References

1. Roger A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, (2007) pp. 11-15.
2. G. Leversha, *The Geometry of the Triangle*, UKMT (2013).
3. H. S. M. Coxeter, *Introduction to Geometry*, John Wiley and Sons, (1989) pp. 21-22.

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101.40 Dandelin circles

The object of this Note is to show how the two-dimensional analogue of a familiar three-dimensional configuration reveals relations between the radii of the in- and e-circles of a triangle.

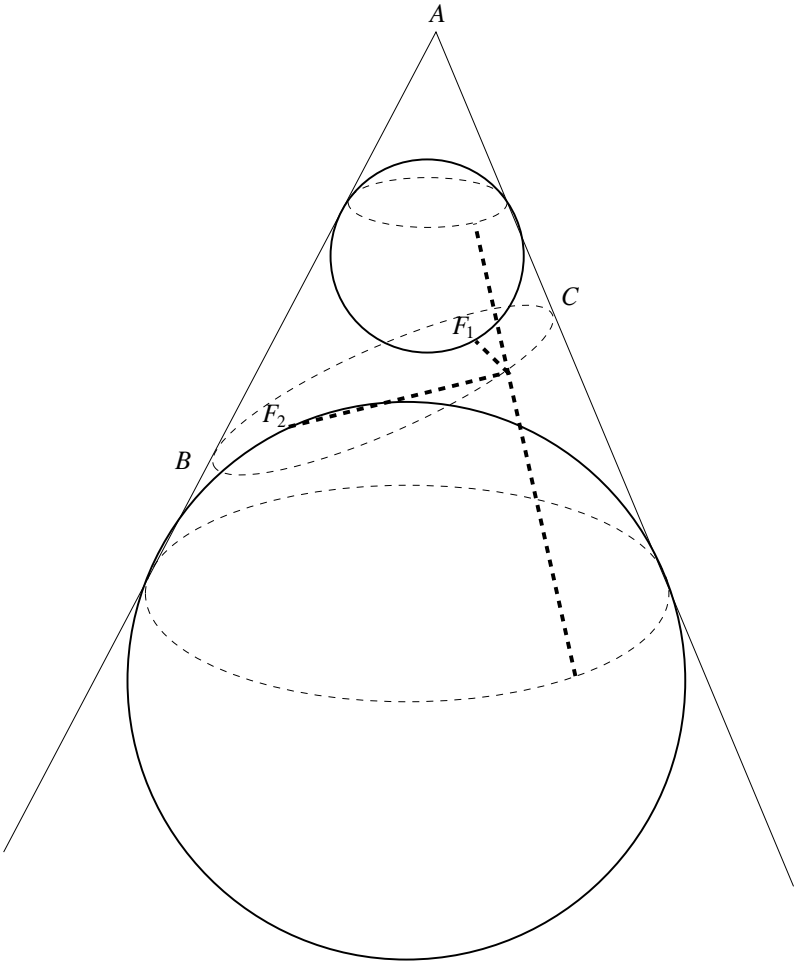


FIGURE 1

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Readers will be familiar with the construction due to Pierre Dandelin (1822) [1, 2]: two spheres are inscribed in a cone (or, more generally, a hyperboloid of one sheet) and touch at points F_1, F_2 a plane cutting the cone in an ellipse. Because tangents from a point to a circle are equal, and those from a point on the ellipse to the circles of tangency of the two spheres have a constant total by symmetry, so too do those from the point to F_1, F_2. There are therefore two points within an ellipse from which the combined distance to a point upon it is constant [Figure 1].

If we take a section containing F_1, F_2 and the sphere centres, we have Figure 2.

By analogy with the three-dimensional case, we have:

$$L_1L_2 = CF_1 + CF_2 = 2CF_1 + F_1F_2$$
$$M_1M_2 = BF_1 + BF_2 = 2BF_2 + F_1F_2.$$

But

$$L_1L_2 = M_1M_2 \Rightarrow CF_1 = BF_2 \text{ and } CF_2 = BF_1. \quad (1)$$

We shall call this 'the Dandelin result'.

We recognise Π_2 as the in-circle and Π_2 as one of the three e-circles of triangle ABC.

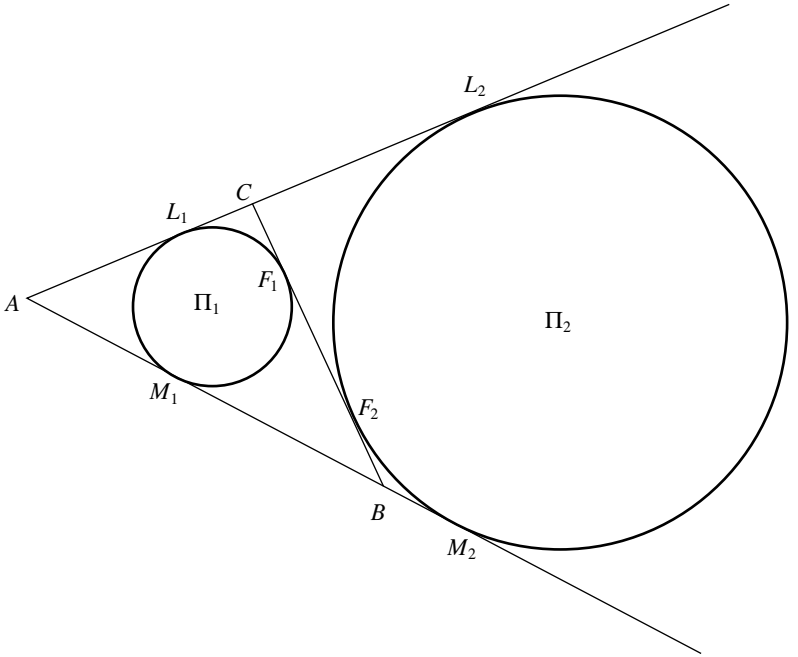


FIGURE 2

Figure 3 shows a triangle, its in-circle and three e-circles, with points of tangency prominently marked.

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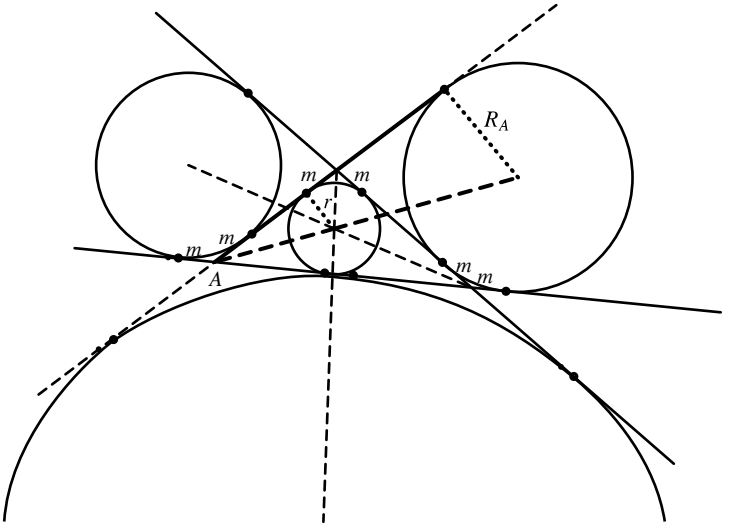


FIGURE 3

By virtue of the Dandelin result and the tangents-from-a-point property, three sets of three pairs of equal lengths, k, l, m, arise. We have just picked out the occurrences of m in the figure. These three lengths make up the

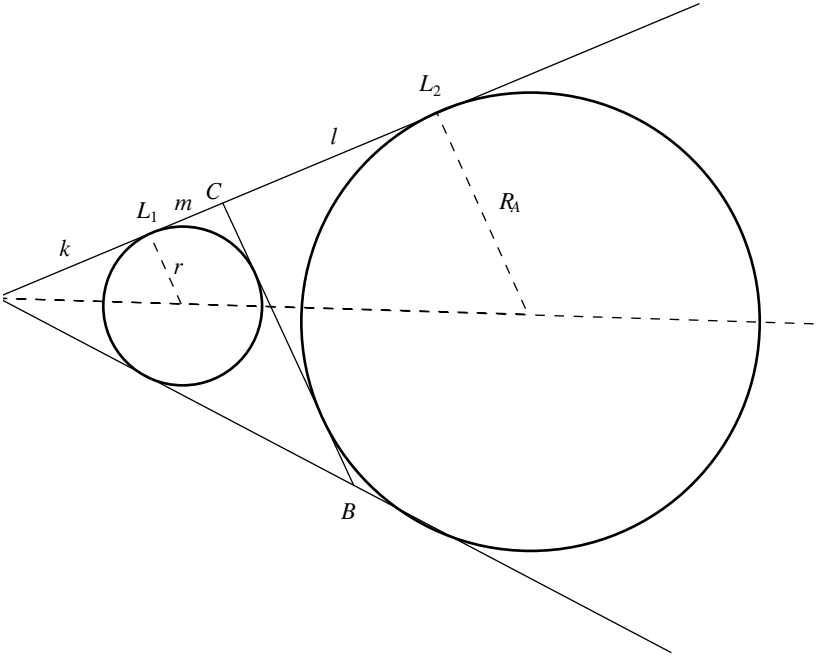


FIGURE 4

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triangle side lengths in pairs and all three make up the distance from a vertex to the point of tangency with the opposite e-circle, which is therefore constant. The interior angle bisectors are shown by dashed lines. Two similar right triangles are picked out in bold and reproduced in Figure 4.

We have

$$r = \frac{kR_A}{k + l + m} \tag{2}$$

and similar expressions involving R_B and R_C.

The area, Δ, of the triangle is the product of the semi-perimeter and r, whence

$$\Delta = (k + l + m)r = kR_A = lR_B = mR_C. \tag{3}$$

In the case of a right angle at A, $r = k$, so we have

$$\Delta = rR_A, \tag{4}$$

a result students can also derive using only the tangents-from-a-point theorem and the Pythagorean condition.

Acknowledgement

I would like to thank the referee for detailed suggestions which have, I hope, improved the clarity of my presentation.

References

1. Students may consult the Wikipedia entry ‘Dandelin spheres’.
2. For the photograph of an experimental set-up, go to www.magicmathworks.org, then Geomlab, then 7.

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101.41 On conics

The goal of this Note is to demonstrate the equivalence of three definitions of conic sections by means of a geometric construction. This also gives a construction of the directrices of any conic section by means of a pair of compasses and an unmarked straightedge only. Given the elementary nature of the arguments presented here, it is very likely that all the constructions presented below have long been known. This note has been submitted for publication only after the author's failure to find them in the literature on the subject available to him.

The ellipse

Points will be denoted by capital letters. AB is the straight segment between A and B. $|AB|$ is the length of AB. $\overline{AB}$ is the straight line through A

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