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Monotone iterative sequences for non-local elliptic problems

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In this paper we establish an existence and uniqueness result for a class of non-local elliptic differential equations with the Dirichlet boundary conditions, which, in general, do not accept a maximum principle. We introduce one monotone sequence of lower–upper pairs of solutions and prove uniform convergence of that sequence to the actual solution of the problem, which is the unique solution for some range of λ (the parameter of the problem). The convergence of the iterative sequence is tested through examples with an order of convergence greater than 1.

Key words: Non-Local Problems; Maximum Principle; Elliptic Differential Equations.

1 Introduction

In this paper we consider a class of non-local elliptic partial differential equations of the form

$$P(u) = \nabla^2 u + \lambda \frac{f(u)}{(h(u))^p} = 0, \quad \text{in } \Omega, \tag{1.1}$$

subject to linear boundary condition of the Dirichlet type,

$$\mathscr{B}(u) = r(x), \quad \text{on } \partial\Omega, \tag{1.2}$$

where $p \geqslant 1$, and $h(u) = \int_\Omega f(u)dx$. Here, Ω is a compact smooth domain, $\partial\Omega$ is the boundary of Ω, ∇^2 denotes the Laplacian operator, which depends on Ω, and λ is the non-negative parameter of the problem. Furthermore, we assume that $f(\cdot)$ is a positive monotone function, while $r(\cdot)$ is also assumed to be a smooth function. Non-local problem (1.1)–(1.2) appears in many applications. When f is decreasing, it models the steady state temperature of flows through a material subject to a fixed potential difference [18]. A form of $f(u)$ that represents the temperature-dependent electrical resistivity can be

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$f(u) = e^{-u}$ or $f(u) = \frac{1}{(1+u)^2}$, as discussed in [4, 14]. When $f(u) = e^u$ and $p = 1$, problem (1.1)–(1.2) describes the stationary state arising in the occurrence of shear bands in metals being deformed under high strain rates for $N = 1$ [5, 7], in the theory of gravitational equilibrium of polytropic stars [17], as well as in the study of the biological phenomena of chemotaxis [28], for $N \geqslant 2$. An interesting feature of the system is the existence of a blow-up parameter value, λ^*, such that the problem (1.1)–(1.2) has a unique solution for $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$. In the latter case the solution $U(x,t)$ of the time-dependent is expected to exhibit the phenomena of finite-time blow-up, i.e. $||U(\cdot,t)||_\infty \to \infty$ as $t \to T^* < \infty$. This critical parameter value, λ^*, depends on the function $f(u)$ and on the value of p, as well as on the domain Ω and on the type of boundary conditions, see [6, 8]. Many estimates for λ^* were derived using numerical and analytical approaches, see [4, 12, 18, 19, 22, 27].

The method of upper and lower solutions is an effective tool in studying non-linear elliptic and parabolic partial differential equations. It has been used to prove the existence and uniqueness of solutions, to explore multiplicity and bifurcation of solutions, to estimate the blow-up parameters and to study the stability of solutions. For the elliptic equation

$$P(u) = \nabla^2 u + \lambda g(u) = 0, \qquad \text{in } \Omega, \tag{1.3}$$

$$\mathscr{B}(u) = r(x), \quad \text{on } \partial\Omega, \tag{1.4}$$

we have the following definition of upper and lower solutions:

Definition 1.1 *A function $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is called an upper solution for (1.3)–(1.4) if it satisfies the differential inequality*

$$P(v) = \nabla^2 v + \lambda g(v) \leqslant 0, \quad \text{in } \Omega,$$

with

$$\mathscr{B}(v) \geqslant r(x), \quad \text{on } \partial\Omega.$$

Analogously, a function $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is called a lower solution if it satisfies the reversed inequalities.

In addition, if $w \leqslant v$ in Ω, we say that w and v are ordered lower and upper solutions.

The maximum principle does not hold if $g(u)$ is increasing, but the constructing ordered upper and lower solutions guarantee the existence of a solution for the problem (1.3)–(1.4), without the condition on g to be decreasing [10] (Theorem 3.1). If $g(u)$ satisfies the one-sided Lipschitz condition, then we can define two convergent monotone sequences of upper and lower solutions. Moreover, if $g(u)$ is decreasing, then the two sequences converge to the exact solution of the problem (see [23, pp. 99]). The two sequences are generated sequentially and do not depend on each other. That is, we start with an upper solution and use it to generate a decreasing sequence of upper solutions, and we obtain an increasing sequence of lower solutions starting with a lower solution, see [23]. Similar sequences were introduced for elliptic and hyperbolic systems with non-local boundary conditions [1, 16]. However, the above results are not valid, in general, for non-local

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problems. For instance, the existence of ordered upper and lower solutions does not guarantee that the actual solution is lying between them as shown in [3], see also [18]. This is due to the lack of monotonicity caused by the presence of the non-local term. The non-linear term $g(u) = \frac{f(u)}{(h(u))^p}$ does not appear to satisfy any monotonicity property, even for the simple linear case with $f(u) = u$, we can't see the behaviour of $g(u)$. Also, we cannot extend previous results to deal with the non-local problem (1.1)–(1.2), as $g(u)$ does not satisfy the one-sided Lipschitz condition. Therefore, different approaches have been used to study the problem. The existence and uniqueness of solutions for some cases of non-local elliptic equations with the homogeneous Dirichlet boundary conditions have been studied in [9] using the Galerkin method, and in [26] using the fixed point theory and Green's function.

However, in order to obtain existence and uniqueness results for problem (1.1)–(1.2) via comparison arguments, we introduce the notion of comparison functions different from lower and upper solutions. The notion of lower–upper solutions has also been studied for local elliptic systems in [23], but our approach is a bit different due to the existence of the non-local term. Indeed, we first define the notion of lower–upper pairs of solutions of problem (1.1)–(1.2), then we derive the unique solution, for a range of parameter λ, by constructing an iterative sequence of lower–upper pairs of solutions.

This paper is organised as follows. In Section 2, we present the notion of ordered lower–upper pairs of solutions and obtain a monotone iterative sequence of such pairs of solutions. We then apply the maximum principle to prove the convergence of that sequence to the actual solution of the problem. We discuss in details the case where f is decreasing and study in brief the case where f is increasing. In Section 3, we present numerical examples to explore the convergence of the presented method. Finally, in Section 4 we close up with some concluding remarks.

2 Ordered lower–upper pairs of solutions

In this section, we construct convergent lower–upper pairs of solutions for the problem (1.1)–(1.2). We will consider two cases: (i) f is decreasing, and (ii) f is increasing. We present the details for the case f is decreasing in subsection 2.1 and highlight the needed adjustments for the case when f is increasing in subsection 2.2.

2.1 The case f is decreasing

We start with the following definition of lower–upper pair of solutions for the problem (1.1)–(1.2), when f is decreasing.

Definition 2.1 A pair of functions $(w, v) \in (C^2(\Omega) \cap C(\overline{\Omega}))^2$ is called a lower–upper pair of solutions to (1.1)–(1.2) if they satisfy the following inequalities:

$$\nabla^2 w + \lambda \frac{f(v)}{(h(w))^p} \geqslant 0, \quad \text{in } \Omega, \qquad \mathscr{B}(w) \leqslant r(x) \ \text{ on } \partial\Omega, \tag{2.1}$$

$$\nabla^2 v + \lambda \frac{f(w)}{(h(v))^p} \leqslant 0, \quad \text{in } \Omega, \qquad \mathscr{B}(v) \geqslant r(x) \ \text{ on } \partial\Omega. \tag{2.2}$$

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If $w \leqslant v$ in Ω, we say that the pair (w, v) is an *ordered* lower–upper pair of solutions to (1.1)–(1.2).

Remark 2.1 In this case, since the maximum principle is applicable, lower and upper solutions are ordered, though they are not so useful for our purposes as in the study of the local problem. On the other hand, lower and upper solutions are not necessarily ordered when f is increasing, see [18].

Given an ordered lower–upper pair of solutions (w, v), we define the set $[w, v] = \{z \in C^2(\Omega) : w \leqslant z \leqslant v\}$. We have the following lemma.

Lemma 2.1 *Consider the non-local elliptic equation (1.1)–(1.2) with $(w^{(0)}, v^{(0)})$ an ordered lower–upper pair of solutions as in Definition 2.1. Let $w^{(1)}$ and $v^{(1)}$ be the solutions to the following:*

$$\nabla^2 w^{(1)} + \lambda \frac{f(v^{(0)})}{(h(w^{(0)}))^p} = 0, \quad \text{in } \Omega, \tag{2.3}$$

$$\mathscr{B}(w^{(1)}) = \mathscr{B}(w^{(0)}) \leqslant r(x), \quad \text{on } \partial\Omega, \tag{2.4}$$

$$\nabla^2 v^{(1)} + \lambda \frac{f(w^{(0)})}{(h(v^{(0)}))^p} = 0, \quad \text{in } \Omega, \tag{2.5}$$

$$\mathscr{B}(v^{(1)}) = \mathscr{B}(v^{(0)}) \geqslant r(x), \quad \text{on } \partial\Omega. \tag{2.6}$$

Then we have

(i) $w^{(1)}, v^{(1)} \in [w^{(0)}, v^{(0)}]$.

(ii) $(w^{(1)}, v^{(1)})$ *is an ordered lower–upper pair of solutions to (1.1)–(1.2).*

Proof For (i), we need to show that $w^{(0)} \leqslant w^{(1)} \leqslant v^{(0)}$ and $w^{(0)} \leqslant v^{(1)} \leqslant v^{(0)}$. We show that the former and similar argument can be applied to prove the latter one. To this end, subtract equation (2.1) from (2.3) to get

$$\begin{aligned} \nabla^2(w^{(1)} - w^{(0)}) &\leqslant 0, \quad \text{in } \Omega, \\ \mathscr{B}(w^{(1)} - w^{(0)}) &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

which imply, by the maximum principle, that $w^{(1)} - w^{(0)} \geqslant 0$ or $w^{(0)} \leqslant w^{(1)}$. Similarly, subtract equation (2.2) from (2.3) to get

$$\begin{aligned} \nabla^2(w^{(1)} - v^{(0)}) &\geqslant \lambda \left[\frac{f(w^{(0)})}{(h(v^{(0)}))^p} - \frac{f(v^{(0)})}{(h(w^{(0)}))^p} \right], \quad \text{in } \Omega, \\ \mathscr{B}(w^{(1)} - v^{(0)}) &\leqslant 0, \quad \text{on } \partial\Omega. \end{aligned}$$

As $w^{(0)} \leqslant v^{(0)}$ and f is decreasing, we have $\nabla^2(w^{(1)} - v^{(0)}) \geqslant 0$. By the maximum principle, $w^{(1)} - v^{(0)} \leqslant 0$ or $w^{(1)} \leqslant v^{(0)}$. This proves that $w^{(1)} \in [w^{(0)}, v^{(0)}]$.

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For (ii), we need to show that the pair $(w^{(1)}, v^{(1)})$ satisfies equations (2.1) and (2.2) and $w^{(1)} \leqslant v^{(1)}$. First we show that $w^{(1)} \leqslant v^{(1)}$ in Ω. Subtract equation (2.5) from (2.3) to obtain

$$\nabla^2(w^{(1)} - v^{(1)}) = \lambda \left[\frac{f(w^{(0)})}{(h(v^{(0)}))^p} - \frac{f(v^{(0)})}{(h(w^{(0)}))^p} \right] \geqslant 0, \tag{2.7}$$

as $w^{(0)} \leqslant v^{(0)}$ and f is decreasing. From equations (2.4) and (2.6), we have

$$\mathscr{B}(w^{(1)} - v^{(1)}) \leqslant 0, \quad \text{on } \partial\Omega. \tag{2.8}$$

By the maximum principle, equations (2.7) and (2.8) imply that $w^{(1)} - v^{(1)} \leqslant 0$, and hence $w^{(1)} \leqslant v^{(1)}$. Next, we have

$$\nabla^2 w^{(1)} + \lambda \frac{f(v^{(1)})}{(h(w^{(1)}))^p} = \lambda \left[\frac{f(v^{(1)})}{(h(w^{(1)}))^p} - \frac{f(v^{(0)})}{(h(w^{(0)}))^p} \right] \geqslant 0, \quad \text{in } \Omega, \tag{2.9}$$

$$\nabla^2 v^{(1)} + \lambda \frac{f(w^{(1)})}{(h(v^{(1)}))^p} = \lambda \left[\frac{f(w^{(1)})}{(h(v^{(1)}))^p} - \frac{f(w^{(0)})}{(h(v^{(0)}))^p} \right] \leqslant 0, \quad \text{in } \Omega, \tag{2.10}$$

as, by (i), $w^{(0)} \leqslant w^{(1)}$, $v^{(1)} \leqslant v^{(0)}$ and f is decreasing. From equations (2.4) and (2.6), it is clearly seen that $\mathscr{B}(w^{(1)}) \leqslant r(x)$ and $\mathscr{B}(v^{(1)}) \geqslant r(x)$ on $\partial\Omega$. Hence, the pair $(w^{(1)}, v^{(1)})$ is an ordered lower–upper pair of solutions to (1.1)–(1.2), which completes the proof. □

Based on the results of Lemma 2.1, we have the following result.

Proposition 2.1 *Let $(w^{(0)}, v^{(0)})$ be an ordered lower–upper pair of solutions to (1.1)–(1.2). Consider $w^{(k)}$ and $v^{(k)}$, $k \geqslant 1$, to be the solutions to*

$$\nabla^2 w^{(k)} + \lambda \frac{f(v^{(k-1)})}{(h(w^{(k-1)}))^p} = 0, \quad \textit{in } \Omega, \tag{2.11}$$

$$\mathscr{B}(w^{(k)}) = \mathscr{B}(w^{(k-1)}) \leqslant r(x), \quad \textit{on } \partial\Omega, \tag{2.12}$$

$$\nabla^2 v^{(k)} + \lambda \frac{f(w^{(k-1)})}{(h(v^{(k-1)}))^p} = 0, \quad \textit{in } \Omega, \tag{2.13}$$

$$\mathscr{B}(v^{(k)}) = \mathscr{B}(v^{(k-1)}) \geqslant r(x), \quad \textit{on } \partial\Omega. \tag{2.14}$$

Then we have

(i) *$(w^{(k)}, v^{(k)}), k \geqslant 1$, is a sequence of lower–upper pair of solutions to (1.1)–(1.2), with*

$$w^{(0)} \leqslant w^{(1)} \leqslant \cdots \leqslant w^{(k-1)} \leqslant w^{(k)} \leqslant v^{(k)} \leqslant v^{(k-1)} \leqslant \cdots \leqslant v^{(1)} \leqslant v^{(0)}.$$

(ii) *The sequences $w^{(k)}$ and $v^{(k)}$ converge uniformly to $\underline{w}$ and $\underline{v}$, respectively, with $\underline{w} \leqslant \underline{v}$ in Ω.*

(iii) *For any solution $u \in (w^{(0)}, v^{(0)})$ of (1.1)–(1.2) we have $\underline{w} \leqslant u \leqslant \underline{v}$.*

Proof The proof of (i) is easily obtained using Lemma 2.1 and induction arguments. For (ii), we have that the sequence $v^{(k)}$ is decreasing monotonically and bounded below by $w^{(0)}$,

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hence convergent. Similarly, the sequence $w^{(k)}$ is increasing monotonically and bounded above by $v^{(0)}$, hence convergent as well. As $w^{(k)} \leqslant v^{(k)}$ for each $k \geqslant 1$, we have $\underline{w} \leqslant \underline{v}$, where $\underline{w} = \lim_{k\to\infty} w^{(k)}$ and $\underline{v} = \lim_{k\to\infty} v^{(k)}$. Moreover, as $w^{(k)}$ and $v^{(k)}$ are continuous functions defined on a compact set Ω, by Dini's theorem [24], the convergence is uniform.

For (iii) we apply induction arguments to show that $w^{(k)} \leqslant u \leqslant v^{(k)}$ for each $k \geqslant 0$. As $u \in (w^{(0)}, v^{(0)})$, then the result is true for $k = 0$. Assuming the result is true for $k-1$ and as f is decreasing, we have

$$\nabla^2(u - w^{(k)}) = \lambda\left(\frac{f(v^{(k-1)})}{(h(w^{(k-1)}))^p} - \frac{f(u)}{(h(u))^p}\right) \leqslant 0,$$

which together with $\mathscr{B}(u - w^{(k)}) = r(x) - \mathscr{B}(w^{(k)}) \geqslant 0$ proves that $w^{(k)} \leqslant u$. Similarly,

$$\nabla^2(u - v^{(k)}) = \lambda\left(\frac{f(w^{(k-1)})}{(h(v^{(k-1)}))^p} - \frac{f(u)}{(h(u))^p}\right) \geqslant 0,$$

which together with $\mathscr{B}(u - v^{(k)}) = r(x) - \mathscr{B}(v^{(k)}) \leqslant 0$ proves that $u \leqslant v^{(k)}$. □

The above proposition states that if we start with an ordered lower–upper pair of solutions to (1.1)–(1.2), then we can generate a convergent sequence of ordered lower–upper pairs of solutions to (1.1)–(1.2). The next proposition provides the circumstances under which the sequences $w^{(k)}$ and $v^{(k)}$ converge to a common limit.

Proposition 2.2 *Let $(w^{(k)}, v^{(k)})$, $k \geqslant 1$, be as in Proposition 2.1 with $\mathscr{B}(w^{(k)}) = \mathscr{B}(v^{(k)}) = r(x)$. Then*

$$\underline{w} = \underline{v},$$

provided that λ is sufficiently small.

Proof Let $\phi = \underline{w} - \underline{v}$. We shall prove that $\phi = 0$. First, note that since $w^{(k)} \leqslant v^{(k)}$, $\forall\, k \geqslant 0$, we have $\underline{w} \leqslant \underline{v}$, or $\phi \leqslant 0$. Besides, the function ϕ satisfies

$$\nabla^2\phi = \lambda\left[\frac{f(\underline{w})}{(h(\underline{v}))^p} - \frac{f(\underline{v})}{(h(\underline{w}))^p}\right], \quad \text{in } \Omega, \tag{2.15}$$

$$\mathscr{B}(\phi) = 0, \quad \text{on } \partial\Omega. \tag{2.16}$$

Equation (2.15) can be written as

$$\nabla^2\phi = \lambda\left[\frac{f(\underline{w}) - f(\underline{v})}{(h(\underline{v}))^p}\right] + \lambda f(\underline{v})\frac{(h(\underline{w}))^p - (h(\underline{v}))^p}{(h(\underline{w})h(\underline{v}))^p}, \quad \text{in } \Omega, \tag{2.17}$$

which, using the mean value theorem, implies that

$$\nabla^2\phi = \lambda\frac{\left[\int_0^1 f'(\underline{w} + s(\underline{v} - \underline{w})ds\right]}{(h(\underline{v}))^p}\phi + \lambda C f(\underline{v})\frac{\int_\Omega \left[\int_0^1 f'(\underline{w} + s(\underline{v} - \underline{w})ds\right]\phi\, dx}{(h(\underline{v})h(\underline{w}))^p}. \tag{2.18}$$

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The constant C above is given by

$$C = \sum_{i=0}^{p-1} (h(\underline{w}))^{p-i-1} (h(\underline{v}))^i > 0,$$

and is obtained by writing $(h(\underline{w}))^p - (h(\underline{v}))^p$ as

$$(h(\underline{w}))^p - (h(\underline{v}))^p = (h(\underline{w}) - h(\underline{v})) \sum_{i=0}^{p-1} (h(\underline{w}))^{p-i-1} (h(\underline{v}))^i.$$

Equation (2.18) with (2.16) can be written equivalently in the operator form as

$$\mathscr{L}(\phi) - \lambda \mathscr{K}(\phi) = 0, \quad \text{in } \Omega, \tag{2.19}$$

$$\mathscr{B}(\phi) = 0, \quad \text{on } \partial\Omega, \tag{2.20}$$

where the operators $\mathscr{K}$ and $\mathscr{L}$ are given by

$$\mathscr{K}(\phi) = \rho(x) \int_\Omega \mu(y)\phi(y)dy, \tag{2.21}$$

$$\mathscr{L}(\phi) = -\nabla^2 \phi + a(x)\phi, \tag{2.22}$$

for

$$\rho(x) = \frac{Cf(\underline{v}(x))}{(h(\underline{v}(x))h(\underline{w}(x)))^p},$$

$$\mu(y) = -\int_0^1 f'(\underline{w}(y) + s(\underline{v}(y) - \underline{w}(y)))ds,$$

$$a(x) = \lambda \frac{\int_0^1 f'(\underline{w}(x) + s(\underline{v}(x) - \underline{w}(x)))ds}{(h(\underline{v}(x)))^p}.$$

Note that the operator $\mathscr{K}$ is positive, as $\mu(y) > 0$ ($f' < 0$) and $\rho(x) > 0$. By using the results of Allegretto and Barabanova existing in [2], see also [15], for any $N \geqslant 1$ there exists $\lambda_0 > 0$ such that $\phi = 0$ when $0 < \lambda < \lambda_0$, by noting that (2.19)–(2.20) is a linear problem. This proves that $\underline{w} = \underline{v} \in [w_0, u_0]$ for any $0 < \lambda < \lambda_0$. □

The result of Proposition 2.2 implies the following existence and uniqueness result.

Theorem 2.1 *If $(w^{(0)}, v^{(0)})$ is an ordered lower–upper pair of solutions to (1.1)–(1.2) as in Definition 2.1, then there exists $\lambda_0 > 0$ such that problem (1.1)–(1.2) has a unique solution $u \in [w^{(0)}, v^{(0)}]$ for any $0 < \lambda < \lambda_0$.*

Proof The proof comes out easily from Propositions 2.1 and 2.2 as follows. By Proposition 2.2, the sequences $w^{(k)}$ and $v^{(k)}$ converge to the same limit, i.e.

$$\lim_{k\to\infty} w^{(k)} = \lim_{k\to\infty} v^{(k)} = \underline{u},$$

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and hence (by (iii) of Proposition 2.1) the problem has at most one solution in $[w^{(0)}, v^{(0)}]$. From equations (2.11)–(2.14), we obtain

$$\begin{aligned} \nabla^2 \underline{u} + \lambda \frac{f(\underline{u})}{(h(\underline{u}))^p} &= 0, && \text{in } \Omega, \\ \mathscr{B}(\underline{u}) &\leqslant r(x), && \text{on } \partial\Omega, \\ \mathscr{B}(\underline{u}) &\geqslant r(x), && \text{on } \partial\Omega, \end{aligned}$$

which imply that $\underline{u}$ satisfies

$$\begin{aligned} \nabla^2 \underline{u} + \lambda \frac{f(\underline{u})}{(h(\underline{u}))^p} &= 0, && \text{in }, \Omega, \\ \mathscr{B}(\underline{u}) &= r(x), && \text{on } \partial\Omega. \end{aligned}$$

Therefore, the common limit $\underline{u}$ is the unique solution to (1.1)–(1.2) in $[w^{(0)}, v^{(0)}]$, provided that $\lambda < \lambda_0$, where $(w^{(0)}, v^{(0)})$ is an ordered lower–upper pair of solutions to (1.1)–(1.2). □

In the case where f is decreasing, problem (1.1)–(1.2) with the homogeneous Dirichlet boundary conditions has a unique solution for λ sufficiently small.

Lemma 2.2 *[20] The solution curve to problem (1.1)–(1.2) depicted by points (λ, u_λ) does not exhibit any bending in a neighbourhood of $(0,0)$, i.e. there exists $\lambda_1 > 0$ such that (1.1)–(1.2) with $r(x) = 0$ has a unique solution for any $0 < \lambda < \lambda_1$.*

Now Lemma 2.2 implies.

Theorem 2.2 *Let $(w^{(0)}, v^{(0)})$ be an ordered lower–upper pair of solutions to (1.1)–(1.2) and $\lambda_0, \lambda_1 > 0$ be as in Theorem 2.1 and Lemma 2.2. Then the solution $u = \underline{w} = \underline{v} \in [w^{(0)}, v^{(0)}]$ is the unique solution of problem (1.1)–(1.2) with the homogeneous Dirichlet boundary conditions, provided that $0 < \lambda < \min\{\lambda_0, \lambda_1\}$.*

2.2 The case f is increasing

In the case where f is increasing, we modify the definition of lower–upper pair of solutions to the following.

Definition 2.2 *A pair of functions $(w, v) \in (C^2(\Omega) \cap C(\overline{\Omega}))^2$ is called a lower–upper pair of solutions to (1.1)–(1.2) if they satisfy the following inequalities:*

$$\nabla^2 w + \lambda \frac{f(w)}{(h(v))^p} \geqslant 0, \quad \text{in } \Omega, \quad \mathscr{B}(w) \leqslant r(x) \quad \text{on } \partial\Omega, \tag{2.23}$$

$$\nabla^2 v + \lambda \frac{f(v)}{(h(w))^p} \leqslant 0, \quad \text{in } \Omega, \quad \mathscr{B}(v) \geqslant r(x) \quad \text{on } \partial\Omega. \tag{2.24}$$

If $w \leqslant v$ in Ω, we say that the pair (w, v) is an *ordered* lower–upper pair of solutions to (1.1)–(1.2).

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With this new definition of lower–upper of solutions (w, v), Proposition 2.1 is modified to the following.

Proposition 2.3 *Consider the non-local elliptic equation (1.1)–(1.2). Let $(w^{(0)}, v^{(0)})$ be an ordered lower–upper pair of solutions to (1.1)–(1.2), as in Definition 2.2. Let $w^{(k)}$ and $v^{(k)}$, $k \geqslant 1$, be the solutions to*

$$\nabla^2 w^{(k)} + \lambda \frac{f(w^{(k-1)})}{(h(v^{(k-1)}))^p} = 0, \quad \text{in } \Omega, \tag{2.25}$$

$$\mathscr{B}(w^{(k)}) = \mathscr{B}(w^{(k-1)}) \leqslant r(x), \quad \text{on } \partial\Omega, \tag{2.26}$$

$$\nabla^2 v^{(k)} + \lambda \frac{f(v^{(k-1)})}{(h(w^{(k-1)}))^p} = 0, \quad \text{in } \Omega, \tag{2.27}$$

$$\mathscr{B}(v^{(k)}) = \mathscr{B}(v^{(k-1)}) \geqslant r(x), \quad \text{on } \partial\Omega. \tag{2.28}$$

Then we have

(i) *$(w^{(k)}, v^{(k)})$, $k \geqslant 1$, is a sequence of lower–upper pair of solutions to (1.1)–(1.2), with*

$$w^{(0)} \leqslant w^{(1)} \leqslant \cdots \leqslant w^{(k)} \leqslant v^{(k)} \leqslant \cdots \leqslant v^{(1)} \leqslant v^{(0)}, \quad k \geqslant 1.$$

(ii) *The sequences $w^{(k)}$ and $v^{(k)}$ converge uniformly to $\underline{w}$ and $\underline{v}$, respectively, with $\underline{w} \leqslant \underline{v}$ in Ω.*

(iii) *For any solution $u \in (w^{(0)}, v^{(0)})$ of (1.1)–(1.2) we have $\underline{w} \leqslant u \leqslant \underline{v}$.*

Proof The proof of the above proposition follows along the same lines as the proof of Proposition 2.1, utilising the new definition of lower–upper pair of solutions in Definition 2.2 and the fact that f is increasing. □

The sequences $w^{(k)}$ and $v^{(k)}$ still converge to the same limit, i.e. $\underline{w} = \underline{v}$ for λ small enough and an analogous to Proposition 2.2 holds. Note that in this case the operator $\mathscr{K}$ is not a positive operator any more; however, the results in [2,15] are still applicable. This implies, as in Proposition 2.1, the following.

Theorem 2.3 *If $(w^{(0)}, v^{(0)})$ is an ordered lower–upper pair of solutions to (1.1)–(1.2) as in Definition 2.2, then there exists $\lambda_2 > 0$ such that problem (1.1)–(1.2) has a unique solution $u \in [w^{(0)}, v^{(0)}]$ for any $0 < \lambda < \lambda_2$.*

In this case we can also prove that (1.1)–(1.2) associated with the homogeneous Dirichlet boundary conditions has a unique solution for λ sufficiently small. We will need an ordering result, which is provided by the following.

Lemma 2.3 *The solutions in the minimal branch of the local problem*

$$\nabla^2 u + \mu f(u) = 0, \quad \text{in } \Omega, \tag{2.29}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{2.30}$$

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where $\mu = \lambda/(h(u))^p$, *are ordered with respect to the local parameter* μ, *in the sense that*

$$\mu_1 < \mu_2 \quad \textit{implies} \quad u_1(x) = u(x;\mu_1) < u_2(x) = u(x;\mu_2) \quad \textit{for every} \quad x \in \Omega. \tag{2.31}$$

Proof By differentiating equation (2.29) with respect to μ we obtain

$$\begin{aligned} -\nabla^2 z - \mu f'(u) z &= f(u) > 0, \quad x \in \Omega, \\ z(x;\mu) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

where $z = \frac{\partial u}{\partial \mu}$. Note that in the minimal branch $C(x;\mu) = -\mu f'(u(x))$ is bounded in $\overline{\Omega}$, hence the desired result is derived using a refined form of Hopf's Lemma, see also [13, p. 519]. □

Lemma 2.4 *Let* Ω *be a star-shaped domain of* $\mathbb{R}^N$, $N \geqslant 3$. *Assume also that* f *is a* C^1*-super linear function with* $f(0) > 0$, *then there exists* λ_* *such that (1.1)–(1.2) with* $r(x) = 0$ *has a unique solution for any* $0 < \lambda < \lambda_*$.

Proof By Theorem 1 in [25] (see also [21, Theorem 8]), there exists a positive constant μ_* such that local problem (2.29)–(2.30) has a unique solution for $0 < \mu < \mu_*$. On the other hand, any solution $u = u(x;\lambda)$ to (1.1)–(1.2) solves (2.29)–(2.30) with

$$\mu = \frac{\lambda}{h^p(u)} \leqslant \frac{\lambda}{\big(f(0)|\Omega|\big)^p},$$

as f is increasing. We claim that problem (1.1)–(1.2) has a unique solution for $0 < \lambda < \lambda_* := \mu_*(f(0)|\Omega|)^p$. Indeed, let us assume that problem (1.1)–(1.2) has two distinct solutions u_1 and u_2 for some $0 < \lambda < \lambda_*$. We claim that

$$\int_\Omega f(u_1)\,dx = \int_\Omega f(u_2)\,dx. \tag{2.32}$$

Otherwise, if for example,

$$\int_\Omega f(u_1)\,dx < \int_\Omega f(u_2)\,dx,$$

then we have

$$\frac{\lambda}{(f(0)|\Omega|)^p} \geqslant \mu_1 := \frac{\lambda}{(h(u_1))^p} > \mu_2 := \frac{\lambda}{(h(u_2))^p}. \tag{2.33}$$

As $0 < \lambda < \lambda_* = \mu_*(f(0)|\Omega|)^p$, we obtain $0 < \mu_2 < \mu_1 < \mu_*$ by (2.33), hence $u_1(x) = u(x;\mu_1)$ and $u_2(x) = u(x;\mu_2)$ are minimal solutions to the local problem (2.29)–(2.30). It follows by the monotonicity property (2.31) for the minimal solution branch of (2.29)–(2.30) that $u_1(x) > u_2(x)$ for $x \in \Omega$. This implies that

$$\int_\Omega f(u_1)\,dx > \int_\Omega f(u_2)\,dx,$$

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which is a contradiction. Similarly, we can exclude the possibility of

$$\int_\Omega f(u_1)\,dx > \int_\Omega f(u_2)\,dx,$$

hence (2.32) is established.

Now (2.32) implies that the local problem (2.29)–(2.30) has two distinct solutions u_1 and u_2 corresponding to same μ, which contradicts to [25, Theorem 1] and [21, Theorem 8]. Consequently, we derive that (1.1)–(1.2) has a unique solution for $0 < \lambda < \lambda_*$ and the proof is complete. □

Remark 2.2 Lemma 2.4 is also valid for a general symmetric domain, since the existence of μ^* such that local problem (2.29)–(2.30) has a unique solution in the range $(0, \mu^*)$ is guaranteed by Theorem 1 in [25]. A different proof of Lemma 2.4 can be found in [11, Corollary 5].

Again, Lemma 2.4 guarantees the following.

Theorem 2.4 *The solution constructed in Theorem 2.3 is actually the only solution of (1.1)–(1.2) for* $0 < \lambda < \min\{\lambda_*, \lambda_2\}$, *when* Ω *is a star-shaped domain in* $\mathbb{R}^N$, $N \geqslant 3$.

The next section presents some numerical examples to illustrate the effectiveness of the algorithm.

3 Numerical examples

In order to illustrate the convergence of the monotone sequences of lower–upper pairs of solutions derived in this paper, we present some examples for the slab and spherical geometries. We note that the algorithm rests on finding an initial lower–upper pair of solutions $(w^{(0)}, v^{(0)})$ to the problem. For all examples, we measure the error at each iteration by the formula

$$E(k) = \int_\Omega (v^{(k)} - w^{(k)})^2\,dx, \tag{3.1}$$

where $(w^{(k)}, v^{(k)})$ is the generated lower–upper pair of solutions at iteration k. We also approximate numerically the order of convergence, $\alpha = \lim_{k\to\infty} \alpha(k)$, of the algorithm, using

$$\alpha(k) = \log_{10}\left(\frac{E(k)}{E(k+1)}\right). \tag{3.2}$$

The iterates $w^{(k)}$ and $v^{(k)}$, $k \geqslant 1$, are solved numerically for using the built-in function NDSolve in the software package Mathematica.

Example 3.1 *Consider the non-local elliptic equation*

$$u_{xx} + \lambda \frac{e^{-u}}{\int_0^1 e^{-u}dx} = 0, \qquad 0 < x < 1, \tag{3.3}$$

with the homogenous Dirichlet boundary conditions $u(0) = u(1) = 0$.

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As f is decreasing, we use Definition 2.1 to construct an initial lower–upper pair of solutions, $(w^{(0)}, v^{(0)})$. We let $w^{(0)}$ and $v^{(0)}$ to be polynomials of degree 2 that satisfy the same boundary conditions as u. Let

$$w^{(0)} = A(x - x^2) \qquad \text{and} \qquad v^{(0)} = B(x - x^2),$$

where A and B are some constants to be determined. In order for $(w^{(0)}, v^{(0)})$ to be an ordered lower–upper pair solutions to (3.3), they must satisfy

$$P(w^{(0)}) = w_{xx}^{(0)} + \lambda \frac{e^{-v^{(0)}}}{\int_0^1 e^{-w^{(0)}} dx} = -2A + \lambda \frac{e^{-B(x-x^2)}}{\int_0^1 e^{-A(x-x^2)} dx} \geqslant 0$$

and

$$P(v^{(0)}) = v_{xx}^{(0)} + \lambda \frac{e^{-w^{(0)}}}{\int_0^1 e^{-v^{(0)}} dx} = -2B + \lambda \frac{e^{-A(x-x^2)}}{\int_0^1 e^{-B(x-x^2)} dx} \leqslant 0,$$

with $B \geqslant A > 0$.

Using that facts $-\frac{B}{4} \leqslant -B(x - x^2) \leqslant 0$ and $-\frac{A}{4} \leqslant -A(x - x^2) \leqslant 0$, we have

$$P(w^{(0)}) = -2A + \lambda \frac{e^{-B(x-x^2)}}{\int_0^1 e^{-A(x-x^2)} dx} \geqslant -2A + \lambda e^{-B/4}$$

and

$$P(v^{(0)}) = -2B + \lambda \frac{e^{-A(x-x^2)}}{\int_0^1 e^{-B(x-x^2)} dx} \leqslant -2B + \lambda e^{B/4}.$$

Now, let $B \geqslant A > 0$ be solutions (if any) for the system

$$-2A + \lambda e^{-B/4} = 0, \tag{3.4}$$

$$-2B + \lambda e^{B/4} = 0, \tag{3.5}$$

then $w^{(0)} = A(x-x^2)$ and $v^{(0)} = B(x-x^2)$ are ordered lower–upper pair of solutions to (3.3). We consider $\lambda = 1$. Equation (3.5) has two solutions, $B_1 = 0.577685$ and $B_2 = 13.0467$. Then equation (3.4), accordingly, has two solutions, $A_1 = 0.432761$ and $A_2 = 0.0191619$. Thus, we have two possible starting lower–upper of solutions. We choose the solution for $A_1 = 0.432761$ and $B_1 = 0.577685$ because $w^{(0)}$ and $v^{(0)}$ are closer.

A plot of the generated $w^{(k)}$ and $v^{(k)}$ for $k = 0, 1, 2, 3$ is shown in Figure 1, where we can clearly see the fast convergence of algorithm. We note that for this example just after four iterations the error $E(k)$ is of the order 10^{-10}. Table 1 presents the error $E(k)$ at iteration k. The approximated order of convergence $\alpha(k)$ is also displayed in Table 1, from which we can argue that the order of convergence is 1.5.

Example 3.2 *Consider the non-local elliptic equation*

$$u_{xx} + \frac{f(u)}{\int_0^1 f(u) dx} = 0, \quad 0 < x < 1, \tag{3.6}$$

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Table 1. *Error $E(k)$ and $\alpha(k)$, $k = 0, 1, \ldots 6$, for Example 1*

Iter. k	$E(k)$	$\alpha(k)$
0	0.000700098	1.49368
1	0.0000224634	1.49521
2	7.18232×10^{-7}	1.49540
3	2.29545×10^{-8}	1.49541
4	7.33607×10^{-10}	1.49541
5	2.34454×10^{-11}	1.49541
6	7.49292×10^{-13}	1.49541

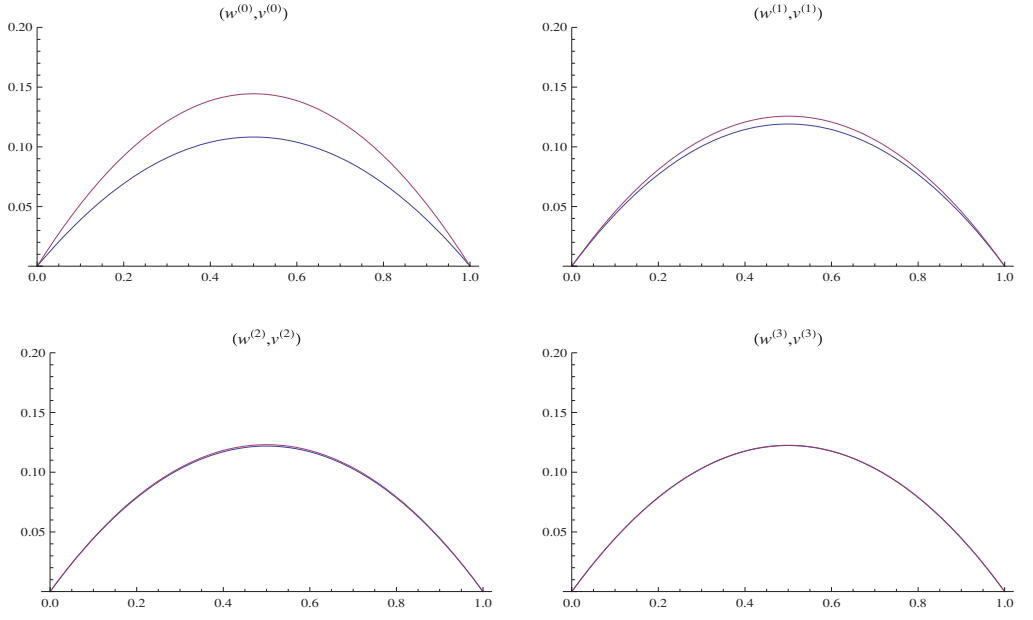


FIGURE 1. (Colour online) A plot of $w^{(k)}$ and $v^{(k)}$, $k = 0, 1, 2, 3$, for Example 1.

with the nonhomogeneous Dirichlet boundary conditions

$$u(0) = 3, \quad u(1) = 1, \tag{3.7}$$

where

$$f(u) = \frac{1}{(1+u)^2}.$$

We start by finding an initial lower–upper pair of solutions $(w^{(0)}, v^{(0)})$. It can be easily verified that

$$w^{(0)}(x) = 3 - 9x + 7x^2, \tag{3.8}$$

$$v^{(0)}(x) = 3 + 5x - 7x^2, \tag{3.9}$$

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Table 2. *Error $E(k)$ and $\alpha(k)$, $k = 0, 1, \ldots 6$, for Example 2*

Iter. k	$E(k)$	$\alpha(k)$
0	6.53333	1.14261
1	0.470467	1.97043
2	0.00503619	1.85278
3	0.0000706847	1.85041
4	9.97509×10^{-7}	1.85025
5	1.40821×10^{-8}	1.85023
6	1.98809×10^{-10}	1.85023

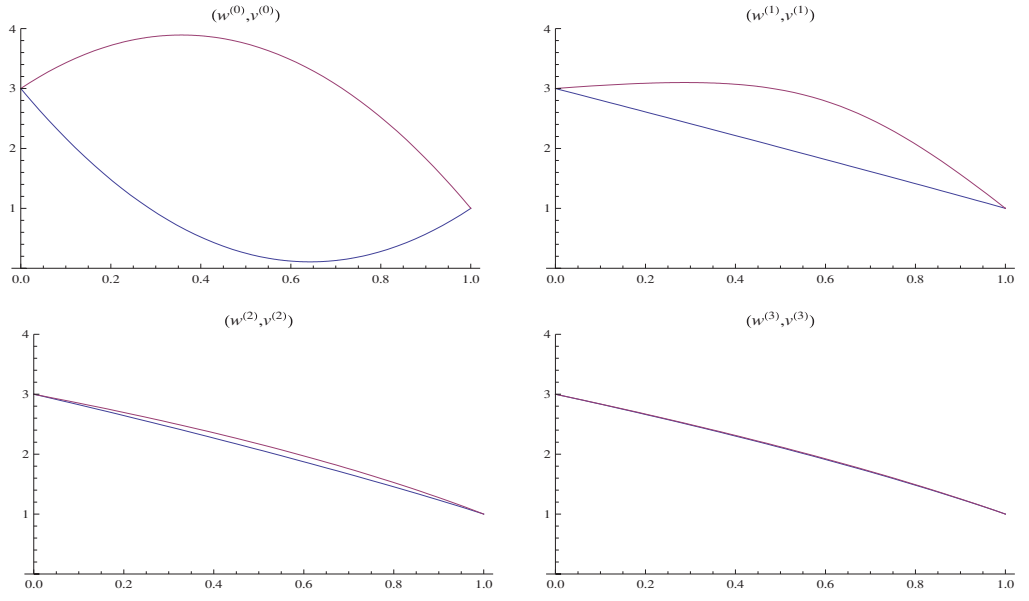


FIGURE 2. (Colour online) A plot of $w^{(k)}$ and $v^{(k)}$, $k = 0, 1, 2, 3$, for Example 2.

constitute a lower–upper pair of solutions to (3.6)–(3.7). Note that there are other possibilities to choose $w^{(0)}$ and $v^{(0)}$. Here we chose them as in (3.8)–(3.9) with $w^{(0)}$ concave up and $v^{(0)}$ concave down as depicted in Figure 2 (top left).

Plots of the generated sequences $w^{(k)}$ and $v^{(k)}$, $k = 1, 2, 3$, are shown in Figure 2. The error $E(k)$ as well as the approximate order of convergence $\alpha(k)$ versus iteration number k are displayed in Table 2. The results in Table 2 show that the iterative algorithm converges fast with an order of convergence of almost 2. The algorithm is also very accurate as depicted in Table 2, where the error $E(k)$ is of the order of 10^{-10} after just six iterations. We remark that for Example 3.2, the initial lower–upper pair of solutions $(w^{(0)}, v^{(0)})$ are farther apart than those for Example 3.1. This explains why it takes more iterations in Example 3.2 than in Example 3.1 to achieve the same order of accuracy.

The next example illustrates the algorithm for the case f is increasing.

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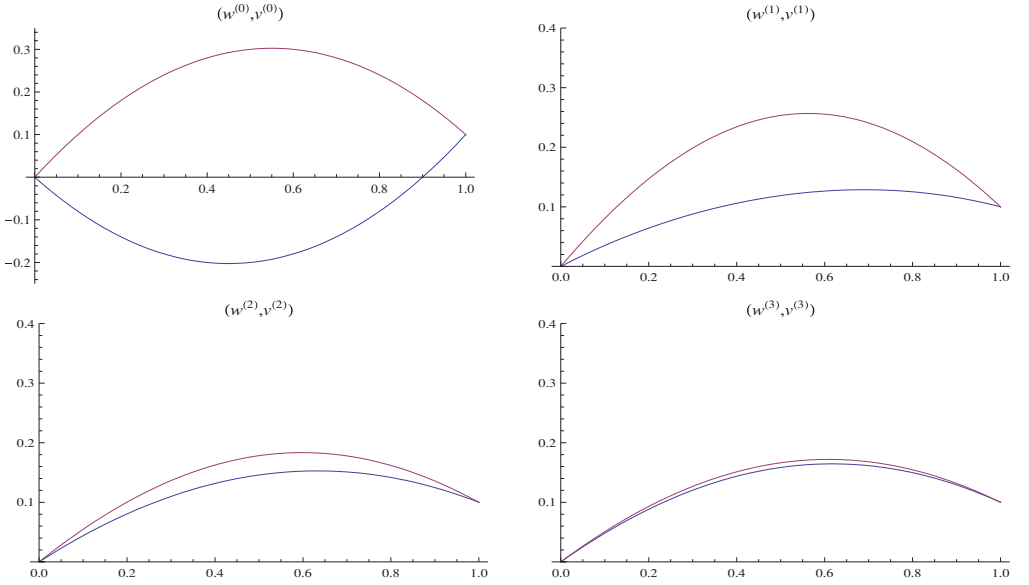


FIGURE 3. (Colour online) A plot of $w^{(k)}$ and $v^{(k)}$, $k = 0, 1, 2, 3$, for Example 3.

Example 3.3 *In this example we consider the non-local elliptic equation*

$$u_{xx} + \frac{f(u)}{\left(\int_0^1 f(u)dx\right)^2} = 0, \quad 0 < x < 1, \tag{3.10}$$

$$u(0) = 0, \quad u(1) = 0.1, \tag{3.11}$$

where $f(u) = e^u$.

Using Definition 2.2, an initial lower–upper pair of solutions to (3.10)–(3.11) could be

$$w^{(0)}(x) = x^2 - \frac{9}{10}x, \tag{3.12}$$

$$v^{(0)}(x) = -x^2 + \frac{11}{10}x. \tag{3.13}$$

The results of the algorithm as applied to this example are displayed in Figure 3 and Table 3. Figure 3 clearly shows the fast convergence of the sequences $w^{(k)}$ and $v^{(k)}$ to the same limit. Table 3 displays in more details the error $E(k)$ and the order of convergence $\alpha(k)$ versus iteration k, for $k = 0, 1, \ldots 8$.

Example 3.4 *In this example we consider the non-local elliptic equation in spherical geometry* (r, θ, ϕ).

$$\nabla^2 u + \lambda \frac{f(u)}{h(u)} = 0, \quad \text{in } \Omega, \tag{3.14}$$

$$u_r(r = 0) = 0, \; u(r = 1) = 0, \tag{3.15}$$

where $\lambda > 0$, $f(u) = e^{-u}$ *and* $h(u) = \int_\Omega f(u)\, dV$.

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Table 3. *Error $E(k)$ and $\alpha(k)$, $k = 0, 1, \ldots, 8$, for Example 3*

Iter. k	$E(k)$	$\alpha(k)$
0	0.133333	1.15088
1	0.00942028	1.23811
2	0.000544441	1.23197
3	0.000031914	1.23224
4	1.86956×10^{-6}	1.23221
5	1.09528×10^{-7}	1.23221
6	6.41671×10^{-9}	1.23221
7	3.75922×10^{-10}	1.23221
8	2.20234×10^{-11}	1.23221

Here the domain Ω is the unit sphere. We assume that the solution u depends only on radius r. As u depends only on r, we have

$$\nabla^2 u = u_{rr} + \frac{2}{r} u_r. \tag{3.16}$$

As mentioned earlier, to start the algorithm we need an initial lower–upper solution pair $w^{(0)}$ and $v^{(0)}$ satisfying the boundary conditions (3.15). Let

$$w^{(0)} = a(1 - r^2), \qquad v^{(0)} = b(1 - r^2)$$

with some constants $0 < a \leqslant b$ to be determined such that

$$\nabla^2 (w^{(0)}) + \lambda \frac{f(v^{(0)})}{h(w^{(0)})} \geqslant 0, \tag{3.17}$$

$$\nabla^2 (v^{(0)}) + \lambda \frac{f(w^{(0)})}{h(v^{(0)})} \leqslant 0. \tag{3.18}$$

First we compute $h(w^{(0)})$ and $h(v^{(0)})$. Using spherical coordinates ($dV = r^2 \sin(\phi) dr d\phi d\theta$), we have

$$\begin{aligned} h(w^{(0)}) = \int_\Omega e^{-a(1-r^2)} dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^2 e^{-a(1-r^2)} \sin(\phi)\, dr\, d\phi\, d\theta \\ &= 4\pi e^{-a} \int_0^1 r^2 e^{ar^2} dr. \end{aligned} \tag{3.19}$$

Similarly,

$$h(v^{(0)}) = 4\pi e^{-b} \int_0^1 r^2 e^{br^2} dr. \tag{3.20}$$

Then, using the fact that

$$\frac{e^\alpha - 1}{3\alpha} = \int_0^1 r^2 e^{\alpha r^3} dr \leqslant \int_0^1 r^2 e^{\alpha r^2} dr \leqslant \int_0^1 r e^{\alpha r^2} dr = \frac{e^\alpha - 1}{2\alpha},$$

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we obtain the following bounds:

$$\frac{4\pi}{3a}(1-e^{-a}) \leqslant h(w^{(0)}) \leqslant \frac{2\pi}{a}(1-e^{-a}), \tag{3.21}$$

$$\frac{4\pi}{3b}(1-e^{-b}) \leqslant h(v^{(0)}) \leqslant \frac{2\pi}{b}(1-e^{-b}). \tag{3.22}$$

Now,

$$\nabla^2(w^{(0)}) + \lambda\frac{f(v^{(0)})}{h(w^{(0)})} = -6a + \lambda\frac{e^{-b(1-r^2)}}{h(w^{(0)})} \geqslant 6a\left[-1 + \frac{\lambda}{12\pi}\frac{e^{-b}}{1-e^{-a}}\right],$$

$$\nabla^2(v^{(0)}) + \lambda\frac{f(w^{(0)})}{h(v^{(0)})} = -6b + \lambda\frac{e^{-a(1-r^2)}}{h(v^{(0)})} \leqslant 6b\left[-1 + \frac{\lambda}{8\pi}\frac{1}{1-e^{-b}}\right].$$

In order for (3.17) and (3.18) to be satisfied, we require that

$$\frac{\lambda}{12\pi}\frac{e^{-b}}{1-e^{-a}} \geqslant 1, \tag{3.23}$$

$$\frac{\lambda}{8\pi}\frac{1}{1-e^{-b}} \leqslant 1. \tag{3.24}$$

As $b > 0$, (3.24) is satisfied for

$$b \geqslant -\ln\left(1 - \frac{\lambda}{8\pi}\right), \tag{3.25}$$

provided $\lambda \leqslant 8\pi$. Setting $b = -\ln\left(1 - \frac{\lambda}{8\pi}\right)$, we find that (3.23) is satisfied for

$$a \leqslant -\ln\left(1 - \frac{\lambda}{12\pi}\left(1 - \frac{\lambda}{8\pi}\right)\right). \tag{3.26}$$

Choosing

$$a = -\ln\left(1 - \frac{\lambda}{12\pi}\left(1 - \frac{\lambda}{8\pi}\right)\right) \qquad \text{and} \qquad b = -\ln\left(1 - \frac{\lambda}{8\pi}\right),$$

we have for any $0 < \lambda < 8\pi$, $w^{(0)} = a(1-r^2)$ and $v^{(0)} = b(1-r^2)$ constitute a lower–upper pair of solutions to (3.14)–(3.15).

For numerical simulations, we chose $\lambda = 5$. Then $a = 0.112322$ and $b = 0.221824$. Plots of $w^{(0)}$ and $v^{(0)}$, as well as those of the generated upper and lower solutions $w^{(k)}$ and $v^{(k)}$, respectively, are shown in Figure 4. As in previous examples, Figure 4 clearly shows the fast convergence of the sequences $w^{(k)}$ and $v^{(k)}$ to the common limit, with an approximate order of convergence $\alpha = 1.5$ as seen in Table 4. Table 4 also displays the error $E(k)$ versus iteration k where it is clear that after just five iterations the error is of the order of 10^{-10}.

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Table 4. *Error $E(k)$ and $\alpha(k)$, $k = 0, 1, \ldots, 6$, for Example 4*

Iter. k	$E(k)$	$\alpha(k)$
0	0.0114804	1.44786
1	0.000409354	1.46332
2	0.0000140856	1.46407
3	4.83844×10^{-7}	1.4643
4	1.66113×10^{-8}	1.46434
5	5.70252×10^{-10}	1.46434

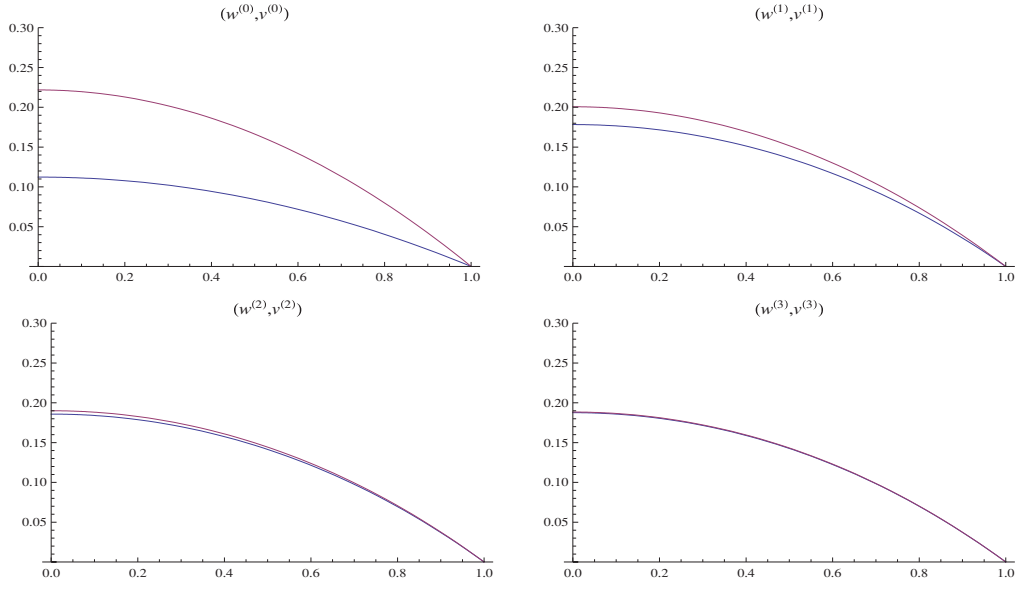


FIGURE 4. (Colour online) A plot of $w^{(k)}$ and $v^{(k)}$, $k = 0, 1, 2, 3$, for Example 4.

4 Concluding remarks

We have applied comparison arguments to study a class of non-local elliptic equations associated with the Dirichlet-type boundary conditions. We discussed the problem for two cases of the non-linear term $f(u)$: monotone decreasing and increasing. For each case we defined the notion of lower–upper pairs of solutions for the studied problem. We then generated sequentially one monotone sequence of lower–upper pairs of solutions, $\{(w^{(i)}, v^{(i)})\}_{i=1}^{\infty}$. The first component $\{w^{(i)}\}_{i=1}^{\infty}$ is monotone increasing and bounded above by $v^{(0)}$ and hence convergent. On the other hand, the second component $\{v^{(i)}\}_{i=1}^{\infty}$ is also convergent as it is monotone decreasing and bounded below by $w^{(0)}$. The maximum principle has been applied to prove the monotonicity of two sequences. We proved that $\{w^{(i)}\}_{i=1}^{\infty}$ and $\{v^{(i)}\}_{i=1}^{\infty}$ converge towards the same limit and hence the existence and uniqueness of solution to problem (1.1)–(1.2) is established for some range of the parameter λ. We note that similar arguments can be applied to prove the existence of

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the same class of non-local problems under the Robin-type conditions. We have tested the constructed algorithm numerically through four applications; the first two and the third one focus on the slab geometry with f decreasing and increasing respectively, while in the last application the sphere geometry is examined when f is decreasing. It is noticed that the iteration scheme converges very fast to the actual solution and the order of convergence (approximated numerically) is around 1.5. Finally, the new approach enriches the literature of analytical and numerical study of the problem. For the analytical part we proved the existence and uniqueness of the solution. Numerically, we developed an accurate numerical scheme to obtain an approximate solution to the problem.

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