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Fitting hyperbolic pants to a three-body problem

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Abstract. Consider the three-body problem with an attractive $1/r^2$ potential. Modulo symmetries, the dynamics of the bounded zero-angular-momentum solutions is equivalent to a geodesic flow on the thrice-punctured sphere, or 'pair of pants'. The sphere is the shape sphere. The punctures are the binary collisions. The metric generating the geodesics is the Jacobi–Maupertuis metric. It is complete, has infinite area, and its ends, the neighborhoods of the punctures, are asymptotically cylindrical. Our main result is that when the three masses are equal then the metric has negative curvature everywhere except at two points (the Lagrange points). A corollary of this negativity is the uniqueness of the $1/r^2$ figure-eight, a complete symbolic dynamics for encoding the collision-free solutions, and the fact that collision solutions are dense within the bound solutions.

1. *Introduction and results*

We study the planar three-body problem with an attractive $1/r^2$ potential. According to the Lagrange–Jacobi identity (equation (3.7) below) every bounded solution must have zero energy and constant moment of inertia I, and conversely, if an initial condition has zero energy and $\dot{I}(0) = 0$ then that solution is bounded. Setting the moment of inertia I equal to a constant defines a 3-sphere in configuration space. Rotations act on this sphere according to the Hopf flow so that the quotient of the 3-sphere by rotations is the 2-sphere or *shape sphere*. See Figure 1. Points of this shape sphere represent oriented similarity classes of triangles. Newton's equations, for solutions with energy zero and with $\dot{I} = 0$, push down to the shape sphere to yield a family of second-order ordinary differential equations (ODEs) parameterized by the angular momentum. These ODEs have singularities at the three points representing the three types of binary collisions. Upon deleting the collision points we arrive at dynamics on the *pair of pants*—the 2-sphere minus three points. When the angular momentum is zero the resulting dynamical system is, after a time reparameterization, the geodesic flow for a certain Riemannian metric on the pair of pants. This metric is the (reduced) Jacobi–Maupertuis metric for energy 0.

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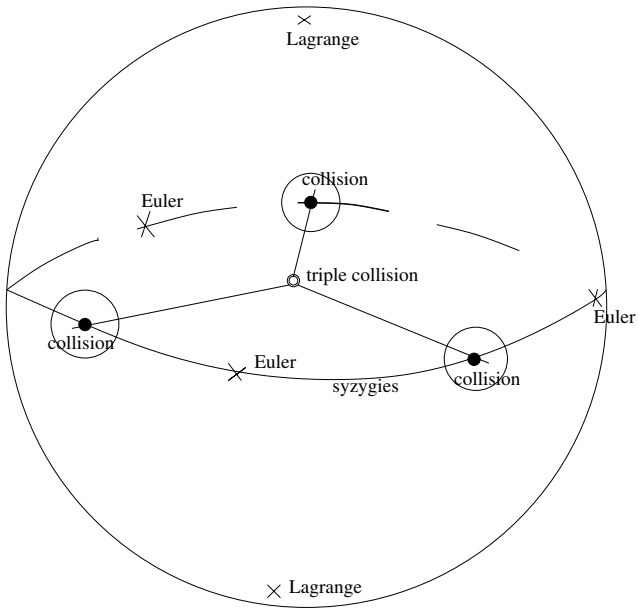


FIGURE 1. The shape sphere.

PROPOSITION 1.1. *Endow the pair of pants with the Jacobi–Maupertuis metric (equations (3.9a, b) below). Modulo rotations, translation and scaling, the set of bounded zero-angular-momentum solutions for the* $1/r^2$ *potential three-body problem are in bijective correspondence with geodesics for this metric. The metric is complete and its ends, the deleted neighborhoods of the three binary collisions, are asymptotic to Euclidean cylinders of positive radii.*

For the proof, see §3.

See Figure 2 for a depiction of the pair of pants.

The Jacobi–Maupertuis metric depends parametrically on the masses of the three bodies by way of the potential (equation (3.1)). Our main result is the following.

THEOREM 1.1. *If all three masses are equal then the Gaussian curvature for the Jacobi–Maupertuis metric on the pair of pants is negative everywhere except at the two Lagrange points (the equilateral triangles), where it is zero.*

For the proof see §4.

One might hope that negativity of the curvature persists for unequal masses. It does not. See §7.

2. *Motivation and dynamical consequences*

2.1. *Periodic orbits and their symbol sequences.* This work began as an attempt to give an analytic proof that the Newtonian ($1/r$ potential) figure-eight solution of Moore, Chenciner and Montgomery [**Moore, ChMont**] is unique. I began with the easier case

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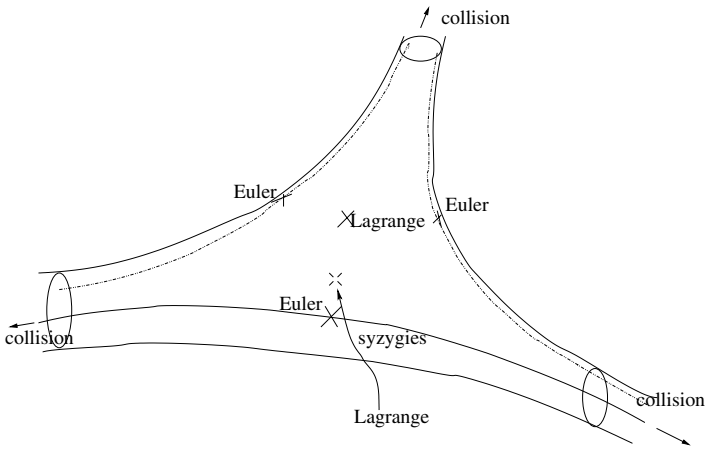


FIGURE 2. The pair of pants.

of the $1/r^2$ eight. The figure-eight is a periodic solution which realizes a certain free homotopy class on the pair of pants. Figure-eights exist for all $1/r^a$ potentials, $a > 0$ [**CGMS, FerrTerr**]. For $a \geq 2$, not only is the free homotopy class of the eight realized, but almost every free homotopy class is realized by a solution. On the other hand, on a compact surface of negative curvature, every free homotopy class has a unique geodesic representing it. Combining these facts suggested the approach of this paper, and Theorem 1.1.

Our pair of pants metric from Theorem 1.1 is neither compact, nor of negative curvature everywhere. But on a complete, non-compact surface of negative curvature if a free homotopy class has a geodesic representative, then that representative is unique. And uniqueness continues to hold if the curvature vanishes on a discrete set of points. (This theorem is fairly well known, and proved in a more general context in §6.4 below, and in particular equation (6.4.3).) We have proved the following.

COROLLARY 2.1. *For the $1/r^2$ equal-mass zero-angular-momentum three-body problem, if a solution realizes a given free homotopy class on the pair of pants, then that solution is unique modulo rotation and scaling. In particular the eight is unique modulo these symmetries.*

In stating the corollary we begged the question of which classes are realized. Every class is realized with the exception of those classes which wind around a single end (see [**MontN**]). [**Gordon**] calls these 'bad' or unrealizable classes 'untied' while the complementary 'good', or realizable classes he calls 'tied', in that they are 'tied' to the collision singularities. On the pants, a bad class can be represented by drawing a small circle, or 'anklet' around one pants leg, and traversing it some number of times. As this 'anklet' is pushed down towards the end of the leg its length decreases. As a result, any minimizing sequence of curves realizing such a class 'falls off' of the leg. (See Theorem 2.2 below.)

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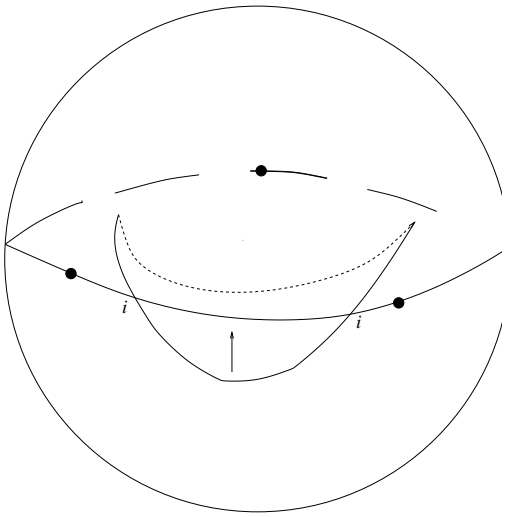


FIGURE 3. Homotoping away a stutter.

We follow [**MontN**] in using syzygies to describe the tied and untied classes. A *syzygy* is a collinear configuration of the three bodies. Syzygies come in three flavors, 1, 2 and 3, depending on which mass is between the other two. (We exclude collisions.) The collinear configurations form the equator of the shape sphere (Figure 1). The three collisions lie on the equator so that deleting them divides the equator into three arcs, labeled 1, 2, 3 according to the flavor of syzygy. A curve on the shape sphere has an associated syzygy sequence: list the syzygies in order. (Assume that the syzygy times are discrete.) The syzygy sequence of a motion of the three bodies is obtained by projecting the motion onto the shape sphere and writing out the syzygy sequence of the curve resulting on the shape sphere.

Periodic curves give rise to periodic sequences. For example, the class in which 1 and 2 circle about each other forever while 3 remains far away has syzygy sequence $\cdots 121212 \cdots$. (This is a 'bad' class.) We subject syzygy sequences to the *no stuttering rule*: if ij are consecutive letters of the sequence, then $i \neq j$. The reason for imposing this rule is that a stutter can be homotoped away. See Figure 3.

A letter j with a plus superscript, as in j^+, denotes that syzygy j occurs by crossing from the upper to the lower hemisphere of the sphere. A j^- means that syzygy j occurs by crossing from the lower to the upper hemisphere. Pluses and minuses must alternate, since the path will alternate between hemispheres. In topological terms, an arc segment with two consecutive pluses, such as $k^- i^+ j^+$ lies entirely in the upper hemisphere and can be homotoped to $k^- j^+$. In dynamical terms, such an arc can never occur since at i^+ it would have to be tangent to the collinear subspace (the equator), but if a solution is tangent to the collinear subspace at a point then it lies completely within the collinear subspace.

It follows from the above considerations, and the topology of the pair of pants, that there is a one-to-one onto correspondence between free homotopy classes and periodic signed

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non-stuttering syzygy sequences. From now on, we will drop the signing indications—the $+, -$ superscripts—for simplicity. For, given an unsigned sequence there are only two ways to decorate it with signs. The untied classes are precisely those with syzygy sequence $\cdots 1212 \cdots$, $2323 \cdots$ or $\cdots 3131 \cdots$. They correspond to curves winding around a single end. We prove in Theorem 2.2 that no bounded zero angular momentum solution realizes these untied classes. Excluding these classes is equivalent to insisting that all three letters occur in the sequence. Thus, Corollary 2.1 asserts that *every periodic non-stuttering syzygy sequence in which all three letters occur is realized by a unique (up to symmetry) relative periodic solution.*

2.2. *Symbolic dynamics and aperiodic syzygy sequences.* We move on to infinite aperiodic syzygy sequences. In the rest of this subsection, 'the problem' means the $1/r^2$ equal-mass zero-angular-momentum three-body problem, and 'solution' means a solution to the problem, i.e. this differential equation. Many of the ideas and results here are adaptations of those pioneered in [**Morse**].

THEOREM 2.1. *Every infinite non-stuttering syzygy sequence with the exception of the untied classes $\cdots iji j \cdots$ is realized by a solution.*

For proof see §6.2. The method is the classical one [**Morse**] of approximation by periodic solutions.

THEOREM 2.2. *If a syzygy sequence ends (begins) with $iji j \cdots$ then any bounded solution which realizes this sequence must end (begin) in the ij collision. The bi-infinite sequence $\cdots iji j \cdots$ is not realized by any solution.*

For proof see §6.3.

It is worth remarking that if a solution suffers collision then it does so in finite Newtonian time, but infinite 'Jacobi time'.

We call the sequences appearing in Theorem 2.2, 'collision sequences'. In more detail we have the following.

Definition. A bi-infinite non-stuttering syzygy sequence $s = \{s_j\}_{j=-\infty}^{+\infty}$ is a forward *collision sequence* if one of its forward tails $\{s_j\}_{j>N}$ contains only two letters. Similarly, we have backward collision sequences. A collision sequence is a sequence which is either a forward or a backward collision sequence. In the contrary case, all three letters must occur in every tail, and the sequence is then called *collision-free*.

Does every solution have a syzygy sequence? If so, is this sequence unique? In [**MontI**] I showed that every bounded non-collinear zero-angular-momentum solution to the Newtonian three-body problem suffers infinitely many syzygies, provided the solution does not tend to triple collision, *and provided* binary collisions are counted as syzygies. (See [**Fuji**] for another proof.) That proof works verbatim for any $1/r^a$ potential, $a > 0$, with the exception that we must exclude binary collisions. (They cannot be regularized.) Thus, every bounded *collision-free* solution has a syzygy sequence. That sequence must be non-stuttering in our equal mass case. To prove that there is no stuttering, use the fact that on a surface of negative curvature any compact geodesic

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arc is the unique length-minimizing curve among all homotopic curves which share its endpoints. (See equation (6.3) and its derivation.) Consequently, an application of the method of reflection as exposed in [**ChMont**] rids us of solution arcs representing stutters, i.e. solution arcs which hit the same equatorial arc twice in a row. These considerations allow us to define a syzygy map from collision-free sequences to infinite non-stuttering syzygy sequences.

THEOREM 2.3. *The syzygy map from bounded solutions to syzygy sequences is a bijection between the set of collision-free solutions, modulo symmetry and time-translation, and the set of bi-infinite non-stuttering collision-free syzygy sequences, modulo shift.*

For proof see §6.4.

Finally, we would like to know how much of phase space (the unit tangent bundle of the pair of pants) is taken up by the non-collision solutions. Not much—see the following.

THEOREM 2.4. *Solutions tending to binary collision are dense within the space of all bounded solutions. Thus, the collision-free solutions have empty interior.*

For proof see §6.5.

Summary. Putting the theorems together gives a rather complete symbolic dynamical picture of our problem, the bounded, zero-angular momentum, equal-mass, $1/r^2$ three-body problem. To begin with, there are no linearly stable periodic orbits, by Theorem 1.1. Theorem 2.3 provides a complete symbolic dynamics picture for the collision-free orbits: they are precisely those orbits whose syzygy sequences are collision-free. The closure of this collision-free set of orbits is the recurrent set. The recurrent set also equals the closure of the set of periodic orbits. There are collision orbits on the boundary of the recurrent set. The situation is similar to the Kepler problem, if we replace 'collision' by 'unbounded'. In the Kepler problem the closure of the space of periodic orbits is the recurrent set and this set contains parabolic orbits on its boundary. These unbounded recurrent orbits just 'barely make it' to infinity, just as in our problem, these boundary collision orbits are 'just barely collision' in that the collision condition $J_1^2 - (m_1 + m_2) \leq 0$ occurring in Appendix A, inequality (A.12), becomes an equality. The complement of our recurrent set consists of orbits tending to 'strongly colliding' orbits. These strongly colliding orbits form an open set. (Appendix A). Finally, the density result, Theorem 2.4, is an analogue of what one *would like to prove* for the honest $1/r$ three-body problem: that the set of solutions tending to infinity (via tight binary pairs) is dense, for fixed energy and angular momentum. M. Herman [**Herman**] calls this density question 'the oldest problem in dynamical systems'.

Loose ends. There are collision orbits left out of our symbol sequence considerations. Collinear solutions are not accounted for. Collinear solutions should either have no syzygy sequence or a continuum of 'i's as their sequence, depending on one's taste. There are exactly six collinear solutions, two for each of the three collision arcs, the two being related by reversing orientation. There are also orbits which head 'straight in' to a collision without the bodies winding around infinitely often. They do so by heading straight in

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to infinity down one of the pants legs. The forward collision sequence of such an orbit simply truncates, and so is not bi-infinite. The simplest of these truncated solutions are the isosceles solutions. The isosceles solution $r_{ij} = r_{ik}$ begins and ends at the jk collision, and has exactly one syzygy in between, the Euler point in which i is at the midpoint of j and k. Its syzygy sequence is the single letter 'i'. The same counting as for the collinear solutions yields six isosceles solutions. Interpolating between the collinear and isosceles solutions which end in ij collision is a one-parameter family of ij collision solutions whose syzygy sequence truncates as it heads straight down the ij end. These interpolating solutions are the λ-curves of equation (3.13), the curves of constant χ, in the λ, χ coordinate system there.

Open questions. Are there any solutions besides the isosceles solutions which have finite syzygy sequences? In particular, are the syzygy sequences of some (or all) of the interpolating λ-curve solutions finite? If there are multiple solutions with finite syzygy sequences, can any finite syzygy sequence be realized by some collision solution? Can distinct collision solutions share the same syzygy sequence?

3. *Set-up and the proof of Proposition 1.1*

Write $x = (x_1, x_2, x_3) \in \mathbb{R}^6$ with $x_i \in \mathbb{R}^2$ for the positions of the three bodies, and $r_{ij} = \|x_i - x_j\|$ for the distances between them. The potential is $-U$ where

$$U = \sum m_i m_j / r_{ij}^2. \tag{3.1}$$

The m_i are the masses. Set

$$K = \sum m_i \|\dot{x}_i\|^2 = \langle \dot{x}, \dot{x} \rangle, \tag{3.2}$$

for twice the kinetic energy. The last equality of (3.2) defines the 'mass inner product' on the three-body configuration space $\mathbb{R}^6$. The total energy

$$H = K/2 - U \tag{3.3}$$

is constant along solutions. The equations of motion, $\ddot{x}_i = -2 \sum_{i \neq j} m_j (x_i - x_j)/r_{ij}^4$, $i = 1, 2, 3$, can be written as the single vector equation

$$\ddot{x} = \nabla U(x), \tag{3.4}$$

where ∇U is defined using the mass inner-product: $dU(x)(v) = \langle \nabla U(x), v \rangle$.

By the standard method of freshman physics, we can, without loss of generality, restrict our considerations to motions for which

$$\sum m_i x_i = 0 \tag{3.5}$$

throughout. This constraint defines a four-dimensional real vector space which can be identified with the two-dimensional complex space $\mathbb{C}^2$ in such a way that counterclockwise rotation of a triangle (x_1, x_2, x_3) by θ radians turns into scalar multiplication of the corresponding complex vector by $\exp(i\theta)$. Set

$$I = \sum m_i m_j r_{ij}^2 \Big/ \sum m_i = \langle x, x \rangle, \tag{3.6}$$

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where the last equality is only true when the center of mass constraint (3.5) is in place. Using $\dot{I} = 2\langle x, \dot{x}\rangle$, $\ddot{I} = 2\langle \dot{x}, \dot{x}\rangle + 2\langle x, \ddot{x}\rangle$ and $\langle x, \nabla U(x)\rangle = -2U(x)$ (by U's homogeneity) we obtain the Lagrange–Jacobi identity:

$$\ddot{I} = 4H, \tag{3.7}$$

valid along any solution. Thus, $I(t) =$ constant. along the solution if and only if $H = 0$ and $\dot{I}(0) = 0$ for that solution.

We will call a solution 'bounded' if the r_{ij} are bounded as functions of time, and do not simultaneously tend to zero, i.e. to triple collision. Now $I \to \infty$ if and only if one of the r_{ij} tends to infinity, and $I \to 0$ if all $r_{ij} \to 0$. It follows from (3.7) that every bounded solution must satisfy $H = 0$, $\dot{I}(0) = 0$, and $I(t) =$ constant.

The scaling symmetry $x(t) \mapsto \lambda^{-1/2} x(\lambda t)$ takes solutions to solutions, preserves zero energy, and takes I to I/λ. Using this scaling, we may, without loss of generality, assume that $I = 1$ in studying bounded solutions. The set $I = 1, \sum m_i x_i = 0$ forms a 3-sphere S^3 in $\mathbb{C}^2$. *We have reduced the study of the bounded solutions to the* $1/r^2$ *problem to a second-order dynamics on this 3-sphere.* It is well known ([**Arn**] or [**AbMar**]) that dynamics on a constant-energy surface $H = E$ is equivalent to geodesic flow for the Jacobi–Maupertuis metric $(E + U)\, ds^2$ where ds^2 is the kinetic energy metric. In our case, $E = 0$, and we restrict the kinetic energy to the sphere $I = 1$. Consequently, the study of bounded solutions is equivalent to the study of geodesics on the 3-sphere under the metric $ds_J^2 = U\, ds^2$ conformal to the standard metric ds^2 on that 3-sphere.

To obtain a metric on the shape sphere, we quotient by the group of rotations. Identify this group with the group S^1 of unit-modulus complex scalars. S^1 acts on our 3-sphere $S^3 \subset \mathbb{C}^2$ by scalar multiplication. The quotient defines the famous Hopf fibration

$$S^3 \to S^3/S^1 = S^2, \tag{3.8}$$

where the base space, S^2, is the *shape sphere*, whose points represent oriented similarity classes of triangles. The projection (3.8), together with the choice of an S^1-invariant metric on S^3, induces a metric on the shape sphere. This quotient metric is defined by declaring the projection (3.8) to be a Riemannian submersion, that is, declaring that the orthogonal complements to the fibers of (3.8) are isometric, under the linearization of (3.8), to the tangent spaces to the shape sphere. If the metric we start with is the kinetic energy metric ds^2, then the induced metric will be called the shape space metric, denoted ds^2_{shape}. The total angular momentum of a solution is zero if and only if that solution is orthogonal to the fibers of (3.8). Orthogonality is measured either with respect to the kinetic energy metric or the Jacobi–Maupertuis metric $U\, ds^2$—the orthogonality condition is the same using either metric because they are conformally related. The projection of a zero-angular-momentum geodesic (for either metric) to the shape sphere is a geodesic (in the corresponding inherited metric) on the shape sphere (see [**Hermann**, Lemma 4.1]). It follows that, modulo the symmetries of rotation and scaling, the bounded zero-angular-momentum solutions are in one-to-one correspondence with the solutions to the induced Jacobi–Maupertuis metric on the shape sphere. (See [**MontN, ChMont, MontR**]. and especially the [**MontR**, appendix] for more details and explicit computations.)

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We can write the Jacobi–Maupertuis metric on the shape sphere as

$$ds_J^2 = U\, ds_{\text{shape}}^2, \tag{3.9a}$$

where U is the (negative) potential (3.1) restricted to $I = 1$ and pushed down by (3.8) to the shape-sphere. The shape sphere metric ds_{shape}^2 is the round metric on a sphere of radius $1/2$:

$$ds_{\text{shape}}^2 = (\tfrac{1}{2})^2[d\phi^2 + \cos^2(\phi)\, d\theta^2], \tag{3.9b}$$

where ϕ is the colatitude—the angle from the equator, and θ labels longitudinal circles on the sphere.

Proof of Proposition 1.1. As discussed above, modulo rotations, translation and scaling, the set of bounded zero-angular momentum solutions for the negative potential (3.1) are in bijective correspondence with geodesics for the Jacobi–Maupertuis metric (3.9) on the shape sphere minus the three binary collision. Under this correspondence, the geodesic flow for the metric corresponds, after a time reparameterization, to the flow defined by Newton's equations. It remains to verify the claims about completeness and the ends.

Completeness. Let $\rho = \rho_{ij}$ be the spherical distance from the ij collision point, as measured in the spherical metric ds_{shape}^2, and let χ be the corresponding angular coordinate centered at that point. We will show that

$$U = \frac{C^2}{\rho^2} + O(1) \tag{3.10a}$$

for some positive constant C^2, while

$$ds_{\text{shape}}^2 = d\rho^2 + (\rho^2 + O(\rho^4))\, d\chi^2 \tag{3.10b}$$

as $\rho \to 0$. It follows that the Jacobi metric has the expansion

$$ds_J^2 = \frac{C^2}{\rho^2}(d\rho^2 + \rho^2\, d\chi^2) + O(1). \tag{3.10c}$$

If we now approach the collision $\rho = 0$ along any curve, then its Jacobi length $\int ds_J$ diverges at least as fast as the integral of $C\sqrt{d\rho^2/\rho^2} = C\, d\rho/\rho$, that is, it diverges logarithmically like $C|\log(\rho)|$ as $\rho \to 0$. Consequently, any curve tending towards 'infinity', i.e. to one of the binary collisions, has infinite length, proving completeness.

To establish (3.10a), it suffices to establish

$$r_{ij} = \frac{1}{\sqrt{\mu_{ij}}} \sin(\rho_{ij}), \tag{3.11}$$

where $\mu_{ij} = m_i m_j/(m_i + m_j)$ is the reduced mass. The other two distances r_{ik}, r_{jk} are bounded away from zero as $r_{ij} \to 0$, due to the constraint $I = 1$ (see (3.6)). Then (3.10a) follows from (3.1) and the Taylor expansion of $\sin(\rho)$. The constant C in (3.10a) is $m_i m_j/\sqrt{\mu_{ij}}$.

To establish (3.11) we work in the full three dimensional shape space, which is the space whose points are oriented congruence classes of planar triangles. The full shape space is

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isometric to the cone over the shape sphere and, consequently, distances d in the full shape space can be obtained from spherical distances together with knowledge of the distance $R = \sqrt{I}$ from the cone point. Write d_{ij} for the distance in the full shape space between an arbitrary point and the ij binary collision ray. Then, we have

$$r_{ij} = \frac{1}{\sqrt{\mu_{ij}}}\, d_{ij} \tag{3.12a}$$

and

$$d_{ij} = R \sin(\rho_{ij}). \tag{3.12b}$$

Upon setting $R = 1$, (3.11) follows immediately. Equations (3.12a, b) can be found in §4 [**MontN**, equations (4.3.15a, b)]. (However, note that there is a typo in equation (4.3.15a). The μ_{ij} in that equation must be replaced by $\sqrt{\mu_{ij}}$.)

To get (3.10b), use the fact that the shape sphere is isometric to the sphere of radius $1/2$ and that the metric on such a sphere is given by

$$ds^2_{\text{shape}} = d\rho^2 + [(1/2)\sin(2\rho)]^2\, d\chi^2 \tag{3.13}$$

in spherical-polar coordinates. Then use the Taylor expansion of $(1/2)\sin(2\rho)$.

Asymptotes to cylinders. We use a more precise version of the expansion (3.10c). Set

$$d\lambda = -\sqrt{U}\, d\rho. \tag{3.14}$$

Fixing χ and integrating (3.14) defines a function $\lambda = \lambda(\rho, \chi)$ such that $\lambda \to \infty$ as the collision $\rho = 0$ is approached. From (3.10a) we have $d\lambda = -C\, d\rho/\rho + O(1)$, from which it follows that

$$\rho = e^{-C\lambda} + o(\rho).$$

The (λ, χ) are coordinates for the end $\rho = 0$. In these coordinates,

$$ds^2_J = d\lambda^2 + f(\lambda, \chi)^2\, d\chi^2, \tag{3.15}$$

where, from (3.13) and (3.9a), we have

$$f^2 = (\tfrac{1}{2}\sin(2\rho))^2 U.$$

Now $(1/2\sin(2\rho))^2 = \sin^2\rho\cos^2\rho$, so that from (3.11) $(1/2\sin(2\rho))^2 = \mu_{ij} r_{ij}^2 \cos^2(\rho)$ and

$$f^2 = \mu_{ij}\cos^2(\rho)\left\{ m_i m_j + m_i m_k \frac{r_{ij}^2}{r_{ik}^2} + m_j m_k \frac{r_{ij}^2}{r_{jk}^2} \right\}, \tag{3.16}$$

where ijk is a permutation of 123. As we approach the collision $\lambda = \infty$, we have $r_{ij} \to 0$ while r_{ik}, r_{jk} remain bounded since we are constrained to $I = 1$. Thus,

$$\lim_{\lambda\to\infty} f = (\mu_{ij} m_i m_j)^{1/2} := K_{ij} > 0. \tag{3.17}$$

Summarizing:

$$ds^2_J = d\lambda^2 + (K_{ij}^2 + O(e^{-2C\lambda}))\, d\chi^2, \tag{3.18}$$

which says the metric asymptotes to a Euclidean cylinder of radius K_{ij} as we approach the ij end. □

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Remark. It follows from equations (7.12, 13) that the Gaussian curvature near the end is negative. But, for any metric of the form (3.15) this curvature is equal to $-(1/f)(\partial^2 f/\partial\lambda^2)$. Thus, for fixed χ, the function $f(\lambda, \chi)$ is a strictly convex function of λ, for all λ from some point on, and, from this point on, $f(\lambda, \chi)$ monotonically decreases to K_{ij}.

4. *Curvature. Proof of Theorem 1.1*

We proceed to the proof of our main result, Theorem 1.1, the negativity of the Gaussian curvature when the masses are equal. The computation proceeds through a series of lemmas. The first is standard and we will not provide the proof.

LEMMA 4.1. *Let a surface be endowed with conformally related metrics ds^2 and $d\bar{s}^2 = U\,ds^2$. Then their curvatures K and $\bar{K}$ are related by*

$$\bar{K} = U^{-1}(K - \tfrac{1}{2}\Delta \log(U))$$

where the Laplacian Δ is with respect to the ds^2 metric.

The curvature K of the standard shape metric $ds^2 = ds^2_{\text{shape}}$ of (3.9b) is $K = 4$. According to Lemma 4.1,

$$\bar{K} = U^{-1}(4 - \tfrac{1}{2}\Delta \log(U)) \tag{4.1}$$

is the desired curvature, the curvature of the Jacobi–Maupertuis metric (3.9a) of Proposition 1.1 and Theorem 1.1. A routine computation yields

$$\Delta(\log(U)) = \frac{U\Delta U - \|\nabla U\|^2}{U^2}. \tag{4.2}$$

Here, and throughout this section, U is considered as a function on the shape sphere, ΔU is its Laplacian with respect to the standard shape space metric ds^2 and $\|\nabla U\|^2$ is the squared length of its gradient with respect to the same metric.

A key to the subsequent computations is to use the squared length coordinates as in [**AlbCh**]

$$s_k = r_{ij}^2, \quad ijk \text{ a permutation of } 123, \tag{4.3}$$

rather than the lengths r_{ij} themselves. Write

$$U_{2n} = \sum 1/r_{ij}^{2n} = \sum 1/s_k^n, \tag{4.4}$$

so that $U = U_2$.

LEMMA 4.2.

$$\Delta U = 8U_4. \tag{4.5}$$

For the proof see §5.2.

LEMMA 4.3.

$$\|\nabla U\|^2 = 4S, \tag{4.6a}$$

where

$$S = 2U_6 - U_4 - 3/2\sum{}' 1/s_i^2 s_j^2 + 2\sum{}' 1/s_i s_j^2 - \sum{}' 1/s_i s_j \tag{4.6b}$$

and where '$\sum'$' means to sum over all indices i, j with $i \neq j$. (For example, $\sum' s_i s_j = 2s_1s_2 + 2s_2s_3 + 2s_3s_1$, twice the second symmetric polynomial in the s_i.)

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For the proof see §5.3.

Proof of the negativity of the curvature. Combining the equations (4.1), (4.2), (4.5) and (4.6a), we find

$$-\bar{K}U^3 = 4UU_4 - 4U^2 - 2S. \tag{4.7}$$

Expand out the first two terms on the right-hand side,

$$\begin{aligned} UU_4 &= \sum 1/s_i \sum 1/s_j^2 \\ &= \sum 1/s_i^3 + \sum{}' 1/s_i s_j^2 \\ &= U_6 + \sum{}' 1/s_i s_j^2, \end{aligned} \tag{4.8}$$

while

$$\begin{aligned} U^2 &= \sum 1/s_i \sum 1/s_j \\ &= \sum 1/s_i^2 + \sum{}' 1/s_i s_j \\ &= U_4 + \sum{}' 1/s_i s_j. \end{aligned} \tag{4.9}$$

Plugging (4.8), (4.9) and equation (4.6b) back into equation (4.7) yields

$$-\bar{K}U^3 = -2U_4 - 2\sum{}' 1/s_i s_j + 3\sum{}' 1/s_i^2 s_j^2. \tag{4.10a}$$

Use the fact that $U_4 + \sum' 1/s_i s_j = \left(\sum 1/s_i\right)^2 = U^2$ to rewrite (4.10a) as:

$$-\bar{K}U^3 = 3\sum{}' 1/s_i^2 s_j^2 - 2U^2. \tag{4.10b}$$

Consequently, $\bar{K} \leq 0$ if and only if

$$3\Big(\sum{}' 1/s_i^2 s_j^2\Big) \geq 2\Big(\sum 1/s_i\Big)^2. \tag{4.11}$$

To prove (4.11), multiply both sides of it by $s_1^2 s_2^3 s_3^2$, thus arriving at $6\big(\sum s_i^2\big) \geq 2\sigma_2^2$ or

$$3\Big(\sum s_i^2\Big) \geq \sigma_2^2, \tag{4.12}$$

where $\sigma_2 = s_1 s_2 + s_2 s_3 + s_3 s_1$ is the second elementary symmetric polynomial in the s_i. (The coefficient 6 arose because for each pair ij there are two terms in the sum $\sum'$. See the parenthetical remark in Lemma 4.3.) To prove (4.12), remember that we are restricting ourselves to the sphere $I = 1$ and that $I = \sum s_i/3$. Thus, we can homogenize the equation by using that $3 = \left(\sum s_i\right)^2/3$ on the sphere. So, the desired inequality now reads

$$\frac{\left(\sum s_i\right)^2}{3}\Big(\sum s_i^2\Big) \geq \sigma_2^2.$$

The two inequalities

$$\sum s_i^2 \geq \sigma_2 \tag{4.13a}$$

and

$$\frac{\left(\sum s_i\right)^2}{3} \geq \sigma_2 \tag{4.13b}$$

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are classical, with equality in *either case* if and only if all the s_i are equal. Here are the proofs. Inequality (4.13a) follows simply upon rearranging the inequality $(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2 \geq 0$. Inequality (4.13b) is a special case of a general inequality among the elementary symmetric polynomials evaluated at positive arguments s_i. See, for example, [**Math**, Appendix A, Table 8, inequality (4)]. Alternatively, expand out $\left(\sum s_i\right)^2 = \sum s_i^2 + 2\sigma_2$ and use inequality (4.13a). Multiplying the two inequalities yields the desired inequality (4.12), with equality if and only if we are at the Lagrange points $s_1 = s_2 = s_3$ of the shape sphere. Since (4.12) is equivalent to the curvature inequality, Theorem 1.1 is proved, modulo the proofs of Lemmas 4.2 and 4.3 which follow in the next section. □

5. *Proofs of Lemmas 4.2 and 4.3*

5.1. *Notation.* To compute ΔU and $\|\nabla U\|^2$ we must express the squared distances $s_k = r_{ij}^2$ of (4.3) in terms of the spherical coordinates (ϕ, θ) of (3.9b). In [**MontI**] I prove that, upon restriction to the sphere $I = 1$,

$$s_k = 1 - \cos(\phi)\gamma_k(\theta), \tag{5.1.1}$$

where

$$\gamma_1(\theta) = \cos(\theta),\ \gamma_2(\theta) = \cos(\theta + 2\pi/3),\ \gamma_3(\theta) = \cos(\theta + 4\pi/3). \tag{5.1.2}$$

The special angles $\theta = 0, 2\pi/3, 4\pi/3$ mark the locations of the three binary collision on the equator $\phi = 0$ of collinear triangles. Later on, we will use the fact that, for any θ, the three planar vectors $(\gamma_k(\theta), \gamma_k'(\theta))$, $k = 1, 2, 3$ form the vertices of an equilateral triangle inscribed in the unit circle. *Here and throughout, we write γ_i' for the derivative $\partial_\theta \gamma_i$ of γ_i with respect to θ.*

5.2. *Proof of Lemma 4.2.* Write $c = \cos(\phi)$, ∂_ϕ for the partial derivative with respect to ϕ and ∂_θ for the partial derivative with respect to θ. Then,

$$\Delta U = \frac{4}{c}\partial_\phi(c\partial_\phi U) + \frac{4}{c^2}(\partial_\theta^2 U). \tag{5.2.1}$$

And

$$\partial_\phi(1/s_i) = -s\gamma_i/s_i^2 \tag{5.2.2a}$$

$$\partial_\theta(1/s_i) = c\gamma_i'/s_i^2. \tag{5.2.2b}$$

Since $U = \sum 1/s_i$, we have

$$\begin{aligned}\partial_\phi(c\partial_\phi U) &= \partial_\phi c \sum(-s\gamma_i)/s_i^2 \\ &= \partial_\phi\left(-cs\sum\gamma_i\right)\Big/ s_i^2 \\ &= (-c^2 + s^2)\sum\gamma_i/s_i^2 + 2cs^2\sum\gamma_i^2/s_i^3.\end{aligned}$$

Thus,

$$\frac{1}{c}\partial_\phi(c\partial_\phi U) = ((s^2 - c^2)/c)\sum\gamma_i/s_i^2 + 2s^2\sum\gamma_i^2/s_i^3, \tag{5.2.3}$$

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and

$$\frac{1}{c^2}\partial_\theta\partial_\theta U = \frac{1}{c^2}\partial_\theta \sum c\gamma_i'/s_i^2$$
$$= \frac{1}{c^2}\sum c\gamma_i''/s_i^2 + 2(c\gamma_i')^2/s_i^3$$
$$= \frac{1}{c}\sum \gamma_i''/s_i^2 + 2\sum(\gamma_i')^2/s_i^3. \tag{5.2.4}$$

Now, $\gamma_i'' = -\gamma_i$, so that

$$\frac{1}{c^2}\partial_\theta\partial_\theta U = -\frac{1}{c}\sum \gamma_i/s_i^2 + 2\sum(\gamma_i')^2/s_i^3. \tag{5.2.5}$$

Adding 4 times (5.2.4) to 4 times (5.2.5) we get

$$\Delta U = 4((s^2 - c^2 - 1)/c)\sum \gamma_i/s_i^2 + 8s^2\sum \gamma_i^2/s_i^3 + 8\sum(\gamma_i')^2/s_i^3. \tag{5.2.6}$$

Now, $s^2 - c^2 - 1 = -2c^2$, so that

$$\Delta U = -8\sum c\gamma_i/s_i^2 + 8\sum[s^2\gamma_i^2 + (\gamma_i')^2]/s_i^3.$$

Recalling that $\gamma_i^2 + (\gamma_i')^2 = 1$ (the vectors (γ_i, γ_i') define an equilateral triangle inscribed in the unit circle), we see that we can replace $(\gamma_i')^2$ by $1 - \gamma_i^2$ in order to obtain

$$s^2\gamma_i^2 + (\gamma_i')^2 = (s^2 - 1)\gamma_i^2 + 1$$
$$= -c^2\gamma_i^2 + 1$$
$$= -(1 - s_i)^2 + 1$$
$$= 2s_i - s_i^2$$
$$= s_i(2 - s_i). \tag{5.2.7}$$

Then, $[s^2\gamma_i^2 + (\gamma_i')^2]/s_i^3 = (2 - s_i)/s_i^2 = (c\gamma_i + 1)/s_i^2$, where I have used $1 - s_i = c\gamma_i$. It follows that

$$\Delta U = -8\sum c\gamma_i/s_i^2 + 8\sum c\gamma_i/s_i^2 + 8\sum(1/s_i^2) = 8U_4,$$

as claimed. □

5.3. *Proof of Lemma 4.3.* We have

$$dU = \partial_\phi U\, d\phi + \partial_\theta U\, d\theta$$
$$= \Big(\sum(-s\gamma_i/s_i^2)\, d\phi + \Big(\sum(c\gamma_i')/s_i^2\Big)\Big)\, d\theta. \tag{5.3.1}$$

Now, $\|\nabla U\|^2 = \|dU\|^2$. The length squared of the covector dU is computed relative to the metric 'g^{ij}' induced on covectors, which, from (3.9), is given at the point (ϕ, θ) by $\|a\, d\phi + b\, d\theta\|^2 = 4(a^2 + (1/c^2)b^2)$, with $c = \cos(\phi)$. It follows that

$$\|\nabla U\|^2 = 4\Big(\sum s\gamma_i/s_i^2\Big)^2 + 4\Big(\sum \gamma_i'/s_i^2\Big)^2$$
$$= 4\Big(\sum s^2\gamma_i^2/s_i^4 + {\sum}' s\gamma_i s\gamma_j/s_i^2 s_j^2 + \sum \gamma_i'^2/s_i^4 + {\sum}' \gamma_i'\gamma_j'/s_i^2 s_j^2\Big)$$
$$= 4\Big(\sum(s^2\gamma_i^2 + \gamma_i'^2)/s_i^4 + {\sum}'(s^2\gamma_i\gamma_j + \gamma_i'\gamma_j')/s_i^2 s_j^2\Big). \tag{5.3.2}$$

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Simplify the numerator in the first summand of the last equation by using (5.2.7). We will simplify the numerator of the second summand by using an analogous identity for $s^2\gamma_i\gamma_j + \gamma_i'\gamma_j'$, $i \neq j$. Indeed, since the vectors (γ_i, γ_i') form the vertices of an equilateral triangle inscribed within the unit circle,we have that $\gamma_i\gamma_j + \gamma_i'\gamma_j' = -1/2$ for $i \neq j$, since $-1/2 = \cos(2\pi/3)$ is the cosine of the central angle defined by any two vertices of an equilateral triangle. Thus,

$$\begin{aligned} s^2\gamma_i\gamma_j + \gamma_i'\gamma_j' &= \gamma_i\gamma_j + \gamma_i'\gamma_j' - c^2\gamma_i\gamma_j \\ &= -1/2 - (1 - s_i)(1 - s_j) \\ &= -3/2 + s_i + s_j - s_i s_j. \end{aligned} \tag{5.3.3}$$

Plugging (5.3.2) and (5.3.3) into (5.3.1), we get

$$\begin{aligned} \|\nabla U\|^2 &= 4\Big(2\sum 1/s_i^3 - \sum 1/s_i^2 - \tfrac{3}{2}\sum{}' 1/s_i^2 s_j^2 + 2\sum{}' 1/s_i s_j^2 - \sum{}' 1/s_i s_j\Big) \\ &= 4S, \end{aligned}$$

as claimed. □

6. *Dynamical consequences*

6.1. *Cutting through R.* Let P denote the pair of pants. Using the spherical shape metric (3.9a), construct three disjoint circles with centers at the three binary collision points. Delete the open discs bounded by these circles to obtain a compact region $R \subset P$ having for its boundary the three disjoint circles. Refer back to Figure 1.

LEMMA 6.1. *Any arc in P whose (finite) syzygy sequence contains all three letters must cut through R.*

Proof. Let c be such a '123' arc. If one of c's syzygies is within R, we are done. Otherwise, all three syzygies lie within the three excised discs. But all three syzygies cannot be in the same disc, since each disc contains exactly two syzygy types. Thus, c must travel from one disc to the other, and in so doing it crosses into R. □

6.2. *Proof of Theorem 2.1.* Let s be syzygy sequence containing all three letters. Approximate s by a sequence w^N, $N = 1, 2, 3$ of periodic sequences as follows. Truncate s to form the finite even-length subword $w_N = s_{-N+1}s_{-N+1}\cdots s_N$. Turn this subword into a periodic sequence $\cdots w_N w_N w_N \cdots$ by repeating it in blocks. If the resulting word has stutters at the join, shift the 'window' we used to form w^N so as to form the word $w_{N,j} = s_{-N+j+1}s_{-N+j}\cdots s_{N+j}$ along with its corresponding periodic word. We can always find a j so that the resulting periodic word, call it w^N, is non-stuttering. The sequence w_N contain all three letters 123 for all N sufficiently large, since s itself contains all three letters. Corollary 2.1 implies that the word w_N is represented by a unique periodic geodesic γ_N. By Lemma 6.1 the γ_N must cut through R. Shifting time (and thus shifting the sequence) if necessary, we may assume that $\gamma_N(0) \in R$. Since R is compact, so is the unit tangent bundle (relative to the Jacobi–Maupertuis metric) to P over R. The sequence of initial conditions $(\gamma_N(0), \dot\gamma_N(0))$, $N \to \infty$ lie in this compact space, so we can find a subsequence which converges to some initial condition (q, v). The geodesic with initial condition (q, v) realizes the infinite sequence s. □

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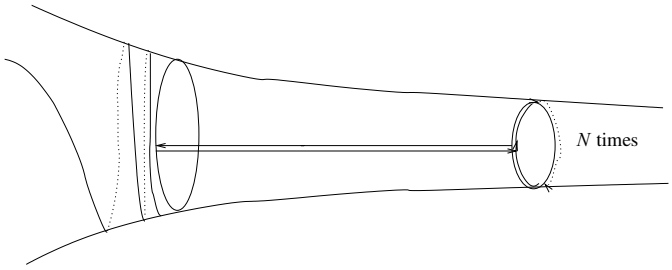


FIGURE 4. Orbit surgery to shorten length of a collision sequence path.

6.3. *Collision sequences. Proof of Theorem 2.2.* Let s be a collision sequence. We may assume, without loss of generality, that it is a forward collision sequence, and that the two letters in its forward tail are 1 and 2. The first part of the theorem asserts that any solution which realizes s must satisfy $r_{12} \to 0$ as the Jacobi time $t \to \infty$. (In Newtonian time the collision occurs in finite time.) As the two bodies get closer, the third body affects them less and less. A straightforward analysis of the two-body $1/r^2$ problem shows that $\liminf r_{12} = 0$ if and only if $\lim r_{12} = 0$. The perturbation of the third body does not affect this assertion. Thus, in order to prove the solution realizing s suffers collision it suffices to show that

$$\liminf r_{12} = 0 \tag{6.3.1}$$

along the solution.

Our proof of (6.3.1) relies on the fact that on a simply connected complete surface of non-negative curvature any compact geodesic arc is the unique minimizer between its endpoints. On a complete non-simply-connected surface such as the pair of pants, this implies that if we have a geodesic arc, then there is no shorter curve having its endpoints and homotopic to it through endpoint-fixing homotopies.

We argue by contradiction. Suppose that some solution $\gamma \subset P$ realizes the collision sequence s but satisfies $\liminf r_{12} = \delta > 0$. We will construct a comparison curve c which has the same endpoints as a (long) arc of γ, is homotopic to this arc through endpoint-fixing homotopies, but which is shorter than γ. The arc will be one whose syzygy sequence is $1212\cdots12$ with N repeats of 12, and N large. See Figure 4 for the picture of this arc and the shorter comparison curve c. The existence of c contradicts the minimality of γ described in the previous paragraph.

To construct c we will use the cylindrical end coordinates (λ, χ) of (3.14), (3.15) associated to the 12-collision. We have

$$\lambda(\rho, \chi) = \int_{\rho_0}^{\rho} (U(s, \chi))^{1/2}\, ds,$$

where (ρ, χ) are spherical-polar coordinates centered at the collision. The (λ, χ) coordinates are valid on the entire sphere minus the '3' equatorial arc and the collision points. The coordinate λ satisfies $\lambda \to \infty$ as collision is approached. The Jacobi metric in these coordinates is

$$ds^2 = d\lambda^2 + f(\lambda, \chi)^2\, d\chi^2, \tag{6.3.2}$$

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where

$$f^2 = \frac{1}{2}\cos^2(\rho)\left\{1 + \frac{s_{12}}{s_{13}} + \frac{s_{12}}{s_{23}}\right\}.$$

The curvature of any metric of the form (6.3.2) is $-(1/f)(\partial^2 f/\partial\lambda^2)$. Since this curvature is negative (Theorem 2.1) and since $\lim_{\lambda\to\infty} f(\lambda, \chi) = 1/\sqrt{2}$ (equation (3.18)) we have that for each fixed χ the function $f(\chi, \lambda)$ decreases monotonically to its infimum $1/\sqrt{2}$ as $\lambda \to \infty$. It follows that $F(\lambda) = \min_\chi f(\lambda, \chi)$ also decreases monotonically to $1/\sqrt{2}$.

Any geodesic arc γ on the pair of pants which realizes the syzygy sequence 12 cannot cross either isosceles circle $r_{12} = r_{23}$ or $r_{12} = r_{13}$. This follows from the minimality property of the arc, discussed above, and the reflection principle (as in [**ChMont**] and the proof of no stuttering in §2, between Theorems 2.2 and 2.3). Reflections about the isosceles circles are isometries of the Jacobi metric, so that any segment of γ which crosses an isosceles circle, then crosses back, can be reflected about that circle so as to form a new arc with the same endpoints as γ and the same homotopy type, contradicting uniqueness. Thus, without loss of generality, we may assume that our long subarc of γ lies entirely in the union of the regions $r_{12} < r_{13}$ and $r_{12} < r_{23}$. In particular, the (λ, χ) coordinates are valid all along our arc.

To construct the desired comparison curve c we will use the fact that F is monotone decreasing in λ and that the number N of 12-crossings of a subarc of γ can be taken arbitrarily large. Since $\inf r_{12} > 0$, by assumption, we have that $\Lambda = \sup\lambda < \infty$ along our curve. Let $\epsilon > 0$ small be given. Choose points $\gamma(s_n), \gamma(t_n)$ $s_n < t_n$, along the arc for which $\lambda(t_n), \lambda(s_n) > \Lambda - \epsilon$ and such that the arc $\gamma[s_n, t_n]$ in between realizes the syzygy sequence $12\cdots12$ with N copies of 12. Our comparison curve c will go 'straight in' to collision until some point with $\lambda_* > \Lambda$, to be determined in the next paragraph, winds around the 12-collision point in the same sense as γ for the same number of syzygies at this fixed value $\lambda = \lambda_*$, and then return headed 'straight out' from collision to the point $\gamma(t_n)$. See Figure 4. 'Straight in' and 'straight out' means that χ is fixed, and only λ varies. During its 'winding around' journey, $\lambda = \lambda_*$ is fixed and χ varies, starting at $\chi = \chi(\gamma(s_n))$, increasing so that c suffers the syzygy sequence $1212\cdots12$ (N times), and then stopping at $\chi = \chi(\gamma(t_n))$ in time for the 'straight out' return segment.

By construction, the curve c shares endpoints with our arc of γ, and is homotopic to it. It remains for us to show that c is shorter than our arc of γ. By the monotonicity and the limiting properties of f we can choose $\lambda_* > \Lambda$ so that

$$\max_\chi f(\lambda_*, \chi) < F(\Lambda) := \min_\chi f(\Lambda, \chi).$$

Choose N so large that

$$\frac{2(\lambda_* - \Lambda) + \epsilon + \pi F(\Lambda)}{N} < \pi(F(\Lambda) - \max_\chi f(\lambda_*, \chi)).$$

It follows that

$$\pi N \max_\chi f(\lambda_*, \chi) + \pi \max_\chi f(\lambda_*, \chi) + 2(\lambda_* - \Lambda) + 2\epsilon < N\pi F(\Lambda).$$

Now consider Figure 4 for the construction of c. The 'in and out' arcs of c have length less than $2(\lambda_* - \Lambda) + 2\epsilon$. The 'around arc' between the N syzygies has length less than

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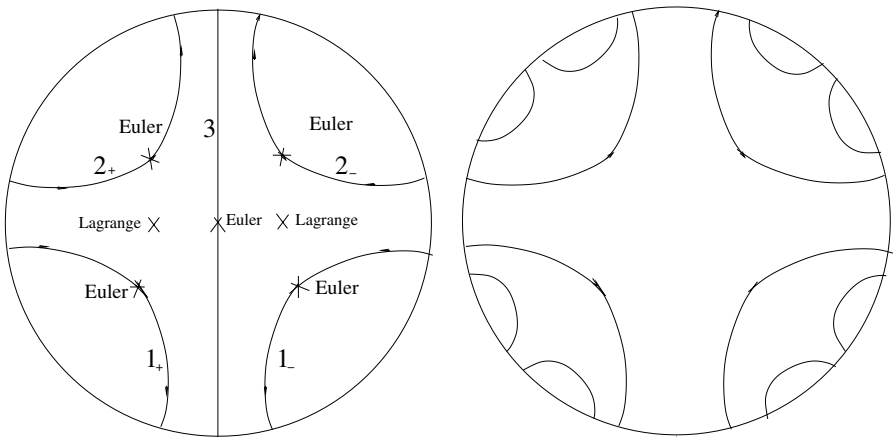


FIGURE 5. The fundamental domain (left) and some of the tiles (right).

$\pi N \max_\chi f(\lambda_*, \chi)$. (Refer again to Figure 4.) The additional $\pi \max_\chi f(\lambda_*, \chi)$ accounts for the fact that $\chi(s_n)$ and $\chi(t_n)$ need not be equal. Thus, the left-hand side of the inequality is greater than the length of c. A similar, but simpler, analysis shows that the right-hand side is smaller than the length of our arc of γ. Thus, c is shorter than the arc of γ, completing the proof of the first assertion of Theorem 2.2.

The same analysis shows that the infinite sequence $\cdots 1212 \cdots$ of all 12's is never realized. For such a realization must be a local minimizer, and the above orbit surgery shows we can always decrease the length of a path by making it closer to collision. □

6.4. *Proof of Theorem 2.3.* To establish the uniqueness of the realizing solutions, we work on the universal cover D of the pair of pants P. Topologically, D is the Poincaré disc and the fundamental group $\Gamma = \pi_1(P)$ (the free group on two letters) acts on D as a Fuchsian group. See Figure 5 and also the book *Indra's Pearls* [**Mumf**]. In Figure 5 we have drawn in a fundamental domain P_0 for P and some of its images under Γ. The boundary of P_0 consists of four circular arcs which are lines relative to the Poincaré metric on D.

Write

$$\pi : D \to P$$

for the covering map, so that the fibers of π are copies of Γ. For the next few paragraphs we will be using the hyperbolic constant negative curvature metric on D. With respect to this metric π is not a local isometry, but no matter. Γ acts on D by hyperbolic isometries, i.e. by Möbius transformations. Γ is freely generated by two elements a and b, one of which, say a, interchanges 1_+ and 1_-, and the other of which, b, interchanges 2_+ and 2_-. These elements act on D as Möbius transformations, and when viewed as acting on the Riemann sphere, they interchange exteriors with interiors of their respective circles. In order to form P out of P_0 glue arc 1_+ to 1_- by a and glue 2_+ to 2_- by b. The projection under π of these boundary arcs forms the arcs 1, 2 of the equator of P. The third arc, 3, is

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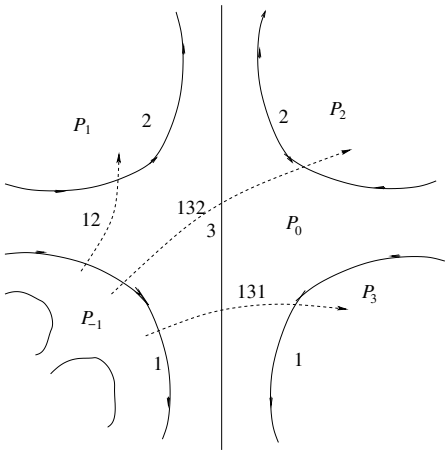


FIGURE 6. Syzygies and tile crossings.

internal to P_0, and separates it into two halves, the northern (+) hemisphere, and southern (−) hemisphere.

The images of γP_0, $\gamma \in \Gamma$, of P_0 tile all of D. Each of the four boundary arcs of such a tile γP_0 is the image under γ of a unique boundary arc of P_0 and we continue to use the old labels from P_0 for the new tile's boundary arcs. Two tiles intersect, if at all, along a common boundary arc. This common arc must be a j_- arc of one tile and a j_+ of the other, $j = 1, 2$.

Suppose, now, that two geodesics realize the same symbol sequence s. Denote by γ, c the lifts of these geodesics to the universal cover D. After translating these curves by elements of Γ we may suppose that both begin in the reference fundamental domain P_0. I claim that the syzygy sequence s uniquely specifies an itinerary $\cdots P_{-2}P_{-1}P_0P_1P_2\cdots$ through which both the curves c and γ must pass. To see this fact, note that each listing of three contiguous tiles $P_{i-1}P_iP_{i+1}$ uniquely represents either two or three letters from a signed syzygy sequence. See Figure 6. This two- or three-letter sequence is obtained by drawing any curve which crosses from P_{i-1} through P_i and into P_{i+1} in the most direct way—i.e. by crossing the internal arc 3 of P_i at most one time. The curve must enter into P_i across one of its bounding arcs, with the choice of P_{i-1} uniquely specifying which. It must leave across another bounding arc and the choice of P_{i+1} uniquely specifies this exit arc. Along the way it either crosses 3 once, or not at all. If it crosses once, then the sequence has three letters with 3 as the middle letter. If it does not cross 3 then the sequence has only two letters, the letters 1 and 2. Since γ and c have the same symbol sequence, and no stutters (they go in the 'most direct way') they must have an identical itinerary, as claimed.

Each tiling domain P_j has within it an inverse image $R_j = \pi^{-1}(R) \cap P_j$ of our compact domain R. By Lemma 6.1, both c and γ must pass through infinitely many of the R_j.

Now, lift the Jacobi–Maupertuis metric from P up to D, using the projection $\pi : D \to P$, thus arriving at a complete non-negatively-curved Γ-invariant metric on D

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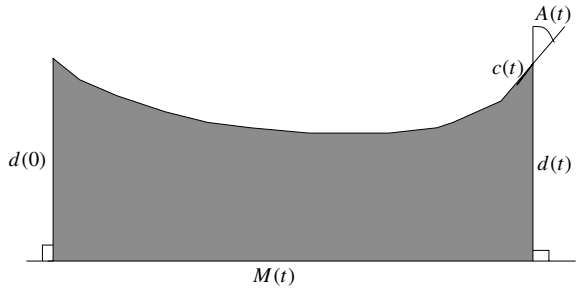


FIGURE 7. Variation of distance.

for which our two curves γ and c are geodesics. The curvature of this metric is zero only at the discrete set of points $\pi^{-1}(L_\pm)$, where $L_\pm$ are the Lagrange points of P. And the map π is a local isometry for this metric. Write

$$h(t) = \operatorname{dist}(\gamma, c(t)) \tag{6.4.1}$$

for the distance between the variable point $c(t)$ on the curve c and the entire geodesic γ. See Figure 7. Now each R_j has finite diameter δ because R is compact, and the two curves pass through the R_j's infinitely often, indeed every time the letters 123 occur contiguously in s. It follows that

$$\lim_{t\to+\infty} \inf h(t) \le \delta \quad \text{and} \quad \lim_{t\to-\infty} \inf h(t) \le \delta. \tag{6.4.2}$$

We will now show that inequality (6.4.2) is impossible unless the two geodesics are in fact the same, in which case $h(t) = 0$ everywhere. We use the formula

$$d^2h/dt^2 = -\sin(A(t)) \int_{d_t} K\, ds, \tag{6.4.3}$$

proved in the following paragraph. In this formula, d_t is the geodesic realizing the distance $h(t)$. It has one endpoint at the point $c(t)$ on c and the other endpoint on γ, which it intersects perpendicularly. See Figure 7. The angle $A(t)$ is the angle of intersection between the geodesics c and d_t at $c(t)$. This angle satisfies $0 < A(t) < \pi$, so that $\sin(A(t)) > 0$. The negativity of K (except at a discrete point set) implies that $h(t)$ is strictly convex: $d^2h/dt^2 > 0$. But, any strictly convex function defined on the real line tends to infinity in one direction or the other. This contradicts (6.4.2). Our two geodesics must be the same. □

Derivation of (6.4.3). The first variation of arclength implies

$$dh/dt = \cos(A(t)). \tag{6.4.4}$$

Let $M(t) \subset D$ denote the quadrilateral whose edges consist of the geodesic arcs d_0, d_t together with the arcs of γ, c which connect d_0 to d_t. According to the Gauss–Bonnet theorem, for any such geodesic quadrilateral Q we have

$$2\pi - \Big(\sum \text{interior angles}\Big) = -\iint_Q K\, dA.$$

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In the case of $M(t)$ the interior angles are $\pi/2, \pi/2, \pi - A(0)$ and $A(t)$. See Figure 7 again. Thus,

$$A(0) - A(t) = -\int\int_{M(t)} K(t)\, dA. \tag{6.4.5}$$

Differentiating (6.4.4) with respect to t yields

$$dA/dt = \int_{d_t} K\, ds. \tag{6.4.6}$$

Now, differentiate (6.4.4) with respect to t, using (6.4.6) to obtain (6.4.3).

6.5. *Proof of Theorem 2.4.* Let c be a bound orbit and $s = \{s_j\}_{j=-\infty}^{\infty}$ its syzygy sequence. Approximate s by a family w^n, $n = 1, 2, 3, \ldots$ of forward collision sequences by replacing the tails s_j, $j > n$ of s by that of the 12-collision sequence. Collapse two letters if necessary when stutters appear at the 'join' $j = n$ of the replacement. The backward tails of the w^n contain all three letters because s is collision-free, so we can apply Theorem 2.1 to realize the w^n by solutions γ_n (not necessarily unique). By Lemma 6.1, the γ_n all pass through R, so, shifting time if necessary, we have that the tangent vectors $v_n = (\gamma_n(0), \dot{\gamma}_n(0))$ are unit vectors with position $\gamma_n(0)$ in R. We now argue as in the proof of Theorem 2.2. By compactness of the set of unit tangent vectors over R, we can form a convergent subsequence of the v_n, which we relabel as v_n, so that $v_n \to v$. Let γ be the curve with initial condition v. The curves γ_n converge, over compact sets, to γ, so that the syzygy sequences w^n must converge to the syzygy sequence of γ. Thus, γ and c share the same syzygy sequence. By Theorem 2.3, $\gamma = c$. Consequently, the unbounded curves γ_n converge to our initial bound curve c. □

7. *Curvature for unequal masses*

In this section we have to prove that Theorem 1.1 is a special case: for most mass distributions the curvature takes on both signs. Take the masses to be general positive numbers m_i. Writing

$$p_i = m_j m_k, \quad s_i = r_{jk}^2$$

for ijk a permutation of 123, we have

$$U = \sum p_i/s_i, \quad I = \frac{1}{M}\sum p_i s_i, \quad \text{where } M = \sum m_i.$$

The Jacobi metric is obtained by multiplying the shape metric ds_m^2 on the shape sphere $I = 1$ by U and restricting it to the sphere $I = 1$. (The subscript 'm' indicates dependence on the masses.) It is conceptually and computationally more straightforward to identify the shape sphere as the space of rays in shape space and to make U a function on the space of rays by making it homogeneous of degree 0 by multiplying it by I. Thus, setting

$$\tilde{U} = IU,$$

our Jacobi metric is

$$ds_J^2 = \tilde{U}\, ds_m^2.$$

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In order to compute, we will use the coordinates ϕ, θ, R of [**MontR**] for shape space. (See, in particular, the notation and computations of §7 there.) Write I_1 for the moment of inertia when all masses are equal to one:

$$I_1 = \sum r_{ij}^2/3. \tag{7.1}$$

The ϕ, θ are spherical coordinates for the I_1 shape sphere, while the radial coordinate $R = \sqrt{I}$ for the mass-dependent I of (3.6). With respect to these coordinates, we have that

$$s_k = I_1(1 - \gamma_k(\theta)\cos(\phi)). \tag{7.2}$$

As before, note that the coordinates θ, ϕ and the functions s_k do not vary as the masses are changed. The metric ds_m^2 when expressed in our coordinates is mass-dependent and is given by

$$ds_m^2 = \lambda^2 ds_1^2, \tag{7.3a}$$

where

$$ds_1^2 = \tfrac{1}{4}(d\phi^2 + \cos(\phi)^2\, d\theta^2) \tag{7.3b}$$

(see [**MontI**, equation (5.6) and Proposition 2]) is the shape sphere metric when all the masses are 1, and where the conformal factor λ is given by

$$\lambda = d(m)I_1/I, \quad d(m) = (3m_1m_2m_3/M)^{1/2} \tag{7.3c}$$

([**MontI**, equation (5.7)]). The metric we are to work with is thus

$$ds_J^2 = \tilde{U}\lambda^2\, ds_1^2. \tag{7.4}$$

Its curvature is given by Lemma 4.1:

$$\bar{K} = \frac{1}{\tilde{U}\lambda^2}\left\{4 - \frac{1}{2}\Delta\log(\tilde{U}\lambda^2)\right\}, \tag{7.5}$$

where the Laplacian Δ is with respect to the metric ds_1^2. The conformal factor is

$$\begin{aligned}\tilde{U}\lambda^2 &= d(m)^2 I U I_1^2/I^2\\ &= d(m)I_1 U(d(m)I_1/I)\\ &= d(m)\hat{U}\lambda,\end{aligned}$$

where

$$\hat{U} = I_1 U.$$

Then,

$$\Delta\log\tilde{U}\lambda^2 = \Delta\log\hat{U} + \Delta\log\lambda.$$

Set

$$\hat{s}_i = s_i/I_1 = 1 - \gamma_k(\theta)\cos(\phi),$$

so that

$$\hat{U} = \sum p_i/\hat{s}_i.$$

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Since $\hat{s}_i$ is equal to the variable s_i as given by equation (5.1.1), and since the Laplacian is the Laplacian of that section, we can continue to compute as per §5. We have

$$\hat{U}\lambda\bar{K} = 4 - \tfrac{1}{2}\Delta\log(\hat{U}) - \tfrac{1}{2}\Delta\log(\lambda).$$

To compute the last term $-(1/2)\Delta\log(\lambda)$ of (7.5) we use the fact that the metrics ds_m^2 and ds_1^2 are conformal through λ^2, and that both have curvature 4. From Lemma 4.1, it follows that

$$4 = \frac{1}{\lambda^2}\left(4 - \frac{1}{2}\Delta\log\lambda^2\right)$$

or

$$4\lambda^2 - 4 = -\Delta\log\lambda.$$

Equation (7.5) can then be rewritten

$$-\hat{U}^3\lambda\bar{K} = -\hat{U}^2(4 - \tfrac{1}{2}\Delta\log(\hat{U})) + \hat{U}^2(2 - 2\lambda^2). \tag{7.6}$$

Computation of $\Delta\hat{U}$. To ease notation, we drop the hats for this subsection, so that U means the function $\hat{U}$ and the s_k means the function $\hat{s}_k$. The structural form of all the formulae and calculations of §5 remains intact provided we insert the weightings p_i in the correct places.

We proceed to the computation of $\Delta\log(\hat{U})$. Equations (4.2) and (5.2.1) continue to hold with $\hat{U}$ in place of U, and equations (5.2.2a,b) hold with $\hat{s}_i$ in place of s_i. The analogues of (5.2.3), (5.2.4) are

$$\frac{1}{c}\partial_\phi(c\partial_\phi U) = ((s^2 - c^2)/c)\sum p_i\gamma_i/s_i^2 + 2s^2\sum p_i\gamma_i^2/s_i^3. \tag{7.7a}$$

And

$$\frac{1}{c^2}\partial_\theta\partial_\theta U = \frac{1}{c}\sum p_i\gamma_i''/s_i^2 + 2\sum p_i(\gamma_i')^2/s_i^3. \tag{7.7b}$$

Now, note that algebraic steps going from (5.2.5) to (5.2.7) apply term by term, so can be carried over verbatim except that the ith term must be multiplied by p_i. Thus (reverting to hats),

$$\Delta\hat{U} = 8\hat{U}_4, \tag{7.8a}$$

where

$$\hat{U}_4 = \sum p_i/\hat{s}_i^2. \tag{7.8b}$$

The first line of (5.3.2) becomes

$$\|\nabla U\|^2 = 4\Big(\sum sp_i\gamma_i/s_i^2\Big)^2 + 4\Big(\sum p_i\gamma_i'/s_i^2\Big)^2.$$

The algebra which follows is essentially the same, leading to

$$\|\nabla U\|^2 = 4S, \tag{7.9a}$$

where

$$S = 2\sum p_i^2/s_i^3 - \sum p_i^2/s_i^3 - 3/2\sum{}' p_ip_j/s_i^2s_j^2 + 2\sum{}' p_ip_j/s_is_j^2 - \sum{}' p_ip_j/s_is_j. \tag{7.9b}$$

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Combining (7.8) and (7.9), according to (4.2) (see also the steps (4.7)–(4.9)), yields the formula:

$$-U^2(4-\tfrac{1}{2}\Delta\log U)=3\sum{}'\, p_ip_j/s_i^2s_j^2-2\Big(\sum p_i/s_i\Big)^2. \tag{7.10}$$

From earlier, we have

$$U^2(2-2\lambda^2)=2\Big(\sum p_i/s_i\Big)^2-2\Big(\sum p_i/s_i\Big)^2d(m)^2\Big(\tfrac{1}{3}\sum s_i\Big)^2\Big/\Big(\sum p_is_i/M\Big)^2. \tag{7.11}$$

(We continue to use s_i in place of $\hat{s}_i$.) Upon adding (7.10) and (7.11), there is a cancellation, yielding:

$$-U^3\lambda^2\bar{K}=3\sum{}'\, p_ip_j/s_i^2s_j^2-2\Big(\sum p_i/s_i\Big)^2d(m)^2\Big(\tfrac{1}{3}\sum s_i\Big)^2\Big/\Big(\sum p_is_i/M\Big)^2.$$

A computation shows that

$$d(m)^2M^2=\tfrac{3}{2}\sum{}'\, p_ip_j.$$

Recall that the s_i (the previous $\hat{s}_i$) satisfy $(\frac{1}{3})\sum s_i=1$. We finally obtain:

$$-U^3\lambda^2\bar{K}=3\left\{\sum{}'\, p_ip_j/s_i^2s_j^2-\sum{}'\, p_ip_j\frac{\big(\sum p_i/s_i\big)^2}{\big(\sum p_is_i\big)^2}\right\}. \tag{7.12}$$

Consequently,

$$\kappa=\left(\frac{\sum' p_ip_j}{s_i^2s_j^2}\right)^{1/2}\frac{\big(\sum p_is_i\big)}{\big(\sum p_i/s_i\big)}-\Big(\sum{}'\, p_ip_j\Big)^{1/2} \tag{7.13}$$

governs the sign of the curvature, with the curvature $\bar{K}$ negative if κ is positive, positive if κ is negative, and zero if κ is zero. We note that κ, and hence the curvature, is zero at the Lagrange point $s_1=s_2=s_3=1$, and that this is true for all choices of the masses p_i.

THEOREM 7.1. *For a Zariski-dense set of mass distributions, the sign of the curvature changes in a neighborhood of the Lagrange point.*

Proof. It suffices to show that for a Zariski-dense set of mass distributions the differential $d\kappa\neq 0$ at the Lagrange point. A differential form $\sum a_i\,ds_i$ represents zero on the shape sphere if and only if it is proportional to $\sum ds_i$, the latter being the differential of the constraint $\sum s_i=3$ satisfied by the s_i (which are the old $\hat{s}_i$). A computation shows that at the Lagrange point $s_i=1$ we have

$$\beta d\kappa=p_1(p_2^2+p_3^2)\,ds_1+p_2(p_1^2+p_3^2)\,ds_2+p_3(p_1^2+p_2^2)\,ds_3\mod\sum ds_i,$$

where β is a non-zero constant. We, thus, want to know whether or not the equality

$$(p_1(p_2^2+p_3^2),\,p_2(p_1^2+p_3^2),\,p_3(p_1^2+p_2^2))=(\lambda,\lambda,\lambda) \tag{$**$}$$

can be satisfied for some λ. The right-hand side of equation $(**)$, being homogeneous of degree 3, defines a polynomial map $\mathbb{RP}^2\to\mathbb{RP}^2$ and we want to know if it is equal to the constant map $[1,1,1]$. Because the map is polynomial, if we can exhibit a single point where the inequality fails then it must fail on a Zariski-dense set. Plugging in $p_1=p_2=1$, $p_3=a$ yields $(p_1(p_2^2+p_3^2),\,p_2(p_1^2+p_3^2),\,p_3(p_1^2+p_2^2))=(1+a^2,\,1+a^2,\,2a)$ which is not proportional to $(1,1,1)$ unless $a=1$. □

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A. *Appendix. An open set of collision orbits*

We prove the following.

THEOREM A.1. *The set of initial conditions within $H = 0$, $I = 1$, $c = 0$ whose solutions tend to a binary collision of type ij has non-empty interior. This fact holds for all positive mass distributions.*

Proof. We use Newtonian time and Jacobi coordinates. For notational simplicity, take $ij = 12$. The Jacobi coordinates are $\zeta_1 = x_1 - x_2$, $\zeta_2 = x_3 - (m_1x_1 + m_2x_2)/(m_1 + m_2)$. The distance to binary collision is

$$r = |\zeta_1|. \tag{A.1}$$

We will exhibit a non-empty open set of initial conditions at time $t = 0$ for which $r(t) = 0$ for some time $t < O(r(0))$.

The Hamiltonian is

$$H = \frac{1}{2}(\mu_1|\dot{\zeta}_1|^2 + \mu_2|\dot{\zeta}_2|^2) - \frac{m_1m_2}{r^2} - W(\zeta_1, \zeta_2), \tag{A.2}$$

where $\mu_1 = m_1m_2/(m_1 + m_2)$, $\mu_2 = m_3(m_1 + m_2)/M$, $W = m_1m_3/s_2 + m_2m_3/s_1$ and the squared distances s_2, s_3 can be expressed $|\zeta_2 + a_i\zeta_1|^2$ in Jacobi coordinates, with mass-dependent non-zero constants a_i. The interaction term W satisfies the estimates

$$|W| \le C_1 + C_2\epsilon, \quad \text{for } r < \epsilon \tag{A.3}$$

$$\left|\frac{\partial W}{\partial \zeta_1}\right| \le C_1 + C_2\epsilon, \quad \left|\frac{\partial W}{\partial \zeta_2}\right| \le C_1 + C_2\epsilon, \quad \text{for } r \le \epsilon \tag{A.4}$$

(ϵ sufficiently small), where C_1, C_2 are constants depending only on the masses.

The equations of motion are

$$\ddot{\zeta}_1 = -(m_1 + m_2)\frac{\zeta_1}{r^4} + \frac{1}{\mu_1}\frac{\partial W}{\partial \zeta_1} \tag{A.5}$$

and

$$\ddot{\zeta}_2 = \frac{1}{\mu_2}\frac{\partial W}{\partial \zeta_2}. \tag{A.6}$$

Write

$$J_1 = \zeta_1 \wedge \dot{\zeta}_1 \tag{A.7}$$

for the angular momentum (up to a factor of μ_1) of the 12 system. We compute

$$\begin{aligned} \dot{J}_1 &= \zeta_1 \wedge \ddot{\zeta}_1 \\ &= \zeta_1 \wedge \frac{1}{\mu_1}\frac{\partial W}{\partial \zeta_1}, \end{aligned}$$

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so that

$$|\dot{J}_1| \leq Cr. \tag{A.8}$$

Because $|\dot{\zeta}_1|^2 = \dot{r}^2 + J_1^2/r^2$ we have that

$$r^2 H = \tfrac{1}{2}(\mu_1 r^2 \dot{r}^2 + \mu_1 J_1^2) - m_1 m_2 + r^2(\tfrac{1}{2}\mu_2 |\dot{\zeta}_2|^2 - W). \tag{A.9}$$

Now, let $\zeta(t) = (\zeta_1(t), \zeta_2(t))$ be a solution satisfying the initial conditions $r(0) < \epsilon$, $H = 0$, $I = 1$, $J = 0$. From (A.6) and (A.4) we have that

$$|\dot{\zeta}_2(t)|^2 \leq |\dot{\zeta}_2(0)|^2 + Ct, \tag{A.10}$$

for $t = O(1)$, provided $r(0) < \epsilon$. Here, C depends only on the masses and ϵ. Letting $r \to 0$, we see from (A.9) that if our solution is to have a collision then we must have

$$\lim r^2 \dot{r}^2 + \lim J_1^2 - 2(m_1 + m_2) = 0, \tag{A.11}$$

where we have used $m_1 m_2/\mu_1 = m_1 + m_2$. But, $r^2\dot{r}^2 \geq 0$, so we must have

$$2(m_1 + m_2) - \lim J_1^2 \geq 0. \tag{A.12}$$

We argue in the reverse. Suppose that $2(m_1 + m_2) - J_1(0)^2$ is sufficiently positive at the initial time $t = 0$, and that $\dot{r}(0) < 0$, then (A.11) forces $r^2\dot{r}^2$ to be positive over a finite time interval. We will show that, upon integration, this will force $r(t) = 0$ in some finite time $t = O(\sqrt{r(0)})$. Note from the bounds (A.10) and the fact that $H = 0$, we have

$$|\mu_1 r^2 \dot{r}^2 + \mu_1 J_1(0)^2 - 2m_1 m_2| \leq K r(0) \tag{A.13}$$

for $0 \leq t \leq 1$ and for as long as $\dot{r}(t) < 0$. Here, the constant K depends only on the masses and $\dot{\zeta}_2(0)$. Dividing by μ_1 and using $m_1 m_2/\mu_1 = m_1 + m_2$, we arrive at

$$|r^2\dot{r}^2 + J_1(0)^2 - 2(m_1 + m_1)| \leq K^* r(0), \tag{A.14}$$

where $K^* = K/\mu_1$.

We now impose the open condition

$$2(m_1 + m_2) - J_1(0)^2 - K^* r(0) > \delta^2 \tag{A.15}$$

on our initial conditions. This will be the open condition of Theorem A.1. The positive constant δ will be constrained further below. It follows from (A.15) and (A.14) that

$$\delta^2 < 2(m_1 + m_2) - J_1(0)^2 - K^* r(0) \leq r^2\dot{r}^2. \tag{A.16}$$

(A.16) together with $\dot{r}(0) < 0$ forces $\dot{r} < 0$ throughout the time interval in question. Thus, $-r\dot{r} > 0$, and so we can take square roots of inequality (A.16) to obtain

$$\delta \leq -r\dot{r}. \tag{A.17}$$

Taking negatives and integrating, we find that $-\delta t \geq (1/2)r(t)^2 - (1/2)r(0)^2$ or

$$r(0)^2 - 2\delta t \geq r(t)^2. \tag{A.18}$$

This forces $r(t) = 0$ for some time t with $t \leq r(0)^2/2\delta$. In order that the collision time t be $o(1)$ it is sufficient to take $\delta = O(r(0))$.

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We have proved that a 12-collision occurs within a time $t = r(0)^2/2\delta$ for all initial conditions satisfying (A.15), $\dot{r}(0) < 0$ and $r(0) < \epsilon$, where ϵ is small enough so that the inequalities (A.3), (A.4) are in force. This set of initial conditions is clearly open. It remains for us to show that this set is non-empty. Consider the *collinear* solution having $H = 0 = J$ and $I = 1$. (There are precisely two such solutions, up to time translation and rotation, one for each arc of the equator which ends in the 12-collision.) These solutions satisfy $J_1 = 0$. In this case (15A) reads $m_1 + m_2 > Kr(0) + \delta$, and so will hold for $r(0)$ small provided only that $\delta < m_1 + m_2$. Since the solution tends to collision it follows that (A.15) is eventually in force along the collinear solution, and hence that our set of initial conditions is non-empty. □

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