$ and $\partial^{[p]}(a) = h\partial(a)$ for some $h \in A$, then $\partial^{[p]} = h\partial$.*

For 1-foliations, let us recall:

Definition 2.4. Let $\mathcal{F}$ be a 1-foliation on a normal variety X. Assume that the divisorial sheaf $\mathcal{O}_X(K_{\mathcal{F}}) = (\bigwedge^{\mathrm{rk}\,\mathcal{F}} \mathcal{F})^{[-1]}$ is $\mathbb{Q}$-Cartier. Then $\mathcal{F}$ is *log canonical (lc)* if for every proper birational morphism $\mu\colon Y \to X$ we can write

$$K_{\mu^*\mathcal{F}} = \mu^* K_{\mathcal{F}} + \sum_{E\colon \mu\text{-exc.}} a(E;\mathcal{F}) \cdot E$$

with

$$a(E;\mathcal{F}) \geq \begin{cases} 0 & \text{if } E \text{ is } \mu^*\mathcal{F}\text{-invariant,} \\ -1 & \text{otherwise.} \end{cases}$$

Definition 2.5. Let $\mathcal{F}$ be a 1-foliation of rank 1 on a normal variety X. Then $\mathcal{F}$ has *only multiplicative singularities* ([3]) if for every $x \in \mathrm{Sing}(\mathcal{F})$, the restriction $\mathcal{F} \otimes \widehat{\mathcal{O}}_{X,x}$ is generated up-to-saturation by a continuous derivation $\partial \in \mathrm{Der}_k^{\mathrm{cont}}\left(\widehat{\mathcal{O}}_{X,x}\right)$ satisfying $\partial^{[p]} = u\partial$ where $u \in \widehat{\mathcal{O}}_{X,x}^{\times}$.

Proposition 2.6 [27, Corollary 1]. *Let X be a regular variety. If $\mathcal{F}$ is a 1-foliation of rank 1, then $\mathcal{F}$ has only multiplicative singularities if and only if $\mathcal{F}$ is lc.*

Pullbacks of 1-foliations along smooth blow-ups can be computed as follows:

Example 2.7. Let $\pi\colon X \to \mathbb{A}^n_{x_1,\dots,x_n}$ be the blow-up of the ideal $(x_1,\dots,x_r)$ for $r \leq n$. The x_1-chart U_1 of the blow-up $\pi^*\colon k[x_1,\dots,x_n] \to k[y_1,\dots,y_n]$ is given by

$$x_1 \mapsto y_1, \quad x_i \mapsto y_1 y_i \ (1 < i \leq r), \quad x_j \mapsto y_j \ (j > r).$$

The pullbacks of derivations on $\mathbb{A}^n$ by $\pi|_{U_1}$ can be computed using the following equalities:

$$\pi^*\partial_{y_1}|_{U_1} = \partial_{y_1} - \sum_{i=2}^{r} \frac{y_i}{y_1}\partial_{y_i}, \quad \pi^*\partial_{x_i}|_{U_1} = \frac{1}{y_1}\partial_{y_i} \ (1 < i \leq r), \quad \pi^*\partial_{x_j}|_{U_1} = \partial_{y_j} \ (j > r).$$

Let us recall some properties of quotients by 1-foliations.

Definition 2.8 [27, Definition 4.2.1]. Let X be a normal variety, $\mathcal{F}$ a 1-foliation on X and $q\colon X \to X/\mathcal{F}$ the quotient morphism. If $\Delta = \sum_i a_i\Delta_i$ is a $\mathbb{Q}$-divisor on X, where Δ_i are prime divisors, then we let

$$\Delta_{X/\mathcal{F}} = \sum_i a_i \varepsilon_i q(\Delta_i)$$

where

$$\varepsilon_i = \begin{cases} 1 & \text{if } \Delta_i \text{ is } \mathcal{F}\text{-invariant,} \\ \frac{1}{p} & \text{otherwise.} \end{cases}$$

[3] The definition [27, Definition 2.4.7] is more general, but for 1-foliations of rank 1 the two definitions agree.

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Proposition 2.9 [27, Proposition 4.2.3]. *Let X be a normal variety, Δ a $\mathbb{Q}$-divisor on X, $\mathcal{F}$ a 1-foliation on X and $q\colon X \to X/\mathcal{F}$ the quotient morphism. Then we have an equality of $\mathbb{Q}$-divisors*

$$q^*(K_{X/\mathcal{F}} + \Delta_{X/\mathcal{F}}) = K_X + \Delta + (p-1)K_{\mathcal{F}}.$$

The following theorem is our main tool to compute the discrepancies over quotient singularities.

Theorem 2.10. *Let (X, Δ) be a normal sub-pair, $\mathcal{F}$ be a 1-foliation on X, and $q\colon (X, \Delta) \to (Y = X/\mathcal{F}, \Delta_Y)$ be the associated quotient morphism.*

1. *If $\mathcal{F}$ is lc and (X, Δ) is sub-lc (respectively sub-klt), then (Y, Δ_Y) is sub-lc (respectively sub-klt).*

Let $\mu\colon Y' \to Y$ be a birational proper morphism. Then:

2. *If X' is the normalization of Y' in $K(X)$, then we have a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{q'} & Y' \\ \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu} \\ X & \xrightarrow{q} & Y \end{array}$$

   where μ' is a proper birational morphism, and q' is the quotient by $(\mu')^*\mathcal{F}$.
3. *If $E \subset X'$ is a prime divisor with image the prime divisor $q'(E) = F \subset Y'$, then*

$$a(F; Y, \Delta_Y) = \begin{cases} a(E; X, \Delta) + (p-1)\cdot a(E; \mathcal{F}) & \text{if } E \text{ is } (\mu')^*\mathcal{F}\text{-invariant,} \\ \frac{1}{p}\left[a(E; X, \Delta) + (p-1)\cdot a(E; \mathcal{F})\right] & \text{otherwise.} \end{cases}$$

Proof. This is proved in [27, Theorem 4.2.5], under the additional assumption that Δ is effective. However, that assumption is not needed in the proof. □

Let us also recall the notion of family of 1-foliations.

Definition 2.11 [27, §5.1]. Let $X \to B$ be a flat morphism between normal varieties with geometrically normal fibers. A *relative 1-foliation* is a 1-foliation on X that is contained in the sub-module $T_{X/B}$ of $T_{X/k}$. If $X \to B$ is smooth, we say that a relative 1-foliation $\mathcal{F} \subset T_{X/B}$ is a *family of 1-foliations* if $T_{X/B}/\mathcal{F}$ is flat over B and for every $b \in B$ the sub-module $\mathcal{F} \otimes \mathcal{O}_{X_b} \hookrightarrow T_{X_b/k(b)}$ is a 1-foliation.

Theorem 2.12 [27, Proposition 5.2.4, Corollary 5.2.5]. *Let $X \to B$ be a smooth morphism over a normal curve B, and $\mathcal{F} \subset T_{X/B}$ be a family of 1-foliations. Then:*

1. *for every $b \in B$ there is a natural morphism $\varphi_b\colon X_b/\mathcal{F}|_{X_b} \to (X/\mathcal{F})_b$,*
2. *φ_b is an isomorphism if and only if $(X/\mathcal{F})_b$ is S_2, and*
3. *if $\mathcal{F}$ has only multiplicative singularities in a neighborhood of X_b then φ_b is an isomorphism.*

Finally, the following lemma will be useful for computations related to base-changes:

Lemma 2.13. *Let $f\colon X \to B$ be a smooth morphism, with B a normal variety. Let $g\colon B' \to B$ be a flat morphism from a normal variety, inducing the cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B. \end{array}$$

Let $\mathcal{F} \subset T_{X/B}$ be a relative 1-foliation, and $\mathcal{F}' = (g')^*\mathcal{F} \hookrightarrow (g')^*T_{X/B} \cong T_{X'/B'}$. Then $\mathcal{F}'$ is a relative 1-foliation, and the natural morphism $X' \to (X/\mathcal{F}) \times_B B'$ is isomorphic over B' to the quotient $X' \to X'/\mathcal{F}'$.

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Proof. This can be checked Zariski-locally on X, B, and B', so we may assume that $X = \operatorname{Spec}(R)$, $B = \operatorname{Spec}(A)$, $B' = \operatorname{Spec}(A')$ and that $\mathcal{F} = \tilde{F}$ is generated by $\partial_1, \dots, \partial_n \in \operatorname{Der}_A(R)$. Let $R' = R \otimes_A A'$. We have a natural map

$$\mathfrak{g}' \colon \operatorname{Der}_R(A) \to \operatorname{Der}_{A'}(R'), \quad \mathfrak{g}'(\psi) \colon r \otimes a' \mapsto \psi(r) \otimes a',$$

and the A'-sub-module F' of $\operatorname{Der}_{A'}(R')$ generated by $\mathfrak{g}'(F)$ sheafifies into $\mathcal{F}'$. We check below that F' is closed under Lie brackets and p-powers. As the $\partial_i' = \mathfrak{g}'(\partial_i)$ generate F' we see that

$$(R')^{F'} = \ker\big((\partial_1', \dots, \partial_n') \colon R' \to (R')^{\oplus n}\big).$$

Similarly we have

$$R^F = \ker\big((\partial_1, \dots, \partial_n) \colon R \to R^{\oplus n}\big).$$

Since A' is flat over A we therefore obtain that $R^F \otimes_A A' = (R')^{F'}$, and the lemma follows.

It remains to check that F' is closed under brackets and p-powers. Let $\xi, \psi \in \mathfrak{g}'(F)$ and $r, s \in R'$. As $\mathfrak{g}'$ commutes with Lie brackets we have $[r\xi, s\psi] = rs \cdot [\xi, \psi] \in F'$. By Hochschild's formula [27, (2.3.0.a)] we see that $(r\xi)^{[p]} \in F'$. Finally, a formula of Jacobson [13, p. 209] shows that $(r\xi + s\psi)^{[p]}$ and $(r\xi)^{[p]} + (s\psi)^{[p]}$ differ by an R'-linear combination of multifold Lie brackets of $r\xi$ and $s\psi$. Thus $(r\xi + s\psi)^{[p]} \in F'$, and we are done. □

3. Non-S_3 isolated MMP singularities

Consider the affine space $\mathbb{A}^n$ over the field k with coordinates $x_1, \dots, x_n$ where $n \geq 2$. We introduce the derivation

$$\partial = \sum_{i=1}^{n} x_i^2 \partial_{x_i} \quad \text{on } \mathbb{A}^n.$$

First we observe that:

Claim 3.1. $\partial^{[p]} = 0$.

Proof. Since the summands $x_i^2 \partial_{x_i}$ commute with each other, we have

$$\partial^{[p]} = \sum_{i=1}^{n} \left(x_i^2 \partial_{x_i}\right)^{[p]}.$$

Since ∂_{x_i} is p-closed, by Hochschild's formula [27, (2.3.0.a)] the scaling $x_i^2 \partial_{x_i}$ is also p-closed. It is easily seen that $(x_i^2 \partial_{x_i})^{[p]}(x_i) = 0$, and by Lemma 2.3 it follows that $(x_i^2 \partial_{x_i})^{[p]} = 0$. Therefore $\partial^{[p]} = 0$. □

In particular $\mathcal{F} = \mathcal{O}_{\mathbb{A}^n} \cdot \partial$ is a 1-foliation on $\mathbb{A}^n$.

Theorem 3.2. *With the notations as above, let $Y = \mathbb{A}^n/\mathcal{F}$ (with $n \geq 2$). Then:*

1. *Y is a normal $\mathbb{Q}$-factorial variety, with a unique singular closed point $\mathbf{0}$;*
2. *$\mathcal{O}_{Y,\mathbf{0}}$ is not S_3 if $n \geq 3$;*
3. *It holds that:*
   - *If $n \leq p - 2$, Y is not lc.*
   - *If $n \geq p - 1$, Y is lc;*
   - *If $n \geq p$, Y is canonical;*
   - *If $n \geq p + 1$, Y is terminal.*

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Proof. By construction Y is normal and $\mathbb{Q}$-factorial: in particular K_Y is $\mathbb{Q}$-Cartier. Since $\mathrm{Sing}(\mathcal{F})$ is the origin $0 \in \mathbb{A}^n$, we see that Y is regular away from the image $\mathbf{0}$ of the origin [27, Lemma 2.5.10]. As the ideal generated by the image of ∂ in the local ring $\mathcal{O}_{\mathbb{A}^n,0}$ is $\mathfrak{m}_{\mathbb{A}^n,0}$-primary, by Claim 3.1 and [27, Lemma 2.5.4] we obtain that $\mathcal{O}_{Y,\mathbf{0}} = \mathcal{O}^{\partial}_{\mathbb{A}^n,0}$ is not S_3.

It remains to analyze the singularity of $(\mathbf{0} \in Y)$ through the glasses of the MMP. To begin with, we blow-up the origin of $\mathbb{A}^n$ to simplify the singularities of $\mathcal{F}$:

$$b\colon X = \mathrm{Bl}_0 \mathbb{A}^n \longrightarrow \mathbb{A}^n.$$

Consider the x_1-patch U_1 of this blow-up:

$$b^*\colon k[x_1,\dots,x_n] \to k[u_1,\dots,u_n], \quad x_1 \mapsto u_1,\ x_i \mapsto u_1u_i\ (i>1).$$

Using the blow-up computations of Example 2.7, we see that on this chart we have

$$\begin{aligned} b^*\partial|_{U_1} &= u_1 \cdot \left(u_1\partial_{u_1} - \sum_{i=2}^{n} u_i\partial_{u_i}\right) + \sum_{i=2}^{n} u_1^2u_i^2 \cdot \frac{1}{u_1}\partial_{u_i} \\ &= u_1 \cdot \underbrace{\left[u_1\partial_{u_1} + \sum_{i=2}^{n}(-u_i+u_i^2)\partial_{u_i}\right]}_{\psi} \end{aligned}$$

Since $b^*\partial$ is p-closed, its scaling ψ is also p-closed. As $\psi(u_1) = u_1$ we obtain by Lemma 2.3 that $\psi^{[p]} = \psi$. Therefore by Proposition 2.6 we obtain that

$$\mathcal{O}_{U_1}\cdot\psi = (b^*\mathcal{F})|_{U_1} \quad \text{is an lc 1-foliation.}$$

The situation is similar on the other charts, and therefore we find that $b^*\mathcal{F}$ is an lc 1-foliation. As $\psi(u_1) \in (u_1)$ we also see that the exceptional divisor $E \subset X$ of b is $b^*\mathcal{F}$-invariant. Finally, as $b^*\partial|_{U_1} = u_1\cdot\psi$ we see that $a(E;\mathcal{F}) = -1$.

Now let $Z = X/b^*\mathcal{F}$ and consider the induced commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{q'} & Z \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \mathfrak{b}} \\ \mathbb{A}^n & \xrightarrow{q} & Y \end{array} \tag{3.1}$$

where the horizontal arrows are the quotient morphisms and where $\mathfrak{b}\colon Z \to Y$ is birational. Let $F \subset Z$ be the unique prime $\mathfrak{b}$-exceptional divisor. By Theorem 2.10 we have

$$a(F;Y) = a(E;\mathbb{A}^n) + (p-1)\cdot a(E;\mathcal{F}) = (n-1) + (p-1)(-1) = n-p.$$

If $n \le p-2$ then $a(F;Y) \le -2$ and Y is not lc. From now on assume that $n \ge p-1$. We have:

$$K_Z + (p-n)F = \mathfrak{b}^*K_Y.$$

This crepant relation shows that Y is lc if and only if $\mathrm{discrep}(Z,(p-n)F) \ge -1$. Since $b^*\mathcal{F}$ is lc, Theorem 2.10 shows that $\mathrm{discrep}(Z,(p-n)F) \ge -1$ as soon as $\mathrm{discrep}(X,(p-n)E) \ge -1$. As (X,E) is log smooth, this holds when $p-n \le 1$ by [16, Corollary 2.11]. This settles the lc case of the theorem.

It remains to study when Y is canonical (resp. terminal): for this we assume that $n \ge p$. Consider an exceptional divisor F' over Y, appearing on a birational model $Z' \to Y$. Replacing Z' by some blow-up, we may assume that $Z' \to Y$ factors through a birational morphism $\mathfrak{f}\colon Z' \to Z$. Let $f\colon X' \to X$ be

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the normalization of $\mathfrak{f}$ in $K(X)$, and $E' \subset X'$ be the prime divisor dominating $F' \subset Z'$. If $f(E') \not\subset E$ then $c_{F'}(Y) \neq \mathbf{0}$, and therefore $a(F';Y) \geq 1$ as Y is regular away from $\mathbf{0}$. Thus we may assume that $f(E') \subset E$. By Theorem 2.10 we have

$$\begin{aligned} a(F';Y) &= a(F';Z,(p-n)F) \\ &= \delta_{E'} \cdot [a(E';X,(p-n)E) + (p-1) \cdot a(E';b^*\mathcal{F})] \end{aligned}$$

where $\delta_{E'} \in \{1, p^{-1}\}$. We must find conditions that guarantee that $a(F';Y) \geq 0$ (resp. > 0), hence the factor $\delta_{E'}$ will not play any role. As (X,E) is log smooth and $p-n \leq 0$, we have $a(E';X,(p-n)E) \geq 1$. Since regular 1-foliations are canonical [27, Lemma 3.0.3], it follows that $a(F';Y) > 0$ unless the center of E' is contained in the singular locus of $b^*\mathcal{F}$: so from now on we assume that $c_X(E')$ is contained in $\mathrm{Sing}(b^*\mathcal{F})$. It is easily seen, using the computations above, that the singular locus of $b^*\mathcal{F}$ is a finite collection of isolated points. Therefore, by Lemma 2.2 we see that $a(E';X) \geq n-1$. Moreover, as $b^*\mathcal{F}$ is lc we have $a(E';b^*\mathcal{F}) \geq -1$. Putting everything together, we get

$$\begin{aligned} \frac{1}{\delta_{E'}} a(F';Y) &= a(E';X) + (n-p) \cdot \mathrm{mult}_{E'}(f^*E) + (p-1) \cdot a(E';b^*\mathcal{F}) \\ &\geq (n-1) + (n-p) + (p-1) \cdot (-1) \\ &= 2(n-p). \end{aligned}$$

So if $n \geq p$ (resp. if $n > p$) it holds that $a(F';Y) \geq 0$ (resp. $a(F';Y) > 0$). This completes the proof. □

Remarks 3.3.

1. The case $n = 3 \geq p$ gives a canonical non-Cohen–Macaulay isolated threefold singularity. It was shown in [2] that klt threefold singularities (with perfect residue fields) are Cohen–Macaulay in characteristic $p > 5$. This is optimal, as there also exist non-Cohen–Macaulay terminal threefold singularities in characteristic 5 [33].
2. In characteristic 0, strictly lc singularities may not be S_3, but this can only happen if they are log canonical centers [16, Corollary 7.21]. If $n = p-1$ in Theorem 3.2 then $\mathbf{0}$ is an lc center of Y, as the proof shows.

We end this section with some further remarks on the singularities ($\mathbf{0} \in Y$) constructed in Theorem 3.2. Unless we specify otherwise, there will be no constraint on the characteristic p nor on $n = \dim Y$.

Proposition 3.4. *If $p > 2$, then the singularity ($\mathbf{0} \in Y$) is never 1-Gorenstein (i.e., K_Y is never Cartier).*

This is false if $p = 2$: for example, in dimension two we obtain the RDP D_4^0, which is Gorenstein.

Proof. Let us use the notation of the proof of Theorem 3.2. We perform a weighted blow-up of the closed point defined by $(u_1, \dots, u_n)$ in $U_1 \subset X$ with weights $(1, p-1, \dots, p-1)$ (see [29, Example 2.4.3]). A (schematic) affine chart of the weighted blow-up is given by

$$k[u_1, \dots, u_n] \to k[v_1, \dots, v_n], \quad u_1 \mapsto v_1,\ u_i \mapsto v_1^{p-1} v_i\ (i \geq 2).$$

The (unique) exceptional divisor E' is given by $(v_1 = 0)$. We have

$$du_1 \wedge \cdots \wedge du_n = v_1^{(n-1)(p-1)} dv_1 \wedge \cdots \wedge dv_n$$

and thus $a(E';X) = (n-1)(p-1)$. Since

$$u_1 \partial_{u_1} = v_1 \partial_{v_1} - \sum_{i=2}^{n} (p-1) v_i \partial_{v_i}, \quad u_i \partial_{u_i} = v_i \partial_{v_i}\ (i \geq 2),$$

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we find

$$\psi = v_1 \cdot \left[\partial_{v_1} + \sum_{i=2}^{n} v_1^{p-2} v_i^2 \partial_{v_i} \right].$$

Therefore $a(E'; b^*\mathcal{F}) = -1$ and E' is *not* invariant with respect to the pullback of $b^*\mathcal{F}$. So if F' is the divisor over Y induced by E', we find:

$$\begin{aligned} a(F'; Y) &= a(F'; Z, (p-n)F) \\ &= \frac{1}{p} \cdot [a(E'; X, (p-n)E) + (p-1) \cdot a(E'; b^*\mathcal{F})] \\ &= \frac{1}{p} \cdot [(n-1)(p-1) - (p-n) + (p-1)(-1)] \\ &= n - 3 + \frac{2}{p}. \end{aligned}$$

Thus when $p > 2$ we see that $a(F'; Y) \notin \mathbb{Z}$. This shows that Y is not 1-Gorenstein. □

For the definition of $W\mathcal{O}$-rationality and Cohen–Macaulayness up to (Frobenius) nilpotence used in the following proposition, and the notions that appear in its proof, we refer to [6].

Proposition 3.5. *The singularities* $(\mathbf{0} \in Y)$ *are* $W\mathcal{O}$*-rational and Cohen–Macaulay up to nilpotence.*

Proof. The property of $W\mathcal{O}$-rationality descends along finite universal homeomorphisms by [9, Proposition 4.4.9]. It evidently holds for $\mathbb{A}^n$, and so it holds for Y.

Let us show that Y is CM up to nilpotence. By [6, Lemma 2.3] the natural morphism $\mathcal{O}_Y \to Rq_*\mathcal{O}_{\mathbb{A}^n}$ is an isomorphism in $D^b(\mathrm{Crys}_Y^F)$. Let $\omega_Y^\bullet$ be a dualizing complex on Y: by [5, Theorem 4.3.5], by applying $\mathrm{RHom}(-, \omega_Y^\bullet)$ to the previous isomorphism we obtain

$$\mathrm{RHom}(Rq_*\mathcal{O}_{\mathbb{A}^n}, \omega_Y^\bullet) \cong \mathrm{RHom}(\mathcal{O}_Y, \omega_Y^\bullet) \quad \text{in } D^b(\mathrm{Crys}_Y^C).$$

Now we have

$$\begin{aligned} \mathrm{RHom}(Rq_*\mathcal{O}_{\mathbb{A}^n}, \omega_Y^\bullet) &\cong Rq_* \,\mathrm{RHom}(\mathcal{O}_{\mathbb{A}^n}, \omega_{\mathbb{A}^n}) \\ &\cong Rq_*\omega_{\mathbb{A}^n} \\ &\cong q_*\omega_{\mathbb{A}^n} \end{aligned}$$

in $D^b(\mathrm{Crys}_Y^C)$, where the first isomorphism holds by [5, Corollary 5.1.7] and the third one holds as q is finite. This shows that $\mathrm{RHom}(\mathcal{O}_Y, \omega_Y^\bullet)$ is supported in a single degree as an element of the category Crys_Y^C. By [6, Lemma 3.2] it follows that $\mathcal{O}_Y$ is CM up to nilpotence as an element of Crys_Y^F. □

Corollary 3.6. *The singularities* $(\mathbf{0} \in Y)$ *are never F-injective.*

Proof. This follows from [27, Lemma 2.5.4], but here is another proof that utilizes the notions introduced above. Since Y is CM up to nilpotence, the action of Frobenius on the local cohomology groups $H^i_{\mathfrak{m}}(\mathcal{O}_{Y,\mathbf{0}})$, where $\mathfrak{m}$ is the maximal ideal and $0 \leq i < \dim \mathcal{O}_{Y,\mathbf{0}}$, is nilpotent (see [6, Lemma 3.2]). Since $\mathcal{O}_{Y,\mathbf{0}}$ is S_2 but not S_3, we have $H^2_{\mathfrak{m}}(\mathcal{O}_{Y,\mathbf{0}}) \neq 0$. Thus the Frobenius action on $H^2_{\mathfrak{m}}(\mathcal{O}_{Y,\mathbf{0}})$ is not injective. This shows that $\mathcal{O}_{Y,\mathbf{0}}$ is not F-injective. □

Proposition 3.7. *There exist proper birational morphisms* $\pi\colon V \to Y$ *where* V *is a regular variety. Moreover, for any such:*

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1. *the Frobenius action on $R^i\pi_*\mathcal{O}_V$ is nilpotent for $i > 0$* ([4]), *and*
2. *the Frobenius trace action on $R^i\pi_*\omega_V$ is nilpotent for $i > 0$.*

Proof. We begin by constructing a specific resolution of singularities and checking the nilpotence statements along it. Consider the diagram (3.1) in the proof of Theorem 3.2. The 1-foliation $b^*\mathcal{F}$ is locally generated by multiplicative derivations with isolated singularities. Thus the quotient $Z = X/b^*\mathcal{F}$ has only isolated μ_p-singularities [27, Proposition 4.1.1]. These toroidal singularities can be resolved using [15, Theorem 11* p.94]: let $\mu\colon V' \to Z$ be the induced resolution of singularities. The composition $\pi' = \mathfrak{b} \circ \mu\colon V' \to Y$ is also a resolution of singularities.

We claim that the sheaves $R^i\mu_*\mathcal{O}_{V'}$ and $R^i\mu_*\omega_{V'}$ vanish for $i > 0$. This can be checked formally on Z, and thus we reduce to the toric situation. Then vanishing holds for $i > 0$ by [15, Theorem 14 p. 52]. From the Leray spectral sequence it follows that

$$R^i\pi'_*\mathcal{O}_{V'} = R^i\mathfrak{b}_*\mathcal{O}_Z \quad \text{and} \quad R^i\pi'_*\omega_{V'} = R^i\mathfrak{b}_*\omega_Z \quad \text{for } i \geq 0.$$

By naturality of the Frobenius, these equalities reduce the nilpotence statements along π' to the nilpotence statements along the morphism $\mathfrak{b}\colon Z \to Y$. By construction the absolute Frobenius morphism $F_Z\colon \mathcal{O}_Z \to F_*\mathcal{O}_Z$ factors through $q'_*\mathcal{O}_X$. Therefore $R^i\mathfrak{b}_*(F_Z)$ factors through $R^i\mathfrak{b}_*(q'_*\mathcal{O}_X) = q_*R^ib_*\mathcal{O}_X$. Since b is the blow-up of a point of $\mathbb{A}^n$ we have $R^ib_*\mathcal{O}_X = 0$ for $i > 0$, which shows that $R^i\mathfrak{b}_*(F_Y)$ is the zero map for $i > 0$. By applying (derived) Hom into ω_Z to the map F_Z, one finds dually that the Frobenius trace $\mathrm{Tr}_Z\colon F_*\omega_Z \to \omega_Z$ factors through $q'_*\omega_X$. As before we have $R^i\mathfrak{b}_*(q'_*\omega_X) = q_*R^ib_*\omega_X = 0$ for $i > 0$, which shows that $R^i\mathfrak{b}_*(\mathrm{Tr}_Z)$ is the zero map for $i > 0$. This proves the nilpotence statements along $\mathfrak{b}$ and π'.

Finally, let $\pi\colon V \to Y$ be another proper birational morphism with V regular. By [8, Theorem 1] we have $R^i\pi_*\mathcal{O}_V \cong R^i\pi'_*\mathcal{O}_{V'}$ and $R^i\pi_*\omega_V \cong R^i\pi'_*\omega_{V'}$ for every $i \geq 0$. So the nilpotence statements along π' imply the corresponding nilpotence statements along π, which concludes the proof. □

4. Locally stable families with non-S_2 special fibers

4.1. First example: $\dim \geq \max\{\mathbf{p}, \mathbf{3}\}$.

Consider the affine space $\mathbb{A}^{n+1}_{x_1,\dots,x_n,y} \times \mathbb{A}^1_t$ and the derivation

$$\partial_m = \sum_{i=1}^{n} x_i^2\partial_{x_i} + t^m\mu(y)\partial_y \quad \text{on } \mathbb{A}^{n+1} \times \mathbb{A}^1, \tag{4.1}$$

where $m \geq 0$ and $\mu(y) \in k[y]$ satisfies $(\mu\partial_y)^{[p]} = 0$ and becomes a unit in a neighborhood of the origin, that is, $\mu(0) \neq 0$.

We let $\mathbb{A} = \mathbb{A}^n_{\mathbf{x}} \times D(\mu(y)) \subset \mathbb{A}^{n+1}$ be the affine open subset where $\mu(y)$ is invertible. We let $\mathcal{F}_m \subset T_{(\mathbb{A}\times\mathbb{A}^1)/k}$ be the sub-module generated by ∂_m.

(We can of course take $m = 1$, $\mu = 1$ and $\mathbb{A} = \mathbb{A}^{n+1}$; but this extra generality will be useful for Theorem 4.7 below.)

Claim 4.1. $\mathcal{F}_m$ is a family of 1-foliations over $\mathbb{A}^1_t$ (Definition 2.11) if and only if $n \geq 2$.

Proof. Clearly $\partial_m \in T_{(\mathbb{A}\times\mathbb{A}^1)/\mathbb{A}^1}$, and we see that $\partial_m^{[p]} = 0$ as in Claim 3.1 (and this survives along any specialization of t). Thus $\mathcal{F}_m$ is a relative 1-foliation over $\mathbb{A}^1_t$, for any $n \geq 1$. It is easily seen that the cokernel of $\mathcal{F}_m \hookrightarrow T_{\mathbb{A}\times\mathbb{A}^1/\mathbb{A}^1}$ is torsion-free, hence flat, over $\mathbb{A}^1$.

[4] This is equivalent to saying that Y has $\mathbb{F}_p$-rational singularities, see [6, Remark 3.5]. Combining this statement, Proposition 3.5 and [6, Proposition 3.8], we obtain that Y satisfies the *Frobenius stable Grauert–Riemenschneider vanishing* [6, Definition 3.6].

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The restriction of $\mathcal{F}_m$ to the generic fiber of the projection $\mathbb{A} \times \mathbb{A}^1_t \to \mathbb{A}^1_t$ is the restriction to $\mathbb{A} \times \operatorname{Spec}(k(t))$ of the module generated by

$$\sum_{i=1}^{n} x_i^2 \partial_{x_i} + t^m \mu(y) \partial_y \in \operatorname{Der}_{k(t)} k(t)[x_1, \ldots, x_n, y],$$

which is clearly a 1-foliation for every $n \geq 1$. A similar computation shows that $\mathcal{F}_m|_{(t=\alpha)}$ is a 1-foliation for every $0 \neq \alpha \in k$ and $n \geq 1$. The only condition is given by the restriction to the central fiber ($t = 0$): indeed, $\mathcal{F}_m|_{(t=0)}$ is generated by

$$\sum_{i=1}^{n} x_i^2 \partial_{x_i} \in \operatorname{Der}_k k[x_1, \ldots, x_n, y],$$

so $\mathcal{F}_m|_{(t=0)}$ is a 1-foliation if and only $n \geq 2$. (For $n = 1$, observe that $x_1^2 \partial_{x_1}$ does not generate a saturated sub-module of $\operatorname{Der}_k k[x_1, \ldots, x_n, y]$.) The statement follows. □

The following lemma explains how to simplify the singularities of $\mathcal{F}_m$.

Lemma 4.2. *We have* $\operatorname{Sing}(\mathcal{F}_m) = V(x_1, \ldots, x_n, t)$ *for* $m \geq 1$, *and* $\mathcal{F}_0$ *is regular. If* $m \geq 1$ *and* $b\colon X = \operatorname{Bl}_{\operatorname{Sing}(\mathcal{F}_m)} \to \mathbb{A} \times \mathbb{A}^1$ *is the blow-up of that singular locus with exceptional divisor E, then:*

1. $a(E; \mathcal{F}_m) = -1$;
2. *E is* $b^*\mathcal{F}_m$*-invariant;*
3. $b^*\mathcal{F}_m$ *is lc on the* x_i*-patches, and isomorphic over* $\mathbb{A}^1_t$ *to* $\mathcal{F}_{m-1}$ *on the t-patch;*
4. *if* $m \geq 2$, *the locus of non-lc singularities S of* $b^*\mathcal{F}_m$ *is closed in X and contained on the t-patch;*
5. $\operatorname{mult}_S E = 1$, $\operatorname{codim}_X S = n + 1$ *and S is disjoint from the strict transform of* $(t = 0) \subset \mathbb{A} \times \mathbb{A}^1_t$.

Proof. The statement about the singular locus is clear. On the x_1-patch U_1 the blow-up b is given by

$$\begin{aligned} b^*\colon k[x_1, \ldots, x_n, y, t][\mu(y)^{-1}] &\longrightarrow k[u_1, \ldots, u_n, y, s][\mu(y)^{-1}], \\ x_1 \mapsto u_1, \quad x_i \mapsto u_1 u_i \ (1 < i \leq n), \quad y \mapsto y, \quad t &\mapsto u_1 s. \end{aligned}$$

Therefore we have

$$\begin{aligned} b^*\partial_m|_{U_1} &= u_1 \cdot \left(u_1 \partial_{u_1} - \sum_{i=2}^{n} u_i \partial_{u_i} - s\partial_s \right) + \sum_{i=2}^{n} u_1^2 u_i^2 \cdot \frac{1}{u_1} \partial_{u_i} + u_1^m s^m \mu(y) \cdot \partial_y \\ &= u_1 \cdot \underbrace{\left[u_1 \partial_{u_1} + \sum_{i=2}^{n} (-u_i + u_i^2) \partial_{u_i} + u_1^{m-1} s^m \mu(y) \partial_y - s\partial_s \right]}_{\psi} \end{aligned}$$

Since $b^*\partial_m|_{U_1}$ is p-closed, so is ψ. As $\psi(u_1) = u_1$, by Lemma 2.3 we obtain that $\psi^{[p]} = \psi$. Thus $b^*\mathcal{F}_m|_{U_1} = \mathcal{O}_{U_1} \cdot \psi$ is lc by Proposition 2.6. Moreover, the above computations show that E is $b^*\mathcal{F}_m$-invariant and that $a(E; \mathcal{F}_m) = -1$. The situation on the other x_i-patches U_i is similar.

On the t-patch V the blow-up b is given by

$$\begin{aligned} b^*\colon k[x_1, \ldots, x_n, y, t][\mu(y)^{-1}] &\longrightarrow k[v_1, \ldots, v_n, y, s][\mu(y)^{-1}], \\ x_i \mapsto s v_i \ (1 \leq i \leq n), \quad y \mapsto y, \quad t &\mapsto s. \end{aligned}$$

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Therefore we have:

$$b^*\partial_m|_V = \sum_{i=1}^{n} v_i^2 s\partial_{v_i} + s^m\mu(y)\partial_y$$
$$= s\cdot\underbrace{\left[\sum_{i=1}^{n} v_i^2\partial_{v_i} + s^{m-1}\mu(y)\partial_y\right]}_{\xi}$$

Under the description $V \cong D_{\mathbb{A}^{n+1}_{v_1,\dots,v_n,y}}(\mu(y)) \times \mathbb{A}^1_s$, we see that ξ corresponds to ∂_{m-1} and thus $b^*\mathcal{F}_m|_V \cong \mathcal{F}_{m-1}$. To finish the proof, it remains to observe that $\mathrm{Sing}(\xi) = V(v_1,\dots,v_n,s) \subset V$ is in fact a closed subset of X, and that the strict transform of $(t=0)$ is disjoint from the patch V. □

Theorem 4.3. *With the notations as above, let $Y = (\mathbb{A}\times\mathbb{A}^1)/\mathcal{F}_m$ with $n \geq 2$ and $m \geq 1$. Then:*

1. *The projection $\mathbb{A}\times\mathbb{A}^1_t \to \mathbb{A}^1_t$ factors through a flat morphism $Y \to \mathbb{A}^1_t$ of relative dimension $n+1$;*
2. *Y_t is smooth for $t \neq 0$, while Y_0 is reduced and non-S_2;*
3. *If $n+1 \geq \max\{p,3\}$, the family $Y \to \mathbb{A}^1$ is locally stable.*

The hypothesis $n \geq 2$ ensures that the 1-foliation $\mathcal{F}_m$ is a family of 1-foliations over $\mathbb{A}^1_t$, see Claim 4.1. The case $(p=2, n=1)$ is considered in Example 4.6 below.

Proof. Since $\partial_m(t) = 0$ we have $k[t] \subset k[x_1,\dots,x_n,y,t]^{\partial_m}$: this proves the first point. On the principal open set $D(t) \subset \mathbb{A}\times\mathbb{A}^1$ the 1-foliation $\mathcal{F}_m|_{D(t)}$ is regular. Thus for $0 \neq \alpha \in k$, by Theorem 2.12 we have

$$Y_\alpha = \mathbb{A}/\mathcal{F}_{m,\alpha} \quad \text{for} \quad \mathcal{F}_{m,\alpha} = \mathcal{O}_{\mathbb{A}}\cdot\left(\sum_{i=1}^{n} x_i^2\partial_{x_i} + \alpha^m\mu(\alpha)\partial_y\right).$$

As $\mathcal{F}_{m,\alpha}$ is regular for $\alpha \neq 0$ we obtain that Y_α is smooth [27, Lemma 2.5.10].

Let us prove that Y_0 is not S_2. For this we reduce to $m=1$. Indeed, consider the cartesian diagram

$$\begin{array}{ccc} \mathbb{A}\times\mathbb{A}^1_t & \xrightarrow{v} & \mathbb{A}\times\mathbb{A}^1_{t'} \\ \downarrow & & \downarrow \\ \mathbb{A}^1_t & \xrightarrow{u} & \mathbb{A}^1_{t'} \end{array}$$

where the vertical arrows are the projections and $u\colon \mathbb{A}^1_t \to \mathbb{A}^1_{t'}$ is given by $u^*(t') = t^m$. By Lemma 2.13 we have $v^*\mathcal{F}_1 = \mathcal{F}_m$ (where $\mathcal{F}_1$ is defined as in (4.1) using t' instead of t) and an $\mathbb{A}^1_t$-isomorphism

$$(\mathbb{A}\times\mathbb{A}^1_{t'})/\mathcal{F}_1 \times_{\mathbb{A}^1_{t'}} \mathbb{A}^1_t \cong Y.$$

Hence it suffices to show that Y_0 is non-S_2 in the case $m=1$. As $\mathcal{F}_1$ is a family of 1-foliations over $\mathbb{A}^1_t$, by Theorem 2.12 it is sufficient to prove that

$$\mathcal{O}^{\partial_1}_{\mathbb{A}\times\mathbb{A}^1,\mathbf{0}} \otimes_{k[t]} k[t]/(t) \subsetneq \left(\mathcal{O}_{\mathbb{A},\mathbf{0}}/(t)\right)^{\bar\partial} \quad \text{with} \quad \bar\partial = \sum_{i=1}^{n} x_i^2\partial_{x_i},$$

where $\mathbf{0} = V(x_1,\dots,x_n,y,t)$. Clearly $\bar\partial(y) = 0$, so it suffices to show that there is no $f \in \mathcal{O}_{\mathbb{A}\times\mathbb{A}^1,\mathbf{0}}$ such that $\partial_1(y+tf) = 0$. If there was such an f, we would have

$$0 = \partial_1(y+tf) = t\mu + \sum_{i=1}^{n} tx_i^2\partial_{x_i}(f) + t^2\mu\partial_y(f).$$

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Taking into account that t is not a zero-divisor in $\mathcal{O}_{\mathbb{A}\times\mathbb{A}^1,\mathbf{0}}$, we would obtain

$$-\mu = \sum_{i=1}^{n} x_i^2 \partial_{x_i}(f) + t\mu\partial_y(f).$$

However, the right-hand side belongs to the prime ideal $(x_1, \dots, x_n, t)$, so this equality cannot hold in the local ring $\mathcal{O}_{\mathbb{A}\times\mathbb{A}^1,\mathbf{0}}$ by our choice of μ. We have reached a contradiction, and hence Y_0 is not S_2 at the image of $\mathbf{0}$.

On the other hand, I claim that Y_0 is (geometrically) reduced. Since Y is S_2, the Cartier divisor Y_0 is S_1 and it is sufficient to show that Y_0 is generically reduced. Observe that Y_0 is irreducible. The 1-foliation $\mathcal{F}_m$ is regular on the open set $U = D(x_1)$, which intersects Y_0 nontrivially. By Theorem 2.12 it follows that the open subset $(U/\mathcal{F}_m|_U)_0$ of Y_0 is the quotient of U_0 by (the restriction of) $\mathcal{F}_{m,0}$. In particular it is normal, so Y_0 is generically reduced.

It remains to show that (Y, Y_0) is lc (in the general case $m \geq 1$). For this, as in the proof of Theorem 3.2, we blow-up $\mathbb{A}\times\mathbb{A}^1$ until the pullback of $\mathcal{F}_m$ has only lc singularities. By Lemma 4.2, this produces a sequence of m smooth blow-ups

$$X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow W = \mathbb{A}\times\mathbb{A}^1$$

such that, if $b_m \colon X_m \to W$ is the composition of the blow-ups, the 1-foliation $b_m^*\mathcal{F}_m$ is lc. Let $E_j \subset X_m$ be the strict transform of the exceptional divisor of $X_j \to X_{j-1}$ (for $1 \leq j \leq m$), and denote $(t = 0) \subset W$ by W_0. We have $a(E_1; W, W_0) = n - 1$, and using Lemma 4.2 we compute

$$\begin{aligned} a(E_2; W, W_0) =& a(E_2; X_1, (X_1 \to W)_*^{-1} W_0 + (1-n)E_1) \\ =& 2n - 1. \end{aligned}$$

Continuing by induction, we find

$$a(E_j; W, W_0) = jn - 1, \quad j = 1, \dots, m.$$

Now let $Z = X_m/b_m^*\mathcal{F}_m$ and consider the induced commutative diagram

$$\begin{array}{ccc} X_m & \longrightarrow & Z \\ \downarrow{\scriptstyle b_m} & & \downarrow{\scriptstyle \mathfrak{b}} \\ \mathbb{A}\times\mathbb{A}^1 & \longrightarrow & Y \end{array}$$

where the horizontal arrows are the quotient morphisms and where $\mathfrak{b} \colon Z \to Y$ is birational. Let $F_j \subset Z$ be the prime divisor which is the image of E_j. Using Lemma 4.2 we get

$$a(E_j; \mathcal{F}_m) = -j, \quad j = 1, \dots, m.$$

Now we use Theorem 2.10: since W_0 is $\mathcal{F}_m$-invariant we find

$$\begin{aligned} a(F_j; Y, Y_0) =& a(E_j; W, W_0) + (p-1) \cdot a(E_j; \mathcal{F}_m) \\ =& j(n-p+1) - 1. \end{aligned}$$

Therefore we have the crepant equation

$$K_Z + \mathfrak{b}_*^{-1} Y_0 - \sum_{j=1}^{m} (j(n-p+1) - 1) F_j = \mathfrak{b}^*(K_Y + Y_0).$$

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To show that (Y, Y_0) is lc it suffices to show that

$$\operatorname{discrep}\left(Z, \mathfrak{b}_*^{-1}Y_0 - \sum_{j=1}^{m}(j(n-p+1)-1)F_j\right) \geq -1.$$

Since $b_m^*\mathcal{F}_m$ is lc, by Theorem 2.10 it suffices to show that

$$\operatorname{discrep}\left(X_m, (b_m)_*^{-1}W_0 - \sum_{j=1}^{m}(j(n-p+1)-1)E_j\right) \geq -1.$$

As $(X_m, (b_m^{-1})_*W_0 + \sum_{j=1}^{m} E_j)$ is log smooth, this holds by [16, Corollary 2.11] as soon as $1 - j(n-p+1) \leq 1$ for every $j \geq 1$. This is the case when $n+1 \geq p$. The proof is complete. □

Remark 4.4. The normalization Y_0^ν of the $n+1$-dimensional central fiber Y_0 in Theorem 4.3 is isomorphic to the product of $\mathbb{A}^1$ with the n-dimensional quotient singularity of Theorem 3.2. In particular, it is $\mathbb{Q}$-factorial, S_3 but non-S_4, non-F-injective, with a singular locus of dimension one. Since $n+1 \geq \max\{p,3\}$ we have $n \geq p-1$ and so Y_0^ν is lc. If $n \geq p$ (resp. $n > p$), it is even canonical (resp. terminal).

Corollary 4.5. *Let* $Y = \mathbb{A}^{n+2}_{x_1,\dots,x_n,y,t}/\langle \sum_{i=1}^{n} x_i^2\partial_{x_i} + t\partial_y\rangle$ *with* $n \geq 2$. *If* $\dim Y = n+2 \geq p+1$, *then* Y *has non-*S_3 *canonical singularities along the image of* $V(x_1,\dots,x_n,t)$.

Proof. The computations in the proof of Theorem 4.3 show that the central fiber Y_0 of $Y \to \mathbb{A}^1_t$ is not S_2 along the image of $V(x_1,\dots,x_n,t)$. Thus Y is non-S_3 along that curve. To show that Y is canonical, as the open set $Y \setminus Y_0$ is regular and thus terminal, it suffices to show that $a(E;Y) \geq 0$ for every divisor E over Y with $c_Y(E) \subseteq Y_0$. As Y_0 is an effective Cartier divisor and (Y, Y_0) is lc we have

$$a(E;Y) = a(E;Y,Y_0) + \operatorname{mult}_E Y_0 \geq -1 + 1 = 0$$

and we are done. □

Example 4.6. Suppose that k has characteristic $p = 2$, and consider the derivation $\partial = x^2\partial_x + t\partial_y$ on $\mathbb{A}^3_{x,y,t}$. Then $\partial^{[2]} = 0$ and that $X = \mathbb{A}^3/\partial$ is a flat $\mathbb{A}^1_t$-scheme given by the spectrum of $k[x^2, y^2, t, tx + x^2y]$: indeed, the latter ring is clearly ∂-invariant, is normal and has degree 4 over $k[x^2, y^2, t^2]$.

It is easily seen that X is isomorphic to an hypersurface in $\mathbb{A}^4$, and thus X is Cohen–Macaulay. Its central fiber $X_0 = \operatorname{Spec} k[x^2, y^2, x^2y] \cong V(W^2 - X^2Y) \subset \mathbb{A}^3_{X,Y,W}$ is demi-normal but not normal: it has a nodal singularity along the curve $(x = 0)$. In fact, this is an inseparable node [28, §3.1], and thus X_0 is semi-normal but not weakly normal [30, Lemma 2.3.7]. Simple computations in the spirit of the proof of Theorem 4.3 show that the pair (X, X_0) is lc but not plt, as the image of the line $(x = 0 = t)$ in X is the singular locus of X_0 and an lc center of (X, X_0). Compare also with the general semi-normality results of [4].

Theorem 4.3 gives a series of local examples. We can compactify them as explained in the next theorem to obtain global ones.

Theorem 4.7. *For each* $p > 0$, *there exists a projective family* $(\mathcal{Y}, \mathcal{B} = \frac{1}{p}\mathcal{H}) \to \mathbb{A}^1$ *of relative dimension* $N \geq \max\{p,3\}$ *such that:*

1. $(\mathcal{Y}, \frac{1}{p}\mathcal{H}) \to \mathbb{A}^1$ *is a stable family,*
2. $\mathcal{Y}_t$ *has only* μ_p*-quotient singularities for* $t \neq 0$, *and*
3. $\mathcal{Y}_0$ *is not* S_2.

Moreover:

- *There exists a finite purely inseparable morphism* $\mathbb{P}^{N-1} \times E \times \mathbb{A}^1 \to \mathcal{Y}$ *of degree* p *over* $\mathbb{A}^1$, *where* E *is a supersingular elliptic curve, and* $\mathcal{H}$ *is the image of a* $\mathbb{Q}$*-divisor* $H_0 \times \mathbb{A}^1 \subset \mathbb{P}^{N-1} \times E \times \mathbb{A}^1$;

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- *If $p = 2$ then we can modify the construction so that $\mathcal{Y}_t$ is regular for every $t \neq 0$;*
- *If C is a normal curve and $C \to \mathbb{A}^1$ is a finite flat morphism, then $\mathcal{Y}_C = \mathcal{Y} \times_{\mathbb{A}^1} C$ is normal and $(\mathcal{Y}_C, \frac{1}{p}\mathcal{H}_C) \to C$ is stable.*

The following lemma will be useful for the proof.

Lemma 4.8. *Consider E a supersingular elliptic curve, $\omega \in H^0(E, T_E)$ a global generator, $\mathbb{A}^{N-1}_{x_1,\dots,x_{N-1}} \subset \mathbb{P}^{N-1}$ a standard chart (with $N \geq 3$), A an integral Noetherian k-algebra and $a \in A$. Then the derivation*

$$\partial_{(0)} = \sum_{i=1}^{N-1} x_i^2 \partial_{x_i} + a\omega \quad \text{on} \quad \mathbb{A}^{N-1} \times E \times \operatorname{Spec}(A)$$

induces a 1-foliation $\mathcal{G}$ of rank 1 on $\mathbb{P}^{N-1} \times E \times \operatorname{Spec}(A)$ such that:

1. *$K_{\mathcal{G}} = \mathcal{O}_{\mathbb{P}^{N-1}}(1) \boxtimes \mathcal{O}_{E \times \operatorname{Spec}(A)}$,*
2. *$\mathcal{G}$ has only multiplicative singularities away from $\{[1 : 0 : \cdots : 0]\} \times E \times V(a)$,*
3. *If $A = k[t]$ and $a = t^m$ with $m \geq 1$, then $\mathcal{G} \subset T_{(\mathbb{P}^{N-1} \times E \times \mathbb{A}^1)/\mathbb{A}^1}$ is a family of 1-foliations.*

Proof. Since E is supersingular we have $\omega^{[p]} = 0$ [14, 12.4.1.3]. An argument similar to Claim 3.1 shows that $\partial_{(0)}$ is p-closed. Therefore it induces a 1-foliation $\mathcal{G}$ of rank 1 on $\mathbb{P}^{N-1} \times E \times \operatorname{Spec}(A)$. We compute the expression of $\partial_{(0)}$ on the other charts. Let $X_0, \dots, X_{N-1}$ be homogeneous coordinates of $\mathbb{P}^{N-1}$: we may assume that $x_i = X_i/X_0$ for $1 \leq i \leq N-1$. If $u_i = X_i/X_{N-1}$ for $0 \leq i < N-1$, then we have

$$x_i = \frac{u_i}{u_0} \ (1 \leq i < N-1), \quad x_{N-1} = \frac{1}{u_0},$$
$$u_i = \frac{x_i}{x_{N-1}} \ (1 \leq i < N-1), \quad u_0 = \frac{1}{x_{N-1}}.$$

Therefore

$$\partial_{x_i} = u_0 \partial_{u_i} \ (1 \leq i < N-1), \quad \partial_{x_{N-1}} = -u_0^2 \partial_{u_0} - \sum_{i=1}^{N-2} u_0 u_i \partial_{u_i}$$

and so

$$\partial_{(0)} = \frac{-1}{u_0} \cdot \underbrace{\left[u_0 \partial_{u_0} + \sum_{i=1}^{N-2} (u_i - u_i^2) \partial_{u_i} - a u_0 \omega \right]}_{\partial_{(N-1)}}. \tag{4.2}$$

This shows that $\mathcal{G}|_{D_+(X_{N-1}) \times E \times \operatorname{Spec}(A)}$ is generated by $\partial_{(N-1)}$ and that $K_{\mathcal{G}}^{-1} = \mathcal{G} \cong \mathcal{O}_{\mathbb{P}^{N-1}}(-1) \boxtimes \mathcal{O}_{E \times \operatorname{Spec}(A)}$. This proves the first assertion.

On $D_+(X_{N-1}) \times E \times \mathbb{A}^1$, notice that $\partial_{(N-1)}$ is p-closed and that $\partial_{(N-1)}(u_0) = u_0$. Therefore $\partial_{(N-1)}^{[p]} = \partial_{(N-1)}$ by Lemma 2.3, which shows that $\mathcal{G}$ has only multiplicative singularities on $D_+(X_{N-1}) \times E \times \mathbb{A}^1$. The computations on the patches $D_+(X_i) \times E \times \mathbb{A}^1$ for $1 \leq i < N-1$ are similar. On $D_+(X_0) \times E \times \mathbb{A}^1$, we see that the singular locus of $\mathcal{G}$ is $\{[1 : 0 : \cdots : 0]\} \times E \times V(a)$. This proves the second assertion. In case $A = k[t]$ and $a = t^m$, we proceed as in Claim 4.1 over each chart to see that $\mathcal{G}$ is a family of 1-foliations if and only if $N \geq 3$, which holds by assumption. □

Remark 4.9. The multiplicative singularities of $\mathcal{G}$ cannot be resolved by smooth blow-ups (unless $p = 2$). Indeed, the singular locus of $\partial_{(N-1)}$ is the union of the closed subsets

$$Z_{\varepsilon_\bullet} = [\varepsilon_0 = 0 : \varepsilon_1 : \dots \varepsilon_{N-2} : 1] \times E \times \operatorname{Spec}(A)$$

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where $\varepsilon_1, \dots, \varepsilon_{N-2} \in \{0,1\}$. It follows from [27, Proposition 4.1.1] that along $Z_{\varepsilon_\bullet}$, and formal-locally on its first factor, the 1-foliation $\mathcal{G}$ is generated by

$$\sum_{i:\ \varepsilon_i=0} \bar{u}_i \partial_{\bar{u}_i} - \sum_{j:\ \varepsilon_j=1} \bar{u}_j \partial_{\bar{u}_j} - a\bar{u}_0\omega$$

where $\bar{u}_\bullet$ are formal parameters of $\mathcal{O}_{\mathbb{P}^{N-1},[\varepsilon_\bullet:1]}$ with $\bar{u}_0 = u_0$. To resolve the singularity of $\mathcal{G}$ along $Z_{\varepsilon_\bullet}$, we need to take the weighted blow-up given by the Rees algebra

$$\sum_{i:\ \varepsilon_i=0} (\bar{u}_i, 1) + \sum_{j:\ \varepsilon_j=1} (\bar{u}_j, p-1).$$

As soon as some weight $p-1 \neq 1$ appears, this blow-up produces a nonschematic tame regular Deligne–Mumford stack whose coarse moduli space has quotient singularities: see [29] for details. So the singularities of $\mathcal{G}$ along $Z_{\varepsilon_\bullet}$ cannot be resolved by usual blow-ups, unless $\varepsilon_\bullet = (0,\dots,0)$ or $p=2$. The situation on the other patches $D_+(X_i) \times E \times \operatorname{Spec}(A)$, for $1 \le i < N-1$, is similar.

Proof of Theorem 4.7. For clarity, we divide the proof into several steps. If $p=2$ we can use a slightly different construction, which we describe in Step 5.

Step 1: Construction of $\mathcal{Y} \to \mathbb{A}^1$. Let E be a supersingular elliptic curve and $\omega \in H^0(E, T_E)$ be a global generator. If $X_0, \dots, X_{N-1}$ are the homogeneous coordinates on $\mathbb{P}^{N-1}$, consider on $D_+(X_0) \times E \times \mathbb{A}^1_t$ the derivation

$$\partial_{(0)} = \sum_{i=1}^{N-1} x_i^2 \partial_{x_i} + t\omega, \quad x_i = X_i/X_0.$$

By Lemma 4.8 it induces a 1-foliation $\mathcal{G}$ on $W = \mathbb{P}^{N-1} \times E \times \mathbb{A}^1$. We let $\mathcal{Y} = (\mathbb{P}^{N-1} \times E \times \mathbb{A}^1)/\mathcal{G}$, with quotient morphism $q\colon W \to \mathcal{Y}$. By construction the projection $W \to \mathbb{A}^1$ factors through a flat projective morphism $\mathcal{Y} \to \mathbb{A}^1$.

Step 2: Construction of the divisor $\mathcal{H}$. The divisor $\mathcal{H}$ is chosen as follows. Take a $\mathbb{Q}$-divisor H on W such that:

- $H = H_0 \times \mathbb{A}^1$ for some $\mathbb{Q}$-divisor H_0 on $\mathbb{P}^{N-1} \times E$;
- $K_W + (p-1)K_{\mathcal{G}} + H$ is ample over $\mathbb{A}^1$,
- $(W, W_\alpha + H)$ is lc for every $\alpha \in \mathbb{A}^1 \setminus \{0\}$,
- the support of H lies in general position with respect to the curve $\{[1:0:\cdots:0]\} \times E \times \{0\}$, and
- H is not $\mathcal{G}$-invariant.

The last three properties are satisfied for a general choice of divisor of the form $H_0 \times \mathbb{A}^1$, thus by choosing H_0 general in a sufficiently ample linear system on $\mathbb{P}^{N-1}$, we see there exists an H with all these properties. Notice that $K_W + (p-1)K_{\mathcal{G}} \cong \mathcal{O}_{\mathbb{P}^{N-1}}(-N+p-1) \boxtimes \mathcal{O}_{E\times\mathbb{A}^1}$ by Lemma 4.8, so because of the second property and on the assumption $N \ge \max\{p,3\}$ we cannot take $H=0$.

We let $\mathcal{H} \subset \mathcal{Y}$ be the prime divisor with support $q(H)$.

Step 3: Behaviour under base-change. Let us immediately observe what happens when we base-change $\mathcal{Y} \to \mathbb{A}^1_t$ along a flat finite morphism $g\colon C \to \mathbb{A}^1_t$. By Lemma 2.13 we have a natural isomorphism

$$\mathcal{Y} \times_{\mathbb{A}^1} C \cong \left(\mathbb{P}^{N-1} \times E \times C\right)/f^*\mathcal{G}$$

where $f\colon \mathbb{P}^{N-1} \times E \times C \to \mathbb{P}^{N-1} \times E \times \mathbb{A}^1$ is the base-change of g. This shows that $\mathcal{Y}_C$ is normal. Observe that $H_C = f^*H$ still satisfies the five properties listed in Step 2. The pullback of $K_{\mathcal{Y}} + \frac{1}{p}\mathcal{H}$

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along $\mathcal{Y}_C \to \mathcal{Y}$ is equal to $K_{\mathcal{Y}_C} + \frac{1}{p}\mathcal{H}_C$: therefore the latter is ample over C as soon as $K_{\mathcal{Y}} + \frac{1}{p}\mathcal{H}$ is ample over $\mathbb{A}^1$. By Proposition 2.9 and the non-$\mathcal{G}$-invariance of H we have

$$q^*\left(K_{\mathcal{Y}} + \frac{1}{p}\mathcal{H}\right) = K_W + (p-1)K_{\mathcal{G}} + H \tag{4.3}$$

which, by choice of H, is ample over $\mathbb{A}^1$.

To complete the proof, it remains to check properties (2) and (3), and that $(\mathcal{Y}_C, \frac{1}{p}\mathcal{H}_C) \to C$ is locally stable. Properties (2) and (3) can be checked after an arbitrary surjective base-change. For local stability, by [18, 2.15.5] and [12] it is enough to check it in case $g = F^r_{\mathbb{A}^1/k}$ is an iterate of the k-linear Frobenius morphism of $\mathbb{A}^1$ ([5]).

Step 4: Local stability and singularities of fibers. To summarize, we have reduced the proof to the following: let $q' \colon W = \mathbb{P}^{N-1} \times E \times \mathbb{A}^1_t \to \mathcal{Y}'$ be the quotient by the 1-foliation $\mathcal{G}'$ induced by the derivation

$$\partial'_{(0)} = \sum_{i=1}^{N-1} x_i^2 \partial_{x_i} + t^{p^r}\omega, \quad \text{on} \quad D_+(X_0) \times E \times \mathbb{A}^1_t.$$

It is endowed with a flat proper morphism $\mathcal{Y}' \to \mathbb{A}^1_t$. Let H' be a $\mathbb{Q}$-divisor on W satisfying the properties listed in Step 2: we let $\mathcal{H}' = q(H')$. We must show that $(\mathcal{Y}', \frac{1}{p}\mathcal{H}') \to \mathbb{A}^1$ is locally stable, that $\mathcal{Y}'_t$ has only μ_p-quotient singularities for $t \neq 0$, and that $\mathcal{Y}'_0$ is not S_2. As we will see, everything is a consequence of Theorem 4.3 and Lemma 4.8.

Step 4.1: Singularity of the central fiber. First, let us prove that $\mathcal{Y}'_0$ is not S_2 along the image through q' of the proper curve $\{[1:0:\cdots:0]\} \times E \times \{0\}$. Indeed, let $z = \{[1:0:\cdots:0]\} \times \{e\} \times \{0\}$ where e is an arbitrary closed point of E. As $\mathcal{Y}'_0$ is a Cartier divisor in $\mathcal{Y}$, it is equivalent to showing that $\mathcal{O}_{\mathcal{Y}',q'(z)}$ is not S_3. I claim that this is étale-local over z. For if $h \colon \operatorname{Spec}(\mathcal{O}^h_{W,z}) \to \operatorname{Spec}(\mathcal{O}_{W,z})$ is the henselization, by [29, Lemma 3.2.1] we have a cartesian diagram

$$\begin{array}{ccc} \operatorname{Spec}(\mathcal{O}^h_{W,z}) & \xrightarrow{h} & \operatorname{Spec}(\mathcal{O}_{W,z}) \\ \downarrow & & \downarrow q' \\ \operatorname{Spec}(\mathcal{O}^h_{W,z})/h^*\mathcal{G}' & \xrightarrow{g} & \operatorname{Spec}(\mathcal{O}_{\mathcal{Y}',q'(z)}) \end{array}$$

where g is an étale morphism. Therefore it suffices to prove that $\operatorname{Spec}(\mathcal{O}^h_{W,z})/h^*\mathcal{G}'$ is not S_3. If $y \in \mathcal{O}^h_{E,e}$ is an étale coordinate at $e \in E$, then an étale-local generator for $\mathcal{G}'$ at z is given by

$$\partial^h = \sum_{i=1}^{p} x_i^2 \partial_{x_i} + t^{p^r}\mu(y)\partial_y$$

where $\mu(y) \in \mathcal{O}^h_{E,e}$ is a unit such that $\omega \otimes \mathcal{O}^h_{E,e} = \mu(y)\partial_y$. Then we apply Theorem 4.3 (or rather its formal-local version at the origin, which is proved similarly) to see that the ∂^h-invariant sub-ring of $\mathcal{O}^h_{W,z}$ is not S_3.

Since $\mathcal{Y}'$ is S_2, the irreducible Cartier divisor $\mathcal{Y}'_0$ is nonetheless S_1. As it is generically reduced by Theorem 4.3, we obtain that $\mathcal{Y}'_0$ is (geometrically) reduced.

[5]Instead of appealing to [18, 12], one could proceed as follows. The local stability is Zariski-local over C', so we reduce to base-changes of the form $\operatorname{Spec}(\mathcal{O}) \to \mathbb{A}^1_t$ where $\mathcal{O}$ is a DVR. If s is a uniformizer of $\mathcal{O}$, write $t = \nu(s)s^m$ where $\nu(s) \in \mathcal{O}^\times$. Then one can prove the analogue of Theorem 4.3 for derivations of the form $\sum_{i=1}^n x_i^2\partial_{x_i} + s^m\nu(s)\mu(y)\partial_y$: the argument is the same, but the bookkeeping is even more cumbersome. The rest of the proof of Theorem 4.7 will then be similar.

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Step 4.2: Local stability. Now we prove that $(\mathcal{Y}', \frac{1}{p}\mathcal{H}') \to \mathbb{A}^1$ is locally stable. First, let $0 \neq \alpha \in k$. As $\mathcal{G}'|_{D(t)}$ is lc and $(W, W_\alpha + H')$ is lc, by Theorem 2.10 we have that $(\mathcal{Y}', \mathcal{Y}'_\alpha + \frac{1}{p}\mathcal{H}')$ is lc along $\mathcal{Y}_\alpha$. Moreover, as $\mathcal{G}'|_{D(t)}$ has only multiplicative singularities by Lemma 4.8, by Theorem 2.12 we obtain that

$$\mathcal{Y}'_\alpha = (\mathbb{P}^{N-1} \times E)/\mathcal{G}'_\alpha$$

where $\mathcal{G}'_\alpha$ is the 1-foliation induced by $\sum_{i=1}^{N-1} x_i^2 \partial_{x_i} + \alpha^{p^r}\omega$. By Lemma 4.8 we see that $\mathcal{G}'_\alpha$ has only multiplicative singularities. Therefore $\mathcal{Y}_\alpha$ has only μ_p-quotient singularities by [27, Proposition 4.1.1].

It remains to prove that $(\mathcal{Y}', \mathcal{Y}'_0 + \frac{1}{p}\mathcal{H}')$ is lc. To this end, we blow-up W to simplify the singularities of $\mathcal{G}'$. By Lemma 4.8 the non-lc singularities of $\mathcal{G}'$ are contained on the chart $D_+(X_0) \times E \times \mathbb{A}^1$, where (étale-locally on the factor E) the 1-foliation $\mathcal{G}'$ is generated by a derivation of the form (4.1) with $m = p^r$. Therefore, the proofs of Lemma 4.2 and Theorem 4.3 show that if we blow-up repeatedly the locus of non-lc singularities of $\mathcal{G}'$, we obtain a birational proper morphism

$$b\colon X = X_{p^r} \longrightarrow W$$

with p^r exceptional divisors $E_1, \dots, E_{p^r} \subset X$, such that $b^*\mathcal{G}'$ is lc, $(X, b_*^{-1}W_0 + \sum_j E_j)$ is log smooth, each E_j is $b^*\mathcal{G}'$-invariant, $a(E_j; \mathcal{G}') = -1$ and $a(E_j; W, W_0) = j(N-1) - 1$. By our choice of H', we also have that $(X, b_*^{-1}(W_0 + H') + \sum_j E_j)$ is log smooth and

$$a(E_j; W, W_0 + H') = a(E_j; W, W_0) \quad \forall j = 1, \dots, p^r.$$

The arguments are now the same as the ones at the end of the proof of Theorem 4.3. Consider the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Z = X/b^*\mathcal{G}' \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \mathfrak{b}} \\ W & \xrightarrow{q'} & \mathcal{Y}'. \end{array}$$

If $F_1, \dots, F_{p^r} \subset Z$ denote the $\mathfrak{b}$-exceptional divisors, then by Theorem 2.10 we have

$$\begin{aligned} a\left(F_j; \mathcal{Y}', \mathcal{Y}'_0 + \frac{1}{p}\mathcal{H}'\right) &= a(E_j; W, W_0 + H') + (p-1) \cdot a(E_j; \mathcal{G}') \\ &= a(E_j; W, W_0) + (p-1)(-j) \\ &= j(N-p) - 1 \end{aligned}$$

Hence we have the crepant equation

$$K_Z + \mathfrak{b}_*^{-1}\left(\mathcal{Y}'_0 + \frac{1}{p}\mathcal{H}'\right) + \sum_{j=1}^{p^r}(1 - j(N-p))F_j = \mathfrak{b}^*\left(K_{\mathcal{Y}'} + \mathcal{Y}'_0 + \frac{1}{p}\mathcal{H}'\right).$$

Notice that $1 - j(N-p) \leq 1$ for every $j \geq 1$ is equivalent to $N \geq p$, which holds as $N = \max\{p, 3\}$. Therefore we have that

$$\operatorname{discrep}\left(X, b_*^{-1}(W_0 + H) + \sum_{j=1}^{p^r}(1 - j(N-p))E_j\right) \geq -1.$$

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Hence by Theorem 2.10 we obtain that

$$\operatorname{discrep}\left(Z, \mathfrak{b}_*^{-1}(\mathcal{Y}_0 + \frac{1}{p}\mathcal{H}) + \sum_{j=1}^{p^r}(1 - j(N-p))F_j\right) \geq -1,$$

and therefore $(\mathcal{Y}', \mathcal{Y}'_0 + \frac{1}{p}\mathcal{H}')$ is lc.

Step 5: the case $p = 2$. It follows from Remark 4.9 that there exists a smooth blow-up $Z \to \mathbb{P}^{N-1}$ whose (reducible) center does not contain $[1 : 0 : \cdots : 0]$, and such that if $b\colon Z \times E \times \mathbb{A}^1 \to W$ is the induced morphism, the birational pullback $b^*\mathcal{G}$ is regular away from the preimage of $\{[1 : 0 : \cdots : 0]\} \times E \times \{0\}$. We take

$$\mathcal{Z} = (Z \times E \times \mathbb{A}^1)/b^*\mathcal{G} \longrightarrow \mathbb{A}^1$$

as the example. By [27, Proposition 5.2.4, Lemma 2.5.10] we have that $\mathcal{Z}_\alpha$ is regular for $\alpha \neq 0$. The rest of the properties are proved exactly as above. This completes the proof. □

Let us make some observations about the preceding proof.

Remark 4.10. The coefficients of H are ≤ 1, so the coefficients of $\frac{1}{p}\mathcal{H}$ are $\leq 1/p$. The extra factor $1/p$ appears because H is chosen to be non-$\mathcal{G}$-invariant. I do not know whether there exists a $\mathcal{G}$-invariant divisor H that satisfies the other properties. On the other hand, the number of irreducible components of H can be any (in particular, $\mathcal{H}$ can be irreducible).

Remark 4.11. If $e \in E$ an arbitrary point then the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{N-1}}(N+p) \boxtimes \mathcal{O}_E(e)$ is ample on $\mathbb{P}^{N-1} \times E$, and $\mathcal{L}^{\otimes n}$ is very ample for every $n \geq 3$. Moreover, for any integer $m \geq 1$, if H_0 is a general element of the $\mathbb{Q}$-linear system $\frac{1}{m}|\mathcal{L}^{\otimes n}|$ then $H_0 \times \mathbb{A}^1$ satisfies the conditions listed in Step 2.

4.2. Second example: dim = **3**, **p** = **3**.

Consider the affine space $\mathbb{A}^3_{x,y,z} \times \mathbb{A}^1_t$ and the derivation

$$\partial_m = y^3\partial_x + x\partial_y + t^m\partial_z, \quad m \geq 1.$$

Using an argument as in Claim 3.1 together with [24, Corollary 6], one checks that $\partial_m^{[p]} = 0$ as soon as $p > 2$. In that case ∂_m generates a 1-foliation $\mathcal{F}_m$ on $\mathbb{A}^3 \times \mathbb{A}^1$. As in Claim 4.1, one sees that $\mathcal{F}_m$ is a family of 1-foliations over $\mathbb{A}^1_t$.

Let us show how to simplify the singularities of $\mathcal{F}_m$. As in Lemma 4.2, we are able to reduce $\mathcal{F}_m$ to $\mathcal{F}_{m-1}$, but we need two blow-ups to achieve this.

Claim 4.12. For $p > 2$, we have $((1 - 2x^2)\partial_x)^{[p]} = 2^{(5p-3)/2}(1 - 2x^2)\partial_x$.

Proof. After the change of variable $u = 1 + \sqrt{2}x$, we reduce to consider the p-th power of the derivation $\partial = \sqrt{2}u(2-u)\partial_u$ on $k[u]$. By a formula of Jacobson [13, p. 209], we have

$$\partial^{[p]} = 2^{p/2}\Big((-u^2\partial_x)^{[p]} + (2u\partial_u)^{[p]} + S\Big)$$

where S is a k-linear combination of $(p-1)$-fold commutators of $-u^2\partial_x$ and $2u\partial_u$. Now we make several observations. First, by Hochschild's formula ∂ is p-closed and so $\partial^{[p]}$ is a $k[u]$-scaling of ∂. As $(-u^2\partial_u)^{[p]} = 0$ and $(2u\partial_u)^{[p]} = 2^p u\partial_u$, it follows that $S \neq 0$. As

$$[u^2\partial_u,\ 2u\partial_u] = -2u^2\partial_u$$

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we see that $S \neq 0$ can only be of the form $-\alpha u^2\partial_u$ for some $\alpha \in k^\times$. Hence $\partial^{[p]} = 2^{p/2}(2^p u\partial_u - \alpha u^2\partial_u)$ is a $k[u]$-scaling of $\partial = 2^{1/2}(2u\partial_u - u^2\partial_u)$, which forces $\alpha = 2^{(5p-3)/2}$. □

Lemma 4.13. *Suppose that $p > 2$. We have $\mathrm{Sing}(\mathcal{F}_m) = V(x, y, t)$ for $m \geq 1$, and $\mathcal{F}_0$ is regular. If $m \geq 1$ and*

$$b_1 \colon Y = \mathrm{Bl}_{(x,y,t)}(\mathbb{A}^3 \times \mathbb{A}^1) \longrightarrow \mathbb{A}^3 \times \mathbb{A}^1$$

with exceptional divisor E, then:

1. *$a(E; \mathcal{F}_m) = 0$,*
2. *E is $b_1^*\mathcal{F}_m$-invariant,*
3. *the locus S of (non-lc) singularities of $b_1^*\mathcal{F}_m$ is closed, $\mathrm{mult}_S E = 1$, $\mathrm{codim}_Y S = 2$ and S is not contained in $(b_1)_*^{-1}(t = 0)$.*

Consider $b_2 \colon \mathrm{Bl}_S Y \longrightarrow Y$ with exceptional divisor F and $b = b_1 \circ b_2$. Then:

1. *$a(F; \mathcal{F}_m) = -1$,*
2. *F is $b^*\mathcal{F}_m$-invariant,*
3. *if $m \geq 2$, the non-lc locus of $b^*\mathcal{F}_m$ is closed, contained in a blow-up chart V of b_2, not contained in the support of $b_*^{-1}(t = 0) + (b_2)_*^{-1}E$, and*
4. *we have an $\mathbb{A}^1_t$-isomorphism $V \cong \mathbb{A}^3 \times \mathbb{A}^1_t$ under which $b^*\mathcal{F}_m|_V$ is isomorphic to $\mathcal{F}_{m-1}$, and the b_2-exceptional divisor corresponds to $(t = 0)$.*

Proof. Let us first consider the blow-up b_1. We have three blow-up charts, which we denote by U_A, U_B, and U_C:

- Chart U_A, given by $x \mapsto u,\ y \mapsto uv,\ z \mapsto z,\ t \mapsto us$. Then

$$b_1^*\partial_m|_{U_A} = u^3v^3\partial_u + (1 - u^2v^4)\partial_v - u^2v^3s\partial_s + u^m s^m\partial_z$$

  which shows that $b_1^*\mathcal{F}_m|_{U_A}$ is a regular 1-foliation. Notice that $b_1^*\partial_m(u) \in (u)$, so E is $b_1^*\mathcal{F}_m$-invariant. Since $b_1^*\partial_m|_{U_A}$ generates $b^*\mathcal{F}_m|_{U_A}$, without having to saturate, we have $a(E; \mathcal{F}_m) = 0$.
- Chart U_B, given by $x \mapsto uv,\ y \mapsto v,\ z \mapsto z,\ t \mapsto vs$. Then

$$b_1^*\partial_m|_{U_B} = (v^2 - u^2)\partial_u + uv\partial_v - us\partial_s + v^m s^m\partial_z$$

  which is singular along the plane $u = v = 0$. These singularities are non-lc.
- Chart U_C, given by $x \mapsto us,\ y \mapsto vs,\ z \mapsto z,\ t \mapsto s$. Then

$$b_1^*\partial_m|_{U_C} = v^3s^2\partial_u + u\partial_v + s^m\partial_z$$

  which is singular along the plane $u = s = 0$. These singularities are non-lc.

We see that the locus $S \subset Y$ of singularities of $b_1^*\mathcal{F}_m$ is smooth, closed, and irreducible of codimension 2. As can be checked on chart U_B, that singular locus is not contained in the strict transform $b_*^{-1}(t = 0)$.

Next let us blow-up S. We have two blow-up charts above U_B (resp. above U_C), which we denote by U_{BA} and U_{BB} (resp. by U_{CA} and U_{CB}):

- Chart U_{BA}, given by $u \mapsto \bar{u},\ v \mapsto \bar{u}\bar{v},\ z \mapsto z,\ s \mapsto s$. Then

$$b^*\partial_m|_{U_{BA}} = \bar{u} \cdot \underbrace{\left[\bar{u}(\bar{v}^2 - 1)\partial_{\bar{u}} + \bar{v}(2 - \bar{v}^2)\partial_{\bar{v}} - s\partial_s + \bar{u}^{m-1}\bar{v}^m s^m\partial_z\right]}_{\psi_{BA}}.$$

  As $\psi_{BA}(s) = -s$, by Lemma 2.3 we see that $\psi_{BA}^{[p]} = -\psi_{BA}$. Thus $b^*\mathcal{F}_m$ is lc on the chart U_{BA}. As $\psi_{BA}(\bar{u}) \in (\bar{u})$ we see that F is $b^*\mathcal{F}_m$-invariant. As $b^*\partial_m = \bar{u} \cdot \psi_{BA}$ we find that $a(F; \mathcal{F}_m) = -1$.

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- Chart U_{BB}, given by $u \mapsto \bar{u}\bar{v},\ v \mapsto \bar{v},\ z \mapsto z,\ s \mapsto s$. Then

$$b^*\partial_m|_{U_{BB}} = \bar{v} \cdot \underbrace{\left[(1-2\bar{u}^2)\partial_{\bar{u}} + \bar{u}\bar{v}\partial_{\bar{v}} - \bar{u}s\partial_s + \bar{v}^{m-1}s'\partial_z\right]}_{\psi_{BB}}.$$

  By Claim 4.12 we see that $\psi_{BB}(u) = 2^{(5p-3)/2}\psi_{BB}(u)$. Thus by Proposition 2.6 we see that $b^*\mathcal{F}_m$ is lc on the chart U_{BB}.
- Chart U_{CA}, given by $u \mapsto \bar{u},\ v \mapsto v,\ z \mapsto z,\ s \mapsto \bar{u}\bar{s}$. Then

$$b^*\partial_m|_{U_{CA}} = \bar{u} \cdot \underbrace{\left[\bar{u}v^3\bar{s}^2\partial_{\bar{u}} + \partial_v + \bar{u}^{m-1}\bar{s}^m\partial_z - v^3\bar{s}^3\partial_{\bar{s}}\right]}_{\psi_{CA}}.$$

  As ψ_{CA} generates a regular 1-foliation, we see that $b^*\mathcal{F}_m$ is regular on the chart U_{CA}.
- Chart U_{CB}, given by $u \mapsto \bar{u}\bar{s},\ v \mapsto v,\ z \mapsto z,\ s \mapsto \bar{s}$. Then

$$b^*\partial_m|_{U_{CB}} = \bar{s} \cdot \underbrace{\left[v^3\partial_{\bar{u}} + \bar{u}\partial_v + \bar{s}^{m-1}\partial_z\right]}_{\psi_{CB}}.$$

  Under the $\mathbb{A}^1_t$-isomorphism $U_{CB} \cong \mathbb{A}^3_{x,y,z} \times \mathbb{A}^1_t$ given by $(\bar{u}, \bar{v}, z, \bar{s}) \mapsto (x, y, z, t)$, we see that ψ_{CB} corresponds to ∂_{m-1}, and thus $b^*\mathcal{F}_m|_{U_{CB}}$ corresponds to $\mathcal{F}_{m-1}$.

The above computations show that the non-lc locus of $b^*\mathcal{F}_m$ is contained in U_{CB}. It is elementary (but tedious) to check that this locus does not intersect the other patches U_{CA}, U_{BA}, and U_{BB}. It remains to observe that the strict transforms of $E \subset Y$ and of $(t = 0) \subset \mathbb{A}^3 \times \mathbb{A}^1_t$ are disjoint from the chart U_{CB}. □

Theorem 4.14. *With the notations as above, assume that $p = 3$ and let $Y = (\mathbb{A}^3 \times \mathbb{A}^1)/\mathcal{F}_1$. Then:*

1. *The projection $\mathbb{A}^3 \times \mathbb{A}^1_t \to \mathbb{A}^1_t$ factors through a flat morphism $Y \to \mathbb{A}^1_t$ of relative dimension 3;*
2. *Y_t is smooth for $t \neq 0$, while Y_0 is reduced and non-S_2;*
3. *$Y \to \mathbb{A}^1$ is locally stable, and Y has canonical non-S_3 singularities along Y_0.*

Moreover, if C is a normal curve and $C \to \mathbb{A}^1_t$ is a finite flat morphism, then $Y \times_{\mathbb{A}^1_t} C$ is normal and $Y_C \to C$ is locally stable.

Proof. The first item is clear. For the remaining ones, as in the proof of Corollary 4.5 and Theorem 4.7, we reduce the proof to the following: if $Y' = (\mathbb{A}^3 \times \mathbb{A}^1)/\mathcal{F}_{p^r}$ then $Y' \to \mathbb{A}^1_t$ is locally stable, has smooth fibers above $t \neq 0$ and a non-S_2 reduced central fiber Y'_0. The statements about the singularities of the fibers are proved as in Theorem 4.3 (and are true whenever $p > 2$). It remains to check that $Y' \to \mathbb{A}^1$ is locally stable above $t = 0$.

As usual by now, we blow-up $W = \mathbb{A}^3 \times \mathbb{A}^1_t$. By Lemma 4.13, there is a sequence of smooth blow-ups

$$b\colon X = X_{2p^r} \longrightarrow W = \mathbb{A}^3 \times \mathbb{A}^1_t$$

with exceptional divisors $E_i, F_i \subset X$ for $i = 1, \ldots, p^r$ such that: $(X, \sum_i E_i + F_i)$ is log smooth, $b^*\mathcal{F}_{p^r}$ has only lc singularities, the E_i's and the F_i's are $b^*\mathcal{F}_{p^r}$-invariant, and

$$a(E_i; \mathcal{F}_{p^r}) = -i + 1, \quad a(F_i; \mathcal{F}_{p^r}) = -i \quad \forall i = 1, \ldots, p^r.$$

Using Lemma 4.13 and proceeding by induction, one computes

$$a(E_i; W, W_0) = 3i - 2, \quad a(F_i; W, W_0) = 3i - 1.$$

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Now let $Z = X/b^*\mathcal{F}_{p^r}$ and denote by $E_i' \subset Z$ (resp. by $F_i' \subset Z$) the image of E_i (resp. of F_i). Then by Lemma 4.13 and Theorem 2.10 we have

$$a(E_i'; Y', Y_0') = a(E_i; W, W_0) + (p-1) \cdot a(E_i; \mathcal{F}_{p^r}) = i(4-p) + p - 3$$

and

$$a(F_i'; Y', Y_0') = a(F_i; W, W_0) + (p-1) \cdot a(F_i; \mathcal{F}_{p^r}) = i(4-p) - 1.$$

As $b^*\mathcal{F}_{p^r}$ is lc, by Theorem 2.10 it suffices to ensure that

$$\operatorname{discrep}\left(Z, -\sum_i a(E_i'; Y', Y_0')E_i' - \sum_i a(F_i'; Y', Y_0')F_i'\right) \geq -1$$

to obtain that (Y', Y_0') is lc. This holds as soon as $i(4-p) \geq 0$ for every $i = 1, \ldots, p^r$, which happens if (and only if) $p = 3$, so the proof is complete. □

Remark 4.15. One can try to compactify the examples of Theorem 4.14 using the method of Theorem 4.7. This is quite delicate, as the 1-foliation $\psi = y^3\partial_x + x\partial_y$ acquires complicated singularities when we compactify. For example, if we regard $\mathbb{A}^2_{x,y}$ as a standard chart of $\mathbb{P}^2$, then on the chart with coordinates $u = 1/x$, $v = y/x$ the 1-foliation induced by ψ is generated by $uv^3\partial_u + (v^4 - u^2)\partial_v$. Similarly, if we regard $\mathbb{A}^2_{x,y}$ has a standard chart of the Hirzebruch surface F_n, then on the chart with coordinates $u = 1/y$, $v = 1/(xy^n)$ the 1-foliation induced by ψ is generated by $u^{2n+5}\partial_u + (v^3 + nu^{2n+4}v)\partial_v$.

A. Appendix: KSBA moduli stacks in positive characteristics

In this appendix we explore the consequences of Theorem 1.3 for KSBA moduli theory. The point is to show that the most straightforward adaptations of the characteristic 0 definitions do not yield satisfactory theories.

As we will see, the point is the unavoidable appearance of non-S_2 fibers as limits of stable families over punctured curves. To formulate this in a precise way we need to define which moduli stacks we are working with (or, equivalently, which families we allow in our theory), and we face two difficulties:

1. Our examples involve boundary divisors with small coefficients, and defining stable families of pairs with such coefficients over general bases is already extremely delicate in characteristic 0;
2. The algebraicity of the stacks we could write down is an open question ([6]), except in the surface case where it should follow from [11].

We go around the first difficulty by using a minimalistic, underdetermined, working definition of the KSBA moduli stacks. We will ignore the second difficulty by imposing algebraicity in our definition: it is natural to expect from the point of view of moduli theory (since we expect boundedness to hold), and on a technical level it allows us to state generic conditions for the objects parametrized by the stacks. In any case, (non-)algebraicity is irrelevant for the phenomenon that we will exhibit.

We work over an algebraically closed field k of characteristic $p > 0$. First, we define one-parameter stable pairs and stable families of pairs, generalizing slightly Definition 2.1.

Definition A.1. A proper pair (X, Δ) over k is *stable* if it is semi-log canonical (slc) ([7]) and if $K_X + \Delta$ is ample.

[6]On the other hand, any reasonable KSBA moduli stack will have a finite diagonal by [25], first paragraph of the proof of Theorem 9.7, and will be separated by Lemma A.3 below.

[7]We refer to [16] and to [28] for the definition of semi-log canonical singularities.

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Definition A.2. Let T be regular k-curve. Let $f\colon X \to T$ be a flat proper pure-dimensional morphism with geometrically reduced fibers. Let Δ be a $\mathbb{Q}$-divisor on X such that $f\colon (X,\Delta) \to T$ is a family of pairs [18, Definition 2.2]. We say that $f\colon (X,\Delta) \to T$ is a *stable family* if:

1. $(X, X_t + \Delta)$ is slc for every closed point $t \in T$, and
2. $K_X + \Delta$ is f-ample.

Lemma A.3 (cf. [25, Lemma 9.4]). *Let T be an affine regular k-curve. Let $(X_i,\Delta_i) \to T (i = 1,2)$ be two stable families over T whose total spaces X_i are normal. Assume that for a fixed closed point $0 \in T$, with complement $T^* = T \setminus \{0\}$, there is a T^*-isomorphism of pairs*

$$\phi^*\colon (X_1,\Delta_1) \times_T T^* \cong (X_2,\Delta_2) \times_T T^*.$$

Then ϕ^* extends to a T-isomorphism $\phi\colon (X_1,\Delta_1) \cong (X_2,\Delta_2)$.

Proof. If Z is the normalization of the closure of the graph of ϕ^*, with projections $\alpha_i\colon Z \to X_i$, let us write for each i

$$\alpha_i^*(K_{X_i} + \Delta_i) = K_Z + \underbrace{\Gamma_{i,-}}_{\leq 0} + \underbrace{\Gamma_{i,+}}_{\geq 0}$$

where $\Gamma_{i,-}$ and $\Gamma_{i,+}$ have no components in common. We have

$$X_i = \operatorname{Proj}_T \bigoplus_{m\geq 0} H^0(X_i, \lfloor m(K_{X_i} + \Delta_i)\rfloor) = \operatorname{Proj}_T \bigoplus_{m\geq 0} H^0(Z, \lfloor m(K_Z + \Gamma_{i,+})\rfloor).$$

Now since $(X_i, \Delta_i + X_{i,0})$ is lc and $X_{i,0}$ is Cartier, if E is an exceptional divisor over X_i with $a(E; X_i, \Delta_i) < 0$ then E dominates T. In particular, every component of $\Gamma_{i,+}$ dominates T. By assumption on ϕ^* it follows that $\Gamma_{1,+} = \Gamma_{2,+}$ and therefore the above Proj description of the X_i yields the extension $\phi\colon (X_1,\Delta_1) \cong (X_2,\Delta_2)$. □

Now we want to give our working definition of KSBA moduli stacks. We will only specify their values on points and regular curves, using the two definitions above. But at least for psychological comfort, it is better to specify what kind of values our stacks take in general. We follow the first step of the approach of [18] (see in particular Chapter 7 there): the objects of interest lie amongst *relative effective Mumford divisors*.

We refer to [18, 4.29, 4.78] for the definition of a relative effective Mumford divisor D supported on a morphism $X \to T$. To simplify the terminology, from now on a relative effective Mumford divisor will refer to the 2-tuple $(X \to T, D)$. As explained in [18, 4.29], given any $h\colon T' \to T$ we can form the pullback $h_X^{[*]}D$ which is again a relative effective Mumford divisor supported on $X_{T'} \to T'$, and this pullback operation is functorial. Thus we can make the following definition:

Definition A.4 (Fibered category of relative Mumford divisors). Fix a (possibly empty) vector of positive rational numbers $\mathbf{c} = (c_1,\dots,c_r) \in (\mathbb{Q}_{>0})^{\oplus r}$. We define the category $\mathcal{M}um_{\mathbf{c}}$ as follows:

- its objects are 2-tuples $\mathcal{D} = (f\colon X \to T, \sum_{i=1}^r c_i D_i)$ where f is a morphism of k-schemes, and each $(f\colon X \to T, D_i)$ is a relative effective Mumford divisor supported on f;
- the class of arrows is generated by the following ones:
  1. if $\mathcal{D} = (X \to T, \sum_{i=1}^r c_i D_i)$ and $\mathcal{D}' = (X' \to T, \sum_{i=1}^r c_i D_i')$ are two relative Mumford divisors over a common base T, then every T-isomorphism $\phi\colon X \cong X'$ sending D_i to D_i' for each i defines an arrow $\phi\colon \mathcal{D} \cong \mathcal{D}'$;
  2. if $\mathcal{D}$ is a relative Mumford divisor over T and $h\colon T' \to T$ is a k-morphism, then there is a pullback morphism $h^{[*]}\mathcal{D} \to \mathcal{D}$.

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We have a forgetful functor $\mathcal{M}um_{\mathbf{c}} \to \mathrm{Sch}_k$ sending $(X \to T, \sum_{i=1}^r c_i D_i)$ to T, the isomorphism ϕ to id_T, and $h^{[*]}\mathcal{D} \to \mathcal{D}$ to h. By functoriality of pullbacks, this makes $\mathcal{M}um_{\mathbf{c}}$ into a fibered category over Sch_k.

With these definitions in place, we can at last introduce our working definition of KSBA moduli stacks.

Terminology A.5. A *stack over k* means a stack in groupoids over the big étale site of k in the sense of [1, 02ZH]. *Algebraic stacks* are understood in the sense of [1, 026N].

Definition A.6 (Potential KSBA moduli stacks). Let $\mathcal{M}$ be an algebraic stack over k. We say that $\mathcal{M}$ is a *potential KSBA moduli stack* if there exist $n \in \mathbb{N}$, $v \in \mathbb{Q}_{>0}$, and $\mathbf{c} = (c_1, \dots, c_r) \in (\mathbb{Q}_{>0})^{\oplus r}$ with the following properties:

1. There exists a fully faithful functor $\mathcal{M} \hookrightarrow \mathcal{M}um_{\mathbf{c}}$ of fibered categories over Sch_k;
2. if K is an algebraically closed field extension of k, then $\mathcal{M}(K)$ is the groupoid of all stable pairs $(X \to \mathrm{Spec}(K), \Delta = \sum_{i=1}^r c_i \Delta_i) \in \mathcal{M}um_{\mathbf{c}}(K)$ with $\dim X = n$ and $(K_X + \Delta)^n = v$;
3. if T is a regular (germ of) k-curve then $\mathcal{M}(T)$ is the set of $(X \to T, \Delta = \sum_{i=1}^r c_i \Delta_i) \in \mathcal{M}um_{\mathbf{c}}(T)$ such that $(X, \Delta) \to T$ is a stable family.

We refer to the array $(n, v, \mathbf{c})$ as the *numerical constants* of $\mathcal{M}$.

Remark A.7. Let us stress that if $\mathcal{M}$ is a potential KSBA moduli stack and T a regular k-curve, then the families in $\mathcal{M}(T)$ are stable families of pairs *whose underlying fibers are demi-normal (in particular reduced and S_2)*. This is forced by the way we compute pullbacks in $\mathcal{M}um_{\mathbf{c}}$ and by the prescription of the values of $\mathcal{M}$ on points.

We introduce two variants of the above definition.

Variant A.8 (Potential KSBA-CM moduli stacks). Let $\mathcal{M}$ be a potential KSBA moduli stack over k. We say that $\mathcal{M}$ is a *potential KSBA-CM moduli stack* if there exists a dense open algebraic sub-stack $\mathcal{M}^{\mathrm{CM}} \subset \mathcal{M}$ such that, whenever $(X, \Delta) \in \mathcal{M}^{\mathrm{CM}}(K)$ for an algebraically closed field K, the variety X is Cohen–Macaulay.

Remark A.9. Because of [10, 12.2.1], we think of a potential KSBA-CM moduli stack as an open sub-stack of a potential KSBA moduli stack. In characteristic 0 we have the following striking picture: if $\mathfrak{M} = \mathfrak{M}^{\mathrm{KSBA}}_{\mathbb{C},n,v,\mathbf{c}}$ is the KSBA moduli stack over $\mathbb{C}$ with numerical constants $(n, v, \mathbf{c})$, defined as in [18, §8.2], then we have a decomposition into *open and closed* sub-stacks

$$\mathfrak{M} = \bigsqcup_{i=2}^{n} \mathfrak{M}(i)$$

where $\mathfrak{M}(i)$ parametrizes pairs whose underlying varieties are S_i but not S_{i+1}. This follows from [20] (see also [19, Corollary 1.3]). In particular $\mathfrak{M}^{\mathrm{CM}} = \mathfrak{M}(n)$ is a connected component of $\mathfrak{M}$.

Variant A.10 (Potential KSBA-F-injective moduli stacks). Let $\mathcal{M}$ be a potential KSBA moduli stack over k. We say that $\mathcal{M}$ is a *potential KSBA-F-injective moduli stack* if there exists a dense open algebraic sub-stack $\mathcal{M}^{F\text{-inj}} \subset \mathcal{M}$ such that, whenever $(X, \Delta) \in \mathcal{M}^{F\text{-inj}}(K)$ for an algebraically closed field K, the variety X is F-injective.

F-injectivity is often described as a positive characteristic analogue of the du Bois condition in characteristic 0: see, for example, [31] and [23].

Remark A.11 (Generic singularities in [17]). We briefly comment on the singularities of a general fiber Y_1^c of the examples of [17], using the notations of that paper. As noted in [17, Lemma 17], Y_1^c is not CM. It is easily seen from the construction that Y_1^c is not klt: a resolution is given by $\tau_Y : Y_1 \to Y_1^c$, and the exceptional locus is the divisor Z_0 which is a log canonical place. We check whether Y_1^c is F-injective

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along $\tau_Y(Z_0)$. Recall that $Z_0 \to \tau_Y(Z_0)$ is the elliptic fibration $\tau\colon S_1 \to \mathbb{P}^1$ [17, Proof of Lemma 17]. Arguing as in [3, Lemma 3.10, Proposition 3.12], we see that for any $y \in \tau_Y(Z_0)$ we have

$$H^2_y(Y_1^c, \mathcal{O}_{Y_1^c}) = H^0_y(Y_1^c, R^1\tau_{Y,*}\mathcal{O}_{Y_1}) = H^0_y(\mathbb{P}^1, R^1\tau_*\mathcal{O}_{S_1}).$$

The fibration τ has one multiple fiber pD_1, and the other fibers are smooth elliptic curves [17, §9]. The torsion of $R^1\tau_*\mathcal{O}_{S_1}$ is concentrated at the image $\mathfrak{y}$ of pD_1, and we have $\dim H^0_{\mathfrak{y}}(\mathbb{P}^1, R^1\tau_*\mathcal{O}_{S_1}) = 1$ [17, Proof of Lemma 17]. By cohomology and base-change, this torsion module embeds into $H^1(pD_1, \mathcal{O}_{pD_1})$, and to check F-injectivity it remains to understand the action of Frobenius on this cohomology group. By [17, Proof of Lemma 17] we have $H^1(pD_1, \mathcal{O}_{pD_1}) = H^1(E, F_{p-1} \oplus \mathcal{O}_E)$ where E is an elliptic curve and F_{p-1} is the unique unipotent indecomposable bundle of rank $p-1$ on E. By induction on the rank of such bundles, we see that the action of Frobenius on $H^1(E, F_{p-1} \oplus \mathcal{O}_E)$ is bijective if and only if E is ordinary. In particular, we may arrange Y_1^c to be F-injective.

With all these preparations, we can finally formulate:

Theorem A.12. *Let $n \geq \max\{p, 3\}$. Then there is a dense subset I of $(0; +\infty)$ such that for every $v \in I$ there exists $\boldsymbol{c} = \boldsymbol{c}(v) \neq \emptyset$ with the following property: there is no proper potential KSBA, nor KSBA-CM/F-injective, moduli stack over k with numerical constants $(n, v, \boldsymbol{c})$.*

Proof. Let $T = \mathbb{A}^1_t$ and consider some family $f\colon (\mathcal{Y}, \mathcal{B}) \to T$ constructed in Theorem 4.7 of relative dimension n. There are many choices for $\mathcal{B}$, and we exploit this later. For now let $T^* = T \setminus \{0\}$ and $f^*\colon (\mathcal{Y}^*, \mathcal{B}^*) \to T^*$ denote the family restricted to the punctured curve. A geometric fiber of f^* has only μ_p-quotient singularities: in particular it is Cohen–Macaulay and F-injective [27, Theorem 1]. So if $\mathcal{M}$ is a potential KSBA (or KSBA-CM/F-injective) moduli stack whose numerical constants match those of $(\mathcal{Y}^*, \mathcal{B}^*) \to T^*$, we must have $[(\mathcal{Y}^*, \mathcal{B}^*) \to T^*] \in \mathcal{M}(T^*)$. Restricting over the generic point of T^*, we obtain $[f]\colon \operatorname{Spec}(k(t)) \to \mathcal{M}$. I claim that there is no finite extension of DVRs $k[t]_{(t)} \hookrightarrow R$ such that we have a commutative diagram over k

$$\begin{array}{ccccc} \operatorname{Spec}(\operatorname{Frac}(R)) & \xrightarrow{\iota} & \operatorname{Spec}(k(t)) & \xrightarrow{[f]} & \mathcal{M} \\ \downarrow & & & \nearrow_{[g]} & \\ \operatorname{Spec}(R) & & & & \end{array}$$

where ι is induced by the DVR extension. This implies nonproperness of $\mathcal{M}$ by [1, 0CLZ].

We proceed by contradiction. Assume that there exists such an extension of DVRs and morphism $[g]\colon \operatorname{Spec}(R) \to \mathcal{M}$. This implies that there exists a stable family $g\colon (\mathcal{Y}', \mathcal{B}') \to \operatorname{Spec}(R)$, belonging to $\mathcal{M}(R)$, whose generic fiber is the base-change of the generic fiber of $(\mathcal{Y}, \mathcal{B}) \to T$ along ι. The scheme $\mathcal{Y}'$ is also normal, since its generic fiber is normal and its special fiber is reduced. Moreover, by Theorem 4.7 the base-change $\mathcal{Y}_R$ is normal. Therefore by Lemma A.3 we actually have an R-isomorphism $(\mathcal{Y}_R, \mathcal{B}_R) \cong (\mathcal{Y}', \mathcal{B}')$. So $(\mathcal{Y}_R, \mathcal{B}_R) \in \mathcal{M}(R)$: but the central fiber of $\mathcal{Y}_R \to \operatorname{Spec}(R)$ is not S_2 by Theorem 4.7, and we obtain a contradiction with the definition of the elements of $\mathcal{M}(R)$ (see Remark A.7).

It remains to prove that as we change the boundary $\mathcal{B}$, the set of volumes $(K_{\mathcal{Y}_t} + \mathcal{B}_t)^n$, for t an arbitrary point in T^*, is dense in the interval $(0; +\infty)$. Recall that the support of $\mathcal{B}_t$ is the image through a degree p quotient map

$$\mathbb{P}^{n-1} \times E \longrightarrow \mathcal{Y}_t$$

of a divisor H_0. By Remark 4.11 it suffices to consider the case where H_0 belongs to the $\mathbb{Q}$-linear systems

$$\frac{1}{s}\left|(\mathcal{O}_{\mathbb{P}^{n-1}}(n+p) \boxtimes \mathcal{O}_E(e))^{\otimes r}\right|$$

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where $r \geq 3$ and $s \geq 1$ are integers. From (4.3) and Lemma 4.8 it follows that

$$\begin{aligned}(K_{\mathcal{Y}_t} + \mathcal{B}_t)^n &= p \cdot (K_{\mathbb{P}^{n-1}\times E} + (p-1)K_{\mathcal{G}_t} + H_0)^n \\ &= p \cdot \left[\mathcal{O}_{\mathbb{P}^{n-1}}\left(p - n - 1 + \frac{r}{s}(n+p)\right) \boxtimes \mathcal{O}_E\left(\frac{r}{s}e\right)\right]^n \\ &= np \cdot \frac{r}{s} \cdot \left(p - n - 1 + \frac{r}{s}(n+p)\right)^n.\end{aligned}$$

We consider this expression as a function v in $r/s \in \mathbb{R}_{>0}$. Then v is continuous, increases to infinity with r/s, and is bounded below by

$$\lim_{\frac{r}{s}\to 0^+} np \cdot \frac{r}{s} \cdot \left(p - n - 1 + \frac{r}{s}(n+p)\right)^n = 0.$$

By continuity the image $I = v(\mathbb{Q}_{>0})$ is dense in $(0; +\infty)$. By construction, for every $v = v(r/s) \in I$ there is a family $(\mathcal{Y}^*, \mathcal{B}^*) \to T^*$ constructed by Theorem 4.7 which belongs to any potential KSBA or KSBA-CM/F-injective moduli stacks with numerical constants $(n, v, \mathbf{c} \neq \emptyset)$ ([8]). This completes the proof. □

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[8] Here $\mathbf{c}$ depends on the choice of H_0: since we can take it to have coefficients $1/s$, one possible value of $\mathbf{c}$ is the 1-vector $(1/sp)$.

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