Proceedings of the Edinburgh Mathematical Society (2026) **69**, 37–59
doi:10.1017/S001309152510093X

TWISTED ROKHLIN PROPERTY FOR MAPPING CLASS GROUPS

PRAVIN KUMAR, APEKSHA SANGHI AND MAHENDER SINGH

Department of Mathematical Sciences, Indian Institute of Science Education and Research (IISER) Mohali, Sector 81, S. A. S. Nagar, P. O. Manauli, Punjab 140306, India

Corresponding author: Apeksha Sanghi, email: apekshasanghi93@gmail.com

(Received 22 April 2024)

Abstract Given an automorphism ϕ of a group G, the map $(g,h) \mapsto gh\phi(g)^{-1}$, defines a left action of G on itself, whose orbits are called the ϕ-twisted conjugacy classes. In this paper, we consider two interesting aspects of this action for mapping class groups, namely, the existence of a dense orbit and the count of orbits. Generalising the idea of the Rokhlin property, a topological group is said to exhibit the twisted Rokhlin property if, for each automorphism ϕ of the group, there exists a ϕ-twisted conjugacy class that is dense in the group. We provide a complete classification of connected orientable infinite-type surfaces without boundaries whose mapping class groups possess the twisted Rokhlin property. Additionally, we prove that the mapping class groups of the remaining surfaces do not admit any dense ϕ-twisted conjugacy class for any automorphism ϕ. This supplements the recent work of Lanier and Vlamis on the Rokhlin property of big mapping class groups. Regarding the count of twisted conjugacy classes, we prove that the number of ϕ-twisted conjugacy classes is infinite for each automorphism ϕ of the mapping class group of a connected orientable infinite-type surface without boundary.

Keywords: big mapping class group, infinite type surface, R_∞-property, Rokhlin property, twisted conjugacy, twisted Rokhlin property

2020 Mathematics subject classification: Primary 57K20, 54H11, Secondary 20E36

1. Introduction

A topological group is said to exhibit the *Rokhlin property* if it contains a dense conjugacy class. From the point of view of dynamics, the Rokhlin property is equivalent to the existence of a dense orbit under the conjugation action of the topological group on itself. This perspective allows us to consider more general actions of the topological group arising from its structure.

Given an automorphism ϕ of a group G, the map $(g,h) \mapsto gh\phi(g)^{-1}$, defines a left action of G on itself. Two elements $x, y \in G$ are said to be *ϕ-twisted conjugate* if they are in the same orbit under this action. The resulting orbits are referred as *ϕ-twisted conjugacy classes*. The group G is said to have the *R_∞-property* if the number of ϕ-twisted conjugacy classes is infinite for each automorphism ϕ of G. The idea of twisted conjugacy arose from the work of Reidemeister [21]. It has deep connection with Nielsen fixed-point theory [6], and appears in Arthur–Selberg theory [23], algebraic geometry

© The Author(s), 2025. Published by Cambridge University Press on behalf of The Edinburgh Mathematical Society.

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

[12], and representation theory [10], to name a few. In recent years, there has been a tremendous amount of work on understanding twisted conjugacy in various classes of groups. See, for example [4, 8], for recent works on twisted conjugacy classes in mapping class groups and braid groups.

In this paper, we combine the concept of the Rokhlin property with that of twisted conjugacy. Let G be a topological group and ϕ an automorphism of the underlying abstract group. We say that G has the *ϕ-twisted Rokhlin property* if G admits a dense ϕ-twisted conjugacy class. Further, G is said to have the *twisted Rokhlin property* if it has the ϕ-twisted Rokhlin property for each automorphism ϕ of the underlying abstract group. Clearly, if a topological group has the twisted Rokhlin property, then it has the Rokhlin property.

The *mapping class group* of a connected orientable surface is the group of isotopy classes of orientation-preserving self-homeomorphisms of the surface. These groups inherit the quotient topology from the compact-open topology on the group of orientation-preserving self-homeomorphisms of the surface. It turns out that this quotient topology is non-discrete if and only if the surface is of infinite-type. The aim of this paper is to classify connected orientable infinite-type surfaces without boundaries whose mapping class groups admit the twisted Rokhlin property. Our approach is built upon the recent work [17] of Lanier and Vlamis, where they classify connected orientable infinite-type surfaces without boundaries whose mapping class groups admit the Rokhlin property (see also [13] for an independent such classification).

The paper is organised as follows. In Section 2, we recall basic results on infinite type surfaces, mapping class groups and curve graphs, that we shall need in the latter sections. In Section 3, we introduce the twisted Rokhlin property of topological groups and make some basic observations. We prove that if ϕ, ψ are two automorphisms of a topological group G such that $\psi \in \mathrm{Inn}(G)\phi$, then G has the ϕ-twisted Rokhlin property if and only if it has the ψ-twisted Rokhlin property (Proposition 3.5). In particular, this implies that the twisted Rokhlin property for an inner automorphism is equivalent to the Rokhlin property of the group. For Polish topological groups, we prove that the ϕ-twisted Rokhlin property is equivalent to the property that, given two non-empty open subsets U and V of G, there exists $g \in G$ such that $U \cap gV\phi(g)^{-1} \neq \emptyset$ (Proposition 3.10). Section 4 contains our results on the twisted Rokhlin property of big mapping class groups. Our main result is the following classification theorem.

Theorem 1.1. *Let S be a connected orientable infinite-type surface without boundary and let $\mathrm{MCG}(S)$ denote its mapping class group. Then the following assertions are equivalent:*

(1) $\mathrm{MCG}(S)$ *has the twisted Rokhlin property.*
(2) $\mathrm{MCG}(S)$ *has the Rokhlin property.*
(3) $\mathrm{MCG}(S)$ *has the ϕ-twisted Rokhlin property for some automorphism ϕ of* $\mathrm{MCG}(S)$.
(4) *S has a unique maximal end and each of its compact subsurfaces is displaceable.*

Along the way, in Section 5, we also investigate the R_∞-property of mapping class groups of surfaces and prove the following result.

Theorem 1.2. *Let S be a connected orientable infinite-type surface without boundary. Then $\mathrm{MCG}(S)$ possesses the R_∞-property.*

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

2. Preliminaries

As the subject resides at the intersection of group theory, topology, and geometry, we offer relevant background information to enhance the readability of the paper.

2.1. Classification of surfaces

A surface is said to be of *finite-type* if its fundamental group is finitely generated; otherwise, it is said to be of *infinite-type*. Throughout the paper, our primary surface S under consideration will be connected, orientable, without boundary and of infinite-type, unless stated otherwise. At some occasions, we shall also need finite-type surfaces, and we write $S^b_{g,n}$ to denote a finite-type surface of genus g with n punctures and b boundary components.

Definition 2.1. *An* exiting sequence *in a surface S is a sequence $\{U_n\}_{n\geq 1}$ of connected open subsets of S admitting the following properties:*

(1) *If $m < n$, then $U_n \subset U_m$.*
(2) *For each $n \in \mathbb{N}$, U_n is not relatively compact.*
(3) *For each $n \in \mathbb{N}$, U_n has compact boundaries.*
(4) *Each relatively compact subset of S is disjoint from all except finitely many U_n's.*

Two exiting sequences $\{U_n\}_{n\geq 1}$ and $\{V_n\}_{n\geq 1}$ are *equivalent* if for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $U_m \subset V_n$ and $V_m \subset U_n$. An equivalence class of an exiting sequence is called an *end* and we denote the set of all ends by $\mathcal{E}(S)$. The set $\mathcal{E}(S)$ can be topologised as follows. Given a subset $U \subset S$ with compact boundary, consider the set

$$U^* = \left\{ [\{U_n\}_{n\geq 1}] \in \mathcal{E}(S) \;\mid\; \text{there exists } n \in \mathbb{Z} \text{ such that } U_n \subset U \right\}$$

Then the desired topology on $\mathcal{E}(S)$ is the topology generated by the basis

$$\{U^* \;\mid\; U \subset S \;\text{ is open with compact boundary}\}.$$

With this topology, the space of ends of a surface is compact, totally disconnected, second countable and Hausdorff. Consequently, the space of ends is homeomorphic to a closed subset of the Cantor set. If $U \subset S$ is an open subset with compact boundary and $e \in U^*$, then we refer to U as a *neighbourhood* of e. Further, every homeomorphism $h : S \to S$ induces a homeomorphism $\mathcal{E}(S) \to \mathcal{E}(S)$ given by

$$[\{U_n\}_{n\geq 1}] \mapsto [\{h(U_n)\}_{n\geq 1}].$$

An end is called *planar* if it has a neighbourhood that can be embedded in the plane. Otherwise, it is called *non-planar*. Let $\mathcal{E}_{np}(S)$ be the subspace of $\mathcal{E}(S)$ consisting of non-planar ends. Clearly, $\mathcal{E}_{np}(S)$ is a closed set in $\mathcal{E}(S)$. The following theorem gives a classification of surfaces that we shall need [22, Theorem 1].

Theorem 2.2. *Let S_1 and S_2 be two surfaces. Let g_1, g_2 and b_1, b_2 represent the genus and the number of boundary components of S_1, S_2, respectively. Then S_1 is homeomorphic*

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

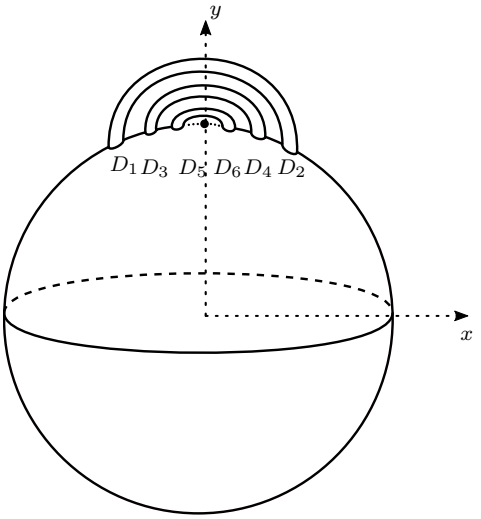


Figure 1. Realization of the Loch Ness monster surface.

to S_2 if and only if $g_1 = g_2$, $b_1 = b_2$, and there exists a homeomorphism from $\mathcal{E}(S_1)$ to $\mathcal{E}(S_2)$ that restricts to a homeomorphism from $\mathcal{E}_{np}(S_1)$ to $\mathcal{E}_{np}(S_2)$.

Without loss of generality, we shall assume throughout that the surface S is symmetric about the xy-plane when embedded in $\mathbb{R}^3$. Since $\mathcal{E}(S)$ is homeomorphic to a closed subset of the Cantor set, it can be embedded on the great circle of intersection of the xy-plane and the unit 2-sphere $\mathbb{S}^2$. Now, take a sequence of disjoint closed disks D_j on $\mathbb{S}^2 \setminus \mathcal{E}(S)$ which are symmetric about the xy-plane such that the set of accumulation points of this sequence is precisely $\mathcal{E}_{np}(S)$. We remove the interiors of the disks D_j and suitably identify in pairs the boundary circles by orientation-reversing homeomorphisms such that the desired resulting surface is again symmetric about the xy-plane. See [22, Theorem 3] for the general construction. For instance, Figure 1 illustrates the construction of Loch Ness monster surface, where the infinite sequence of disjoint closed disks $\{D_i\}_{i\in\mathbb{N}}$ is chosen such that the radii of the disks decreases as it approaches a point of $\mathcal{E}(S)$.

For a connected orientable surface S (not necessarility of infinite-type), let $\mathrm{Homeo}^+(S)$ denote the group of orientation-preserving self-homeomorphisms of S. We define a binary relation $\preceq$ on the space of ends $\mathcal{E}(S)$ of S. For $x, y \in \mathcal{E}(S)$, we say that $y \preceq x$ if for every open neighbourhood U of x, there is an open neighbourhood V of y and a homeomorphism $h \in \mathrm{Homeo}^+(S)$ such that $h(V) \subseteq U$. An end μ is called *maximal* if, for any end e satisfying $\mu \preceq e$, we must have $e \preceq \mu$. Let $\mathcal{M}(S)$ denote the set of all maximal ends of S. Note that for every end $e \in \mathcal{E}(S)$, there exists a maximal end $\mu \in \mathcal{M}(S)$ such that $e \preceq \mu$ [18, Proposition 4.7]. We say that two ends e_1 and e_2 are comparable if either $e_1 \preceq e_2$ or $e_2 \preceq e_1$. The space $\mathcal{E}(S)$ is said to be *self-similar* if given a decomposition $\mathcal{E}(S) = \mathcal{E}_1 \sqcup \cdots \sqcup \mathcal{E}_n$ into a finite disjoint union of clopen subsets, there exists an index $i \in \{1, \ldots, n\}$ such that $\mathcal{E}_i$ contains an open homeomorphic copy of $\mathcal{E}(S)$. The idea of maximality and self-similarity for end spaces is due to [18].

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

A subsurface Σ of S is called *displaceable* if there exists a $h \in \mathrm{Homeo}^+(S)$ such that $\Sigma \cap h(\Sigma) = \emptyset$. We shall use the following result, which follows from [1, Theorem 6.1] and [18].

Theorem 2.3. *If S is a surface in which every compact subsurface is displaceable, then either*

(1) *$\mathcal{M}(S)$ has exactly one element,*
(2) *$\mathcal{M}(S)$ has exactly two elements, or*
(3) *$\mathcal{M}(S)$ is homeomorphic to a Cantor space in which any two maximal ends are comparable.*

Moreover, $\mathcal{E}(S)$ is self-similar in the cases (1) *and* (3).

2.2. Mapping class groups of surfaces

Recall that, the *mapping class group* $\mathrm{MCG}(S)$ of a connected orientable surface S (not necessarily of infinite-type) is the group of isotopy classes of orientation-preserving self-homeomorphisms of S, which preserves the boundary of S point-wise and permutes the punctures, if any. The mapping class group of an infinite-type surface is also referred as the *big mapping class group*.

Let $\mathrm{Homeo}^+(S, \partial(S))$ be the group of orientation-preserving self-homeomorphisms of S fixing the boundary $\partial(S)$ point-wise, and equipped with the compact-open topology. We equip $\mathrm{MCG}(S)$ with the corresponding quotient topology inherited from $\mathrm{Homeo}^+(S, \partial(S))$. Note that the mapping class group of a surface with compact boundary is discrete if and only if the surface is of finite-type. On the other hand, big mapping class groups have uncountably many elements and inherit a non-discrete topology from $\mathrm{Homeo}^+(S, \partial(S))$. The *extended mapping class group* of S is the group $\mathrm{MCG}^{\pm}(S)$ of isotopy classes of, possibly orientation-reversing, self-homeomorphisms of S. We refer the reader to [2] for a survey on both topological and algebraic aspects of big mapping class groups.

For infinite-type surfaces without boundary, the outer automorphism group of the mapping class group is known due to [3, Theorem 1.1].

Proposition 2.4. *If S is a connected orientable infinite-type surface without boundary, then*

$$\mathrm{Out}\left(\mathrm{MCG}^{\pm}(S)\right) = 1 \quad \text{and} \quad \mathrm{Out}(\mathrm{MCG}(S)) \cong \mathbb{Z}_2.$$

As mentioned earlier, we assume that our surface S is symmetric about the xy-plane when embedded in $\mathbb{R}^3$. Thus, the map $\mathfrak{t} : \mathbb{R}^3 \to \mathbb{R}^3$ which sends a point to its mirror image about the xy-plane restricts to an orientation-reversing self-homeomorphism

$$\mathfrak{t} : S \to S.$$

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

Clearly, $\mathfrak{t}$ is an involution of S. For notational convenience, let $\hat{\mathfrak{t}}$ denote the automorphism of $\mathrm{MCG}(S)$ given by

$$\hat{\mathfrak{t}}(h) = \mathfrak{t}h\mathfrak{t},$$

for all $h \in \mathrm{MCG}(S)$. Then $\hat{\mathfrak{t}}$ is a representative of the non-trivial element of $\mathrm{Out}(\mathrm{MCG}(S))$.

For a simple closed curve c on a surface S, let T_c denote the left-hand Dehn twist along c. By abuse of notation, we denote a curve and its isotopy class by the same symbol, and use the same convention for homeomorphisms as well. The following lemma, perhaps well-known, gives a relation between the Dehn twist and its conjugate by a homeomorphism of S. The proof of the following lemma is a slight modification of the arguments of [5, Fact 3.7]. See also [15, Lemma 2.1].

Lemma 2.5. *If h is a homeomorphism of a surface S and c a simple closed curve on S, then*

$$hT_ch^{-1} = T_{h(c)}^{\epsilon},$$

where $\epsilon = 1$ when h is orientation-preserving and $\epsilon = -1$ when h is orientation-reversing.

By an *essential curve* on a surface S, we mean a simple closed curve on S which does not bound a disk, a punctured disk, or an annulus. Let $C(S)$ be the set of isotopy classes of all essential simple closed curves on S. Given a subset A of $C(S)$, we define the subset U_A of $\mathrm{MCG}(S)$ by

$$U_A = \{\phi \in \mathrm{MCG}(S) \mid \phi(a) = a \text{ for all } a \in A\}.$$

It is known that if S has empty boundary and $\mathrm{MCG}(S)$ has trivial center, then the set

$$\mathcal{B} = \{\psi U_A \mid \psi \in \mathrm{MCG}(S) \text{ and } A \text{ is a finite subset of } C(S)\},$$

is a basis for the topology on $\mathrm{MCG}(S)$. Note that, the center of the big mapping class group is always trivial [16, Proposition 2]. The following basic observation shall be used later.

Lemma 2.6. *Suppose that $g, h \in \mathrm{MCG}(S)$. Then $h \in gU_A$ if and only if $hU_A = gU_A$.*

Proof. If $h \in gU_A$, then $h = g\phi$ for some $\phi \in U_A$. Thus, $g = h\phi^{-1} \in hU_A$, and hence $gU_A = hU_A$. Conversely, if $hU_A = gU_A$, then $h \in gU_A$. □

2.3. Curve graphs of surfaces

We say that a simple closed curve c on a surface S is *non-separating* if $S \backslash c$ is connected; otherwise, it is called *separating*. The *intersection number* $i(c, c')$ between two curves c, c' on S is defined to be the minimum number of points of intersections between the

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

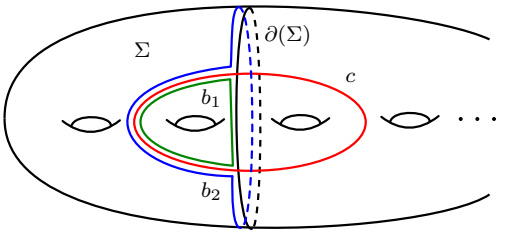


Figure 2. Projections of the curve c on Σ.

representatives in their isotopy classes. A *multicurve* on S is a set of pairwise (non-isotopic) disjoint curves.

The set $C(S)$ can be viewed as a graph whose vertex set is $C(S)$ and there is an edge between two vertices if the corresponding curves on S are disjoint. The graph $C(S)$ is referred as *curve graph* of S. There is a natural metric $d_{C(S)}$ on $C(S)$ by defining the length of each edge as 1. We shall need the following result [19, Proposition 4.6].

Proposition 2.7. *Let $S_{g,n}$ be a surface of genus g with n punctures such that $3g + n \geq 5$. Then there exists a real number $c > 0$ such that, for any pseudo-Anosov $h \in \mathrm{MCG}(S_{g,n})$, $\gamma \in C(S_{g,n})$ and $k \in \mathbb{Z}$, we have*

$$d_{C(S_{g,n})}(h^k(\gamma), \gamma) \geq c|k|.$$

In particular, for any pseudo-Anosov $h \in \mathrm{MCG}(S_{g,n})$ and $\gamma \in C(S_{g,n})$, there exists $\ell \in \mathbb{N}$ such that

$$d_{C(S_{g,n})}(h^m(\gamma), \gamma) > 2,$$

for all $m \geq \ell$.

We say that a subsurface Σ and a simple closed curve c on a surface S have a *non-trivial geometric intersection* if there exist $a \in C(\Sigma)$ such that $i(a, c) \neq 0$.

Definition 2.8. *Let Σ be a subsurface of a surface S such that the embedding of Σ into S induces an embedding of $C(\Sigma)$ into $C(S)$. Let $c \in C(S)$ be such that it has a non-trivial geometric intersection with Σ. Assume that the boundary $\partial(\Sigma)$ and c are in a minimal position. Then a* projection *of c onto $C(\Sigma)$ is defined to be any component $b \in C(\Sigma)$ of the boundary of a regular neighbourhood of $\alpha \cup \partial(\Sigma)$, where α is a component of $c \cap \Sigma$.*

For example, let S be a surface and Σ be a genus two subsurface of S with a single boundary component. Let c be a curve on S as illustrated in Figure 2. Then the curves b_1 and b_2 are two projections of c onto Σ (see Figure 2).

Note that, if each component of $\partial(\Sigma)$ is an essential separating simple closed curve on S, then $C(\Sigma)$ embeds into $C(S)$. Further, for each $c \in C(S)$ such that it has a non-trivial geometric intersection with Σ, such a projection $b \in C(\Sigma)$ always exists, and if $c \in C(\Sigma)$, then $b = c$.

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

Lemma 2.9. [20, Lemma 2.2] *Let Σ be a subsurface of a surface S such that the embedding of Σ into S induces an embedding of $C(\Sigma)$ into $C(S)$. If $c \in C(S)$ and $b, b' \in C(\Sigma)$ are two projections of c, then $d_{C(\Sigma)}(b, b') \leq 2$.*

3. Twisted Rokhlin property

Recall that, a topological group is said to have *Rokhlin property* if it contains a dense conjugacy class. Equivalently, the topological group has a dense orbit under its conjugation action on itself. See, for example, the introduction of [14] for a survey on Rokhlin property.

Example 3.1. Some topological groups with the Rokhlin property are:

(1) The symmetric group on $\mathbb{N}$.
(2) The homeomorphism group of the Cantor set [11, Theorem 2.6].
(3) The group of orientation-preserving homeomorphisms of an even-dimensional sphere [11, Theorem 3.4].
(4) The mapping class group of a connected orientable surface which is either the 2-sphere, 2-sphere with one puncture or infinite-type with genus either zero or infinite, and whose end space is self-similar with a unique maximal end (see [13, Theorem 0.2] or [17, Theorem 6.1]).

Example 3.2. Some topological groups without the Rokhlin property are:

(1) A non-trivial closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ [25, Lemma 5.2].
(2) A non-trivial connected locally compact group [25, Theorem 5.3].

Next we consider the idea of twisted conjugacy.

Definition 3.3. *Let G be a group and ϕ an automorphism of G. Two elements $x, y \in G$ are said to be* ϕ-twisted conjugate*, denoted by $x \sim_\phi y$, if $y = gx\phi(g)^{-1}$ for some $g \in G$. The resulting equivalence classes are called as* ϕ-twisted conjugacy classes*, and we denote such a class by $C_\phi(x)$.*

Note that, given an automorphism ϕ of a group G, the map $G \times G \to G$ given by $(g, h) \mapsto gh\phi(g)^{-1}$ defines a left action of G on itself. Thus, ϕ-twisted conjugacy classes are simply orbits under this action.

Definition 3.4. *Let G be a topological group and ϕ an automorphism of the underlying abstract group G. We say that G has the* ϕ-twisted Rokhlin property *if G admits a dense ϕ-twisted conjugacy class. Further, G is said to have the* twisted Rokhlin property *if it has ϕ-twisted Rokhlin property for each automorphism ϕ of the underlying abstract group G.*

The subsequent proposition significantly simplifies the task of verifying the twisted Rokhlin property for a topological group.

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

Proposition 3.5. *Let G be a topological group, and ϕ, ψ be two automorphisms of G such that $\psi \in \mathrm{Inn}(G)\phi$. Then G has the ϕ-twisted Rokhlin property if and only if it has the ψ-twisted Rokhlin property. In particular, if $\psi \in \mathrm{Inn}(G)$, then G has the ψ-twisted Rokhlin property if and only if it has the Rokhlin property.*

Proof. Let $\psi = \hat{g}\,\phi$, where $\hat{g}$ is the inner automorphism of G induced by the element $g \in G$. We have

$$\begin{aligned} C_\phi(x) &= \{kx\phi(k)^{-1} \mid k \in G\} \\ &= \{kxg^{-1}(g\phi(k)^{-1}g^{-1})g \mid k \in G\} \\ &= \{k(xg^{-1})\,\hat{g}(\phi(k)^{-1})g \mid k \in G\} \\ &= \{k(xg^{-1})\,\psi(k)^{-1}g \mid k \in G\} \\ &= R_g(C_\psi(xg^{-1})), \end{aligned}$$

where $R_g : G \to G$ is right multiplication by g. Since R_g is a homeomorphism of G, it follows that $C_\phi(x)$ is dense in G if and only if $C_\psi(xg^{-1})$ is dense in G. □

Remark 3.6. The preceding lemma implies that if $\mathrm{Aut}(G) = \mathrm{Inn}(G)$, then G has the Rokhlin property if and only if G has the twisted Rokhlin property. In particular, if G is the symmetric group on a countably infinite set, then every automorphism of G is inner, and hence G has the twisted Rokhlin property.

Proposition 3.7. *Let G be a topological group and $\phi \in \mathrm{Aut}(G)$. Suppose that G admits a proper open subgroup H such that $gH\phi(g)^{-1} = H$ for all $g \in G$. Then G does not have the ϕ-twisted Rokhlin property. In particular, if there exists a proper open normal subgroup H, then G does not have the Rokhlin property.*

Proof. The given condition implies that, for each $x \in G$, either $C_\phi(x) \subseteq H$ or $C_\phi(x) \cap H = \emptyset$. Thus, $C_\phi(x) \cap gH = \emptyset$ for some $g \in G$, where gH is open. Therefore, $C_\phi(x)$ is not dense. Hence, G does not have ϕ-twisted Rokhlin property. □

Remark 3.8. By Proposition 2.4, if S is an infinite-type surface, then $\mathrm{Aut}(\mathrm{MCG}^{\pm}(S)) = \mathrm{Inn}(\mathrm{MCG}^{\pm}(S))$. Since $\mathrm{MCG}(S)$ is a proper open normal subgroup of $\mathrm{MCG}^{\pm}(S)$ (being a finite index subgroup), it follows that $\mathrm{MCG}^{\pm}(S)$ does not have the Rokhlin property. Consequently, it does not have the twisted Rokhlin property.

Definition 3.9. *Let G be a topological group and let ϕ be an automorphism of G. Then G has the* ϕ-twisted joint embedding property *(ϕ-TJEP for short) if given any two non-empty open subsets U and V of G, there exists $g \in G$ such that $U \cap gV\phi(g)^{-1} \neq \emptyset$.*

Note that the set $gV\phi(g)^{-1}$ is open whenever V is open. Recall that, a topological group is *Polish* if its underlying topological space is separable and completely metrizable. The next result extends [17, Theorem 2.2] to the ϕ-twisted case.

Proposition 3.10. *Let G be a Polish topological group and let ϕ be an automorphism of G. Then G has the ϕ-twisted Rokhlin property if and only if G has the ϕ-TJEP.*

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

Proof. Suppose that G has the ϕ-TJEP. Since G is Polish, it is second countable. Let $\mathcal{B}$ be a countable basis for G. For each $U \in \mathcal{B}$, define

$$D_U = \cup_{g \in G}\, gU\phi(g)^{-1}.$$

Note that D_U is open. Further, it follows from the ϕ-TJEP that D_U is dense in G. Since G is completely metrizable, it is a Baire space. Hence, the set $D = \cap_{U \in \mathcal{B}}\, D_U$ is dense in G. Let $x \in D$ be an arbitrary element. Then, for each basis member $V \in \mathcal{B}$, there exists $g \in G$ (depending on V) such that $x \in gV\phi(g)^{-1}$. In other words, there exists $v \in V$ such that $v = g^{-1}x\phi(g)$, and hence $v \in V \cap C_\phi(x)$. This proves that $C_\phi(x)$ is dense in G. Conversely, suppose that the ϕ-twisted conjugacy class $C_\phi(x)$ is dense in G. Given non-empty open sets U and V in G, there exist $y, z \in G$ such that $yx\phi(y)^{-1} \in U$ and $zx\phi(z)^{-1} \in V$. Setting $h = yz^{-1}$, it is easy to see that $yx\phi(y)^{-1} \in U \cap hV\phi(h)^{-1}$. Hence, the group G has the ϕ-TJEP. □

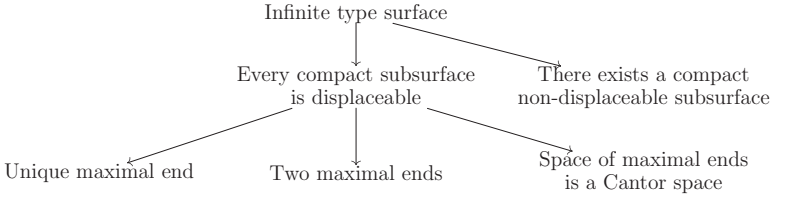
Remark 3.11. It follows from [24, Corollary 5] that if S is an infinite-type surface, then $\mathrm{MCG}(S)$ is a Polish group. Hence, $\mathrm{MCG}(S)$ has the ϕ-twisted Rokhlin property if and only if it has the ϕ-TJEP.

4. Twisted Rokhlin property of big mapping class groups

It is known that the mapping class group of a surface with compact boundary (including no boundary) is discrete if and only if the surface is of finite-type. Thus, the mapping class group of a finite-type surface does not contain any proper dense subset, and consequently no dense ϕ-twisted conjugacy class. Hence, we consider only infinite-type surfaces (without boundary).

Recently, in [17, Theorem 6.1], Lanier and Vlamis have shown that the mapping class group of a connected orientable infinite-type surface without boundary has the Rokhlin property if and only if the surface has genus either zero or infinite, and whose end space is self-similar with a unique maximal end.

If a topological group has the twisted Rokhlin property, then it has the Rokhlin property. Thus, we need to investigate the twisted Rokhlin property only for an infinite-type surface whose genus is either zero or infinite, and whose end space is self-similar with a unique maximal end. Further, we shall also see whether the big mapping class group of the remaining surfaces have the ϕ-twisted Rokhlin property for some automorphism ϕ. Employing the approach outlined in the following flowchart, we establish Theorem 4.2, Proposition 4.4, Theorem 4.7 and Theorem 4.9, which collectively lead to our main theorem.



https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

4.1. When the surface has a unique maximal end

In this subsection, we assume that every compact subsurface of S is displaceable and $\mathcal{M}(S)$ is a singleton set, say, $\mathcal{M}(S) = \{\mu\}$. It follows from Theorem 2.3 that the end space $\mathcal{E}(S)$ is self-similar. Notice that, in this case, S is either planar or has infinite genus. Consider a separating simple closed curve c on the surface S. Define Ω_c to be the component of $S \setminus c$ such that $\mu \in \Omega_c^*$ and Σ_c to be the complement of Ω_c in S. Note that Σ_c has c as its unique boundary component. Let G be the subset of $\mathrm{MCG}(S)$ consisting of elements admitting representatives that restricts to the identity on Ω_c for some separating simple closed curve c. Note that the set

$$\{\Omega_c^* \mid c \text{ is a separating simple closed curve on } S\},$$

is a neighbourhood basis for μ, and consequently G is a subgroup of $\mathrm{MCG}(S)$.

Let S be a surface where every compact subsurface is displaceable, and let S have a unique maximal end, denoted μ. We choose a separating simple closed curve b_1 on S such that $\mathfrak{t}(\Sigma_{b_1}) = \Sigma_{b_1}$. In particular, $\mathfrak{t}(b_1) = b_1$, and hence b_1 meets the xy-plane in precisely two points. By the proof of [17, Proposition 4.2], Ω_{b_1} has a homeomorphic copy of Σ_{b_1}. Let $\Sigma_{\tilde{b}_1}$ be this homeomorphic copy, where $\tilde{b}_1$ is a separating simple closed curve in Ω_{b_1}. If $\mathfrak{t}(\Sigma_{\tilde{b}_1}) \neq \Sigma_{\tilde{b}_1}$, then we cut S along the curves b_1 and $\tilde{b}_1$. Thus, we can write $S = \Sigma_{b_1} \cup \Sigma_{\tilde{b}_1} \cup \Omega_{b_1,\tilde{b}_1}$, where $\Omega_{b_1,\tilde{b}_1} = \Omega_{b_1} \setminus \Sigma_{\tilde{b}_1}$. We choose realisations Σ and Ω of $\Sigma_{\tilde{b}_1}$ and $\Omega_{b_1,\tilde{b}_1}$ respectively, on the unit 2-sphere in $\mathbb{R}^3$ such that the following hold:

- $\mathfrak{t}(\Sigma) = \Sigma$ and $\mathfrak{t}(\Omega) = \Omega$.
- $\Sigma_{b_1}, \Sigma, \Omega$ have disjoint interiors.
- If α and β are the two boundary components of Ω, then $\mathfrak{t}(\alpha) = \alpha$ and $\mathfrak{t}(\beta) = \beta$.
- If γ is the boundary component of Σ, then we have $\mathfrak{t}(\gamma) = \gamma$.

This implies that the curves α, β, and γ meet the xy-plane in precisely two points. Thus, we can assume that $b_1 = \alpha$ and $\beta = \gamma$ as curves in $\mathbb{R}^3$. Let S_{b_1} be the surface obtained from Σ_{b_1}, Σ and Ω by identifying b_1 with α (denote the resulting curve again by b_1) and β with γ (denote the resulting curve by b_1'). Then we see that S_{b_1} is homeomorphic to S with $\mathfrak{t}(S_{b_1}) = S_{b_1}$, $\mathfrak{t}(\Sigma_{b_1}) = \Sigma_{b_1}$ and $\mathfrak{t}(\Sigma_{b_1'}) = \Sigma_{b_1'}$. See below Figure 3.

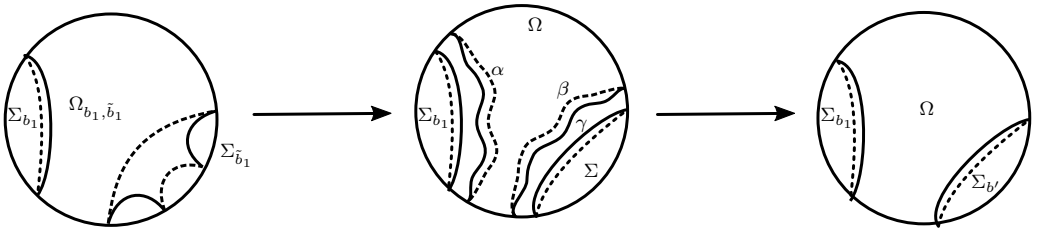


Figure 3. Different realisation of infinite type surface with unique maximal end.

Now, choose a separating simple closed curve b_2 on S_{b_1} such that $\Sigma_{b_1} \cup \Sigma_{b_1'} \subset \Sigma_{b_2}$ and $\mathfrak{t}(\Sigma_{b_2}) = \Sigma_{b_2}$. By the proof of [17, Proposition 4.2], Ω_{b_2} has a homeomorphic copy,

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

say $\Sigma_{\tilde{b}_2}$, of Σ_{b_2}. By repeating the preceding process, we get another surface S_{b_2} from S_{b_1} without perturbing Σ_{b_2} such that $\mathfrak{t}(S_{b_2}) = S_{b_2}$ and $\mathfrak{t}(\Sigma_{b'_2}) = \Sigma_{b'_2}$. By repeating this process, for each i, we get a surface S_{b_i} from $S_{b_{i-1}}$ without perturbing Σ_{b_i} such that $\mathfrak{t}(S_{b_i}) = S_{b_i}$ and $\mathfrak{t}(\Sigma_{b'_i}) = \Sigma_{b'_i}$. Define $\widetilde{S} = \cup_{i=1}^{\infty}\Sigma_{b_i}$. Then, we see that $\widetilde{S}$ is homeomorphic to S such that $\mathfrak{t}(\Sigma_{b_i}) = \Sigma_{b_i}$, $\mathfrak{t}(\Sigma_{b'_i}) = \Sigma_{b'_i}$, $\Sigma_{b_i} \cap \Sigma_{b'_i} = \emptyset$, $\Sigma_{b_i} \cong \Sigma_{b'_i}$ and $\Sigma_{b_i} \cup \Sigma_{b'_i} \subset \Sigma_{b_{i+1}}$ for each i. We now present an extension of [17, Proposition 4.2].

Proposition 4.1. *Let S be a surface where every compact subsurface is displaceable, and let S have a unique maximal end, denoted μ. Assume that S has a realisation such that $S = \cup_{i=1}^{\infty}\Sigma_{b_i}$ as above. Let G be the subgroup of* $\mathrm{MCG}(S)$ *defined above. Then the following assertions holds:*

(1) *The subgroup G is dense in* $\mathrm{MCG}(S)$.
(2) *For each separating simple closed curve b_i, there exists $g \in \mathrm{Homeo}^+(S)$ such that $g(\Sigma_{b_i}) \subset \Omega_{b_i}$ and $g\mathfrak{t}g^{-1}\mathfrak{t}$ is identity on Σ_{b_i}.*

Proof. Assertion (1) follows from [17, Proposition 4.2]. Recall, for each b_i, there exists a separating curve b'_i such that $\Sigma_{b_i} \cap \Sigma_{b'_i} = \emptyset$, $\Sigma_{b_i} \cong \Sigma_{b'_i}$, $\mathfrak{t}(\Sigma_{b_i}) = \Sigma_{b_i}$ and $\mathfrak{t}(\Sigma_{b'_i}) = \Sigma_{b'_i}$. Further, choose a separating simple closed curve c_i such that the curves $\{b_i, b'_i, c_i\}$ bounds a pair of pants (a subsurface homeomorphic to the two-sphere with three disks removed) and $\{b_i, b'_i\} \subset \Sigma_{c_i}$. For each subsurface Σ of S with $\mathfrak{t}(\Sigma) = \Sigma$, define

$$\Sigma^+ := \Sigma \cap \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\} \quad \text{and} \quad \Sigma^- := \Sigma \cap \{(x, y, z) \in \mathbb{R}^3 \mid z \leq 0\}.$$

We choose a homeomorphism $h : \Sigma_{c_i} \to \Sigma_{c_i}$ fixing $\partial(\Sigma_{c_i})$ identically such that $h(\Sigma_{b_i}) = \Sigma_{b'_i}$ and $h(\Sigma^+_{b_i}) = \Sigma^+_{b'_i}$. In fact, such a homeomorphism can be defined by extending a homeomorphism on the pair of pants bounded by the curves $\{b_i, b'_i, c_i\}$ which takes the curve b_i to b'_i and fix c_i point-wise. Define $f : S \to S$ by

$$f(x) = \begin{cases} h(x) & \text{if } x \in \Sigma^+_{b_i}, \\ \mathfrak{t}h\mathfrak{t}(x) & \text{if } x \in \Sigma^-_{b_i}, \\ h(x) & \text{if } x \in \Sigma_{c_i} \setminus \Sigma_{b_i}, \\ x & \text{if } x \in S \setminus \Sigma_{c_i}. \end{cases}$$

Thus $g := f^{-1} \in \mathrm{Homeo}^+(S)$ such that $g^{-1}(\Sigma_{b_i}) \subset \Omega_{b_i}$. Consequently, we have $g(\Sigma_{b_i}) \subset \Omega_{b_i}$. Now, if $x \in \Sigma^+_{b_i}$, then $\mathfrak{t}(x) \in \Sigma^-_{b_i}$, and hence

$$g\mathfrak{t}g^{-1}\mathfrak{t}(x) = g\mathfrak{t}\mathfrak{t}h\mathfrak{t}(\mathfrak{t}(x)) = gh(x) = gf(x) = x.$$

Similarly, if $x \in \Sigma^-_{b_i}$, then $\mathfrak{t}(x) \in \Sigma^+_{b_i}$, and hence $g\mathfrak{t}g^{-1}\mathfrak{t}(x) = x$. This shows that $g\mathfrak{t}g^{-1}\mathfrak{t}$ is identity on Σ_{b_i}, which establishes assertion (2). □

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

Theorem 4.2. *Let S be a surface where every compact subsurface is displaceable and S has a unique maximal end. Then* $\mathrm{MCG}(S)$ *possesses the twisted Rokhlin property.*

Proof. We write $S = \cup_{i=1}^{\infty}\Sigma_{b_i}$ as above. It follows from [17, Theorem 2.2 and Lemma 4.3] that $\mathrm{MCG}(S)$ has the Rokhlin property. Consequently, by Proposition 3.5, $\mathrm{MCG}(S)$ has the ϕ-twisted Rokhlin property for each inner automorphism ϕ of $\mathrm{MCG}(S)$. By Proposition 2.4, we have $\mathrm{Out}(\mathrm{MCG}(S)) \cong \mathbb{Z}_2$. Thus, in view of Propositions 3.5 and 3.10, it is enough to prove that $\mathrm{MCG}(S)$ has the $\hat{\mathfrak{t}}$-TJEP.

Let U_1 and U_2 be non-empty open subsets of $\mathrm{MCG}(S)$. Without loss of generality, we assume that U_1 and U_2 are the basis elements of $\mathrm{MCG}(S)$. Thus, there exist finite subsets $A_1, A_2 \subset C(S)$ and $h_1, h_2 \in \mathrm{MCG}(S)$ such that $U_1 = h_1U_{A_1}$ and $U_2 = h_2U_{A_2}$. By Proposition 4.1(1), the subgroup G is dense in $\mathrm{MCG}(S)$. Thus, by Lemma 2.6, we can assume that $h_1, h_2 \in G$. Since $h_1, h_2 \in G$, there exist separating simple closed curves c_1, c_2 on S such that h_1 and h_2 admit representatives that restrict to the identity on Ω_{c_1} and Ω_{c_2}, respectively. Choose a separating simple closed curve b_i such that $c_1, c_2 \subset \Sigma_{b_i}$ and each curve in $A_1 \cup A_2$ has a representative in Σ_{b_i}. By Proposition 4.1(2), there exists $g \in \mathrm{Homeo}^+(S)$ such that $g(\Sigma_{b_i}) \subset \Omega_{b_i}$ and $g\mathfrak{t}g^{-1}\mathfrak{t}$ is identity on Σ_{b_i}. Note that $g(\Sigma_{b_i}) \subset \Omega_{b_i}$ implies that $g^{-1}(\Sigma_{b_i}) \subset \Omega_{b_i}$. We have

$$gU_2\hat{\mathfrak{t}}(g)^{-1} = gh_2U_{A_2}\hat{\mathfrak{t}}(g)^{-1} = gh_2\hat{\mathfrak{t}}(g)^{-1}\big(\hat{\mathfrak{t}}(g)U_{A_2}\hat{\mathfrak{t}}(g)^{-1}\big) = gh_2\hat{\mathfrak{t}}(g)^{-1}U_{\hat{\mathfrak{t}}(g)(A_2)}. \tag{1}$$

Recall that, the element h_1 admits a representative that restricts to the identity on Ω_{c_1}. Since $\Omega_{b_i} \subset \Omega_{c_1}$, this representative of h_1 restricts to the identity on Ω_{b_i}. Similarly, since $g\mathfrak{t} = \mathfrak{t}g$ on Σ_{b_i} and $g^{-1}(\Sigma_{b_i}) \subset \Omega_{b_i}$, it follows that the element $gh_2\hat{\mathfrak{t}}(g)^{-1}$ also has a representative that restricts to the identity on Σ_{b_i}. Thus, we have the following:

(i) h_1 and $gh_2\hat{\mathfrak{t}}(g)^{-1}$ commute (since they admit representatives with complementary supports),
(ii) $h_1 \in U_{\hat{\mathfrak{t}}(g)(A_2)}$ (since $\hat{\mathfrak{t}}(g)(A_2) \subset \hat{\mathfrak{t}}(g)(\Sigma_{b_i}) \subset \Omega_{b_i}$),
(iii) $gh_2\hat{\mathfrak{t}}(g)^{-1} \in U_{A_1}$ (since $A_1 \subset \Sigma_{b_i}$).

Taking $h = h_1(gh_2\hat{\mathfrak{t}}(g)^{-1})$, we see that

$$h \in h_1U_{A_1} = U_1 \ \ (\text{using } (iii))$$

and

$$h \in gh_2\widehat{\mathfrak{t}}(g)^{-1}U_{\hat{\mathfrak{t}}(g)(A_2)} \ \ (\text{using } (i) \text{ and } (ii)).$$

In view of (1), we have $h \in gU_2\hat{\mathfrak{t}}(g)^{-1}$. Hence, $h \in U_1 \cap gU_2\hat{\mathfrak{t}}(g)^{-1}$, which completes the proof. □

4.2. When the surface has exactly two maximal ends

In this subsection, we assume that every compact subsurface of our surface S is displaceable and $\mathcal{M}(S)$ has exactly two elements. We claim that $\mathrm{MCG}(S)$ does not have

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

the ϕ-twisted Rokhlin property for any $\phi \in \mathrm{Aut}(\mathrm{MCG}(S))$. In view of Proposition 3.5, it is enough to consider the automorphism $\hat{\mathfrak{t}}$ of $\mathrm{MCG}(S)$. Recall that, there is an action of $\mathrm{MCG}(S)$ on $\mathcal{E}(S)$ that keeps $\mathcal{M}(S)$ invariant.

Lemma 4.3. *The stabiliser* $\mathrm{Stab}(\mu)$ *of a maximal end is a proper open normal subgroup of* $\mathrm{MCG}(S)$ *such that* $h\,\mathrm{Stab}(\mu)\hat{\mathfrak{t}}(h)^{-1} = \mathrm{Stab}(\mu)$ *for all* $h \in \mathrm{MCG}(S)$.

Proof. It follows from [17, Lemma 5.4] that $\mathrm{Stab}(\mu)$ is a proper open normal subgroup of $\mathrm{MCG}(S)$. If $f \in h\,\mathrm{Stab}(\mu)\hat{\mathfrak{t}}(h)^{-1}$, then $f = hf'\mathfrak{t}h^{-1}\mathfrak{t}$ for some $f' \in \mathrm{Stab}(\mu)$. Since $\mathrm{Stab}(\mu)$ is normal, we see that

$$f(\mu) = hf'\mathfrak{t}h^{-1}\mathfrak{t}(\mu) = hf'\mathfrak{t}h^{-1}(\mu) = hf'h^{-1}(\mu) = \mu.$$

Hence, $h\,\mathrm{Stab}(\mu)\hat{\mathfrak{t}}(h)^{-1} \subseteq \mathrm{Stab}(\mu)$ for all $h \in \mathrm{MCG}(S)$. The converse inclusion can be proved in a similar way. □

Proposition 4.4. *Let* S *be a surface where every compact subsurface is displaceable, and* $\mathcal{M}(S)$ *contains exactly two elements. Then* $\mathrm{MCG}(S)$ *does not have the* ϕ*-twisted Rokhlin property for any* $\phi \in \mathrm{Aut}(\mathrm{MCG}(S))$.

Proof. The proof follows directly from Propositions 3.5, 3.7 and Lemma 4.3. □

4.3. When the space of maximal ends is a Cantor space

This subsection deals with the case when every compact subsurface of S is displaceable and $\mathcal{M}(S)$ is a Cantor space. The following result is a special case of [5, Theorem 14.1].

Proposition 4.5. *Let* $S_{g,n}$ *be a surface of genus* g *with* n *punctures and let* $\{a, b\}$ *be a filling system of curves for* $S_{g,n}$. *Consider the subgroup* $\langle T_a, T_b\rangle$ *of* $\mathrm{MCG}(S)$ *and the representation* $\rho : \langle T_a, T_b\rangle \to PSL(2,\mathbb{R})$ *given by*

$$\rho(T_a) = \begin{pmatrix} 1 & i(a,b) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(T_b) = \begin{pmatrix} 1 & 0 \\ -i(a,b) & 1 \end{pmatrix}.$$

Let $f \in \langle T_a, T_b\rangle$ *such that* $\rho(f)$ *is hyperbolic in* $PSL(2,\mathbb{R})$. *Then* f *is pseudo-Anosov.*

The following lemma will be used in the main result of this subsection.

Lemma 4.6. *Consider the surface* $S_{0,6}$ *such that it is symmetric about the* xy*-plane and all its punctures lie on the* xy*-plane. Then there exists a mapping class* $f \in \mathrm{MCG}(S_{0,6})$ *such that* f *permutes the punctures cyclically and* $f\mathfrak{t}f\mathfrak{t}$ *is pseudo-Anosov.*

Proof. Let $b_1, \ldots, b_6$ be the punctures on $S_{0,6}$. Let r be the order six homeomorphism of $S_{0,6}$ such that $r(b_i) = b_{i+1}$ for $1 \le i \le 5$ and $r(b_6) = b_1$. Let $\mathfrak{t}$ denote a map that sends a point of $S_{0,6}$ to its mirror image about the xy-plane, c be the simple closed curve on $S_{0,6}$ as shown in Figure 4 and $d = \mathfrak{t}(c)$. Then $f = rT_cT_d^{-2}T_c^3T_d^{-4}$ is an orientation-preserving homeomorphism of $S_{0,6}$. Note that $r(c) = \mathfrak{t}(c) = d$ and $r\mathfrak{t} = \mathfrak{t}r$ on $S_{0,6}$. Furthermore, $f(b_i) = b_{i+1}$ for $1 \le i \le 5$ and $f(b_6) = b_1$.

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

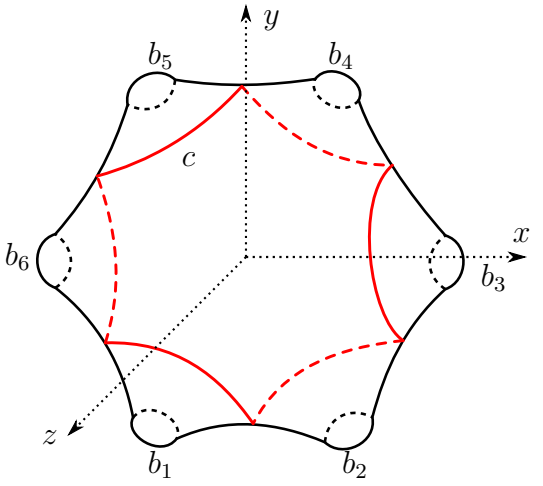


Figure 4. The curve c on $S_{0,6}$ such that $\{c, \mathfrak{t}(c)\}$ fills $S_{0,6}$.

We see that

$$\begin{aligned}(f\mathfrak{t}f\mathfrak{t})^3 &= (rT_cT_d^{-2}T_c^3T_d^{-4}rT_d^{-1}T_c^2T_d^{-3}T_c^4)^3, \quad \text{since} \quad r\mathfrak{t} = \mathfrak{t}r \text{ and } \mathfrak{t}T_c\mathfrak{t} = T_{\mathfrak{t}(c)}^{-1}\\ &= (T_dT_c^{-2}T_d^3T_c^{-4}T_d^{-1}T_c^2T_d^{-3}T_c^4r^2)^3, \quad \text{since} \quad rT_cr^{-1} = T_{r(c)} = T_d\\ &= (T_dT_c^{-2}T_d^3T_c^{-4}T_d^{-1}T_c^2T_d^{-3}T_c^4)^3.\end{aligned}$$

Since $i(c,d) = 6$, by Proposition 4.5, we have

$$\rho(T_c) = \begin{pmatrix}1 & 6\\ 0 & 1\end{pmatrix} \quad \text{and} \quad \rho(T_d) = \begin{pmatrix}1 & 0\\ -6 & 1\end{pmatrix}.$$

This gives

$$\rho((f\mathfrak{t}f\mathfrak{t})^3) = \begin{pmatrix}-67685042634532711678300 31 & -16281842128186331925122976 0\\ 41172474347137850845138560 & 99041634791684755574056372 9\end{pmatrix},$$

which is hyperbolic since the trace of $\rho\big((f\mathfrak{t}f\mathfrak{t})^3\big)$ is greater than 2. Hence, $f\mathfrak{t}f\mathfrak{t}$ is a pseudo-Anosov map. □

Theorem 4.7. *Let S be a surface where every compact subsurface is displaceable, and $\mathcal{M}(S)$ is homeomorphic to a Cantor space. Then,* $\mathrm{MCG}(S)$ *does not possess the ϕ-twisted Rokhlin property for any $\phi \in \mathrm{Aut}(\mathrm{MCG}(S))$.*

Proof. By Proposition 3.5 and [17, Lemma 4.8], it is enough to prove that $\mathrm{MCG}(S)$ does not have the $\hat{\mathfrak{t}}$-twisted Rokhlin property. By results in [17, Proposition 4.6] (or [18, Lemma 4.18]), we choose a compact subsurface Σ of S such that

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

(i) Σ is planar,
(ii) Σ has six boundary components, say $b_1, b_2, b_3, b_4, b_5, b_6$,
(iii) each b_i is a separating curve of S,
(iv) $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6$ are the six components of $S \setminus \Sigma$ such that Ω_i^* is homeomorphic to $\mathcal{E}(S)$,
(v) $\mathfrak{t}(\Sigma) = \Sigma$, $\mathfrak{t}(b_i) = b_i$ and $\mathfrak{t}(\Omega_i) = \Omega_i$ for each $1 \leq i \leq 6$.

By Lemma 4.6, let $f' \in \text{Homeo}^+(\Sigma)$ such that $f'(b_i) = b_{i+1}$ for $1 \leq i \leq 5$ and $f'(b_6) = b_1$ and $f'\mathfrak{t}f'\mathfrak{t}$ is pseudo-Anosov. Using condition (iv), there is an extension $f \in \text{MCG}(S)$ of f'. Recall that a subset A of $C(S)$ is called a *stable Alexander system* if U_A is the center of $\text{MCG}(S)$. Choose a stable Alexander system A for Σ. Let us take $U = fU_A$ and $V = U_{\{b_1\}}$. We claim that $U \cap gV\hat{\mathfrak{t}}(g)^{-1} = \emptyset$ for all $g \in \text{MCG}(S)$. For an element $g \in \text{MCG}(S)$, we consider the following cases:

Case 1: Suppose that $g(b_1)$ has a representative which is disjoint from Σ. In this case, $g(b_1) \in \Omega_i$ for some i. If $h \in gV\hat{\mathfrak{t}}(g)^{-1}$, then using condition (v), we obtain $\hat{\mathfrak{t}}(g)(b_1) \in \Omega_i$ and $h(\hat{\mathfrak{t}}(g)(b_1)) = g(b_1) \in \Omega_i$. Hence, we have $h(\Omega_i) \cap \Omega_i \neq \emptyset$. And, if $h \in U$, then $h(\Omega_i) \cap \Omega_i = \emptyset$ since $h(b_i) = b_{i+1}$. This proves that $U \cap gV\hat{\mathfrak{t}}(g)^{-1} = \emptyset$.

Case 2: Suppose that $g(b_1)$ has a non-trivial geometric intersection with Σ. Let $c \in C(\Sigma)$ be a projection of $g(b_1)$ onto Σ. If $h \in U \cap gV\hat{\mathfrak{t}}(g)^{-1}$, then $htg(b_1) = h(\hat{\mathfrak{t}}(g)(b_1)) = g(b_1)$. This gives

$$htht g(b_1) = htg(b_1) = g(b_1),$$

and consequently $(ht\mathfrak{t}ht)^n(g(b_1)) = g(b_1)$ for all $n \in \mathbb{N}$. Since $h \in U$, it follows that $(ht\mathfrak{t}ht)^n(c) \in C(\Sigma)$ is also a projection of $g(b_1)$ onto Σ for all $n \in \mathbb{N}$. Hence, by Lemma 2.9, we have

$$d_{C(\Sigma)}((ht\mathfrak{t}ht)^n(c), c) \leq 2 \tag{2}$$

for all $n \in \mathbb{N}$. Since $h \in U$, we have $ht\mathfrak{t}ht = f\mathfrak{t}f\mathfrak{t}$ on Σ, and hence the map $ht\mathfrak{t}ht$ is pseudo-Anosov on Σ. By Proposition 2.7, there exists $m \in \mathbb{N}$ such that

$$d_{C(\Sigma)}\big((f\mathfrak{t}f\mathfrak{t})^m(c), c\big) > 2$$

for all $c \in C(\Sigma)$. This contradicts (2). Hence, we must have $U \cap gV\hat{\mathfrak{t}}(g)^{-1} = \emptyset$ for all $g \in \text{MCG}(S)$. Hence, by Proposition 3.10, the proof of the theorem is complete. □

4.4. When the surface admits a compact non-displaceable subsurface

Finally, we consider the case when S admits a compact non-displaceable subsurface.

Lemma 4.8. *Consider the surface $S_{g,n}$ of genus $g \geq 0$ with n punctures, where $n > 5$ is even when $g = 0$. Suppose that $S_{g,n}$ is symmetric about the xy-plane with all the punctures lying on the xy-plane. Then there exists an essential simple closed curve c on $S_{g,n}$ such that the collection $\{c, \mathfrak{t}(c)\}$ fills the surface. Moreover, there exists an orientation-preserving homeomorphism f of $S_{g,n}$ such that $f\mathfrak{t}f\mathfrak{t}$ is pseudo-Anosov.*

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

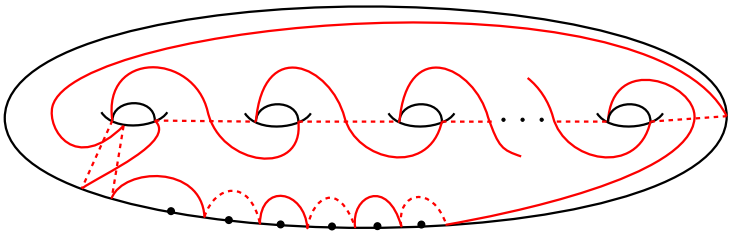


Figure 5. The curve c on $S_{g,n}$ such that $\{c, \mathfrak{t}(c)\}$ fills $S_{g,n}$.

Proof. If $g>0$, then the curve c as shown in Figure 5 together with $\mathfrak{t}(c)$ fills the surface $S_{g,n}$.

If $g=0$, then choose the simple closed curve c to be as depicted in Figure 4 for $S_{0,6}$. Take $f=T_c$, where c is the curve as above. By using Penner's construction [5, Theorem 14.4], the homeomorphism $f\mathfrak{t}f\mathfrak{t}=T_cT_{\mathfrak{t}(c)}^{-1}$ is pseudo-Anosov since the collection $\{c, \mathfrak{t}(c)\}$ fills $S_{g,n}$. □

Theorem 4.9. *Let S be a surface containing a compact non-displaceable subsurface. Then* $\mathrm{MCG}(S)$ *does not possess the ϕ-twisted Rokhlin property for any $\phi \in \mathrm{Aut}(\mathrm{MCG}(S))$.*

Proof. In view of Proposition 3.5 and [17, Theorem 3.1], it is enough to prove that $\mathrm{MCG}(S)$ does not have the $\hat{\mathfrak{t}}$-twisted Rokhlin property. Let Σ be a compact non-displaceable subsurface of S. By enlarging Σ, we can ensure that it is connected, has a stable Alexander system, satisfies $\mathfrak{t}(\Sigma)=\Sigma$, and its boundary $\partial(\Sigma)$ consists of at least six components with an even number of boundary components, each being essential and separating. Furthermore, Σ remains a finite-type subsurface of S.

We claim that there exist open subsets U and V of $\mathrm{MCG}(S)$ such that $U\cap gV\hat{\mathfrak{t}}(g)^{-1}=\emptyset$ for all $g\in\mathrm{MCG}(S)$. Let $A\subset C(S)$ be a stable Alexander system for Σ. It follows from the proof of Lemma 4.8 that there is an orientation-preserving homeomorphism $f':\Sigma\to\Sigma$ which fixes $\partial(\Sigma)$ point-wise such that $f'\mathfrak{t}f'\mathfrak{t}$ is pseudo-Anosov. Thus, by Proposition 2.7, there exists $m\in\mathbb{N}$ such that

$$d_{C(\Sigma)}\big((f'\mathfrak{t}f'\mathfrak{t})^m(a), a\big) > 2 \tag{3}$$

for all $a\in C(\Sigma)$. Let $f\in\mathrm{MCG}(S)$ denote the extension of f'. Take $U=fU_A$ and $V=U_A$. Suppose that there exists $g\in\mathrm{MCG}(S)$ such that $h\in fU_A\cap gU_A\hat{\mathfrak{t}}(g)^{-1}$. Since $h\in gU_A\hat{\mathfrak{t}}(g)^{-1}=g\hat{\mathfrak{t}}(g)^{-1}U_{\hat{\mathfrak{t}}(g)(A)}$, we have

$$h\mathfrak{t}g\mathfrak{t}(c)=g(c)$$

for all $c\in A$. Thus, we obtain

$$h\mathfrak{t}h\mathfrak{t}(g(c))=h\mathfrak{t}h\mathfrak{t}g\mathfrak{t}\mathfrak{t}(c)=h\mathfrak{t}g(\mathfrak{t}(c))=g(c) \tag{4}$$

for all $c\in A$. Since Σ is non-displaceable in S, there exists a curve $c\in C(\Sigma)$ such that $g(c)$ and Σ have a non-trivial geometric intersection. Let b denote a projection of $g(c)$

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

onto Σ. By (4), $(h\mathfrak{t}h\mathfrak{t})^n(b)$ is also a projection of $g(c)$ onto $(h\mathfrak{t}h\mathfrak{t})^n(\Sigma) = \Sigma$. By Lemma 2.9, we have

$$d_{C(\Sigma)}((h\mathfrak{t}h\mathfrak{t})^n(b), b) \leq 2$$

for all $n \in \mathbb{N}$. Since $h \in fU_A$, by (3), we have

$$d_{C(\Sigma)}((h\mathfrak{t}h\mathfrak{t})^m(b), b) = d_{C(\Sigma)}((f\mathfrak{t}f\mathfrak{t})^m(b), b) > 2$$

for some m, which is the contradiction. Hence, we conclude that $fU_A \cap gU_A\hat{\mathfrak{t}}(g)^{-1} = \emptyset$ for all $g \in \mathrm{MCG}(S)$. Hence, by Proposition 3.10, the proof of the theorem is complete. □

The outcomes from this section lead to the following theorem.

Theorem 4.10. *Let S be a connected orientable infinite-type surface without boundary. Then the following assertions are equivalent:*

(1) $\mathrm{MCG}(S)$ *has the twisted Rokhlin property.*
(2) $\mathrm{MCG}(S)$ *has the Rokhlin property.*
(3) $\mathrm{MCG}(S)$ *has the ϕ-twisted Rokhlin property for some automorphism ϕ of* $\mathrm{MCG}(S)$.
(4) *S has a unique maximal end and each of its compact subsurfaces is displaceable.*

Proof. (1) $\implies$ (2) follows from the definition of the twisted Rokhlin property. (2) $\implies$ (3) follows by taking ϕ to be the identity map. For (3) $\implies$ (4), suppose that $\mathrm{MCG}(S)$ has the ϕ-twisted Rokhlin property for some automorphism ϕ of $\mathrm{MCG}(S)$. Since S is an infinite-type surface, by Theorem 4.9, each compact subsurface of S must be displaceable. Further, by Theorem 2.3, Theorem 4.2, Proposition 4.4, and Theorem 4.7, it follows that S has a unique maximal end. Finally, (4) $\implies$ (1) follows directly from Theorem 4.2. □

5. R_∞-property of big mapping class groups

In this section, we investigate the number of twisted conjugacy classes in mapping class groups.

Definition 5.1. *Let G be a group and $\phi \in \mathrm{Aut}(G)$. The* Reidemeister number *of ϕ, denoted by $R(\phi)$, is the number of ϕ-twisted conjugacy classes in G. The group G is said to have the* R_∞-property *if $R(\phi)$ is infinite for each automorphism ϕ of G.*

The R_∞-property for the mapping class groups of finite-type surfaces has been investigated in [8, Theorem 4.3]. It has been proven that the mapping class group of a closed orientable surface has the R_∞-property if and only if the surface is not the 2-sphere. A partial result for infinite-type surfaces has been established in [4, Theorem 3.4], for instance, it is proven that if a connected orientable infinite-type surface without boundary admits a non-displaceable compact subsurface, then its mapping class group possesses the R_∞-property. The aim of this section is to provide a complete answer for infinite-type surfaces through a general approach.

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

Let S be a connected orientable infinite-type surface without boundary. Recall that, $\mathrm{Out}(\mathrm{MCG}(S)) \cong \mathbb{Z}_2$. In fact, the group is generated by the coset of the restriction of the inner automorphism $\hat{f}$ of $\mathrm{MCG}^{\pm}(S)$ induced by an orientation reversing homeomorphism f of S of order two [3, Theorem 1.1]. We use the same notation $\hat{f}$ to denote its restriction on $\mathrm{MCG}(S)$.

The subsequent lemma provides a connection between $\hat{f}$-twisted conjugacy classes in $\mathrm{MCG}(S)$ and conjugacy classes of $\mathrm{MCG}^{\pm}(S)$ lying in the coset $\mathrm{MCG}(S)f$. See [7, Lemma 2.1] for a general result.

Lemma 5.2. *Let S be a connected orientable infinite-type surface without boundary and f an orientation reversing homeomorphism of S. Then, two elements $g, h \in \mathrm{MCG}(S)$ are $\hat{f}$-twisted conjugate in $\mathrm{MCG}(S)$ if and only if gf and hf are conjugate in $\mathrm{MCG}^{\pm}(S)$.*

Proof. Suppose that there exists $r \in \mathrm{MCG}(S)$ such that $h = rg\hat{f}(r)^{-1} = rgfr^{-1}f^{-1}$. But, this gives $hf = r(gf)r^{-1}$, that is, gf and hf are conjugate in $\mathrm{MCG}^{\pm}(S)$. Conversely, suppose that there exists $s \in \mathrm{MCG}^{\pm}(S)$ such that $hf = s(gf)s^{-1}$. If $s \in \mathrm{MCG}(S)$, then $h = sgfs^{-1}f^{-1} = sg\hat{f}(s)^{-1}$, and we are done. If $s \notin \mathrm{MCG}(S)$, then $s = tf$ for some $t \in \mathrm{MCG}(S)$. In this case, we have

$$h = t(fgf)f^{-1}t^{-1}f^{-1} = t(fgf^{-1})\hat{f}(t)^{-1}.$$

Hence, h and (fgf^{-1}) are $\hat{f}$-twisted conjugate in $\mathrm{MCG}(S)$. Since $(fgf^{-1}) = g^{-1}g(fgf^{-1}) = g^{-1}g\hat{f}(g^{-1})^{-1}$, the elements (fgf^{-1}) and g are $\hat{f}$-twisted conjugate in $\mathrm{MCG}(S)$. Consequently, h and g are $\hat{f}$-twisted conjugate in $\mathrm{MCG}(S)$, and the proof is complete. □

Lemma 5.3. *Let S be a connected orientable surface.*

(1) *Let $\{a_1, \ldots, a_l\}$ and $\{b_1, \ldots, b_k\}$ be multicurves on S. Let $p_1, \ldots, p_l$ and $q_1, \ldots, q_k$ be non-zero integers. If*

$$T_{a_1}^{p_1}T_{a_2}^{p_2}\cdots T_{a_l}^{p_l} = T_{b_1}^{q_1}T_{b_2}^{q_2}\cdots T_{b_k}^{q_k}$$

in $\mathrm{MCG}(S)$, then $l = k$ and $\{T_{a_1}^{p_1}, \ldots, T_{a_l}^{p_l}\} = \{T_{b_1}^{q_1}, \ldots, T_{b_k}^{q_k}\}$ [5, Lemma 3.17].

(2) *Let a and b be essential simple closed curves on S. Then $T_a^k = T_b^l$ if and only if $a = b$ and $k = l$ [5, Section 3.3].*

Lemma 5.4. [9, Corollary 3.2] *Let G be a group, $\phi \in \mathrm{Aut}(G)$ and $g \in G$. Then $R(\phi\,\hat{g}) = R(\phi)$. In particular, $R(\hat{g})$ is the number of conjugacy classes in G.*

Proposition 5.5. *Let S be a connected orientable infinite-type surface without boundary such that either $g \geq 2$ (can be infinite) or every compact subsurface of S is displaceable. Then there exists an essential simple closed curve c and an orientation reversing homeomorphism f of S of order two such that $\{c, f(c)\}$ forms a multicurve.*

Proof. We have the following two cases:

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

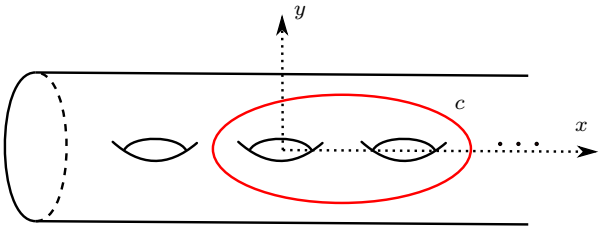


Figure 6. The curve c such that $\{c, \mathfrak{t}(c)\}$ forms a multicurve.

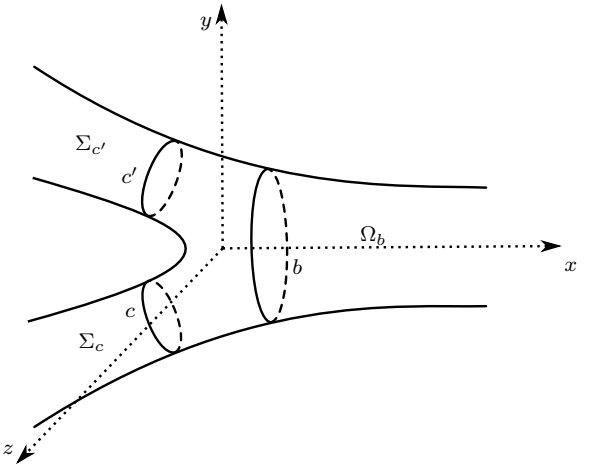


Figure 7. The curve c such that $\{c, f(c) = c'\}$ forms a multicurve.

Case 1: Suppose that $g \geq 2$ (including infinite genus). We assume that the surface S is symmetric about the xy-plane when embedded in $\mathbb{R}^3$. Then the orientation reversing homeomorphism $\mathfrak{t}$ and the curve c shown in Figure 6 have the property that $\{c, \mathfrak{t}(c)\}$ forms a multicurve.

Case 2: Suppose that every compact subsurface of S is displaceable. In this case, the genus of S is either zero or infinite. The latter possibility is already considered in Case 1. So, we assume that the genus is zero, that is, S is planar. In view of Theorem 2.3, we have three possibilities.

If S has a unique maximal end, then by the proof of [17, Proposition 4.2], there exist curves $\{c, c', b\}$ which bound a pair of pants P such that $\{c, c'\} \subset \Sigma_b$ and $\Sigma_c \cong \Sigma_{c'}$. We realise the subsurface P to be symmetric about the xz-plane such that c and c' do not intersect the xz-plane. We then glue the subsurfaces Σ_c, $\Sigma_{c'}$, and Ω_b to P along c, c', and b, respectively, such that the resulting surface S is symmetric about the xz plane (see Figure 7). Now, we take f to be the orientation reversing homeomorphism of S which maps a point on S to its mirror image about the xz-plane. Then the collection $\{c, f(c) = c'\}$ forms a multicurve.

If S has two maximal ends, then by [17, Lemma 5.4], there is a homeomorphism of S permuting the two maximal ends. In particular, such a homeomorphism maps an open

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

neighbourhood of one end onto an open neighbourhood of the other. Without loss of generality, we can choose an open neighbourhood U of one of the ends to be such that its boundary is a separating simple closed curve, and U and its image V have disjoint closures. We can realise our surface $S \setminus (U \cup V)$ in such a way that it is symmetric about the xz-plane and $\partial(S \setminus (U \cup V))$ does not intersect the xz-plane. Now, we glue the neighbourhoods U and V back onto $S \setminus (U \cup V)$ along the boundary curves such that the resulting surface S is symmetric about the xz-plane. Let f be the orientation reversing homeomorphism of S, which maps a point on S to its mirror image about the xz-plane. Then f is of order two and permutes the two maximal ends. If c is a curve in a neighbourhood of one of the maximal ends, then $\{c, f(c)\}$ forms a multicurve. If S is such that the space $\mathcal{M}(S)$ of maximal ends is homeomorphic to the Cantor space, then by the proof of Theorem 4.7, there exist curves $\{b_1, \ldots, b_6\}$ bounding S_0^6, which is symmetric about the xz-plane (see Figure 4). By using a similar argument as above, we can realise our surface S to be symmetric about the xz-plane. We take f to be the orientation reversing homeomorphism of S which maps a point of S to its mirror image about the xz-plane. In this case, $\{b_1, f(b_1) = b_5\}$ forms a multicurve. □

Proposition 5.6. *Let S be a connected orientable infinite-type surface without boundary. Let f be an orientation reversing homeomorphism of S of order two such that there exists an essential simple closed curve c for which $\{c, f(c)\}$ is a multicurve. Then the right coset $\mathrm{MCG}(S)f$ contains infinitely many conjugacy classes of the group $\mathrm{MCG}^{\pm}(S)$.*

Proof. We claim that, for each $n \in \mathbb{N}$, the element $T_c^n f$ represents a distinct conjugacy class in $\mathrm{MCG}^{\pm}(S)$. We set $d := f(c)$. Suppose that $T_c^n f$ is conjugate to $T_c^m f$ in $\mathrm{MCG}^{\pm}(S)$ for some $m, n \geq 1$. This implies that $(T_c^n f)^2 = T_c^n T_d^{-n}$ is conjugate to $(T_c^m f)^2 = T_c^m T_d^{-m}$ in $\mathrm{MCG}^{\pm}(S)$. Thus, there exists $g \in \mathrm{MCG}^{\pm}(S)$ such that

$$T_c^n T_d^{-n} = g T_c^m T_d^{-m} g^{-1} = T_{g(c)}^{\epsilon m} T_{g(d)}^{-\epsilon m},$$

where

$$\epsilon = \begin{cases} 1 & \text{if } g \text{ is orientation preserving,} \\ -1 & \text{if } g \text{ is orientation reversing.} \end{cases}$$

By Lemma 5.3, this is possible only when $m = n$. □

Theorem 5.7. *Let S be a connected orientable infinite-type surface without boundary. Then $\mathrm{MCG}(S)$ has the R_∞-property.*

Proof. It follows from Lemmas 2.5 and 5.3 that if c is an essential simple closed curve on S, then T_c^m is conjugate to T_c^n in $\mathrm{MCG}(S)$ if and only if $m = n$. Thus, $\mathrm{MCG}(S)$ has infinitely many conjugacy classes. If S admits a non-displaceable compact subsurface of genus 0 or 1, then the theorem follows from [4, Theorem 3.4]. So, we assume that either every compact subsurface of S is displaceable, or S has genus at least two. Since $\mathrm{Out}(\mathrm{MCG}(S)) \cong \mathbb{Z}_2$, in view of Lemma 5.4, it is enough to show that $R(\hat{f})$ is infinite for an orientation reversing homeomorphism f of S of order two. By Proposition 5.5, we can

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

take f to be such that there exists an essential simple closed curve c for which $\{c, f(c)\}$ forms a multicurve. By Proposition 5.6, $\mathrm{MCG}(S)f$ contains infinitely many conjugacy classes of the group $\mathrm{MCG}^{\pm}(S)$. Finally, Lemma 5.2 implies that $R(\hat{f})$ is infinite, which completes the proof. □

6. Further directions

We conclude with the following questions that arose naturally during our investigation.

(1) Our findings indicate that when S is an infinite-type surface without boundary, the mapping class group $\mathrm{MCG}(S)$ either exhibits the ϕ-twisted Rokhlin property for all automorphisms ϕ, or it lacks the ϕ-twisted Rokhlin property for each automorphism ϕ. We do not know an example of a non-abelian topological group G which admits a pair of automorphisms ϕ, ψ such that G has the ϕ-twisted Rokhlin property, but not the ψ-twisted Rokhlin property? In general, it would be interesting to find such examples in topological groups that admit 'sufficiently large' outer automorphism groups.
(2) Suppose that H is a (closed) subgroup of a topological group G such that each automorphism of H is the restriction of an inner automorphism of G. Is it true that H has the Rokhlin property if and only if it has the twisted Rokhlin property?
(3) Which other interesting topological groups admit the twisted Rokhlin property?

Acknowledgements The authors are grateful to the anonymous referee(s) for many valuable suggestions which have improved the exposition of the paper. Pravin Kumar is supported by the PMRF fellowship at IISER Mohali. Mahender Singh is supported by the Swarna Jayanti Fellowship grants DST/SJF/MSA-02/2018-19 and SB/SJF/2019-20/04. Apeksha Sanghi acknowledges post-doctoral fellowship from the grant SB/SJF/2019-20/04.

Competing interests The authors declare that they have no competing interests and no data are associated with this paper.

References

1. T. Aougab, P. Patel and N. G. Vlamis, Isometry groups of infinite-genus hyperbolic surfaces, *Math. Ann.* **381**(1-2) (2021), 459–498.
2. J. Aramayona and N. Vlamis, Big mapping class groups: an overview. In *The Tradition of Thurston–Geometry and Topology*, pp. 459–496 (Springer, Cham, 2020).
3. J. Bavard, S. Dowdall and K. Rafi, Isomorphisms between big mapping class groups, *Int. Math. Res. Not. IMRN*(10) (2020), 3084–3099.
4. S. Bhunia and S. Krishna, Twisted conjugacy in big mapping class groups, *Topology Appl.* **324** (2023), 13.
5. B. Farb and D. Margalit, A primer on mapping class groups. *Princeton Mathematical Series*, Volume 49, p. xiv+472 (Princeton University Press, Princeton, NJ, 2012).
6. A. Fel'shtyn, Dynamical zeta functions, Nielsen theory and Reidemeister torsion, *Mem. Amer. Math. Soc.* **147**(699) (2000), xii+146.

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press

7. A. Fel'shtyn, New directions in Nielsen–Reidemeister theory, *Topology Appl.* **157**(10-11) (2010), 1724–1735.
8. A. Fel'shtyn and D. L. Gonçalves, Twisted conjugacy classes in symplectic groups, mapping class groups and braid groups (with an appendix written jointly with Francois Dahmani, *Geom. Dedicata* **146** (2010), 211–223).
9. A. Fel'shtyn, Y. Leonov and E. Troitsky, Twisted conjugacy classes in saturated weakly branch groups, *Geom. Dedicata* **134** (2008), 61–73.
10. F. Gantmacher, Canonical representation of automorphisms of a complex semi-simple Lie group, *Rec. Math. (Moscou)* **5**(47) (1939), 101–146.
11. E. Glasner and B. Weiss, The topological Rohlin property and topological entropy, *Amer. J. Math.* **123**(6) (2001), 1055–1070.
12. A. Grothendieck. Formules de Nielsen-Wecken et de Lefschetz en géométrie algébrique, in: *Séminaire de Géométrie Algébrique du Bois-Marie 1965–66 SGA 5* (*Lecture Notes in Mathematics*), Springer, Berlin, 1977, pp. 407–441Volume 569.
13. J. Hernández, M. Hrušák, I Morales, A. Randecker, M. Sedano and F. Valdez, Conjugacy classes of big mapping class groups, *J. Lond. Math. Soc.* **106**(2) (2022), 1131–1169.
14. A. S. Kechris and C. Rosendal, Turbulence, amalgamation, and generic automorphisms of homogeneous structures, *Proc. Lond. Math. Soc.* (**3**) **94**(2) (2007), 302–350
15. M. Korkmaz, Generating the surface mapping class group by two elements, *Trans. Amer. Math. Soc.* **357**(8) (2005), 3299–3310.
16. J. Lanier and M. Loving, Centers of subgroups of big mapping class groups and the Tits alternative, *Glas. Mat. Ser. III* **55(75)**(1) (2020), 85–91.
17. J. Lanier and N. G. Vlamis, Mapping class groups with the Rokhlin property, *Math. Z.* **302**(3) (2022), 1343–1366.
18. K. Mann and K. Rafi, Large scale geometry of big mapping class groups, *Geom. Topol.* **27** (2023), 2237–2296.
19. H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. I. Hyperbolicity, *Invent. Math.* **138**(1) (1999), 103–149.
20. H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. II. Hierarchical structure, *Geom. Funct. Anal.* **10**(4) (2000), 902–974.
21. K. Reidemeister, Automorphismen von Homotopiekettenringen, *Math. Ann.* **112** (1936), 586–593.
22. I Richards, On the classification of noncompact surfaces, *Trans. Amer. Math. Soc.* **106** (1963), 259–269.
23. S. Shokranian, *The Selberg-Arthur Trace formula, Based on Lectures by James Arthur* (*Lecture Notes in Mathematics*), p. viii+97 (Springer-Verlag, Berlin, 1992).
24. N. G. Vlamis, Notes on the topology of mapping class groups. http://qcpages.qc.cuny.edu/nvlamis/Papers/AIM-Notes.pdf.
25. P. Wesolek, Conjugacy class conditions in locally compact second countable groups, *Proc. Amer. Math. Soc.* **144**(1) (2016), 399–409.

https://doi.org/10.1017/S001309152510093X Published online by Cambridge University Press